GROUP RUNS AND MODIFIED GROUP RUNS CONTROL CHARTS FOR MONITORING LINEAR REGRESSION PROFILES

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Abstract

Profile monitoring is a critical tool for manufacturing industries to evaluate and maintain quality performance, as well as detect faults. The process of profile monitoring involves observing how variables interact with one another throughout a given period. This enables the understanding of any changes in their functional relationship over time. Generally, control charts are used for monitoring profiles. This paper proposes two new methods to enhance the monitoring of simple and multiple linear regression profiles in Phase II. The proposed methods are based on group runs (GR) and modified group runs (MGR) control charting schemes. The procedure to obtain optimal design parameters for the proposed methods is discussed in detail. The effectiveness of the suggested techniques is assessed through the ARL standard. The study found that the proposed GR and MGR monitoring methods displayed superior performance compared to other available monitoring methods in the literature. A real-life example is illustrated using proposed GR and MGR charting schemes.

Keywords: Statistical process control**,** Control chart, Average run length, Linear regression models, profile monitoring.

1. Introduction

Statistical process control (SPC) plays a crucial role in enhancing the quality and productivity of manufacturing processes by creating effective methods. The control chart serves as a valuable tool in SPC for monitoring the quality of manufactured products. In particular, Shewhart-type, exponentially weighted moving average (EWMA) charts and cumulative sum (CUSUM) control charts are frequently utilized to monitor the quality of single-variable characteristics. In various instances, the manufacturing product quality depends on multiple factors; thus, a multivariate control chart is employed for such scenarios. Some of the commonly used multivariate control charts include Hotelling's T^2 chart, multivariate EWMA chart, and multivariate CUSUM chart. In many practical situations, there is a functional relationship between a variable being studied and other factors that can explain it. In these situations, the quality of the manufacturing product is effectively monitored by using a profile. Profile monitoring involves the surveillance of how the study variable relates to one or several explanatory variables. When the response variable has a linear relationship with only one explanatory variable, it is a simple linear regression profile. However, if the response variable has a linear relationship with more than one explanatory variable, it is a multiple linear regression profile. The practice of profile monitoring utilizes control charts and is implemented to ascertain whether any alterations have occurred in the established functional relation, with respect to time. The use of profile monitoring in both Phase I and II have become increasingly important in various industries as it allows for the early detection of shifts in a process and contributes to the maintenance of quality control. During the initial stage, denoted as Phase I, there is a lack of knowledge regarding the parameters governing each profile. As such, these values are appraised through a reliable and consistent method. Moving on to Phase II, the profiles' parameters have been distinctly determined beforehand and serve as reference points for detecting any alterations in the process being analyzed.

There are various charting techniques in academic literature to track simple and multiple linear regression data for Phase I and II scenarios. Kang and Albin [1] used simple linear profiles to calibrate semiconductor manufacturing equipment. They monitored the relationship between gas flow and pressure in the chamber Kang and Albin [1] proposed common charting techniques to supervise linear profiles by employing the multivariate Hotelling's T^2 and EWMA/R methods. Kim et al. [2] devised a Phase II monitoring strategy, known as the EWMA-3 chart, which employs three individual univariate EWMA regression control charts to monitor the intercept, slope, and variance of errors in linear profiles. Their findings indicate that the EWMA-3 method is more effective than the methods of Kang and Albin [1] in Phase II analysis. Woodall et al. [3] addressed the key concerns regarding using control charts for monitoring process and product quality profiles. They have also given a review of the literature on profile monitoring in SPC. Gupta et al. [4] presented a Shewhart-based simple linear profile method called the Shewhart-3 chart. Saghaei et al. [5] suggested CUSUM-3 method to monitor simple linear profile parameters. Woodall [6] presented a review of linear profile monitoring methods. Riaz et al. [7] developed the Assorted-3 control charting strategy for monitoring parameters of simple linear profiles. To monitor multiple linear profiles, several methods have been developed by researchers, including Zou et al. [8], Zhang et al. [9], Zou et al. [10] and Amiri et al. [11]. Further, the T^2 control chart method by Kang and Albin [1] can also be used for Phase II monitoring of multiple linear regression profiles. Maleki et al. [12] provided an overview of profile monitoring papers published from 2008 to 2018.

