ON AN IMPATIENT CONSUMER QUEUE WITH SECONDARY SERVICE, MULTIPLE VACATIONS AND SERVER BREAKDOWNS

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Abstract

This study presents a limited buffer secondary service queue with multiple vacations and server breakdowns. The model under consideration includes two types of impatient policies: balking and reneging. After the completion of the essential primary service, only few consumers choose to proceed with secondary service with a certain probability. During the active period of the server, it is subject to breakdown and the broken down server is immediately sent for repair. Further, the server will go on vacation as soon as there are no waiting consumers in the queue. On returning from a vacation, if the system is still empty the server leaves for another vacation and continues to do so until atleast one consumer is found at a vacation termination epoch. The model is analyzed under steady-state conditions and the explicit expressions of various performance indices are evaluated. A few numerical results illustrate how the model parameters have an effect on the performance metrics.

Keywords: Balking, reneging, secondary service, multiple vacations, breakdown, repair.

1. Introduction

Vacation models address a very significant category of the real-world congestion scenarios that are seen in both day-to-day living and in industrial settings. In the context of queueing, the period of time during which the server is not available is referred to as a vacation. During the active period, the server operates at maximum capacity, but while it is on vacation, it does not carry out any tasks. The numerous adaptable implementations encourage us to investigate queuing systems with server vacations that can be exploited in some beneficial method when coping with congestion problems in various frameworks. These systems may be used in a variety of contexts. Over the course of more than two decades, number of scholars and practitioners have investigated vacation models of many sorts. Their goals have been either to find solutions to specific queueing issues at hand or to acquire a knowledge of the stochastic processes that evolve as a result of these models. Excellent studies on these vacation models have been done by Doshi [8, 9], Takagi [21], Tian and Zhang [22], Jau-Chaun Ke et al. [15], Panta et al. [18], etc.

Consumers typically have less patience when waiting for service because they value their time. In the research that has been done on queueing, impatience of consumers has been examined mostly in the context of consumers abandoning the queue because of either a prolonged wait that they had previously experienced or a lengthy wait that they predicted would occur upon arrival. In many service or operational contexts, such impatience is frequently noticed through the acts of consumers "balking" or "reneging" from waiting in a queue. Altman and Yechiali [1] undertook an analysis of consumer dissatisfaction in server vacation queues. A study on priority queues with impatient consumers has been presented by Foad and Baris [10]. Yue et al. [27] conducted research on the effects of synchronised vacations and impatient consumers in a multi-server queue. Jamol Pender [14] computed a novel approximation for single-server queues with abandonment based on the truncated normal distribution. Ammar [3] came up with the time dependent results of an $M/M/1$ vacation queue that included an anticipating server and impatient consumers. Sampath and Liu [19] conducted the research to determine how impatience of customers affected the performance of an $M/M/1$ queueing system subject to waiting server and differential vacations. A Bernoulli feedback queueing system with K-variant vacations, waiting server and impatient consumers has been considered by Amina and Guendouzi [2]. Mathematical evaluation of the $M/M/C$ vacation queuing model with a waiting server and dissatisfied consumers has been carried out by Ganesh and Ghimire [11].

Numerous malfunctions in the service providing facility are a major source of service interruptions in various manufacturing processes. In such circumstances, service will not be provided to the waiting consumers until the service providing facility is repaired. Such breakdowns in service are typical in commercial settings like factories and phone booths, as well as in the use of mechanical technologies like electronic computers. William et al. [26] examined a queueing model with vacations where the service station may experience a breakdown while it is in operation. Queuing system with fixed capacity and vacancies as well as server breakdowns has been dealt by Ghimire and Ritu [12]. An $M^X/G/1$ queue with server breakdowns and repairs has been analyzed by Djamila et al. [7]. Hanumantha Rao et al. [13] has studied an impatient consumer two-phase queuing system with server breakdowns and delayed repair. Under the T-policy, the investigation of two phase queue with breakdowns and vacations has been carried out by Khalid and Lotfi [17]. A bulk service queue with server breakdowns and repairs had been investigated by Bharathidass et al. [4]. Srinivas et al. [20] researched a server breakdown queueing system with repairs and vacations.

In the modelling of a wide variety of congestion issues that arise during real-life activities, queueing systems that include the provision of a secondary service (SS) play an essential role. In queueing models that include SS, the server offers the primary service (PS) to all of the arriving consumers. However, after the PS has been completed, only few consumers choose to receive SS, according to a predetermined probability. Take the banking sector as an illustration; among the primary duties imposed upon any bank are the receipt and dispensing funds in the form of deposits, withdrawals, loans, advances, etc. Printing of passbooks, issue of checks, lockers, etc., fall under the category of secondary services that banks conduct in addition to their primary duties. Kalyanaraman and Pazhani [16] analyzed a single server queue with optional service and server vacations. Uma and Punniyamoorthy [23] have investigated a single-server bulk queue with vacations, balking and secondary service. A bulk arrival queue with SS and server breakdowns has been researched by Charan and Sandeep [5]. Charan et al. [6] have studied the effects of secondary services and service disruptions on bulk queues. The time dependent behaviour of a bulk service queueing system with optional service and impatient consumers has been studied by Vijaya Laxmi and Andwilile [24]. A Markovian secondary service queue operating under triadic policy has been considered by Vijaya Laxmi et al. [25].

The current article deals with a finite capacity Markovian queue with secondary service, multiple vacations, breakdowns and impatient consumers. The paradigm under consideration has many real-world uses in places as diverse as communications networks, manufacturing systems, cloud computing, customer service centres, etc. Consider the following example of a call centre that provides services to customers. The call centre employs customer support agents to handle inbound calls received at the call centre from customers seeking assistance. Customers after receiving the required assistance and are prepared to pay an extra price can take advantage of the premium service option that is available through the contact centre. During times of low call volume or off-peak hours, it is possible that the server can schedule for a vacation during which they will be offline in order to conserve money and resources. Occasionally, there may be brief disruptions in service as a result of server experiencing technical problems or being required to undergo maintenance. It is always possible that customers contacting the call centre may be impatient and reluctant to wait in the queue for lengthy periods of time. In this practical application, the call centre aims to manage server breakdowns effectively, offers an optional service to customers who value quicker support and allows server vacations during low demand to optimize resource usage. By doing so, the call centre enhances the overall customer experience and reduces the likelihood of impatient customers abandoning the queue.

