π-Power Half Logistic-G Family: Applications to Medical and Traffic Data

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Abstract

This research article introduces a novel family of distributions achieved through the methodology of the π-power transformation technique. The study focuses on one specific member that is inverse Weibull distribution within this family, which showcases a hazard function exhibiting distinct J, reverse-J, bathtub, or monotonically increasing shapes. The article explores the essential characteristics of this distribution and employs the maximum likelihood estimation (MLE) method to estimate its associated parameters. To evaluate the accuracy of the estimation procedure, a simulation experiment is conducted, revealing a decrease in biases and mean square errors as sample sizes increase, even when working with small samples.

Furthermore, the practical application of the proposed distribution is demonstrated by analyzing two real medical and traffic datasets. By employing model selection criteria and conducting goodness-of-fit test statistics, the article establishes that the proposed model surpasses existing models in performance. The application of this research work can be significant in various fields where modeling and analyzing hazard functions or survival data are essential, while also making contributions to probability theory and statistical inferences.

Keywords: π-Power transformation, Half logistic, Reliability, Pivotal quantity

1. Introduction

Statistical models play a crucial role in representing and analyzing datasets in practical applications. While traditional distributions such as Weibull, Lomax, gamma, log-normal, exponential, beta, etc. have been widely used, they may not always provide a satisfactory fit for complex datasets. To address this limitation, researchers have been actively working on developing new models that offer greater adaptability and generality. These advancements often involve techniques such as exponentiation and the T-X approach to generate more flexible distributions. In this research article, we concentrate on an alternative approach called the π-power transformation (PPT) family, which was introduced by Lone and Jan [14]. The PPT family provides a distinct blend of high skewness and flexibility to the base distribution. The authors specifically examined the Pie-exponentiated Weibull as a member of this family. Prior to its introduction, the alpha power transformation (APT) approach had gained significant popularity among researchers in the fields of probability theory and survival analysis. Using the APT technique, numerous authors have put forward new generalized models and distribution families. For instance, Nassar et al. [20] employed the APT technique to define the new family of distributions using log transformation. Mead et al. [18] have further studied the APT family by providing some mathematical properties which were not provided in Mahdavi and Kundu [15]. Further, Lomax distribution was transformed using APT by Maruthan and Venkatachalam [17]. Ihtisham et al. [9] and Ihtisham et al. [10] studied the Pareto and inverse Pareto distributions using the APT approach, with the inverse Pareto distribution being applied to model real data related to extreme values. Similarly, Hozaien et al. [8] and Klakattawi and Aljuhani [12] introduced new models using the APT family of distributions. Alotaibi et al. [1] have introduced a new distribution as a weighted form of the

APT method while Gomma et al. [6] introduced the Alpha power of the power Ailamujia distribution, which offers a flexible hazard function. They utilized this distribution to model COVID-19 datasets from Italy and the UK. Furthermore, Nassar et al. [22] defined a new family utilizing the quantile function of the APT family whose cumulative distribution function (CDF) is

$$F(t)=\frac{\log\left[1+(\alpha-1)\,G\left(t;\phi\right)\right]}{\log\left(\alpha\right)};\quad t>0,\alpha>0,\alpha\neq 1,$$

where $G\left(t;\phi\right)$ is the CDF and $\phi$is the parameter space of base distribution. Similarly, Elbatal et al. [5] introduced another new APT family whose CDF is

$$F\left(x\right)=\frac{\alpha^{G(x)}G\left(x\right)}{\alpha};\alpha>0,x\in\Re.$$

Similarly, Kyurkchiev [13] has introduced a family of distribution based on the Verhulst logistic function and its CDF is

$$F\left(x\right)=\frac{2G\left(x\right)}{1-G\left(x\right)};x\in\Re.$$

Another new method for transformation can be found in Kavya and Manoharan [11] and the CDF of this transformation is

$$F\left(x\right)=\frac{e}{e-1}\left\{1-e^{-G(x)}\right\};x\in\Re.$$

Also using the APT method Mandouch et al. [16] have reported a new two-parameter family of distributions whose CDF is

$$F\left(x\right)=\frac{\alpha^{kW\{G(x)\}}-1}{\alpha-1};\alpha>0,\alpha\neq 1,x\in\Re.$$

Lone and Jan [14] have introduced another new family using the concept of the APT family and named it the Pie-Exponentiated transformed (PET) family whose CDF is

$$F\left(x\right)=\frac{\pi^{\{G(x)\}^{\alpha}}-1}{\pi-1};\alpha>0,x\in\Re.$$

Hence, researchers are continuously developing and exploring new models and families of distributions to better capture the characteristics of complex datasets. The PET family has emerged as a popular approach, offering increased skewness and flexibility to the base distribution. These advancements have led to the proposal of various generalized models and distributions, which have been successfully applied to a range of datasets, including those related to COVID-19, reliability engineering, and extreme values. Building upon the concept of the PET, we have introduced a novel method to enhance existing distributions by incorporating a logistic form of CDF of any continuous distribution, which we refer to as the π-power half logistic-G (π-PHL-G) family of distributions. This new family offers increased robustness compared to other compound probability distributions and demonstrates great potential for modeling real-life datasets. The suggested family possesses two parameters that enable it to capture a broader range of characteristics exhibited by a dataset, including skewness, kurtosis, failure rate, and mathematically tractable. This enhanced flexibility allows for a more accurate representation of complex data patterns and distributional properties. By considering the π-PHL-G family, researchers, and practitioners can better account for the intricate nature of real-world datasets, leading to improved modeling outcomes. Among the members of the π-PHL-G family, one distribution stands out as particularly noteworthy-the inverse Weibull distribution. The inverse Weibull distribution has long been employed in reliability theory and life testing due to its ability to capture failure rates and survival probabilities effectively see [19, 24, 25]. With the integration of the π-PHL-G framework, the Weibull distribution can be further adapted and refined to better align with the unique characteristics observed in various applications. We have organized the remaining sections of this paper as follows; π-PHL-G family is introduced in Section 2, while its particular member, the π-PHL-Weibull distribution, is presented in Section 3. Some statistical properties are discussed in Section 4, and in Section 5, we discuss statistical inferences of the π-PHLIW distribution. The simulation experiment, application, and conclusion of the suggested model are presented in Sections 6, 7, and 8 respectively.

