MODELING OF RELIABILITY AND SURVIVAL DATA WITH EXPONENTIATED GENERALIZED INVERSE LOMAX DISTRIBUTION

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Abstract

In this paper, a new four-parameter distribution is developed and studied by combining the properties of the exponentiated generalized-G family of distributions and the features of the Inverse Lomax distribution. The newly developed distribution is called the exponentiated generalized inverse Lomax distribution that extends the classical inverse Lomax distribution. The shape of the hazard rate function is very flexible because it possesses increasing, decreasing, and inverted (upside-down) bathtub shapes. Some important characteristics of the exponentiated generalized inverse Lomax distribution are derived, including moments, moment generating function, survival function, hazard function and order statistics. The method of maximum likelihood estimation is used to obtain estimates of the unknown parameters of the new model. The application of the new model is based on two real-life data sets used to show the modeling potential of the proposed distribution. The exponentiated generalized inverse Lomax distribution turns out to be the best by capturing important details in the structure of the data sets considered.

Keywords: MLE, carbon fibers, non-linear, inverse Lomax, cancer

I. Introduction

Different distributions have been put out to extend existing distributions and act as models for numerous applications on data from various real-life scenarios. There are many ways to accomplish this, but the most popular one is to give the baseline distributions more flexibility by employing families of distributions.

The Lomax (Lx) distribution, also known as the Pareto type II distribution, is a special case of the generalized beta distribution of the second kind [1]. The distribution can be observed in many application areas, including actuarial science, economics, biological sciences, engineering, lifetime and reliability modeling, and so on [2]. According to [3], this distribution can be used as a substitute for survival issues and life-testing in engineering and survival analysis.

A significant lifetime distribution that can be used as a good substitute for well-known distributions like gamma, inverse Weibull, Weibull, and Lomax distributions is the Inverse Lomax (ILx) distribution. Because its hazard rate can be decreasing and upside-down bathtub shaped, it has a variety of uses in modeling diverse sorts of data, including economics and actuarial sciences data. The Inverse Lomax (ILx) distribution's unique characteristics and applications make it a valuable tool for statisticians, data analysts, and researchers dealing with a wide range of data sets.

A random variable X is said to have an ILx distribution if its cdf and pdf are given as

$$G(x)=\left[1+\frac{c}{x}\right]^{-d} \tag{1}$$

$$g(x)=cdx^{-2}\left[1+\frac{c}{x}\right]^{-d-1} \tag{2}$$

It was demonstrated in [4] that the ILx distribution is a member of the inverted family of distributions. They also discovered that the ILx distribution is a very flexible model in analyzing situations with a realized non-monotonic failure rate. For more details on the extensions of ILx distribution, readers are referred to [5], [6], [7], [8], [9], [10], [11], [12], [13], among others.

A new class of exponentiated generalized distribution that extends the exponentiated-G class was proposed by [14]. Given a continuous cdf $G(x)$, the cdf and pdf of exponentiated generalized class of distributions are given as

$$F(x)=\left[1-\left[1-G(x)\right]^{a}\right]^{b} \tag{3}$$

$$f(x)=abg(x)\left[1-G(x)\right]^{a-1}\left[1-\left[1-G(x)\right]^{a}\right]^{b-1} \tag{4}$$

In real-world applications, it is often impractical to generate data with a high failure rate, which can limit the applicability of the inverse Lomax distribution. Consequently, it becomes necessary to introduce additional flexibility into the inverse Lomax distribution to accommodate various hazard function shapes. The primary objective of this extension is to leverage the advantages offered by the exponentiated generalized-G class of distributions, as introduced by [14]. This extension is designed to enhance the modeling capabilities of the baseline distribution, ultimately leading to improved goodness-of-fit. When compared to the baseline model, the newly extended model demonstrates a wider range of hazard function shapes and provides superior fits for diverse types of datasets.

II. Methods

2.1 Exponentiated Generalized Inverse Lomax (EGILx) Distribution

This section introduces a novel continuous probability distribution function known as the Exponentiated Generalized Inverse Lomax (EGILx) distribution. It includes plots of its probability density function (pdf) and cumulative distribution function (cdf) to evaluate the characteristics of this new distribution. By substituting (1) into (3) and (2) into (4), we obtain the cdf and pdf of the EGILx distribution, respectively, as follows:

$$F(x)=\left[1-\left[1-\left[1+\frac{c}{x}\right]^{-d}\right]^{a}\right]^{b} \tag{5}$$

$$f(x)=abcdx^{-2}\left[1+\frac{c}{x}\right]^{-d-1}\left[1-\left[1+\frac{c}{x}\right]^{-d}\right]^{a-1}\left[1-\left[1-\left[1+\frac{c}{x}\right]^{-d}\right]^{a}\right]^{b-1} \tag{6}$$

Where $x\geq 0$, $c>0$ is the scale parameter and $a,b,d>0$ are the shape parameters.