Owing to the practical application as one mentioned above, we study an $M/M/1/N$ multiple vacation queue with secondary service, breakdowns and impatient consumers. The vacation durations, secondary service durations and breakdown times are assumed to be follow exponential distribution. Balking and reneging are the two forms of consumer impatience which have been included in the current article. Both the forms of consumer impatience are considered to be state dependent. Using iterative approach, the model's steady-state results are achieved. The expected system size, expected balking rate, expected reneging rate and other performance parameters are reported. Through a limited number of numerical experiments, the parameter influence on the performance indices is demonstrated.

The remaining sections of the paper are structured as follows. A detailed explanation of the model has been provided in Section 2. In Section 3, we reported the results of the steady-state model under discussion. Section 4 provides different metrics by which the model's effectiveness may be evaluated. In Section 5, some numerical findings illustrating the impact of the model parameters on the performance metrics are shown and in Section 6, conclusions are drawn.

2. Model Overview

Consider an impatient consume $M/M/1/N$ queue with multiple vacations, secondary service, and server breakdowns.

- Consumers arrive one at a time according to Poisson process with rate λ. Arriving consumers make a decision whether to be a part of the queue or not depending on the queue size. Let b_n be the probability of joining the queue and $1 - b_n$ be the probability of not joining the queue, where n denotes the number of consumers in the system. We also, assume that $b_0 = 1, b_{n+1} \leq b_n$ and $b_N = 0$.
- Consumers who join the queue wait for certain period of time, T, which follows exponential distribution with parameter α. If the service does not start before this time, he may leave due to impatience. The average reneging rate of a consumer is taken as $(n-1)\alpha$ $(n \geq 0)$, where n represents the number of consumers in the system.
- Consumers who enter the system are served according to FCFS discipline by a single server. All the consumers receive a primary service (PS) and exit from the system with probability ω while only few consumers may opt for secondary service (SS) with probability $\bar{\omega} = 1 - \omega$. The service durations during PS and SS follow exponential distribution with parameters μ_1 and μ_2, respectively.
- Under the multiple vacation policy, the server will take a series of breaks in the form of vacations in between two consecutive busy times, and it will continue to do so until it locates a waiting consumer in the system. The vacation durations are also assumed to follow exponential distribution with parameter σ.
- The server is subject to breakdown both during PS and SS with rate β. The broken down server is immediately sent for repair. The repair times are exponentially distributed with rate δ.

3. Steady-state analysis

At steady-state, let

- $\pi_{n,0}$ - Probability of n consumers in the system and the server in vacation,
- $\pi_{n,1W}$ - Probability of n consumers in the system and server in working state during PS,
- $\pi_{n,1B}$ - Probability of n consumers in the system and server in breakdown state during PS,
- $\pi_{n,2W}$ - Probability of n consumers in the system and server in working state during SS,
- $\pi_{n,2B}$ - Probability of n consumers in the system and server in breakdown state during SS.

Using the Markov theory, the set of steady-state equations may be obtained as

$$\begin{aligned}
\lambda\pi_{0,0} &= \omega\mu_1\pi_{1,1W} + \mu_2\pi_{1,2W}, & (1)\\
(\lambda b_n + \sigma + (n-1)\alpha)\,\pi_{n,0} &= \lambda b_{n-1}\pi_{n-1,0} + n\alpha\pi_{n+1,0}, 1 \le n \le N-1, & (2)\\
(\sigma + (N-1)\alpha)\,\pi_{N,0} &= \lambda b_{N-1}\pi_{N-1,0}, & (3)\\
(\lambda b_1 + \beta + \mu_1)\,\pi_{1,1W} &= (\omega\mu_1 + \alpha)\,\pi_{2,1W} + \sigma\pi_{1,0} + \mu_2\pi_{2,2W} + \delta\pi_{1,1B}, & (4)\\
(\lambda b_n + \beta + \mu_1 + (n-1)\alpha)\,\pi_{n,1W} &= (\omega\mu_1 + n\alpha)\,\pi_{n+1,1W} + \mu_2\pi_{n+1,2W} + \sigma\pi_{n,0} + \delta\pi_{n,1B}, & \\
&\quad + \lambda b_{n-1}\pi_{n-1,1W}, 2 \le n \le N-1, & (5)\\
(\beta + \mu_1 + (N-1)\alpha)\,\pi_{N,1W} &= \sigma\pi_{N,0} + \delta\pi_{N,1B} + \lambda b_{N-1}\pi_{N-1,1W}, & (6)\\
(\lambda b_1 + \beta + \mu_2)\,\pi_{1,2W} &= \alpha\pi_{2,2W} + \bar{\omega}\mu_1\pi_{1,1W} + \delta\pi_{1,2B}, & (7)\\
(\lambda b_n + \beta + \mu_2 + (n-1)\alpha)\,\pi_{n,2W} &= n\alpha\pi_{n+1,2W} + \bar{\omega}\mu_1\pi_{n,1W} + \delta\pi_{n,2B} + \lambda b_{n-1}\pi_{n-1,2W}, & \\
&\quad 2 \le n \le N-1, & (8)\\
(\beta + \mu_2 + (N-1)\alpha)\,\pi_{N,2W} &= \bar{\omega}\mu_1\pi_{N,1W} + \delta\pi_{N,2B} + \lambda b_{N-1}\pi_{N-1,2W}, & (9)\\
(\lambda b_1 + \delta)\,\pi_{1,1B} &= \alpha\pi_{2,1B} + \beta\pi_{1,1W}, & (10)\\
(\lambda b_n + \delta + (n-1)\alpha)\,\pi_{n,1B} &= n\alpha\pi_{n+1,1B} + \beta\pi_{n,1W} + \lambda b_{n-1}\pi_{n-1,1B}, 2 \le n \le N-1 & (11)\\
(\delta + (N-1)\alpha)\,\pi_{N,1B} &= \beta\pi_{N,1W} + \lambda b_{N-1}\pi_{N-1,1B}, & (12)\\
(\lambda b_1 + \delta)\,\pi_{1,2B} &= \alpha\pi_{2,2B} + \beta\pi_{1,2W}, & (13)\\
(\lambda b_n + \delta + (n-1)\alpha)\,\pi_{n,2B} &= n\alpha\pi_{n+1,2B} + \beta\pi_{n,2W} + \lambda b_{n-1}\pi_{n-1,2B}, 2 \le n \le N-1 & (14)\\
(\delta + (N-1)\alpha)\,\pi_{N,2B} &= \beta\pi_{N,2W} + \lambda b_{N-1}\pi_{N-1,2B}. & (15)
\end{aligned}$$