2. π-PHL-G Family and Some Important Functions

Let $Y \sim \pi$-PHL-G family, then the CDF and PDF of π-PHL-G family $U(y;\Psi)$ and $u(y;\Psi)$ for $y \in \Re$, and $\Psi > 0$ is vector of parameters are defined as

$$U(y;\Psi) = \frac{\pi^{\left(\frac{2T(y;\Psi)}{1+T(y;\Psi)}\right)} - 1}{\pi - 1}; \quad y \in \Re. \tag{1}$$

$$u(y;\Psi) = \frac{(\log \pi)}{\pi - 1} \pi^{\left(\frac{2T(y;\Psi)}{1+T(y;\Psi)}\right)} \frac{2t(y;\Psi)}{[1 + T(y;\Psi)]^2}; \quad y \in \Re. \tag{2}$$

where $T(y;\Psi)$ and $t(y;\Psi)$ are the CDF and PDF of any continuous distribution and $\bar{T}(y;\Psi)$ is the reliability function. Further reliability and hazard functions of π-PHL-G family can be expressed as

$$R(y;\Psi) = 1 - \left\{ \frac{\pi^{\left(\frac{2T(y;\Psi)}{1+T(y;\Psi)}\right)} - 1}{\pi - 1} \right\}; \quad y \in \Re.$$

$$h(y;\Psi) = \frac{(\log \pi)}{\pi - 1} \pi^{\left(\frac{2T(y;\Psi)}{1+T(y;\Psi)}\right)} \frac{2t(y;\Psi)}{[1 + T(y;\Psi)]^2} \left[1 - \left\{ \frac{\pi^{\left(\frac{2T(y;\Psi)}{1+T(y;\Psi)}\right)} - 1}{\pi - 1} \right\} \right]^{-1}.$$

2.1. Qunatile function and Random deviation

$$Q_Y(p) = T^{-1} \left\{ \frac{\log\left((\pi - 1)\, p + 1\right)}{\left(2 \log \pi - \log\left((\pi - 1)\, p + 1\right)\right)} \right\}. \tag{3}$$

and

$$y = T^{-1} \left\{ \frac{\log\left((\pi - 1)\, u + 1\right)}{\left(2 \log \pi - \log\left((\pi - 1)\, u + 1\right)\right)} \right\}.$$

2.2. Linear form of π-PHL-G distribution

The CDF defined in Equation 6 can be expressed in the linear form as

$$U(y;\Psi) = \frac{1}{\pi - 1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{2^i (\log \pi)^i}{i!} \binom{i}{j} T^{i+j}(y;\Psi) - \frac{1}{\pi - 1}. \tag{4}$$

Now differentiating Equation 5 with respect to y we get the linear form of PDF as

$$u(y;\Psi) = \frac{1}{\pi - 1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{2^i (i+j) (\log \pi)^i}{i!} \binom{i}{j} T^{i+j-1}(y;\Psi) t(y;\Psi).$$

$$u(y;\Psi) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Delta_{ij} T^{i+j-1}(y;\Psi) t(y;\Psi), \tag{5}$$

where $\Delta_{ij} = \left(\frac{1}{\pi-1}\right) \frac{2^i (i+j)(\log \pi)^i}{i!} \binom{i}{j}$.

3. π-Power Half Logistic Inverse Weibull (π-PHLIW) Distribution

Let Y be a continuous random variable following the inverse Weibull distribution, then the CDF and PDF are

$$T(y;\Psi) = e^{-\beta y^{-\delta}}.$$

$$t(y;\Psi) = \beta \delta y^{-(\delta+1)} e^{-\beta y^{-\delta}}.$$

Now using Equation 3 as a base distribution, we introduce a new distribution π-PHLIW distribution as a special member having CDF

$$U(y;\beta,\delta)=\frac{\pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)}-1}{\pi-1} \quad ;\beta,\delta>0, y>0. \tag{6}$$

The PDF of the π-PHLIW distribution can be expressed as

$$u(y;\beta,\delta)=\frac{2\beta\delta(\log\pi)}{\pi-1}\pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)}\frac{y^{-(\delta+1)}e^{-\beta y^{-\delta}}}{\left[1+e^{-\beta y^{-\delta}}\right]^2}; \quad y>0. \tag{7}$$

Now some key functions like reliability and hazard function of the π-PHLIW distribution can be presented as

$$R(y;\beta,\delta)=1-\frac{1}{\pi-1}\left\{\pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)}-1\right\}; \quad y>0.$$

$$h(y;\beta,\delta)=\frac{2\beta\delta(\log\pi)}{\pi-1}\pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)}\frac{y^{-(\delta+1)}e^{-\beta y^{-\delta}}}{\left[1+e^{-\beta y^{-\delta}}\right]^2}\left[1-\left\{\frac{\pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)}-1}{\pi-1}\right\}\right]^{-1}.$$

3.1. Quantile function and Random deviation

The quantile function for the suggested distribution can be obtained by inverting the CDF defined in Equation 6 as

$$Q_Y(p)=\left[\log\left\{\frac{\log\left((\pi-1)\,p+1\right)}{\left(2\log\pi-\log\left((\pi-1)\,p+1\right)\right)}\right\}^{-1/\beta}\right]^{-1/\delta}. \tag{8}$$

also, random number deviate can be expressed as

$$y=\left[\log\left\{\frac{\log\left((\pi-1)\,u+1\right)}{\left(2\log\pi-\log\left((\pi-1)\,u+1\right)\right)}\right\}^{-1/\beta}\right]^{-1/\delta}.$$

The π-PHLIW distribution has a density plot that can take on a diversity of shapes, including symmetrical, left-skewed, right-skewed, or decreasing, and Figure 1 (left) shows some examples of these shapes. The HRF, on the other hand, can take on the shapes of an increasing, a j, or a reverse-j, and Figure 1 (right) shows some examples of these shapes.

4. Statistical properties of π-PHLIW distribution

4.1. Linear form of PDF of π-PHLIW distribution

After some mathematics using Equation 7, the PDF of π-PHLIW distribution can be obtained in a linear form as

$$u(y;\beta,\delta)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}y^{-(\delta+1)}e^{-(i+j)\beta y^{-\delta}}, \tag{9}$$

where $\Delta_{ij}^{*}=\beta\delta\Delta_{ij}$.