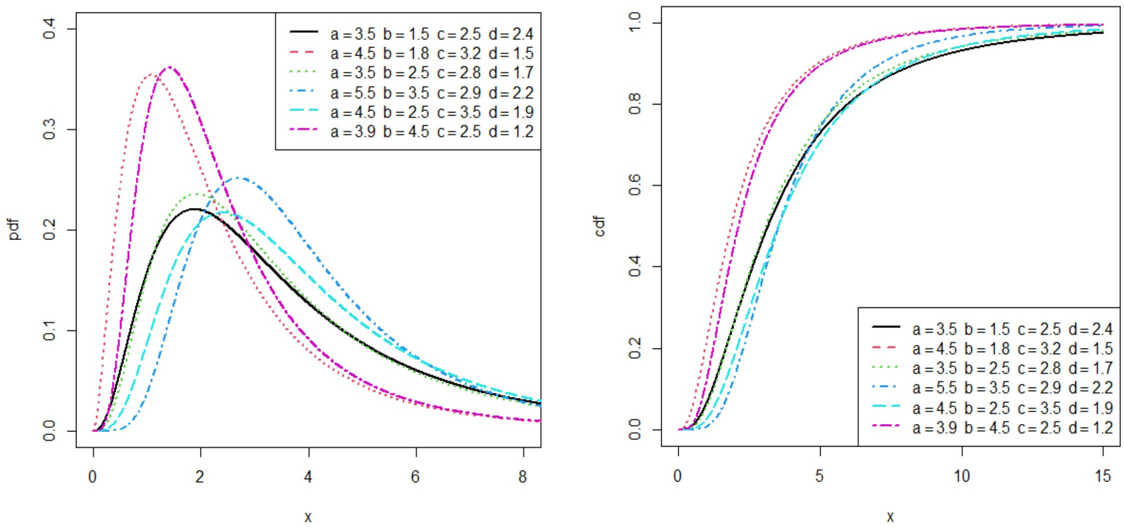

Figure 1: *Plots of pdf and cdf of EGILx distribution*

2.2 Expansion of Density

Using the generalized bionomial expansion

$$(1-x)^{p-1}=\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(p)}{i!\Gamma(p-i)}x^i \tag{7}$$

The pdf of EGILx distribution in (6) can be expressed in mixture form in terms of Lx densities as

$$f(x)=abcdx^{-2}\left[1+\frac{c}{x}\right]^{-d-1}\left[1-\left[1+\frac{c}{x}\right]^{-d}\right]^{a-1}\left[1-\left[1-\left[1+\frac{c}{x}\right]^{-d}\right]^{a}\right]^{b-1}$$

$$\left[1-\left[1-\left[1+\frac{c}{x}\right]^{-d}\right]^{a}\right]^{b-1}=\sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\left[1-\left[1+\frac{c}{x}\right]^{-d}\right]^{ai}$$

$$\left[1-\left[1+\frac{c}{x}\right]^{-d}\right]^{a(i+1)-1}=\sum_{j=0}^{\infty}(-1)^j\binom{a(i+1)-1}{j}\left[1+\frac{c}{x}\right]^{-dj}$$

$$f(x)=abcd\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}x^{-2}\left[1+\frac{c}{x}\right]^{-d(j+1)-1} \tag{8}$$

where

$$\int_0^{\infty}x^{r-2}\left[1+\frac{c}{x}\right]^{-d(j+1)-1}dx=c^rB\left[(1-r),d(j+1)+r\right]$$

On expanding the cdf, we have

$$\left[F(x)\right]^h=\left[1-\left[1-\left[1+\frac{c}{x}\right]^{-d}\right]^{a}\right]^{bh}$$

$$\left[1-\left[1-\left[1+\frac{c}{x}\right]^{-d}\right]^{a}\right]^{bh}=\sum_{k=0}^{h}(-1)^{k}\binom{bh}{k}\left[1-\left[1+\frac{c}{x}\right]^{-d}\right]^{ak}$$

$$\left[1-\left[1+\frac{c}{x}\right]^{-d}\right]^{ak}=\sum_{w=0}^{\infty}(-1)^{w}\binom{ak}{w}\left[1+\frac{c}{x}\right]^{-dw}$$

$$\left[F(x)\right]^{h}=\sum_{k=0}^{h}\sum_{w=1}^{\infty}(-1)^{k+w}\binom{ak}{w}\binom{bh}{k}\left[1+\frac{c}{x}\right]^{-dw} \tag{9}$$

2.3 Properties of EGILx Distribution

This section derives some statistical properties of the EGILx distribution including moments, moment generating function, survival function, hazard function, quantile functions, and order statistics.