The steady-state probabilities are obtained by solving the above system of equations recursively as shown below.

$$\begin{aligned}
\pi_{n,0} &= r_n\pi_{N,0}, 1 \le n \le N,\\
\pi_{n,2B} &= (d_n + s_n k_{13} + t_n k_{14} + z_n k_{15} + \gamma_n k_{16})\,\pi_{N,0}, 1 \le n \le N,\\
\pi_{n,2W} &= (l_n + y_n k_{13} + w_n k_{14} + x_n k_{15} + m_n k_{16})\,\pi_{N,0}, 1 \le n \le N,\\
\pi_{n,1W} &= (g_n + p_n k_{13} + o_n k_{14} + f_n k_{15} + h_n k_{16})\,\pi_{N,0}, 1 \le n \le N,\\
\pi_{n,1B} &= (q_n + \chi_n k_{13} + c_n k_{14} + v_n k_{15} + u_n k_{16})\,\pi_{N,0}, 1 \le n \le N,
\end{aligned}$$

where

$$\begin{aligned}
r_N &= s_N = w_N = f_N = u_N = 1,\\
t_N &= z_N = d_N = \gamma_N = x_N = y_N = l_N = m_N = g_N = h_N = o_N = p_N = v_N = q_N = c_N = 0,\\
z_{N-1} &= d_{N-1} = \gamma_{N-1} = m_{N-1} = l_{N-1} = o_{N-1} = p_{N=1} = q_{N-1} = c_{N-1} = \chi_N = \chi_{N-1} = 0\\
d_{N-2} &= \gamma_{N-2} = p_{N-2} = c_{N-2} = \chi_{N-2} = \chi_{N-3} = 0,\\
r_{N-1} &= \frac{\sigma + (N-1)\alpha}{\lambda b_{N-1}},\ s_{N-1} = \frac{\delta + (N-1)\alpha}{\lambda b_{N-1}},\ t_{N-1} = -\frac{\beta}{\lambda b_{N-1}}, w_{N-1} = \frac{\beta + \mu_2 + (N-1)\alpha}{\lambda b_{N-1}},
\end{aligned}$$