Various methods have been suggested to improve conventional control charts in the literature of SPC. The synthetic control chart is one such approach that seeks to enhance the effectiveness of traditional control charts. Wu and Spedding [13] first developed synthetic chart technique which combines the features of both Shewhart chart and CRL chart. Bourke [14] developed the CRL control chart for monitoring nonconforming fractions. Wu and Spedding [13] demonstrated that the synthetic $\bar{X}$ chart outperforms the Shewhart $\bar{X}$ chart. Synthetic control charts have proven to be effective in detecting shifts in both univariate and multivariate processes, which has led many researchers to focus on designing them to enhance their shift detection capabilities. Numerous research papers on synthetic control charts exist in the literature. Some of these are Chen and Huang [15], Huang and Chen [16], Ghute and Shirke [14], Ghute and Shirke [18], Ghute and Shirke [19], Aparisi and de Luna [20], Rajmanya and Ghute [21]. Rakitzis et al. [22] reviewed over 100 scholarly articles based on synthetic-type control charts. In literature, the Group Runs (GR) method is recommended in literature as an upgraded version of the synthetic method for identifying changes in process parameters. Gadre and Rattihalli [23] devised GR chart to enhance shift detection ability of Shewhart $\bar{X}$ chart and synthetic $\bar{X}$ chart. The GR control chart scheme combines Shewhart $\bar{X}$ chart and an extended version of the CRL chart and it shows more acceptable performance than the Shewhart $\bar{X}$ chart and synthetic $\bar{X}$ chart. More related work on GR charts can be seen in Gadre et al. [24], Gadre and Kakade [25], Gadre and Kakade [26], Chong et al. [27], Khilare and Shirke [28]. Gadre and Rattihalli [29] developed Modified Group Runs (MGR) scheme to detect the shift in process mean of the normally distributed process. The results of their study suggest that the MGR chart is more efficient than Shewhart, synthetic, and GR charts. Gadre and Kakade [26] introduced multivariate GR and MGR control charts for monitoring process mean vector of normally distributed process. It was shown that, the proposed multivariate versions of GR and MGR charts detect changes in the process mean vector faster than the Hotelling's T^2 chart and synthetic T^2 chart proposed by Ghute and Shirke [19].

By getting motivation of improved performance from GR and MGR charts, we implement these charting schemes to the T^2 method proposed by Kang and Albin [1] for monitoring simple and multiple linear regression profiles. The objective of this paper is to propose GR and MGR charting methods for improved monitoring of linear profiles and we expect that the proposed methods will provide a better option for the detection of a wide range of shifts in the model parameters of both simple and multiple linear regression models in Phase II monitoring. In this paper, we propose group runs based T^2 method (denoted as GR-T^2) and modified group runs based T^2 method (denoted as MGR-T^2) for monitoring simple and multiple linear profiles in Phase II. The performance of the proposed methods for monitoring simple linear profiles is compared with the T^2 method by Kang and Albin [1] and Shewhart-3 method of Gupta et al. [4]. There are other methods in the literature based on EWMA and CUSUM control charts for monitoring profiles. But for a fair comparison, we have selected only T^2 and Shewhart-3 methods because our proposed method GR-T^2 method is Shewhart based method and Shewhart-3 approach is Shewhart type also Hotelling's T^2 chart is multivariate extension of the univariate $\bar{X}$ chart. The performance of the proposed methods for monitoring multiple linear profiles is also compared with the T^2 method. The performance of the proposed GR-T^2 and MGR-T^2 methods is evaluated using Monte Carlo simulations in terms of average run length (ARL) criterion under sustained shifts of different magnitudes in the regression parameters and error standard deviation of simple and multiple linear profile monitoring.

The rest of this paper is organized as follows: The model for simple linear profiles under consideration is presented in Section 2. In Section 3, we discuss multiple linear regression model to be considered in this study for monitoring profiles in Phase II. Our proposed GR-T^2 and MGR-T^2 methods for monitoring simple and multiple linear profiles in Phase II are described in Sections 4 and 5 respectively. Section 6 of this study evaluates the effectiveness of our suggested approach in comparing with traditional methods for monitoring simple linear and multiple linear profiles, using the average run length standard. A demonstrative example is provided in Section 7. The conclusions are given in Section 8.

2. Simple linear regression profile model

This section will analyze a simple linear regression model that utilizes two parameters to demonstrate how the response variable Y is a function of an explanatory variable X. Let Y_{ij} is the i^{th} observation of the response variable in the j^{th} profile and X_i is the corresponding explanatory variable, which is assumed to be known constant in each profile ($i = 1,2, \ldots, n; j = 1,2, \ldots$). If the relationship between the response variable and explanatory variable can be accurately shown through a simple linear regression model, then the model should be created in the following way,

$$Y_{ij} = A_{0j} + A_{1j} X_i + \varepsilon_{ij} \quad (1)$$

where, $\beta_0, \beta_1 and\ \varepsilon_{ij}$ represents intercept, slope and error terms respectively. It is assumed that $\varepsilon_{ij}{}'$ s are independent and identically distributed (i.i.d.) normal random variables with mean 0 and variance σ^2.The regression coefficients, $\beta_0\ and\ \beta_1$ are assumed to be known. Furthermore, it is presupposed that the X values remain constant and invariant in every sample. The ordinary least squares estimators of $\beta_0\ and\ \beta_1$ for the j^{th}sample are given by