2.3.1 Moments

Using moments, some of the most important features and characteristics of a distribution such as tendency, dispersion, skewness and kurtosis can be studied. The rth ordinary moment of the EGILx distribution can be written from (8) as

$$E\left(X^{r}\right)=\int_{0}^{\infty}x^{r}f\left(x\right)dx \tag{10}$$

$$=abcd\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\int_{0}^{\infty}x^{r-2}\left[1+\frac{c}{x}\right]^{-d(j+1)-1}dx$$

$$\int_{0}^{\infty}x^{r-2}\left[1+\frac{c}{x}\right]^{-d(j+1)-1}dx=c^{r}B\left[(1-r),d(j+1)+r\right]$$

$$E\left(X^{r}\right)=abc^{r}d\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}B\left[(1-r),d(j+1)+r\right] \tag{11}$$

the mean of the EGILx distribution can be obtained by setting r = 1 in (11)

2.3.2 Moment generating function (mgf)

Following the process of moments, the mgf is obtained as

$$M_{x}(t)=abd\sum_{m=0}^{\infty}\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\frac{(tc)^{m}}{m!}B\left[(1-m),d(j+1)+m\right] \tag{12}$$

2.3.3 Reliability function (rf)

$$\mathrm{R}(x)=1-\left[1-\left[1-\left[1+\frac{c}{x}\right]^{-d}\right]^{a}\right]^{b} \tag{13}$$

2.3.4 Hazard Function (hf)

$$H(x)=\frac{abcdx^{-2}\left[1+\frac{c}{x}\right]^{-d-1}\left[1-\left[1+\frac{c}{x}\right]^{-d}\right]^{a-1}\left[1-\left[1-\left[1+\frac{c}{x}\right]^{-d}\right]^{a}\right]^{b-1}}{1-\left[1-\left[1-\left[1+\frac{c}{x}\right]^{-d}\right]^{a}\right]^{b}} \tag{14}$$

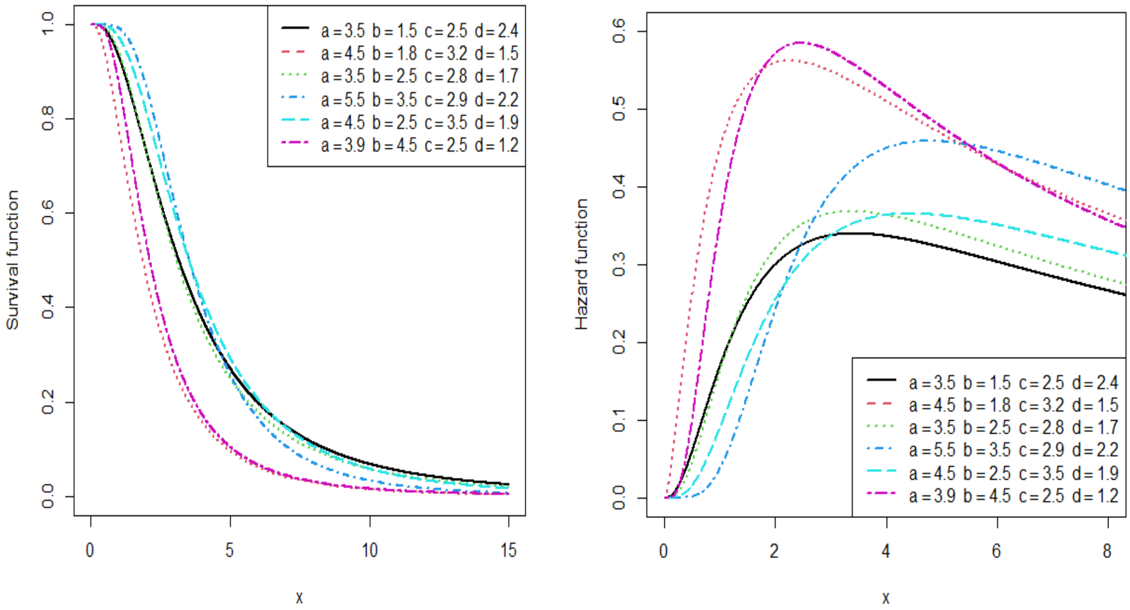

Figure 2: *Plots of survival function and hazard function of EGILx distribution*

2.3.5 Quantile Function (qf)

The quantile function (qf) of a probability distribution is the real solution of the equation $F(x)=u$, for $0 \le u \le 1$, and $F(x)$ is the cumulative distribution function (CDF) of the random variable x
For EGILx distribution the quantiles are given by

$$x=Q(u)=\frac{c}{1-\left[1-\left[1-u^{\frac{1}{b}}\right]^{\frac{1}{a}}\right]^{\frac{-1}{d}}} \tag{15}$$