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\begin{aligned}
x_{N-1} &= -\frac{\bar{\omega}\mu_1}{\lambda b_{N-1}}, u_{N-1} = \frac{\delta+(N-1)\alpha}{\lambda b_{N-1}}, v_{N-1} = -\frac{\beta}{\lambda b_{N-1}},\ f_{N-1} = \frac{\beta+\mu_1+(N-1)\alpha}{\lambda b_{N-1}},\\
g_{N-1} &= -\frac{\sigma}{\lambda b_{N-1}},\ h_{N-1} = -\frac{\delta}{\lambda b_{N-1}}/\\
r_n &= \left(\frac{\lambda b_{n+1}+\sigma+n\alpha}{\lambda b_n}\right) r_{n+1} - \left(\frac{(n+1)\alpha}{\lambda b_n}\right) r_{n+2}, n = N-2, N-3, \ldots, 1,\\
s_n &= \left(\frac{\lambda b_{n+1}+\delta+n\alpha}{\lambda b_n}\right) s_{n+1} - \left(\frac{\beta}{\lambda b_n}\right) y_{n+1} - \left(\frac{(n+1)\alpha}{\lambda b_n}\right) s_{n+2}, n = N-2, N-3, \ldots, 1,\\
t_n &= \left(\frac{\lambda b_{n+1}+\delta+n\alpha}{\lambda b_n}\right) t_{n+1} - \left(\frac{\beta}{\lambda b_n}\right) w_{n+1} - \left(\frac{(n+1)\alpha}{\lambda b_n}\right) t_{n+2}, n = N-2, N-3, \ldots, 1,\\
z_n &= \left(\frac{\lambda b_{n+1}+\delta+n\alpha}{\lambda b_n}\right) z_{n+1} - \left(\frac{\beta}{\lambda b_n}\right) x_{n+1} - \left(\frac{(n+1)\alpha}{\lambda b_n}\right) z_{n+2}, n = N-2, N-3, \ldots, 1,\\
d_n &= \left(\frac{\lambda b_{n+1}+\delta+n\alpha}{\lambda b_n}\right) d_{n+1} - \left(\frac{\beta}{\lambda b_n}\right) l_{n+1} - \left(\frac{(n+1)\alpha}{\lambda b_n}\right) d_{n+2}, n = N-2, N-3, \ldots, 1,\\
\gamma_n &= \left(\frac{\lambda b_{n+1}+\delta+n\alpha}{\lambda b_n}\right) \gamma_{n+1} - \left(\frac{\beta}{\lambda b_n}\right) m_{n+1} - \left(\frac{(n+1)\alpha}{\lambda b_n}\right) \gamma_{n+2}, n = N-2, N-3, \ldots, 1,\\
w_n &= \left(\frac{\lambda b_{n+1}+\beta+\mu_2+n\alpha}{\lambda b_n}\right) w_{n+1} - \left(\frac{\delta}{\lambda b_n}\right) t_{n+1} - \left(\frac{\bar{\omega}\mu_1}{\lambda b_n}\right) o_{n+1} - \left(\frac{(n+1)\alpha}{\lambda b_n}\right) w_{n+2},\\
&\quad n = N-2, N-3, \ldots, 1,\\
x_n &= \left(\frac{\lambda b_{n+1}+\beta+\mu_2+n\alpha}{\lambda b_n}\right) x_{n+1} - \left(\frac{\delta}{\lambda b_n}\right) z_{n+1} - \left(\frac{\bar{\omega}\mu_1}{\lambda b_n}\right) f_{n+1} - \left(\frac{(n+1)\alpha}{\lambda b_n}\right) x_{n+2},\\
&\quad n = N-2, N-3, \ldots, 1,\\
y_n &= \left(\frac{\lambda b_{n+1}+\beta+\mu_2+n\alpha}{\lambda b_n}\right) y_{n+1} - \left(\frac{\delta}{\lambda b_n}\right) s_{n+1} - \left(\frac{\bar{\omega}\mu_1}{\lambda b_n}\right) p_{n+1} - \left(\frac{(n+1)\alpha}{\lambda b_n}\right) y_{n+2},\\
&\quad n = N-2, N-3, \ldots, 1,\\
l_n &= \left(\frac{\lambda b_{n+1}+\beta+\mu_2+n\alpha}{\lambda b_n}\right) l_{n+1} - \left(\frac{\delta}{\lambda b_n}\right) d_{n+1} - \left(\frac{\bar{\omega}\mu_1}{\lambda b_n}\right) g_{n+1} - \left(\frac{(n+1)\alpha}{\lambda b_n}\right) l_{n+2},\\
&\quad n = N-2, N-3, \ldots, 1,\\
m_n &= \left(\frac{\lambda b_{n+1}+\beta+\mu_2+n\alpha}{\lambda b_n}\right) m_{n+1} - \left(\frac{\delta}{\lambda b_n}\right) \gamma_{n+1} - \left(\frac{\bar{\omega}\mu_1}{\lambda b_n}\right) h_{n+1} - \left(\frac{(n+1)\alpha}{\lambda b_n}\right) m_{n+2},\\
&\quad n = N-2, N-3, \ldots, 1,\\
f_n &= \left(\frac{\lambda b_{n+1}+\beta+\mu_1+n\alpha}{\lambda b_n}\right) f_{n+1} - \left(\frac{\omega\mu_1+(n+1)\alpha}{\lambda b_n}\right) f_{n+2} - \left(\frac{\mu_2}{\lambda b_n}\right) x_{n+2} - \left(\frac{\delta}{\lambda b_n}\right) v_{n+1},\\
&\quad n = N-2, N-3, \ldots, 1,\\
g_n &= \left(\frac{\lambda b_{n+1}+\beta+\mu_1+n\alpha}{\lambda b_n}\right) g_{n+1} - \left(\frac{\omega\mu_1+(n+1)\alpha}{\lambda b_n}\right) g_{n+2} - \left(\frac{\mu_2}{\lambda b_n}\right) l_{n+2} - \left(\frac{\delta}{\lambda b_n}\right) q_{n+1},\\
&\quad -\left(\frac{\sigma}{\lambda b_n}\right) r_{n+1}, n = N-2, N-3, \ldots, 1,\\
h_n &= \left(\frac{\lambda b_{n+1}+\beta+\mu_1+n\alpha}{\lambda b_n}\right) h_{n+1} - \left(\frac{\omega\mu_1+(n+1)\alpha}{\lambda b_n}\right) h_{n+2} - \left(\frac{\mu_2}{\lambda b_n}\right) m_{n+2} - \left(\frac{\delta}{\lambda b_n}\right) u_{n+1},\\
&\quad n = N-2, N-3, \ldots, 1,\\
o_n &= \left(\frac{\lambda b_{n+1}+\beta+\mu_1+n\alpha}{\lambda b_n}\right) o_{n+1} - \left(\frac{\omega\mu_1+(n+1)\alpha}{\lambda b_n}\right) o_{n+2} - \left(\frac{\mu_2}{\lambda b_n}\right) w_{n+2} - \left(\frac{\delta}{\lambda b_n}\right) c_{n+1},\\
&\quad n = N-2, N-3, \ldots, 1,\\
p_n &= \left(\frac{\lambda b_{n+1}+\beta+\mu_1+n\alpha}{\lambda b_n}\right) p_{n+1} - \left(\frac{\omega\mu_1+(n+1)\alpha}{\lambda b_n}\right) p_{n+2} - \left(\frac{\mu_2}{\lambda b_n}\right) y_{n+2} - \left(\frac{\delta}{\lambda b_n}\right) \chi_{n+1},\\
&\quad n = N-2, N-3, \ldots, 1,\\
u_n &= \left(\frac{\lambda b_{n+1}+\delta+n\alpha}{\lambda b_n}\right) u_{n+1} - \left(\frac{(n+1)\alpha}{\lambda b_n}\right) u_{n+2} - \left(\frac{\beta}{\lambda b_n}\right) h_{n+1}, n = N-2, N-3, \ldots, 1,
\end{aligned}
$$