$$\hat{\beta}_{0j} = \bar{Y}_j - \hat{\beta}_{1j}\bar{X} \text{ and } \hat{\beta}_{1j} = \frac{S_{xy(j)}}{S_{xx}} \quad (2)$$

where, $\bar{Y}_j = \frac{1}{n}\sum_{i=1}^{n} Y_{ij}$, $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X$, $S_{xy(j)} = \sum_{i=1}^{n}(X_i - \bar{X})Y_{ij}$ and $S_{xx} = \sum_{i=1}^{n}(X_i - \bar{X})^2$. The least square estimators $\hat{\beta}_{0j}$ and $\hat{\beta}_{1j}$ have a bivariate normal distribution with mean vector $\boldsymbol{U} = (\beta_0, \beta_1)'$ and variance-covariance matrix,

$$\Sigma = \begin{bmatrix} \sigma_0^2 & \sigma_{01} \\ \sigma_{01} & \sigma_1^2 \end{bmatrix} \tag{3}$$

$\sigma_0^2 = \sigma^2(n^{-1} + \frac{\bar{X}^2}{S_{xx}})$ and $\sigma_1^2 = \sigma^2/S_{xx}$ are the variances of $\hat{\beta}_{0j}$ and $\hat{\beta}_{1j}$ respectively and $\sigma_{01} = -\sigma^2\bar{X}^2/S_{xx}$ is the covariance between $\hat{\beta}_{0j}$ and $\hat{\beta}_{1j}$.

Kang and Albin [1] proposed the Hotelling's T^2 chart for monitoring intercept and slope of simple linear regression model. The vector of sample estimators $\boldsymbol{Z_j} = (\hat{\beta}_{0j}, \hat{\beta}_{1j})'$ for sample j is computed and then T^2 statistic is computed as

$$T_j^2 = (\boldsymbol{Z_j} - \boldsymbol{U})'\boldsymbol{\Sigma^{-1}}(\boldsymbol{Z_j} - \boldsymbol{U}) \tag{4}$$

where, $\boldsymbol{Z_j} = (\hat{\beta}_{0j}, \hat{\beta}_{1j})'$ is vector of estimated values of intercept and slope for j^{th} sample, $\boldsymbol{U} = (\beta_0, \beta_1)'$ and Σ is defined by Eq. (3). When the process is under control, T_j^2 follows a chi-square distribution with 2 degrees of freedom. Therefore, the upper control limit for the chart is $UCL = \chi^2_{2,\alpha}$, where $\chi^2_{2,\alpha}$ is the upper 100α percentage point of chi-square distribution with 2 degrees of freedom. When the process is not stable, the Hotelling's T^2statistic follows a non-central chi-square distribution with non-centrality parameter $\tau = n(\lambda\sigma + \delta\sigma\bar{X})^2 + (\delta\sigma)^2 S_{xx}$, where λ and δ are the shifts in the intercept and slope of the model given in Eq. (1).

3. Multiple linear regression profile model

In this section, the profile is represented by a multiple linear regression model. We consider a multiple linear regression model in $(p + 1)$ parameters to model the response variable Yas a function of p explanatory variables $X_{1,}X_{2,} \dots, X_p$. Let us assume that for the j^{th} random sample collected over time, we have n observations given as $X_{i1,}X_{i2,} \dots, X_{ip}$ and $Y_{ij} = i = 1,2, \dots, n$, where p is the number of explanatory variables. The relation between response variable and explanatory variables is characterized by a multiple linear regression model as

$$\boldsymbol{Y_j} = \boldsymbol{X\beta_j} + \boldsymbol{\varepsilon_j} \tag{5}$$

where $\boldsymbol{Y_j}$ is $n \times 1$ vector of response variables for the j^{th} sample, $\boldsymbol{X}$ **is** $n \times (p + 1)$ matrix of explanatory variables, $\boldsymbol{\beta_j}$ is $(p + 1) \times 1$ vector of regression parameters and $\boldsymbol{\varepsilon_j}$ is a $n \times 1$ vector of error terms which are assumed to be independently and identically distributed normal variables with mean zero and known variance σ^2. When $p = 1$, the model of multiple linear profiles reduces to a simple linear profile. In addition, the X values are assumed to be fixed and constants for each sample. The least squares estimator of $\boldsymbol{\beta_j}$ is given as

$$\widehat{\boldsymbol{\beta_j}} = (\boldsymbol{X'X})^{-1}\boldsymbol{X'Y} \tag{6}$$

and variance-covariance matrix of sample estimates of regression parameters is given as

$$\Sigma = cov(\widehat{\boldsymbol{\beta}}) = (\boldsymbol{X'X})^{-1}\sigma^2 \tag{7}$$