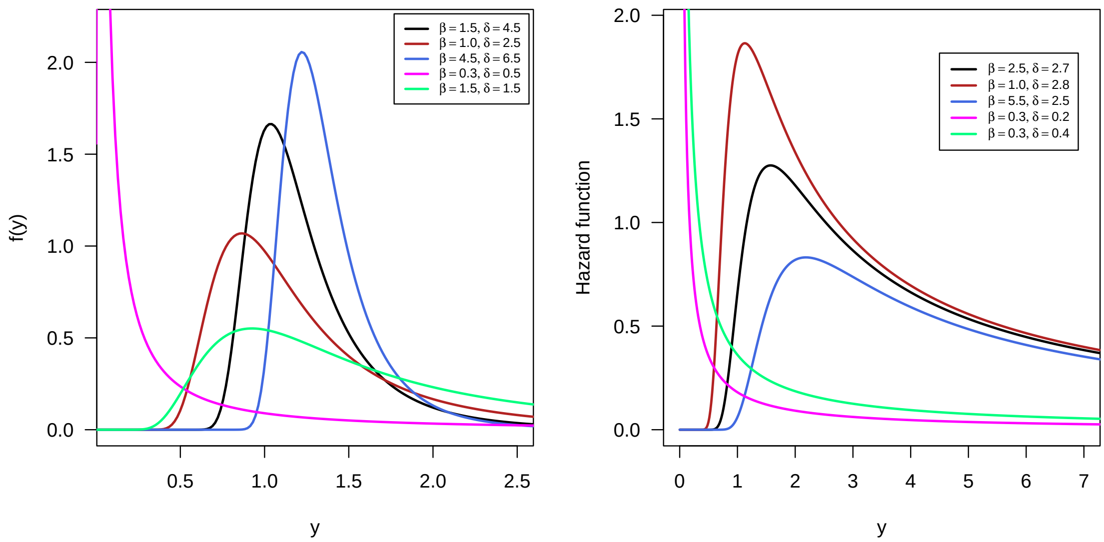

Figure 1: *Shapes of PDF and HRF of π-PHLIW distribution.*

4.2. Moments

The r^{th} moment of π-PHLIW distribution is

$$\begin{aligned} E\left[Y^r\right] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\int_0^{\infty} y^{r-\delta-1}e^{-(i+j)\beta y^{-\delta}}dy \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\int_0^{\infty} t^{-\frac{r}{\delta}+1-1}e^{-(i+j)\beta t}dt \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\delta^{-1}\Gamma\left(-\frac{r}{\delta}+1\right)}{\left\{(i+j)\beta\right\}^{-\frac{r}{\delta}+1}} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\delta^{-1}\Gamma\left(\frac{\delta-r}{\delta}\right)}{\left\{(i+j)\beta\right\}^{\frac{\delta-r}{\delta}}}; \quad \delta > r. \end{aligned} \tag{10}$$

Now mean and variance of π-PHLIW distribution can be expressed as

$$E\left[Y\right] = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\delta^{-1}\Gamma\left(\frac{\delta-1}{\delta}\right)}{\left\{(i+j)\beta\right\}^{\frac{\delta-1}{\delta}}}; \quad \delta > 1.$$

and

$$E\left[Y^2\right] = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\delta^{-1}\Gamma\left(\frac{\delta-2}{\delta}\right)}{\left\{(i+j)\beta\right\}^{\frac{\delta-2}{\delta}}}; \quad \delta > 2.$$

$$\begin{aligned} V[Y] &= E\left[Y^2\right] - \left[E\left(Y\right)\right]^2 \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\delta^{-1}\Gamma\left(\frac{\delta-2}{\delta}\right)}{\left\{(i+j)\beta\right\}^{\frac{\delta-2}{\delta}}} - \left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\delta^{-1}\Gamma\left(\frac{\delta-1}{\delta}\right)}{\left\{(i+j)\beta\right\}^{\frac{\delta-1}{\delta}}}\right]^2; \delta > 2. \end{aligned}$$

4.3. Moment Generating Function (MGF)

For any real number t, the MGF of π-PHLIW distribution can be defined as

$$M_Y(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta_{ij}^{*}\frac{t^k}{k!}\int_0^{\infty} y^{r-(\delta+1)}e^{-(i+j)\beta y^{-\delta}}dy$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta_{ij}^{*}\frac{t^k}{k!}\int_0^{\infty} t^{-\frac{r}{\delta}+1-1}e^{-(i+j)\beta t}dt$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta_{ij}^{*}\frac{t^k}{k!}\frac{\delta^{-1}\Gamma\left(-\frac{r}{\delta}+1\right)}{\{(i+j)\beta\}^{-\frac{r}{\delta}+1}}$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta_{ij}^{*}\frac{t^k}{k!}\frac{\delta^{-1}\Gamma\left(\frac{\delta-r}{\delta}\right)}{\{(i+j)\beta\}^{\frac{\delta-r}{\delta}}};\delta > r.$$

4.4. Characteristic Function

For any real number t, the characteristic function of π-PHLIW distribution can be defined as

$$\Phi_Y(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta_{ij}^{*}\frac{(vt)^k}{k!}\int_0^{\infty} y^{r-(\delta+1)}e^{-(i+j)\beta y^{-\delta}}dy$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta_{ij}^{*}\frac{(vt)^k}{k!}\int_0^{\infty} t^{-\frac{r}{\delta}+1-1}e^{-(i+j)\beta t}dt$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta_{ij}^{*}\frac{(vt)^k}{k!}\frac{\delta^{-1}\Gamma\left(-\frac{r}{\delta}+1\right)}{\{(i+j)\beta\}^{-\frac{r}{\delta}+1}}$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta_{ij}^{*}\frac{(vt)^k}{k!}\frac{\delta^{-1}\Gamma\left(\frac{\delta-r}{\delta}\right)}{\{(i+j)\beta\}^{\frac{\delta-r}{\delta}}};\delta > r.$$

where $v = \sqrt{-1}$.

4.5. Incomplete moment

The incomplete moment for π-PHLIW distribution is given by

$$M_r(z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\int_0^{z} y^{r-\delta-1}e^{-(i+j)\beta y^{-\delta}}dy$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\delta^{-1}\left[\gamma\left(\frac{\delta-r}{\delta},(i+j)\beta z^{-\delta}\right)\right]}{\{(i+j)\beta\}^{\frac{\delta-r}{\delta}}};\delta > r.$$

where $\gamma(.)$ incomplete gamma function.

4.6. Mean Residual Life

The Mean residual life for π-PHLIW distribution is given by

$$\overline{M(z)} = \frac{1}{F(z)}\left[\mu - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\int_0^{z} y^{-\delta}e^{-(i+j)\beta y^{-\delta}}dy\right] - z$$
$$= \frac{1}{F(z)}\left[\mu - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\delta^{-1}\left[\gamma\left(\frac{\delta+1}{\delta},(i+j)\beta z^{-\delta}\right)\right]}{\{(i+j)\beta\}^{\frac{\delta+1}{\delta}}}\right] - z$$

where $\gamma(.)$ incomplete gamma function.