$$
\begin{aligned}
v_n &= \left(\frac{\lambda b_{n+1}+\delta+n\alpha}{\lambda b_n}\right) v_{n+1} - \left(\frac{(n+1)\alpha}{\lambda b_n}\right) v_{n+2} - \left(\frac{\beta}{\lambda b_n}\right) f_{n+1}, n = N-2, N-3, \ldots, 1, \\
q_n &= \left(\frac{\lambda b_{n+1}+\delta+n\alpha}{\lambda b_n}\right) q_{n+1} - \left(\frac{(n+1)\alpha}{\lambda b_n}\right) q_{n+2} - \left(\frac{\beta}{\lambda b_n}\right) g_{n+1}, n = N-2, N-3, \ldots, 1, \\
c_n &= \left(\frac{\lambda b_{n+1}+\delta+n\alpha}{\lambda b_n}\right) c_{n+1} - \left(\frac{(n+1)\alpha}{\lambda b_n}\right) c_{n+2} - \left(\frac{\beta}{\lambda b_n}\right) o_{n+1}, n = N-2, N-3, \ldots, 1, \\
\chi_n &= \left(\frac{\lambda b_{n+1}+\delta+n\alpha}{\lambda b_n}\right) \chi_{n+1} - \left(\frac{(n+1)\alpha}{\lambda b_n}\right) \chi_{n+2} - \left(\frac{\beta}{\lambda b_n}\right) p_{n+1}, n = N-2, N-3, \ldots, 1, \\
k_1 &= \frac{\alpha z_2 + \beta x_1 - (\lambda b_1+\delta) z_1}{(\lambda b_1+\delta)\gamma_1 - \alpha\gamma_2 - \beta m_1},\ k_2 = \frac{\alpha d_2 + \beta l_1 - (\lambda b_1+\delta) d_1}{(\lambda b_1+\delta)\gamma_1 - \alpha\gamma_2 - \beta m_1}, \\
k_3 &= \frac{\alpha t_2 + \beta w_1 - (\lambda b_1+\delta) t_1}{(\lambda b_1+\delta)\gamma_1 - \alpha\gamma_2 - \beta m_1},\ k_4 = \frac{\alpha s_2 + \beta y_1 - (\lambda b_1+\delta) s_1}{(\lambda b_1+\delta)\gamma_1 - \alpha\gamma_2 - \beta m_1}, \\
k_5 &= \alpha u_2 + \beta h_1 - (\lambda b_1+\delta) u_1,\ k_6 = \frac{k_2 k_5 + \alpha q_2 + \beta g_1 - (\lambda b_1+\delta) q_1}{(\lambda b_1+\delta) v_1 - \alpha v_2 - \beta f_1 - k_1 k_5}, \\
k_7 &= \frac{k_3 k_5 + \alpha c_2 + \beta o_1 - (\lambda b_1+\delta) c_1}{(\lambda b_1+\delta) v_1 - \alpha v_2 - \beta f_1 - k_1 k_5},\ k_8 = \frac{k_4 k_5 + \alpha \chi_2 + \beta p_1 - (\lambda b_1+\delta)\chi_1}{(\lambda b_1+\delta) v_1 - \alpha v_2 - \beta f_1 - k_1 k_5}, \\
k_9 &= \alpha x_2 + \bar{\omega}\mu_1 f_1 + \delta z_1 - (\lambda b_1 + \mu_2 + \beta) x_1,\ k_{10} = \alpha m_2 + \bar{\omega}\mu_1 h_1 + \delta\gamma_1 - (\lambda b_1 + \mu_2 + \beta) m_1, \\
k_{11} &= \frac{k_9 k_6 + \alpha l_2 + \bar{\omega}\mu_1 g_1 + \delta d_1 + k_2 k_{10} + k_1 k_6 k_{10} - (\lambda b_1 + \mu_2 + \beta) l_1}{(\lambda b_1 + \mu_2 + \beta) w_1 - \alpha w_2 - \bar{\omega}\mu_1 o_1 - \delta t_1 - k_9 k_7 - k_3 k_{10} - k_1 k_7 k_{10}}, \\
k_{12} &= \frac{k_9 k_8 + \alpha y_2 + \bar{\omega}\mu_1 p_1 + \delta s_1 + k_4 k_{10} + k_1 k_8 k_{10} - (\lambda b_1 + \mu_2 + \beta) y_1}{(\lambda b_1 + \mu_2 + \beta) w_1 - \alpha w_2 - \bar{\omega}\mu_1 o_1 - \delta t_1 - k_9 k_7 - k_3 k_{10} - k_1 k_7 k_{10}}, \\
num &= (k_6 + k_7 k_{11}) \left((\omega\mu_1 + \alpha) f_2 + \mu_2 x_2 + \delta v_1 - (\lambda b_1 + \beta + \mu_1) f_1\right) \\
&\quad + (k_2 + k_3 k_{11} + k_1 k_6 + k_1 k_7 k_{11}) \left((\omega\mu_1 + \alpha) h_2 + \mu_2 m_2 + \delta u_1 - (\lambda b_1 + \beta + \mu_1) h_1\right) \\
&\quad + (\omega\mu_1 + \alpha)(g_2 + o_2 k_{11}) + \mu_2 (l_2 + w_2 k_{11}) + \delta (q_1 + c_1 k_{11}) \\
&\quad + \sigma r_1 - (\lambda b_1 + \beta + \mu_1)(g_1 + o_1 k_{11}), \\
den &= (k_8 + k_7 k_{12}) \left((\lambda b_1 + \beta + \mu_1) f_1 - (\omega\mu_1 + \alpha) f_2 - \mu_2 x_2 - \delta v_1\right) + (k_4 + k_3 k_{12} \\
&\quad + k_8 k_1 + k_1 k_7 k_{12}) \left((\lambda b_1 + \beta + \mu_1) h_1 - (\omega\mu_1 + \alpha) h_2 - \mu_2 m_2 - \delta u_1\right) + (\lambda b_1 + \beta + \mu_1) \\
&\quad \times (p_1 + o_1 k_{12}) - (\omega\mu_1 + \alpha)(p_2 + o_2 k_{12}) - \mu_2 (y_2 + w_2 k_{12}) - \delta (\chi_1 + c_1 k_{12}) \\
k_{13} &= \frac{num}{den}, \\
k_{14} &= k_{11} + k_{12} k_{13},\ k_{15} = k_6 + k_7 k_{14} + k_8 k_{13},\ k_{16} = k_2 + k_1 k_{15} + k_3 k_{14} + k_4 k_{13}
\end{aligned}
$$