On setting $u=0.5$ in (15), the median of the EGILx distribution can be obtained.

2.3.6 Order Statistics

Let $X_1, X_2, ...X_n$ be n independent random variable from the EGILx distribution and let $X_{(1)}, X_{(2)}, ...X_{(n)}$ be their corresponding order statistic. Let $F_{r:n}(x)$ and $f_{r:n}(x)$, $r = 1,2,3,...,n$ denote the cdf and pdf of the r^{th} order statistics $X_{r:n}$ respectively. The pdf of the r^{th} order statistics of $X_{r:n}$ is given as

$$f_{r:n}(x) = \frac{f(x)}{B(r, n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} F(x)^{v+r-1} \tag{16}$$

The pdf of r^th^ order statistic for distribution is obtained by replacing h with v+r-1 in cdf expansion. We have

$$f_{r:n}(x) = abcdx^{-2} \frac{1}{B(r, n-r+1)} \sum_{v=0}^{n-r} \sum_{i,j,w=0}^{\infty} \sum_{k=0}^{v+r-1} (-1)^{i+j+w+k+v} \binom{n-r}{v} \binom{b-1}{i} \binom{a(i+1)-1}{j} \binom{ak}{w} \binom{b(v+r-1)}{k} \left[1+\frac{c}{x}\right]^{-d(j+1+w)-1} \tag{17}$$

The pdf of the minimum order statistic of the distribution is obtained by setting r=1 in (17) as

$$f_{1:n}(x) = abcdx^{-2} n \sum_{v=0}^{n-1} \sum_{i,j,w=0}^{\infty} \sum_{k=0}^{v} (-1)^{i+j+w+k+v} \binom{n-1}{v} \binom{b-1}{i} \binom{a(i+1)-1}{j} \binom{ak}{w} \binom{bv}{k} \left[1+\frac{c}{x}\right]^{-d(j+1+w)-1} \tag{18}$$

Also, the pdf of the maximum order statistic of the distribution is obtained by setting r = n in (17) as

$$f_{n:n}(x) = abcdx^{-2} n \sum_{k=0}^{v+n-1} \sum_{i,j,w=0}^{\infty} (-1)^{i+j+w+k+v} \binom{b-1}{i} \binom{a(i+1)-1}{j} \binom{ak}{w} \binom{b(v+n-1)}{k} \left[1+\frac{c}{x}\right]^{-d(j+1+w)-1} \tag{19}$$

2.4 Maximum Likelihood Estimation (MLE)

In this section, the estimation of the unknown parameters for the EGILx distribtuion by MLE method. Let $X_1, X_2, ...X_n$ be random variables of the EGILx distribution of size n. Then the sample log-likelihood function of the EGILx distribution is obtained as

$$\log(L) = n\log(a) + n\log(b) + n\log(c) + n\log(d) - 2\sum_{i=1}^{n} \log(x_i) - (d+1)\sum_{i=1}^{n} \log\left[1+\frac{c}{x_i}\right] + (a-1)\sum_{i=1}^{n} \log\left[1-\left[1+\frac{c}{x_i}\right]^{-d}\right] + (b-1)\sum_{i=1}^{n} \log\left[1-\left[1-\left[1+\frac{c}{x_i}\right]^{-d}\right]^{a}\right] \tag{20}$$