4.7. Order Statistics

Let $y_i (i = 1, 2, ..., n) \sim \pi - PHLIW(y; \beta, \delta)$ with CDF $U(y_i; \beta, \delta)$ and PDF $u(y_i; \beta, \delta)$. If $u_r(y)$ denote the PDF of rth order statistic $Y_{(r)}$, then their CDF and PDF are given by $U_r(y) = I_{U(y)}(r, n-r+1)$

$$u_r(y) = \frac{d}{dy}\left[U_r(y)\right] = \frac{d}{dy}\left[I_{U(y)}(r, n-r+1)\right] = \frac{1}{B(r, n-r+1)} U^{r-1}(y) u(y) \left[1 - U(y)\right]^{n-r}.$$

$$u_r(y) = \frac{1}{B(r, n-r+1)} \frac{2\beta\delta(\log\pi)}{\pi - 1} \pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)} \frac{y^{-(\delta+1)} e^{-\beta y^{-\delta}}}{\left[1 + e^{-\beta y^{-\delta}}\right]^2} \left[\frac{1}{\pi - 1}\left\{\pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)} - 1\right\}\right]^{r-1}$$
$$\left[1 - \left\{\frac{1}{\pi - 1}\left\{\pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)} - 1\right\}\right\}\right]^{n-r}.$$

The CDF and PDF of first order statistic $Y_{(1)}$ are given by

$$U_1(y) = 1 - \left[1 - \left\{\frac{1}{\pi - 1}\left\{\pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)} - 1\right\}\right\}\right]^{n}; y > 0.$$

$$u_1(y) = \frac{2n\beta\delta(\log\pi)}{\pi - 1} \pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)} \frac{y^{-(\delta+1)} e^{-\beta y^{-\delta}}}{\left[1 + e^{-\beta y^{-\delta}}\right]^2} \left[1 - \left\{\frac{1}{\pi - 1}\left\{\pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)} - 1\right\}\right\}\right]^{n-1}; y > 0.$$

The CDF and PDF of first order statistic $Y_{(n)}$ are given by

$$U_n(y) = \left[\frac{1}{\pi - 1}\left\{\pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)} - 1\right\}\right]^{n}; y > 0.$$

$$u_n(y) = \frac{2n\beta\delta(\log\pi)}{\pi - 1} \pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)} \frac{y^{-(\delta+1)} e^{-\beta y^{-\delta}}}{\left[1 + e^{-\beta y^{-\delta}}\right]^2} \left[\frac{1}{\pi - 1}\left\{\pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)} - 1\right\}\right]^{n-1}; y > 0.$$

The Joint PDF of r^{th} and sth order statistics is given by

$$u_{rs}(x, y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} U^{r-1}(x).u(x)\left[U(y) - U(x)\right]^{s-r-1} u(y).\left[1 - U(y)\right]^{n-s}$$

$$u_{rs}(x, y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \pi^{\left\{\left(\frac{2e^{-\beta x^{-\delta}}}{1+e^{-\beta x^{-\delta}}}\right)+\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)\right\}} \frac{e^{-\beta(x^{-\delta}+y^{-\delta})}(xy)^{-(\delta+1)}}{\left[1 + e^{-\beta y^{-\delta}}\right]^2\left[1 + e^{-\beta x^{-\delta}}\right]^2}$$
$$\left[\frac{2\beta\delta(\log\pi)}{\pi - 1}\right]^2 \left[\frac{1}{\pi - 1}\left\{\pi^{\left(\frac{2e^{-\beta x^{-\delta}}}{1+e^{-\beta x^{-\delta}}}\right)} - 1\right\}\right]^{r-1}$$
$$\left[\left\{\frac{1}{\pi - 1}\left\{\pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)} - 1\right\}\right\} - \left\{\frac{1}{\pi - 1}\left\{\pi^{\left(\frac{2e^{-\beta x^{-\delta}}}{1+e^{-\beta x^{-\delta}}}\right)} - 1\right\}\right\}\right]^{s-r-1}$$
$$\left[1 - \left\{\frac{1}{\pi - 1}\left\{\pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)} - 1\right\}\right\}\right]^{n-s}; x > 0, y > 0.$$

The Joint PDF of the 1^{st} and n^{th} order statistics are given by

$$u_{1n}(x, y) = n(n-1)\left[U(y) - U(x)\right]^{n-2} u(x).u(y)$$

$$u_{1n}(x, y) = n(n-1)\left(\frac{2\beta\delta(\log\pi)}{\pi - 1}\right)^2 \left[\left\{\frac{1}{\pi - 1}\left\{\pi^{\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)} - 1\right\}\right\} - \left\{\frac{1}{\pi - 1}\left\{\pi^{\left(\frac{2e^{-\beta x^{-\delta}}}{1+e^{-\beta x^{-\delta}}}\right)} - 1\right\}\right\}\right]^{n-2}$$
$$\pi^{\left\{\left(\frac{2e^{-\beta x^{-\delta}}}{1+e^{-\beta x^{-\delta}}}\right)+\left(\frac{2e^{-\beta y^{-\delta}}}{1+e^{-\beta y^{-\delta}}}\right)\right\}} \frac{e^{-\beta(x^{-\delta}+y^{-\delta})}(xy)^{-(\delta+1)}}{\left[1 + e^{-\beta y^{-\delta}}\right]^2\left[1 + e^{-\beta x^{-\delta}}\right]^2}; x > 0, y > 0$$

4.8. System Reliability Function

4.8.1 Series System:

Consider a system with n independent components, each component follows $\pi - PHLIW(y;\beta,\delta)$ distribution. Let's assume $T_i(i=1,2,...,n) \sim \pi - PHLIW(y;\beta,\delta)$ with CDF $U(t_i;\beta,\delta)$ and PDF $u(t_i;\beta,\delta)$, then the system reliability for linear consecutive (series system) is given by

$$R_S(t) = \prod_{i=1}^{n} R_i(t) = \left[1 - \left\{\frac{1}{\pi-1}\left\{\pi^{\left(\frac{2e^{-\beta t^{-\delta}}}{1+e^{-\beta t^{-\delta}}}\right)} - 1\right\}\right\}\right]^n \quad ;\beta,\delta > 0, t > 0.$$

The CDF system reliability for linear consecutive (series system) is given by

$$F_S(t) = 1 - \prod_{i=1}^{n} R_i(t) = 1 - \left[1 - \left\{\frac{1}{\pi-1}\left\{\pi^{\left(\frac{2e^{-\beta t^{-\delta}}}{1+e^{-\beta t^{-\delta}}}\right)} - 1\right\}\right\}\right]^n \quad ;\beta,\delta > 0, t > 0. \tag{11}$$