Finally, $\pi_{0,N}$ is computed from $\sum_{n=0}^{N} \pi_{n,0} + \sum_{n=1}^{N} (\pi_{n,1W} + \pi_{n,1B} + \pi_{n,2W} + \pi_{n,2B}) = 1$ as

$$
\begin{aligned}
\pi_{0,N} &= 1/(r_0 + \sum_{n=1}^{N} [k_{13}(s_n + y_n + \chi_n) + k_{14}(t_n + w_n + o_n + c_n) + k_{15}(z_n + x_n + f_n + v_n) + \\
&\quad k_{16}(\gamma_n + m_n + h_n + u_n) + d_n + l_n + g_n + q_n + r_n]).
\end{aligned}
$$

4. Performance Indices

Under this section various performance indices are presented. The expected system length ($E[L]$) is given by

$$
\begin{aligned}
E[L] &= \sum_{n=1}^{N} n \left(\pi_{n,0} + \pi_{n,1W} + \pi_{n,1B} + \pi_{n,2W} + \pi_{n,2B}\right) \\
&= \sum_{n=1}^{N} n \big((r_n + d_n + l_n + g_n + q_n) + k_{13}(s_n + y_n + p_n + \chi_n) + k_{14}(t_n + w_n + o_n + c_n) \\
&\quad + k_{15}(z_n + x_n + f_n + v_n) + k_{16}(\gamma_n + m_n + h_n + u_n)\big) / (r_0 + \sum_{n=1}^{N} [k_{13}(s_n + y_n + \chi_n)
\end{aligned}
$$

$$+k_{14}(t_n+w_n+o_n+c_n)+k_{15}(z_n+x_n+f_n+v_n+k_{16}(\gamma_n+m_n+h_n+u_n)$$
$$+d_n+l_n+g_n+q_n+r_n)$$

The probabilities that the server is in vacation (P_v), busy with PS (P_{1w}), in breakdown state during PS (P_{1b}), busy with SS (P_{2w}) and the probability that the server is in breakdown state during SS (P_{2b}) are, respectively, given by

$$\begin{aligned}
P_v &= \sum_{n=0}^{N}\pi_{n,0},\\
&= \sum_{n=0}^{N} r_n/(r_0+\sum_{n=1}^{N}[k_{13}(s_n+y_n+\chi_n)+k_{14}(t_n+w_n+o_n+c_n)+k_{15}(z_n+x_n+f_n+v_n)\\
&\quad +k_{16}(\gamma_n+m_n+h_n+u_n)+d_n+l_n+g_n+q_n+r_n),\\
P_{1b} &= \sum_{n=1}^{N}\pi_{n,1b}\\
&= \sum_{n=1}^{N}(q_n+\chi_n k_{13}+c_n k_{14}+v_n k_{15}+u_n k_{16})/(r_0+\sum_{n=1}^{N}[k_{13}(s_n+y_n+\chi_n)+k_{14}(t_n+w_n+\\
&\quad o_n+c_n)+k_{15}(z_n+x_n+f_n+v_n)+k_{16}(\gamma_n+m_n+h_n+u_n)+d_n+l_n+g_n+q_n+r_n),\\
P_{1w} &= \sum_{n=1}^{N}\pi_{n,1w}\\
&= \sum_{n=1}^{N}(g_n+p_n k_{13}+o_n k_{14}+f_n k_{15}+h_n k_{16})/(r_0+\sum_{n=1}^{N}[k_{13}(s_n+y_n+\chi_n)+k_{14}(t_n+w_n\\
&\quad +o_n+c_n)+k_{15}(z_n+x_n+f_n+v_n)+k_{16}(\gamma_n+m_n+h_n+u_n)+d_n+l_n+g_n+q_n+r_n),\\
P_{2w} &= \sum_{n=1}^{N}\pi_{n,2w}\\
&= \sum_{n=1}^{N}(l_n+y_n k_{13}+w_n k_{14}+x_n k_{15}+m_n k_{16})/(r_0+\sum_{n=1}^{N}[k_{13}(s_n+y_n+\chi_n)+k_{14}(t_n+w_n+\\
&\quad o_n+c_n)+k_{15}(z_n+x_n+f_n+v_n)+k_{16}(\gamma_n+m_n+h_n+u_n)+d_n+l_n+g_n+q_n+r_n),\\
P_{2b} &= \sum_{n=1}^{N}\pi_{n,2b}\\
&= \sum_{n=1}^{N}(d_n+s_n k_{13}+t_n k_{14}+z_n k_{15}+\gamma_n k_{16}))/(r_0+\sum_{n=1}^{N}[k_{13}(s_n+y_n+\chi_n)+k_{14}(t_n+w_n\\
&\quad +o_n+c_n)+k_{15}(z_n+x_n+f_n+v_n)+k_{16}(\gamma_n+m_n+h_n+u_n)+d_n+l_n+g_n+q_n+r_n).
\end{aligned}$$

The expected balking rate (B_r), expected reneging rate (R_r) and the expected rate of losing a consumer (L_r) are given by

$$\begin{aligned}
B_r &= \sum_{n=1}^{N}\lambda(1-b_n)(\pi_{n,0}+\pi_{n,1W}+\pi_{n,1B}+\pi_{n,2W}+\pi_{n,2B})\\
&= \sum_{n=1}^{N}\lambda(1-b_n)\big((d_n+l_n+g_n+q_n)+k_{13}(s_n+y_n+p_n+\chi_n)+k_{14}(t_n+w_n+o_n+c_n)\\
&\quad +k_{15}(z_n+x_n+f_n+v_n)+k_{16}(\gamma_n+m_n+h_n+u_n)\big)/(r_0+\sum_{n=1}^{N}[k_{13}(s_n+y_n+\chi_n)+k_{14}(t_n\\
&\quad +w_n+o_n+c_n)+k_{15}(z_n+x_n+f_n+v_n+k_{16}(\gamma_n+m_n+h_n+u_n)+d_n+l_n+g_n+q_n+r_n),
\end{aligned}$$