The Hotelling's T^2 statistic for j^{th} sample is computed by Eq. (4), where $\boldsymbol{Z_j} = (\hat{\beta}_{0j}, \hat{\beta}_{1j}, \dots, \hat{\beta}_{pj})'$ is the vector of estimated regression parameters and $U = (\beta_0, \beta_1, \dots, \beta_p)'$ is the vector of known regression parameters. The upper control limit of Hotelling's T^2 chart is given as $\chi^2_{p+1,\alpha}$. The theoretical ARL values of the T^2chart calculated as follows,

$$ARL = \frac{1}{P(T_j^2 > \chi^2_{p+1,\alpha})} \tag{8}$$

4. The GR control chart for monitoring linear regression profiles

In this section, we present the design structure of the proposed GR-T^2 chart for monitoring simple and multiple linear regression profiles. Following the work of Gadre and Ratihalli [23], in order to increase the detection ability of the Hotelling's T^2 chart, GR-T^2 chart combines the T^2 chart with an improved version of the CRL chart. The T^2 chart has only upper control limit (UL) and CRL chart has only the lower control limit L. Let Y_r be the r^{th} group based CRL and L be the lower limit of the CRL chart.

Operation of GR-T^2 Chart

Following Gadre and Rattihalli [23], the operation of the GR-T^2 chart is outlined by the following steps

1. Decide the upper control limit UL for the T^2 chart and lower control limit L of the CRL chart.
2. Take a group of n items (sample of size n) at each inspection point j and compute the chart statistic say T_j^2.
3. If $T_j^2 \leq UL$, then the group is considered as a conforming group and control flow goes back to step 2. Otherwise, the group is considered as nonconforming and control flow goes to the next step.
4. To determine the CRL (Y_r), count the number of T_j^2 samples between nonconforming groups.
5. If $Y_r > L$ the process is thought to be under control, and control flow goes back to step 2. If $Y_1 \leq L$ or two successive $Y_r \leq L$ and $Y_{r+1} \leq L$, for $r = 2,3, \ldots$ for the first time, the process is thought to be out-of-control, and control flow proceeds to the next step.
6. Indicate the out-of-control signal.
7. An assignable cause should be searched and take corrective action should be taken to remove it.

Let the expected number of groups (samples) required for a GR-T^2 chart to detect a shift of magnitude d in process mean vector be denoted by $ARL_{GR}(d)$. Following Gadre and Rattihalli [23], the ARL measure for the GR-T^2 chart is as follows:

$$ARL_{GR}(d) = \frac{1}{P(d)} \times \frac{1}{\{1-[1-P(d)]^L\}^2} \tag{9}$$

where, $P(d)$ is probability of detecting nonconforming group (sample) when shift of magnitude d is occurred.

$$P(d) = P[T^2 > UL/d \neq 0] = 1 - F_{(p+1,\lambda^2)}(UL) \tag{10}$$

where, $F_{(p+1,\lambda^2)}(.)$ is the cumulative distribution function of chi-square distribution with $p+1$ degrees of freedom and non-centrality parameter λ^2. The value of λ^2 is given as

$$\lambda^2 = (\mu_1 - \mu)'\Sigma^{-1}(\mu_1 - \mu) \tag{11}$$

where, $\boldsymbol{\mu_1}$ is vector of shifted regression parameters and $\boldsymbol{\mu}$ is vector of in-control regression parameters.

If $d = 0$, the in-control ARL of the GR-T^2 chart is given as

$$ARL_{GR}(0) = \frac{1}{P(0)} \times \frac{1}{\{1-[1-P(0)]^L\}^2} \tag{12}$$

where,
$$P(0) = P[T^2 > UL/d = 0] = 1 - F_{(p+1)}(UL) \tag{13}$$

The optimal design of GR-T^2chart depends on the desired in-control ARL, $ARL_{GR}(0)$ and out-of-control ARL, $ARL_{GR}(d^*)$. Here, d^* is the magnitude of shift considered large enough to seriously impair the quality of the products; the corresponding $ARL_{GR}(d^*)$ should be as small as possible. The GR control chart is designed by solving an optimization problem. The objective function to be minimized is

$$ARL_{GR}(d^*) = \frac{1}{P(d^*)} \times \frac{1}{\{1-[1-P(d^*)]^L\}^2} \tag{14}$$

subject to the equality constraint in Eq. (12).

Optimal Design Procedure of GR-T^2 Chart

We present the optimal design to obtain the optimal parameters (L, UL) for the GR-T^2 chart that result in minimum $ARL_{GR}(d^*)$ value, subject to in-control ARL $ARL_{GR}(0)$ which is at least equal to 200.