Differentiating the Equation 11, the PDF system reliability for linear consecutive (series system) is given by

$$f_S(t) = \frac{dF_S(t)}{dt} = \frac{2n\beta\delta(\log\pi)}{\pi-1}\pi^{\left(\frac{2e^{-\beta t^{-\delta}}}{1+e^{-\beta t^{-\delta}}}\right)}\frac{t^{-(\delta+1)}e^{-\beta t^{-\delta}}}{\left[1+e^{-\beta t^{-\delta}}\right]^2}\left[1 - \left\{\frac{1}{\pi-1}\left\{\pi^{\left(\frac{2e^{-\beta t^{-\delta}}}{1+e^{-\beta t^{-\delta}}}\right)} - 1\right\}\right\}\right]^{n-1}$$

The Hazard function system reliability for linear consecutive (series system) is obtained by the ratio of PDF and Reliability function of linear consecutive (series system).

$$h_S(t) = \frac{f_S(t)}{R_S(t)} = \frac{2n\beta\delta(\log\pi)}{\pi-1}\pi^{\left(\frac{2e^{-\beta t^{-\delta}}}{1+e^{-\beta t^{-\delta}}}\right)}\frac{t^{-(\delta+1)}e^{-\beta t^{-\delta}}}{\left[1+e^{-\beta t^{-\delta}}\right]^2}\left[1 - \left\{\frac{1}{\pi-1}\left\{\pi^{\left(\frac{2e^{-\beta t^{-\delta}}}{1+e^{-\beta t^{-\delta}}}\right)} - 1\right\}\right\}\right]^{-1}.$$

4.8.2 Parallel System:

Consider a system with n independent components, each component follows $\pi - PHLIW(y;\beta,\delta)$ distribution. Lets assume $T_i(i=1,2,...,n) \sim \pi - PHLIW(y;\beta,\delta)$ with CDF $U(t_i;\beta,\delta)$ and PDF $u(t_i;\beta,\delta)$, then the reliability function for parallel system is given by

$$R_P(t) = 1 - \prod_{i=1}^{n}(1 - R_i(t)) = 1 - \left[\frac{1}{\pi-1}\left\{\pi^{\left(\frac{2e^{-\beta t^{-\delta}}}{1+e^{-\beta t^{-\delta}}}\right)} - 1\right\}\right]^n \quad ;\beta,\delta > 0, t > 0.$$

The CDF for the parallel system is given by

$$F_P(t) = \left[\frac{1}{\pi-1}\left\{\pi^{\left(\frac{2e^{-\beta t^{-\delta}}}{1+e^{-\beta t^{-\delta}}}\right)} - 1\right\}\right]^n \quad ;\beta,\delta > 0, t > 0.$$

Differentiating the above equation, the PDF formula for the parallel system is given by

$$f_P(t) = \frac{2n\beta\delta(\log\pi)}{\pi-1}\pi^{\left(\frac{2e^{-\beta t^{-\delta}}}{1+e^{-\beta t^{-\delta}}}\right)}\frac{t^{-(\delta+1)}e^{-\beta t^{-\delta}}}{\left[1+e^{-\beta t^{-\delta}}\right]^2}\left[\frac{1}{\pi-1}\left\{\pi^{\left(\frac{2e^{-\beta t^{-\delta}}}{1+e^{-\beta t^{-\delta}}}\right)} - 1\right\}\right]^{n-1}$$

The Hazard function for a parallel system is obtained by the ratio of PDF and the Reliability function of a parallel system.

$$h_P(t) = \frac{\frac{2n\beta\delta(\log\pi)}{\pi-1}\pi^{\left(\frac{2e^{-\beta t^{-\delta}}}{1+e^{-\beta t^{-\delta}}}\right)}\frac{t^{-(\delta+1)}e^{-\beta t^{-\delta}}}{\left[1+e^{-\beta t^{-\delta}}\right]^2}\left[\frac{1}{\pi-1}\left\{\pi^{\left(\frac{2e^{-\beta t^{-\delta}}}{1+e^{-\beta t^{-\delta}}}\right)} - 1\right\}\right]^{n-1}}{1 - \left[\frac{1}{\pi-1}\left\{\pi^{\left(\frac{2e^{-\beta t^{-\delta}}}{1+e^{-\beta t^{-\delta}}}\right)} - 1\right\}\right]^n}.$$

5. Statistical Inference

5.1. Estimation

The parameters of the suggested model are estimated using the maximum likelihood method (MLE). Let $y_i(i = 1, 2, ..., m) \sim \pi - PHLIW(y; \beta, \delta)$ with PDF $u(y_i; \beta, \delta)$then the log-likelihood function can be calculated as

$$\ell\left(\underline{y}; \beta, \delta\right) = n\log\left(2\beta\delta(\log\pi)\right) - n\log\left(\pi - 1\right) + (\log\pi)\sum_{i=1}^{n}\left(\frac{2e^{-\beta y_i^{-\delta}}}{1+e^{-\beta y_i^{-\delta}}}\right) - (\delta+1)\sum_{i=1}^{n}\log y_i - \beta\sum_{i=1}^{n}y_i^{-\delta} - 2\sum_{i=1}^{n}\log\left(1+e^{-\beta y_i^{-\delta}}\right). \tag{12}$$

Differentiating Equation 12 with respect to associated parameters, we get the first-order derivatives as

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} - (\log\pi)\sum_{i=1}^{n}\frac{2y_i^{-\delta}e^{-\beta y_i^{-\delta}}}{\left\{1+e^{-\beta y_i^{-\delta}}\right\}^2} - \sum_{i=1}^{n}y_i^{-\delta} + 2\sum_{i=1}^{n}\frac{y_i^{-\delta}e^{-\beta y_i^{-\delta}}}{\left\{1+e^{-\beta y_i^{-\delta}}\right\}}.$$

$$\frac{\partial\ell}{\partial\delta} = \frac{n}{\delta} + (\log\pi)\sum_{i=1}^{n}\frac{2\beta y_i^{-\delta}\left(\log y_i\right)e^{-\beta y_i^{-\delta}}}{\left\{1+e^{-\beta y_i^{-\delta}}\right\}^2} - \sum_{i=1}^{n}\log y_i + \beta\sum_{i=1}^{n}y_i^{-\delta}\left(\log y_i\right) - 2\sum_{i=1}^{n}\frac{y_i^{-\delta}\left(\log y_i\right)\beta e^{-\beta y_i^{-\delta}}}{\left\{1+e^{-\beta y_i^{-\delta}}\right\}}$$

By solving the above two non-linear equations using suitable software one can obtain the estimates under the MLE method.