$$
\begin{aligned}
R_r &= \sum_{n=1}^{N}(n-1)\alpha\left(\pi_{n,0}+\pi_{n,1W}+\pi_{n,1B}+\pi_{n,2W}+\pi_{n,2B}\right)\\
&= \sum_{n=1}^{N}(n-1)\alpha\big((d_n+l_n+g_n+q_n)+k_{13}(s_n+y_n+p_n+\chi_n)+k_{14}(t_n+w_n+o_n+c_n)\\
&\quad +k_{15}(z_n+x_n+f_n+v_n)+k_{16}(\gamma_n+m_n+h_n+u_n)\big)/\big(r_0+\sum_{n=1}^{N}[k_{13}(s_n+y_n+\chi_n)+k_{14}(t_n\\
&\quad +w_n+o_n+c_n)+k_{15}(z_n+x_n+f_n+v_n+k_{16}(\gamma_n+m_n+h_n+u_n)+d_n+l_n+g_n+q_n+r_n),\\
L_r &= B_r+R_r\\
&= \sum_{n=1}^{N}\left(\lambda(1-b_n)+(n-1)\alpha\right)\big((d_n+l_n+g_n+q_n)+k_{13}(s_n+y_n+p_n+\chi_n)+k_{14}(t_n+w_n\\
&\quad +o_n+c_n)+k_{15}(z_n+x_n+f_n+v_n)+k_{16}(\gamma_n+m_n+h_n+u_n)\big)/\big(r_0+\sum_{n=1}^{N}[k_{13}(s_n+y_n+\chi_n)\\
&\quad +k_{14}(t_n+w_n+o_n+c_n)+k_{15}(z_n+x_n+f_n+v_n+k_{16}(\gamma_n+m_n+h_n+u_n)+d_n+l_n+g_n\\
&\quad +q_n+r_n).
\end{aligned}
$$

5. Numerical Results

The impact of the various model parameters on the performance indices is presented in this section. The arbitrary choice of the model parameters for the purpose of numerical results is $N = 10, \lambda = 1.9, \omega = 0.3, \alpha = 0.5, \mu_1 = 2.9, \mu_2 = 2.5, \sigma = 2.0, \beta = 1.5, \delta = 1.2$. Table 1 displays the steady-state probabilities for the above chosen set of parameters. The table also presents the corresponding performance measures $E[L], P_v, P_{1w}, P_{1b}, P_{2w}, P_{2b}, B_r, R_r$ and L_r.

Table 1: *Steady-state probability distributions*

n	$\pi_{n,0}$	$\pi_{n,1B}$	$\pi_{n,1W}$	$\pi_{n,2B}$	$\pi_{n,2W}$
0	0.043328	–	–	–	–
1	0.022608	0.027649	0.038451	0.015641	0.019548
2	0.010840	0.055023	0.052865	0.037740	0.036276
3	0.004760	0.064685	0.050222	0.050595	0.041409
4	0.001911	0.055825	0.037802	0.048113	0.035437
5	0.000702	0.038317	0.023580	0.035631	0.024374
6	0.000236	0.021832	0.012510	0.021605	0.013991
7	0.000073	0.010615	0.005749	0.011075	0.006871
8	0.000021	0.004492	0.002317	0.004912	0.002940
9	0.000000	0.001682	0.000819	0.001923	0.001112
10	0.000000	0.000572	0.000236	0.00068	0.000379
$E[L] = 3.145050,\ P_v = 0.084489,\ P_{1W} = 0.224555,$ $P_{1b} = 0.280694,\ P_{2w} = 0.182339,\ P_{2b} = 0.227923,$ $B_r = 0.062196,\ R_r = 1.095870,\ L_r = 1.158070$					

The impact of arrival rate (λ) on $P_v, P_{1w}, P_{1b}, P_{2w}$ and P_{2b} is depicted in Figure 1. From the figure, it is evident that except P_v, all other values P_{1w}, P_{1b}, P_{2w} and P_{2b} increase with the increase of λ. The reason behind this nature is that as λ increases the number of consumers in the system increase due to which the server cannot leave for a vacation.

The changes in $P_v, P_{1w}, P_{1b}, P_{2w}$ and P_{2b} with β is shown in Figure 2. The probability of the server being in breakdown state both during PS and SS, P_{1b} and P_{2b}, respectively, increase with the increase of β which is obvious. Due to this the remaining probabilities P_v, P_{1w} and P_{2w} decrease with the growth of β as evident from the graph.