The optimal design procedure for the GR-T^2 chart is described as follows:

1. Specify p, d^*, Σ and $ARL_{GR}(0)$.
2. Obtain $P(0)$ by solving Eq. (12) numerically. From this value of $P(0)$, obtain the value of UL

using Eq. (13).

3. From the current values of L and UL compute $P(d^*)$ using Eq. (10) and then compute $ARL_{GR}(d^*)$ using Eq. (14)
4. If $ARL_{GR}(d^*)$ has been reduced then increase L by 1 and go back to Step 3. Otherwise, go to the next step.
5. Take the current L and UL as the final values for which $ARL_{GR}(d^*)$ is minimum.

5. The MGR control chart for monitoring linear regression profiles

The MGR chart proposed by Gadre and Rattihalli [29] is an extension of the GR chart with the inclusion of warning limit in the extended CRL scheme. Following Gadre and Rattihalli [29], we develop the procedure related to our proposed MGR-T^2 chart. The MGR-T^2 chart is divided into two parts. The first part assesses group conformity using a T^2-based technique, while the second analyzes process status through a group runs approach. This component has two levels of inspection.

- T^2 based procedure: If the value of T^2 statistic for a group of n units falls outside the upper control limit UL, declare the group as nonconforming; otherwise it is treated as a conforming group. based procedure: If the value of T^2 statistic for a group of n units falls outside the upper control limit UL, declare the group as nonconforming; otherwise it is treated as a conforming group.
- Group runs based procedure: Let Y_r denote the r^{th} (r=1,2,…) group based CRL. In other words, it is the number of groups inspected between $(r-1)^{th}$ and r^{th} nonconforming group.
- The group runs based procedure declares the process out-of-control if $Y_1 \leq L_2$ or for some $r(>1)$, $Y_r \leq L_1$ and $Y_{r+1} \leq L_2$ for the first time. Here L_1 is a warning limit and L_2 is lower limit of the CRL sub-chart chart.

Following Gadre and Rattihalli [29], the ARL expression for MGR-T^2 chart to detect a shift of magnitude d in the process mean vector is given as

$$ARL_{MGR}(d) = \frac{1}{P(d)} \times \frac{[1+Q(d)^{L_2} - Q(d)^{L_1}]}{[1-Q(d)^{L_1}][1-Q(d)^{L_2}]} \tag{15}$$

where, $Q(d) = 1 - P(d)$ and $P(d)$ is given in Eq. (10).
If $d = 0$, the In-Control ARL of the MGR-T^2 chart is given as

$$ARL_{MGR}(0) = \frac{1}{P(0)} \times \frac{[1+Q(0)^{L_2} - Q(0)^{L_1}]}{[1-Q(0)^{L_1}][1-Q(0)^{L_2}]} \tag{16}$$

The MGR-T^2 chart is designed by solving an optimization problem. The objective function to be minimized is

$$ARL_{MGR}(d^*) = \frac{1}{P(d^*)} \times \frac{[1+Q(d^*)^{L_2} - Q(d^*)^{L_1}]}{[1-Q(d^*)^{L_1}][1-Q(d^*)^{L_2}]} \tag{17}$$

subject to the equality constraint in Eq. (16).
Optimal design procedure of MGR-T^2 Chart
We present the optimal design to obtain the optimal parameters $(L_1, L_2, UL$ for the MGR-T^2 chart that result in minimum $ARL_{MGR}(d)$ value, subject to In-Control $ARL_{MGR}(0)$ which is at least equal to 200.
The optimal design procedure for the MGR-T^2 chart is described as follows

1. Specify p, d^*, Σ and $ARL_{MGR}(0)$.
2. Initialize L_1 as 1.
3. Initialize L_2 as 1.
4. Obtain $P(0)$ by solving Eq. (16) numerically. From this value of $P(0$ obtain the value of UL using Eq. (13).
5. From the current values of L_1, L_2 and UL compute $P(d^*)$ using Eq. (10) and then compute $ARL_{MGR}(d^*)$ using Eq. (17).
6. If $ARL_{MGR}(d^*)$ has been reduced then increase L_2 by 1 and go back to Step 4. Otherwise, go to

the next step.

7. If L_1 or the value of $ARL_{MGR}(d^*)$ has been reduced then increase L_1 by 1 and go back to Step 3; else go to the next step.
8. Take the values of L_1, L_2 and UL as the final values for which $ARL_{MGR}(d^*)$ is minimum.

6. Performance comparison

In this section, the proposed GR-$T^2$and MGR-T^2control chart methods for Phase II monitoring of linear regression profile processes were evaluated through simulation results, which demonstrate their effectiveness in detecting out-of-control conditions in simple linear and multiple linear regression profile processes. ARL is a performance measure used in this study for the evaluation of the proposed GR T^2 and MGR T^2 methods.