5.2. Cramer-Rao (CR) Inequality

If $T(y_1, ..., y_n)$ is an unbiased estimator for $k(\omega)$, a function of parameter ω, then

$$Var[T(y_1, ..., y_n)] \geqslant \frac{\left\{\frac{d}{d\omega}k(\omega)\right\}^2}{E\left(\frac{\partial}{\partial\omega}logL\right)} = \frac{\left\{k'(\omega)\right\}^2}{I(\omega)},$$

where $I(\omega)$ is the information on ω, supplied by the sample. To define CR lower bound (CRLB) for β when δ is supposed to be known, then CRLB for an unbiased estimator $T_1(y_1, ..., y_n)$ of a parameter β is given by $\frac{1}{I(\beta)}$, where

$$I(\beta) = -E\left[\frac{\partial^2\ell}{\partial\beta^2}\right] = \frac{n}{\beta^2} + (\log\pi)\sum_{i=1}^{n}E\left\{\frac{\partial}{\partial\beta}\left\{\frac{2y_i^{-\delta}e^{-\beta y_i^{-\delta}}}{\left\{1+e^{-\beta y_i^{-\delta}}\right\}^2}\right\}\right\} - 2\sum_{i=1}^{n}E\left\{\frac{\partial}{\partial\beta}\left\{\frac{y_i^{-\delta}e^{-\beta y_i^{-\delta}}}{\left\{1+e^{-\beta y_i^{-\delta}}\right\}}\right\}\right\}$$

and

$$\frac{\partial^2\ell}{\partial\beta^2} = -\frac{n}{\beta^2} - (\log\pi)\sum_{i=1}^{n}\frac{\partial}{\partial\beta}\left\{\frac{2y_i^{-\delta}e^{-\beta y_i^{-\delta}}}{\left\{1+e^{-\beta y_i^{-\delta}}\right\}^2}\right\} + 2\sum_{i=1}^{n}\frac{\partial}{\partial\beta}\left\{\frac{y_i^{-\delta}e^{-\beta y_i^{-\delta}}}{\left\{1+e^{-\beta y_i^{-\delta}}\right\}}\right\}$$

Again CRLB for δ when β is supposed to be known, then CRLB for an unbiased estimator $T_2(y_1, ..., y_n)$ of a parameter δ is given by $\frac{1}{I(\delta)}$, where

$$I(\delta) = -E\left[\frac{\partial^2\ell}{\partial\delta^2}\right] = \frac{n}{\delta^2} - (\log\pi)\sum_{i=1}^{n}E\left\{\frac{\partial}{\partial\delta}\left\{\frac{2\beta y_i^{-\delta}\left(\log y_i\right)e^{-\beta y_i^{-\delta}}}{\left\{1+e^{-\beta y_i^{-\delta}}\right\}^2}\right\}\right\} + \beta\sum_{i=1}^{n}E\left\{y_i^{-\delta}\left(\log y_i\right)^2\right\} + 2\sum_{i=1}^{n}E\left\{\frac{\partial}{\partial\delta}\left\{\frac{y_i^{-\delta}\left(\log y_i\right)\beta e^{-\beta y_i^{-\delta}}}{\left\{1+e^{-\beta y_i^{-\delta}}\right\}}\right\}\right\}$$

And

$$\frac{\partial^2 \ell}{\partial \delta^2} = -\frac{n}{\delta^2} + (\log \pi) \sum_{i=1}^{n} \frac{\partial}{\partial \delta} \left\{ \frac{2\beta y_i^{-\delta} (\log y_i) e^{-\beta y_i^{-\delta}}}{\left\{1 + e^{-\beta y_i^{-\delta}}\right\}^2} \right\} - \beta \sum_{i=1}^{n} y_i^{-\delta} (\log y_i)^2 - 2 \sum_{i=1}^{n} \frac{\partial}{\partial \delta} \left\{ \frac{y_i^{-\delta} (\log y_i) \beta e^{-\beta y_i^{-\delta}}}{\left\{1 + e^{-\beta y_i^{-\delta}}\right\}} \right\}.$$

5.3. Asymptotical Properties

A consistent solution of the likelihood equation is asymptotically normally distributed about true value θ_0. Thus, $\hat{\theta}$ is asymptotically $N\left(\theta_0, \frac{1}{I(\theta_0)}\right)$ as $n \to \infty$, Now $\hat{\beta}$ is asymptotically $N\left(\beta, \frac{1}{I(\beta)}\right)$ as $n \to \infty$ where

$$I(\beta) = -E\left[\frac{\partial^2 \ell}{\partial \beta^2}\right] = \frac{n}{\beta^2} + (\log \pi) \sum_{i=1}^{n} E\left\{ \frac{\partial}{\partial \beta} \left\{ \frac{2 y_i^{-\delta} e^{-\beta y_i^{-\delta}}}{\left\{1 + e^{-\beta y_i^{-\delta}}\right\}^2} \right\} \right\} - 2 \sum_{i=1}^{n} E\left\{ \frac{\partial}{\partial \beta} \left\{ \frac{y_i^{-\delta} e^{-\beta y_i^{-\delta}}}{\left\{1 + e^{-\beta y_i^{-\delta}}\right\}} \right\} \right\}.$$

$\hat{\delta}$ is asymptotically $N\left(\delta, \frac{1}{I(\delta)}\right)$ as $n \to \infty$ where

$$I(\delta) = -E\left[\frac{\partial^2 \ell}{\partial \delta^2}\right] = \frac{n}{\delta^2} - (\log \pi) \sum_{i=1}^{n} E\left\{ \frac{\partial}{\partial \delta} \left\{ \frac{2\beta y_i^{-\delta} (\log y_i) e^{-\beta y_i^{-\delta}}}{\left\{1 + e^{-\beta y_i^{-\delta}}\right\}^2} \right\} \right\} + \beta \sum_{i=1}^{n} E\left\{ y_i^{-\delta} (\log y_i)^2 \right\}$$

$$+ 2 \sum_{i=1}^{n} E\left\{ \frac{\partial}{\partial \delta} \left\{ \frac{y_i^{-\delta} (\log y_i) \beta e^{-\beta y_i^{-\delta}}}{\left\{1 + e^{-\beta y_i^{-\delta}}\right\}} \right\} \right\}$$

Pivotal Quantity (PQ): Let $y_i (i = 1, ..., m) \sim \pi - PHLIW(y; \beta, \delta)$ with CDF $U(y_i; \beta, \delta)$ then pivotal quantity is defined as

$$-2 \sum_{i=1}^{n} \ln\left[U(y_i; \beta, \delta)\right] \sim \chi^2_{2n} \quad \Rightarrow -2 \sum_{i=1}^{n} \ln\left[1 - U(y_i; \beta, \delta)\right] \sim \chi^2_{2n}$$

$$PQ = -2 \sum_{i=1}^{n} \ln\left[\frac{1}{\pi - 1}\left\{\pi^{\left(\frac{2e^{-\beta y_i^{-\delta}}}{1 + e^{-\beta y_i^{-\delta}}}\right)} - 1\right\}\right] \sim \chi^2_{2n}$$

$$\Rightarrow PQ = -2 \sum_{i=1}^{n} \ln\left[1 - \left\{\frac{1}{\pi - 1}\left\{\pi^{\left(\frac{2e^{-\beta y_i^{-\delta}}}{1 + e^{-\beta y_i^{-\delta}}}\right)} - 1\right\}\right\}\right] \sim \chi^2_{2n}.$$