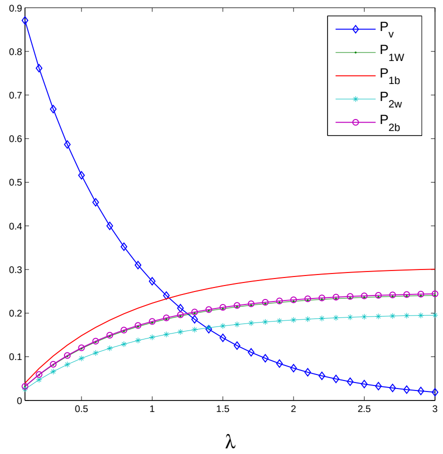

Figure 1: *Impact of λ on $P_v, P_{1w}, P_{1b}, P_{2w}, P_{2b}$*

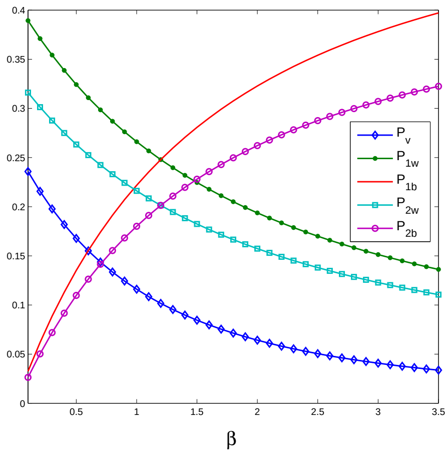

Figure 2: *Changes in $P_v, P_{1w}, P_{1b}, P_{2w}, P_{2b}$ with β*

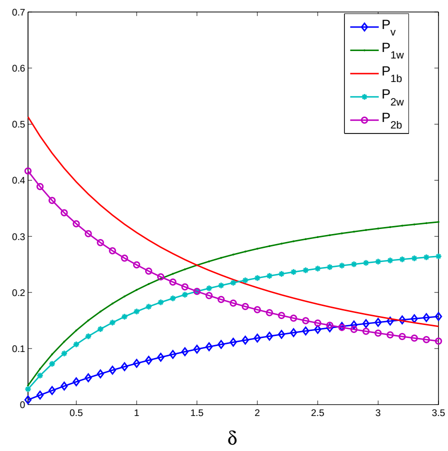

Figure 3: *Influence of δ on $P_v, P_{1w}, P_{1b}, P_{2w}, P_{2b}$*

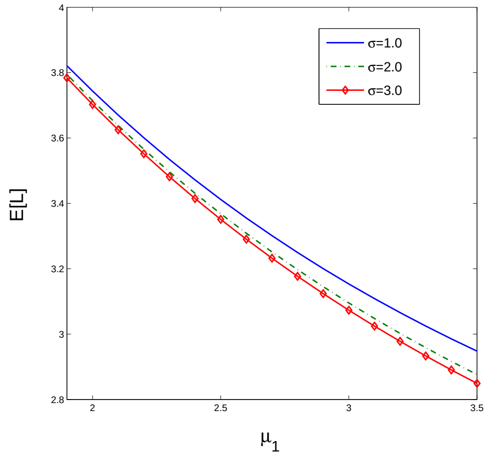

Figure 4: *Impact of μ_1 on $E[L]$ for different σ*

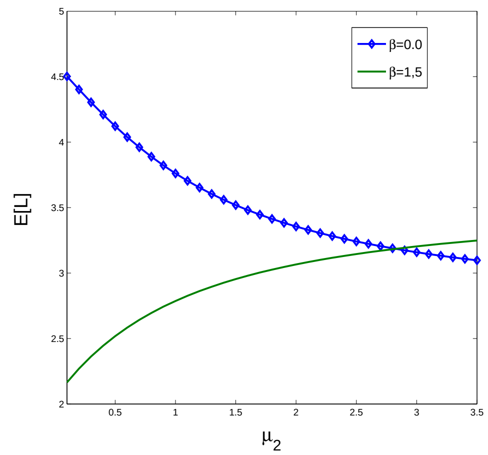

Figure 5: *Effect of μ_2 on $E[L]$ with and without server breakdowns*

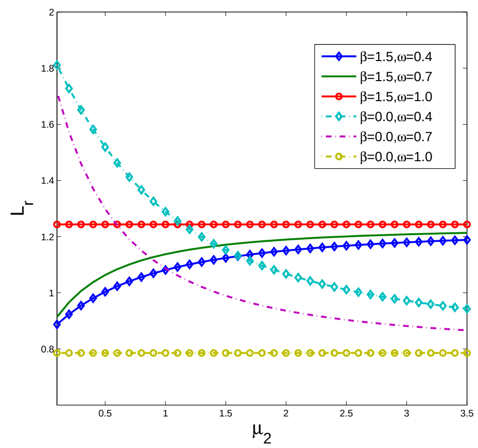

Figure 6: *Effect of μ_2 on L_r for different ω with and without server breakdowns*

The influence of δ on $P_v, P_{1w}, P_{1b}, P_{2w}$ and P_{2b} is displayed in Figure 3. In contrary to Figure 2, P_{1b} and P_{2b} decrease with the increase of δ while P_v, P_{1w} and P_{2w} increase with δ.

The effect of the service rate during PS (μ_1) on the expected system length ($E[L]$)is depicted in Figure 4 for different vacation rates σ. As evident from the figure, the expected system length decline with the increment of μ_1 for any choice of vacation rate (σ). Further, for a constant μ_1 value, $E[L]$ diminish with the growth in σ.

Figure 5 exhibits the effect of μ_2 on the expected system length ($E[L]$) in models with ($\beta = 1.5$) and without ($\beta = 0.0$) server breakdowns. In models without server breakdown, $E[L]$ diminishes with μ_2 as intuitively expected. However, this trend is reversed in models with server breakdown because there will no service during breakdown period leading to the increase in system length.

The impact of μ_2 on the expected rate of losing a consumer (L_r) in models with ($\beta = 1.5$) and without ($\beta = 0.0$) server breakdowns for different ω is revealed in Figure 6. It may be perceived that, for any ω ($\neq 1$), L_r drops with the growth of μ_2 in models without breakdown ($\beta = 0.0$) while L_r grows with the growth of μ_2 in models with breakdown ($\beta = 1.5$). Further, $\omega = 1.0$ implies that no client is opting for SS and hence, μ_2 has no impact on L_r as a result L_r remains the same for any μ_2 in both the models. Furthermore, with the increase of ω, the number of consumers leaving the system increase resulting in the decrease of L_r in models without breakdown while this trend gets reversed in models with breakdown.

6. Conclusions

An impatient consumer queue with secondary service, vacations and server breakdowns has been examined in this research. A wide range of real-time systems, including production and manufacturing systems, computer and communication networks, inventory and distribution systems, and others, may employ the described approach. The state-dependent nature of consumers' impatience in a secondary service queue with server breakdowns and vacations are the key contributions of this article. We used a recursive approach to obtain the steady-state probabilities. To illustrate the effect of the system parameters, numerical data is presented as table and graphs. This study might be expanded to include a renewal input impatient consumer queue with SS and working vacations and is a topic for further research.

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