6.1. Performance comparison of simple linear regression profiles

In sub-section 6.1 of study, we aim to evaluate and compare the average run length performance of two proposed control chart methods for monitoring simple linear regression profiles, namely the GR T^2 and MGR T^2, with two existing methods. These existing methods include the control chart methods proposed by Kang and Albin [1], as well as the Shewhart-3 method suggested by Gupta et al. in [4], which are used as benchmarks for comparison. This study aims to provide valuable insights into the effectiveness of these control chart methods for monitoring simple linear regression profiles, which will aid practitioners in choosing an appropriate method for their specific needs. To compare the performance of proposed methods with above mentioned methods, the GR-T^2 and MGR-T^2 control charts are designed such that in-control ARL is approximately 200. For our study purpose, we have used in-control simple linear profile model which is given by Kang and Albin [1] as,

$$Y = 3 + 2X + \varepsilon \tag{18}$$

with ε follows i.i.d. normal random variables with mean zero and variance one and values of explanatory variable are fixed as $X = 2,4,6\ and\ 8$, following Kang and Albin [1]. The optimal design parameters and control limits for the proposed GR T^2 and MGR T^2 methods under $p = 1, n = 4$ and $d^* = 1$are given in Table 1 in order to achieve an overall ARL of 200.

Table 1: *Optimal design parameters for GR-T² and MGR-T²control chart*

Control chart	Optimal design parameters
GR T^2	$L = 16, UL = 6.9248,$
MGR T^2	$L_1 = 1, L_2 = 31, UL = 6.2459$

The shifts in the β_0, β_1 and σ considered in the study are presented in Table 2. These are same as the example discussed by Kang and Albin [1]. For performance evaluation of proposed GR-T^2 and MGR-T^2 methods to monitor simple regression profiles, we used 50000 simulation runs to obtain out-of-control ARL under different shifts in the β_0, β_1 and σ of the model given in Eq. (18). The ARL performance of above mentioned methods for monitoring simple linear regression profiles is given by Riaz et al. [7]. The out-of-control ARL values of proposed GR T^2 and MGR T^2 and other methods for detecting shifts in regression parameters of a simple linear regression model are presented in Table 3.

From Table 3, we observe that under the intercept shift from β_0 to $\beta_0 + \lambda\sigma$, the proposed GR-T^2 and MGR-T^2 methods produces smaller out-of-control ARL than the $T^2$and Shewhart-3 methods for entire range of shifts in the β_0 parameter. Similarly, under the slope shift from β_1 to $\beta_1 + \delta\sigma$, the proposed GR-T^2 and MGR-T^2 methods consistently produces smaller out-of-control ARL than the T^2method and Shewhart-3 method for entire range of shifts in the β_1 parameter. Finally, for monitoring the error standard deviation shifts from $\sigma \ to \ \gamma\sigma$, the proposed GR-T^2 and MGR-T^2 methods consistently produces shorter out-of-control average run lengths than the T^2 and Shewhart-3 methods for entire range of shifts in the slope parameter. Therefore, we conclude that both the GR-T^2 and MGR-T^2 methods are suitable for monitoring different shifts in the β_0, β_1 and σ of the simple linear regression model.

Table 2: *Shifts considered for various methods*

Type of shift	Notation	Values of the shift
	Simple linear profiles	
Shift in β_0	β_0 to $\beta_0 + \lambda\sigma$	$\lambda = 0.2, 0.4, 0.6, \ldots, 2.0$
Shift in β_1	β_1 to $\beta_1 + \delta\sigma$	$\delta = 0.025, 0.05, 0.075, \ldots, 0.25$
Shift in σ	$\sigma \ to \ \gamma\sigma$	$\gamma = 1.2, 1.4, 1.6, \ldots, 3.0$
	Multiple linear profiles	
Shift in β_0	β_0 to $\beta_0 + \lambda_0\sigma$	$\lambda_0 = 0.2, 0.4, 0.6, \ldots, 2.0$
Shift in β_1	β_1 to $\beta_1 + \lambda_1\sigma$	$\lambda_1 = 0.02, 0.04, 0.06, \ldots, 0.20$
Shift in σ	$\sigma \ to \ \gamma\sigma$	$\gamma = 1.1, 1.2, 1.3, \ldots, 2.0$

Table 3: *Performance comparison under the shifts in intercept, slope and error variance*