Let $x_i (i = 1, ..., m) \sim \pi - PHLIW(x; \beta, \delta)$ and $y_i (i = 1, ..., m) \sim \pi - PHLIW(y; \beta, \delta)$ are two independent random variable with CDF $U(x_i; \beta, \delta)$ and $U(y_i; \beta, \delta)$ respectively, then $\frac{PQ_1}{PQ_2} \sim Beta_2(m, n)$ and $\frac{PQ_1}{PQ_1 + PQ_2} \sim Beta_1(m, n)$ and $\frac{n}{m}\frac{PQ_1}{PQ_2} \sim F(m, n)$, where

$$PQ_1 = -2 \sum_{i=1}^{n} \ln\left[\frac{1}{\pi - 1}\left\{\pi^{\left(\frac{2e^{-\beta x_i^{-\delta}}}{1 + e^{-\beta x_i^{-\delta}}}\right)} - 1\right\}\right].$$

and

$$PQ_2 = -2 \sum_{i=1}^{n} \ln\left[\frac{1}{\pi - 1}\left\{\pi^{\left(\frac{2e^{-\beta y_i^{-\delta}}}{1 + e^{-\beta y_i^{-\delta}}}\right)} - 1\right\}\right].$$

5.4. Confidence interval for Large Sample

Under certain conditions, the first derivative of the logarithm of the likelihood function w. r. to parameter θ viz., $\frac{\partial \log L}{\partial \theta}$, is asymptotically normal with mean zero and variance given by:

$$Var\left(\frac{\partial \log L}{\partial \theta}\right) = E\left(\frac{\partial \log L}{\partial \theta}\right)^2 = -E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right)$$

Hence for large n, $Z = \frac{\frac{\partial}{\partial\theta}\log L}{\sqrt{Var\left(\frac{\partial}{\partial\theta}\log L\right)}} \sim N(0,1)$ The result enables us to obtain a confidence interval for the parameter θ in a large sample. Thus for a large sample, the confidence interval for θ with confidence coefficient $(1-c)\%$ is obtained by converting the inequalities in $P(|Z| \leqslant \gamma_c) = 1-c$ where γ_c is given by $\frac{1}{2\pi}\int_{-\gamma_c}^{\gamma_c} \exp\left(-t^2/2\right)dt = 1-c$. Thus confidence interval for β and δ are given by $\widehat{\beta} \pm SE\left(\widehat{\beta}\right)$ and $\widehat{\delta} \pm SE\left(\widehat{\delta}\right)$ at the confidence coefficient $(1-c)\%$.

6. Simulation

In our research study, we employed the maxLik R package, developed by Henningsen and Toomet [7], to generate samples from the quantile function described in Equation 8 for various combinations of parameters of the $\pi - PHLIW$ distribution. The MLEs were then computed for each sample using the `maxLik()` function and the BFGS algorithm. This analysis allowed us to investigate issues related to parameter estimation and determine the direction and magnitude of bias in the MLEs, whether it be overestimation or underestimation.

In our simulation, we utilized sample sizes ranging from 100 to 400 with increments of 100. The entire process was repeated 1000 times in order to obtain estimates of the mean square error (MSE). The Biases and MSEs for the four different parameter combinations are presented in Tables 1 to 4. The results demonstrate that as the sample size increases, the corresponding Biases and MSEs decrease toward zero. This finding suggests that the MLE method exhibits asymptotic efficiency, and consistency, and follows the invariance property.

Table 1: *Bias and MSE for* $(\beta = 1.25, \delta = 0.5)$

Sample size	Bias		MSE	
	β	δ	β	δ
100	0.008	0.0085	0.0153	0.0018
200	0.001	0.0042	0.0072	9.00E-04
300	0.0035	0.0015	0.005	5.00E-04
400	0.0016	0.0026	0.0041	4.00E-04

Table 2: *Bias and MSE for* $(\beta = 0.75, \delta = 0.75)$

Sample size	Bias		MSE	
	β	δ	β	δ
100	0.0015	0.0102	0.0077	0.0036
200	-9.00E-04	0.0057	0.0035	0.0018
300	0.0032	0.0019	0.0025	0.0012
400	2.00E-04	0.0031	0.0018	9.00E-04

Table 3: *Bias and MSE for* $(\beta = 0.5, \delta = 1.25)$

Sample size	Bias		MSE	
	β	δ	β	δ
100	0.0035	0.0164	0.0046	0.0109
200	-0.0011	0.0083	0.0022	0.0055
300	0.0005	0.0031	0.0015	0.0036
400	-0.0018	0.0051	0.0011	0.0024

Table 4: *Bias and MSE for* $(\beta = 0.25, \delta = 2.5)$

Sample size	Bias		MSE	
	β	δ	β	δ
100	-6.00E-04	0.0207	0.0019	0.0149
200	0.0012	0.0056	9.00E-04	0.0068
300	7.00E-04	0.005	7.00E-04	0.005
400	-8.00E-04	0.0055	5.00E-04	0.0038

7. Application

In this section, we demonstrate the application of the π-PHLIW distribution using two real datasets. The datasets utilized for applying the suggested distribution are presented below.
Data set I:
A real data set of the relief time of 20 patients taking an analgesic is provided in this section and can be found in Chaudhary et al. [3]. Data are as follows: 1.4, 1.1, 1.7, 1.3, 1.8, 1.9, 2.2, 1.6, 2.7, 1.7, 1.8, 4.1, 1.2, 1.5, 3, 1.4, 2.3, 1.7, 2.0, 1.6
Data set II:
We consider the TRAFFIC data set given by Bain and Engelhardt [2] which represents 128 observations on times, in seconds, between the arrival of vehicles at a particular location on a road: "0.2, 0.5, 0.8, 0.8, 0.8, 1.0, 1.1, 1.2, 1.2, 1.2, 1.2, 1.2, 1.3, 1.4, 1.5, 1.5, 1.6, 1.6, 1.6, 1.7, 1.8, 1.8, 1.8, 1.8, 1.8, 1.9, 1.9, 1.9, 1.9, 1.9, 1.9, 1.9, 2.0, 2.1, 2.1, 2.2, 2.3, 2.3, 2.4, 2.4, 2.5, 2.5, 2.5, 2.6, 2.6, 2.7, 2.8, 2.8, 2.9, 3.0, 3.0, 3.1, 3.2, 3.4, 3.7, 3.9, 3.9, 3.9, 4.6, 4.7, 5.0, 5.1, 5.6, 5.7, 6.0, 6.0, 6.1, 6.6, 6.9, 6.9, 7.3, 7.6, 7.9, 8.0, 8.3, 8.8, 8.8, 9.3, 9.4, 9.5, 10.1, 11.0, 11.3, 11.9, 11.9, 12.3, 12.9, 12.9, 13.0, 13.8, 14.5, 14.9, 15.3, 15.4, 15.9, 16.2, 17.6, 20.1, 20.3, 20.6, 21.4, 22.8, 23.7, 24.7, 29.7, 30.6, 31.0, 33.7, 34.1, 34.7, 36.8, 40.1, 40.2, 41.3, 42.0, 44.8, 49.8, 51.7, 55.7, 56.5, 58.1, 70.5, 72.6, 87.1, 88.6, 91.7, 119.8, 125.3"