Differentiating (20) with respect to each parameter and equationg them zero yields the MLE as

$$\frac{\partial L}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\log\left[1-\left[1+\frac{c}{x_i}\right]^{-d}\right]-(b-1)\sum_{i=1}^{n}\frac{\left[1-\left[1+\frac{c}{x_i}\right]^{-d}\right]^{a}\log\left[1-\left[1+\frac{c}{x_i}\right]^{-d}\right]}{1-\left[1-\left[1+\frac{c}{x_i}\right]^{-d}\right]^{a}}=0 \tag{21}$$

$$\frac{\partial L}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\log\left[1-\left[1-\left[1+\frac{c}{x_i}\right]^{-d}\right]^{a}\right]=0 \tag{22}$$

$$\frac{\partial L}{\partial c}=\frac{n}{c}-(d+1)\sum_{i=1}^{n}\frac{1}{\left[1+\frac{c}{x_i}\right]x_i}+d(a-1)\sum_{i=1}^{n}\frac{\left[1+\frac{c}{x_i}\right]^{-d-1}}{\left[1-\left[1+\frac{c}{x_i}\right]^{-d}\right]x_i}+ad(b-1)\sum_{i=1}^{n}\frac{\left[1-\left[1+\frac{c}{x_i}\right]^{-d}\right]^{a-1}\left[1+\frac{c}{x_i}\right]^{-d-1}}{\left[1-\left[1-\left[1+\frac{c}{x_i}\right]^{-d}\right]^{a}\right]x_i}=0 \tag{23}$$

$$\frac{\partial L}{\partial d}=\frac{n}{d}-\sum_{i=1}^{n}\log\left[1+\frac{c}{x_i}\right]+(a-1)\sum_{i=1}^{n}\frac{\left[1+\frac{c}{x_i}\right]^{-d}\log\left[1+\frac{c}{x_i}\right]}{1-\left[1+\frac{c}{x_i}\right]^{-d}}+a(b-1)\sum_{i=1}^{n}\frac{\left[1-\left[1+\frac{c}{x_i}\right]^{-d}\right]^{a-1}\left[1+\frac{c}{x_i}\right]^{-d}\log\left[1+\frac{c}{x_i}\right]}{1-\left[1-\left[1+\frac{c}{x_i}\right]^{-d}\right]^{a}}=0 \tag{24}$$

Equations (21) to (24) lack a straightforward analytical representation, rendering them intractable. Consequently, we must employ non-linear parameter estimation techniques through iterative methods.

III. Results

3.1 Applications

In this section, we provide two applications to real-life data sets to illustrate the importance and flexibility of the EGILx distribution compared to some classical distributions such as inverse Weibull (IW), inverse Lomax (ILx) and Lomax (Lx) distributions.

The first data set represents the breaking stress of carbon fibers of 50 mm length (GPa) was reported by [15]. This data was used by [16]. The data are: 0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

The second data set was given by [17] and it represents the remission times (in months) of a random sample of one hundred and twenty-eight (128) bladder cancer patients. The data set is as follows: 0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Table 1: *The models' MLEs and performance requirements based on data set 1*

Models	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{d}$
EGILx	68.2326	1.2151	4.6461	4.2988
IW	-	-	2.0356	1.6479
Lx	-	-	0.2884	1.5594
ILx	-	-	0.0139	164.8019

Table 2: *The models' MLEs and performance requirements based on data set 2*

Models	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{d}$
EGILx	2.2598	63.3711	21.6622	0.0938
IW	-	-	3.2589	0.7522
Lx	-	-	0.0083	13.9032
ILx	-	-	2.0033	2.4610

Table 3: *The Performance requirements based on data set 1*

Models	ll	AIC	AICc	BIC	HQIC
EGILx	-89.2197	186.4394	187.951	195.1900	189.9003
IW	-121.1949	246.3898	246.5803	250.7697	248.1203
Lx	-149.8545	303.7089	303.8994	308.0882	305.43.94
ILx	-136.1274	276.2548	276.4453	280.6342	277.9853

Table 4: *The Performance requirements based on data set 2*

Models	ll	AIC	AICc	BIC	HQIC
EGILx	-414.2379	836.4759	836.8017	847.8840	841.1111
IW	-444.0008	892.0015	892.0975	897.7056	894.3191
Lx	-416.8329	837.6658	837.7618	848.0126	842.0654
ILx	-424.6757	853.3514	853.4474	859.0555	855.6690

IV. Discussion

In this paper, we introduce a novel four-parameter distribution called the Exponentiated Generalized Inverse Lomax distribution and conduct a comprehensive study of its properties. We delve into various mathematical and statistical aspects of this newly developed model, including moments, moment generating functions, reliability functions, quantile functions, and order statistics. Furthermore, we investigate and present the probability density functions of both the maximum and minimum order statistics. To estimate the unknown parameters of this distribution, we employ the maximum likelihood estimation method, which allows us to determine their values effectively. To demonstrate the practical utility of our proposed model, we provide insights into its performance by applying it to two real-life datasets. Our analysis reveals that the Exponentiated Generalized Inverse Lomax distribution outperforms other lifetime models considered in this study when applied to the provided datasets, underscoring its potential for improved modeling and prediction in real-world applications.

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