Method	Shift in $\beta_0(\lambda)$									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
T^2	137.7	63.5	28.0	13.2	6.9	4.0	2.6	1.8	1.5	1.2
Shewhart-3	151.4	78.0	33.3	15.5	7.7	4.3	2.7	1.9	1.5	1.2
GR-T^2	106.8	30.4	10.0	4.8	2.9	2.0	1.6	1.3	1.1	1.1
MGR-T^2	89.7	17.9	6.5	3.8	2.5	1.8	1.5	1.2	1.1	1.1
Method	Shift in $\beta_1(\delta)$									
	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2	0.225	0.25
T^2	166.0	105.6	60.7	34.5	20.1	12.2	7.8	5.2	3.7	2.7
Shewhart-3	178.3	125.0	79.2	46.7	27.9	17.1	10.9	7.1	5.0	3.6
GR-T^2	146.2	68.5	28.4	12.9	7.0	4.5	3.2	2.4	1.9	1.6
MGR-T^2	133.7	50.0	16.5	7.8	5.1	3.6	2.7	2.2	1.8	1.5
Method	Shift in $\sigma(\gamma)$									
	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
T^2	39.6	14.9	7.9	5.1	3.8	3.0	2.5	2.2	2.0	1.8
Shewhart-3	40.1	13.5	6.5	4.0	2.8	2.2	1.8	1.6	1.5	1.4
GR-T^2	18.2	6.5	3.9	2.9	2.4	2	1.8	1.7	1.6	1.5
MGR-T^2	10.8	4.9	3.4	2.6	2.2	1.9	1.7	1.6	1.5	1.4

6.2. Performance comparison of multiple linear regression profiles

In this section, we extend the T^2 method of Kang and Albin [1] to monitor multiple linear regression profiles. Further to improve performance of T^2 method, we apply the concept of GR and MGR charting schemes to the T^2 method. We compare ARL performance of the proposed GR-T^2, MGR-T^2 and T^2 to monitor multiple linear profiles in Phase II.

To compare the performance of proposed GR-T^2 and MGR-T^2 methods with T^2 method all the methods are designed such that in-control ARL is approximately 200. For our study purpose, we have used in-control multiple linear profile model used by Amiri et al. [11] as

$$Y = 3 + 2X_{1+}X_2 + X_3 + \varepsilon \quad (19)$$

where, $\beta_0 = 3, \beta_1 = 2, \beta_2 = 1$ and $\beta_3 = 1$. The error terms are independent normal random variables with mean 0 and known variance of $\sigma^2 = 1$. Following Amiri et al. [11], the values of explanatory variables X_1, X_2 and X_3 are given in the following transpose matrix.

$$X' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 & 2 & 4 & 6 & 8 \\ 1 & 4 & 3 & 2 & 1 & 4 & 3 & 2 \\ 1 & 3 & 2 & 4 & 4 & 3 & 2 & 4 \end{bmatrix}$$

The optimal design parameters of the proposed GR-T^2 and MGR-T^2 methods under $p = 3, n = 8$ and $d^* = 1$ are provided in Table 4 in order to achieve an overall ARL of 200.

Table 4: *Optimal design parameters for GR-T^2 and MGR-T^2 control chart*

Control chart	Optimal design parameters
GR T^2	$L = 20, UL = 10.2978$
MGR T^2	$L_1 = 1, L_2 = 31, UL = 6.2459,$

The shifts in regression parameters β_0, β_1 and σ considered in the study for multiple linear regression profiles are also presented in Table 2. These are same as the example discussed by Amiri et al. [11].For performance evaluation of proposed GR-T^2 and MGR-T^2 methods to monitor multiple regression profiles, we used 50000 simulation runs to obtain out-of-control ARL under different shifts in the regression parameters and error standard deviation of the model given in Eq. (19). The out-of-control ARL values of T^2, GR-T^2 and MGR-T^2 methods for detecting shifts in regression parameters and error standard deviation of a multiple linear regression model are presented in Table 5.

Table 5: The simulated out-of-control ARL values under the shifts in regression parameters of multiple linear regression model when $p = 3, n = 8$

Method	λ_0 (shift in β_0)										
	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
T^2	200.0	126.0	48.0	17.5	7.2	3.5	2.1	1.5	1.2	1.1	1.0
GR-T^2	200.0	89.2	19.8	6.4	3.2	2.0	1.4	1.2	1.1	1.0	1.0
MGR-T^2	200.0	64.8	11.5	5.2	2.8	1.8	1.3	1.1	1.0	1.0	1.0
Method	λ_1 (shift in β_1)										
	0.0	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
T^2	200.0	172.0	116.3	69.2	39.4	22.6	13.3	8.2	5.3	3.7	2.7
GR-T^2	200.0	153.2	77.8	33.6	15.3	8.1	5.1	3.5	2.6	2.0	1.6
MGR-T^2	200.0	138.4	53.5	18.6	9.5	6.2	4.3	3.1	2.4	1.9	1.5
Method	γ (shift in σ)										
	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
T^2	200.0	64.6	27.1	14.2	8.4	5.3	3.6	2.6	2.0	1.6	1.3
GR-T^2	200.0	32.5	11.4	6.3	4.4	3.3	2.7	2.3	2.0	1.8	1.7
MGR-T^2	200.0	18.3	7.9	5.2	3.8	3.0	2.5	2.1	1.9	1.7	1.6

From Table 5, we observe that the proposed GR-T^2 and MGR-T^2 methods are superior methods in detecting shifts in the parameters of multiple linear regression model. Under the error variance shifts, proposed methods are better than T^2 method except in very large shifts in which performance of all methods is approximately same.

7. An example

In this section, an example is illustrated by using proposed GR-T^2 and MGR-T^2 methods. Dataset is taken from Gupta et al. [4]. The dataset contains measurements of line widths on ten photo mask reference standards. These measurements are used to keep track of the linear calibration profiles of optical imaging systems. Simple linear regression profile for monitoring data given in Table 6 is as follows: $Y_{ij} = 0.2817 + 0.9767X_i$. Residual standard deviation is 0.06826 micrometers. The estimates of regression coefficients β_{0j} and β_{1j} are calculated using ordinal least square method.

Table 6: *Line-Width Measurements and* T^2 *values*

Day	Position	X	Y	$\hat{\beta}_{0j}$	$\hat{\beta}_{1j}$	T^2
1	L	0.76	1.12			
1	M	3.29	3.49	0.3194	0.9862	4.73
1	U	8.89	9.11			
2	L	0.76	0.99			
2	M	3.29	3.53	0.2891	0.9693	0.80
2	U	8.89	8.89			
3	L	0.76	1.05			
3	M	3.29	3.46	0.2726	0.9824	0.40
3	U	8.89	9.02			
4	L	0.76	0.76			
4	M	3.29	3.75	0.1149	1.0406	38.45
4	U	8.89	9.3			
5	L	0.76	0.96			
5	M	3.29	3.53	0.2279	0.9935	2.36
5	U	8.89	9.05			
6	L	0.76	1.03			
6	M	3.29	3.52	0.2847	0.9827	0.81
6	U	8.89	9.02			

GR-T^2 and MGR-T^2 control charts for data given in Table 6 are plotted in Figure 1. From Figure 1, it is noted that on the fourth day, the value of T^2 statistic is 38.45 which is greater than UL of both the GR-T^2 and MGR-T^2control charts. First conforming run length (Y_1) does not satisfy the criteria of in-control for both the GR-T^2 and MGR-T^2control charts. Therefore, charts give an out-of-control signal on fourth day.

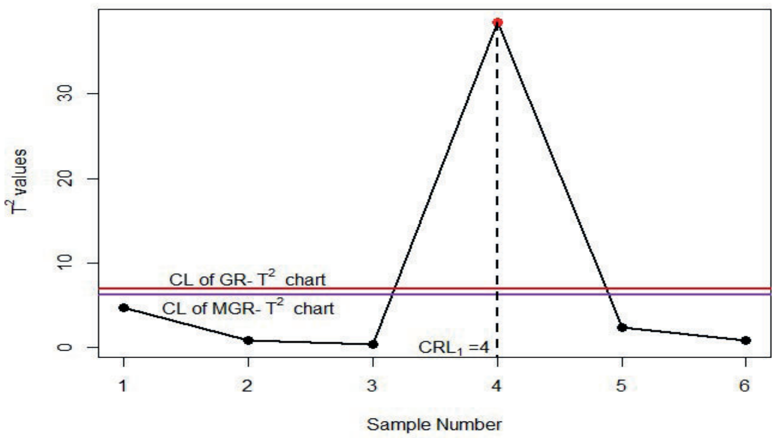

Figure 1: *GR-*T^2 *and MGR-*T^2 *control chart for Line-Width Measurements*

8. Conclusions

In this paper, we proposed two methods namely, GR-T^2 and MGR-T^2 to monitor simple and multiple linear regression profiles in Phase II. The performance of the proposed methods under simple linear profile monitoring is compared with the existing Shewhart-type methods namely T^2 method by Kang and Albin [1] and Shewhart-3 method by Gupta et al. [4] in terms of average run length criterion. From the numerical results, it is shown that the GR-T^2 and MGR-T^2 methods are very effective for detecting shifts in intercept, slope and error standard deviation. In addition, the performance of proposed methods in detecting shifts in the regression parameters and error standard deviation of multiple linear regression profiles is better than the T^2 method except very large shifts in which performance of all the methods is approximately same. Furthermore, the MGR-T^2 method has better performance than the GR-T^2 method for monitoring simple and multiple linear regression profiles. Hence, due to the effectiveness of the MGR-T^2 method, it can be more suitable for monitoring simple and multiple linear regression profiles.

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