7.1. Model Analysis

We computed several well-known goodness-of-fit statistics to analyze both data sets I and II. The fitted models were evaluated using various metrics, including the log-likelihood value (-2logL), Akaike information criterion (AIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (AD), Kolmogrov-Smirnov (KS), and Cramer-von Mises (CVM) with corresponding p-values. All the essential computations and graphical plots were performed using the R software Wickham and Grolemund [26] and the R Core Team [23]. To compare the fitting capability of the $\pi - PHLIW$ model, we have selected several models such as the Lindley Weibull (LW) Cordeiro et al. [4], alpha power exponential (APE) Mahdavi and Kundu [15], Weibull, APT-Weibull (APTW) Nassar et al. [21], and new APT-Weibull (NAPTW) Elbatal et al. [5]. We have presented the KS and PP plots which provide an estimate of the CDF based on both data sets under study in Figures 2 and 3 (left for dataset-I, right for dataset-II). The estimated values of the parameters and their associated standard errors (SE) for both datasets were presented in Tables 5 and 7, which were obtained using the MLE method. Additionally, Tables 5 and 7 showcase model selection criteria such as log-likelihood, HQIC, and AIC, and goodness of fit statistics such as KS along with p-value for both datasets. Our observations show that the π-PHLIW model has the least statistics compared to the LW, APE, Weibull, APTW, and NAPTW distributions, along with the corresponding highest p-values. This indicates that the PiPHLIW model is more flexible and provides a good fit. Furthermore, we have provided graphical illustrations of the fitted models in Figures 6 and 8, which support our findings that the π-PHLIW model outperforms the other candidate models.

8. Conclusion

In this study, we have introduced an innovative distribution family called the π-power half logistic-G family. Drawing inspiration from the PPT methodology, we selected the Inverse Weibull distribu-

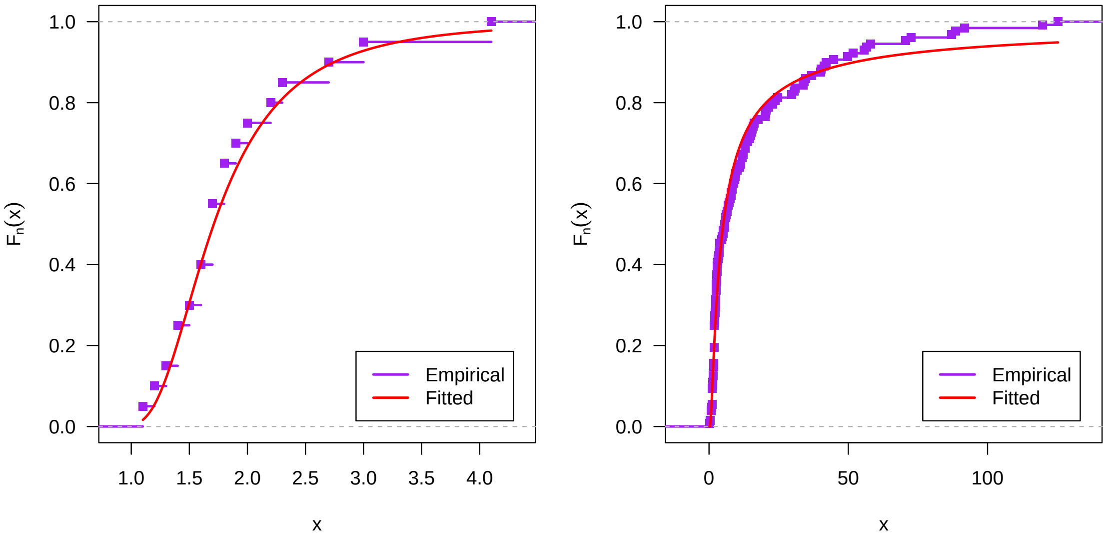

Figure 2: *Graphs of KS plot of $\pi - PHLIW$ distribution (left for dataset-I, right for dataset-II)*

Table 5: *Estimated parameters using the MLE method for the data set-I*

Model	parameter	SE	parameter	SE	parameter	SE
π-PHLIW(β, δ)	6.0338	1.9673	3.8496	0.6916	–	–
LW(α, β, λ)	9.2825	21.6417	2.0201	0.3020	0.0053	0.0241
APE(α, λ)	229.1815	16.7718	1.2110	0.1415	–	–
Weibull(α, β)	0.1216	0.0563	2.7869	0.4274	–	–
APTW$(\alpha, \delta, \lambda)$	93.1808	5.0649	1.6946	0.2693	0.6935	0.1681
NAPTW(α, β, λ)	105.5443	172.8892	1.5803	0.2237	0.8243	0.0895

tion as the foundation for this new family. The π-PHLIW distribution offers a wide range of hazard function shapes, including increasing, bathtub, J-shaped, and reverse-J-shaped. By employing the maximum likelihood estimation technique, we explored the statistical properties of this distribution and estimated its parameters. To assess the accuracy of our estimation method, we conducted a Monte Carlo simulation. The results revealed that the mean square errors decrease as the sample size increases, even when dealing with small samples. To demonstrate the practical utility of the π-PHLIW distribution, we applied it to two real engineering datasets. Through model selection criteria and goodness-of-fit tests, we compared its performance against seven existing models. Our findings strongly support the superiority of the π-PHLIW distribution over the alternative models, suggesting its potential application in various fields, such as medical and life sciences, reliability engineering, actuarial science, and survival analysis. Additionally, the π-power transformation family of distributions holds promise as a foundation for developing novel models in the future.

Conflict of interest: The authors declare that there is no conflict of interest.

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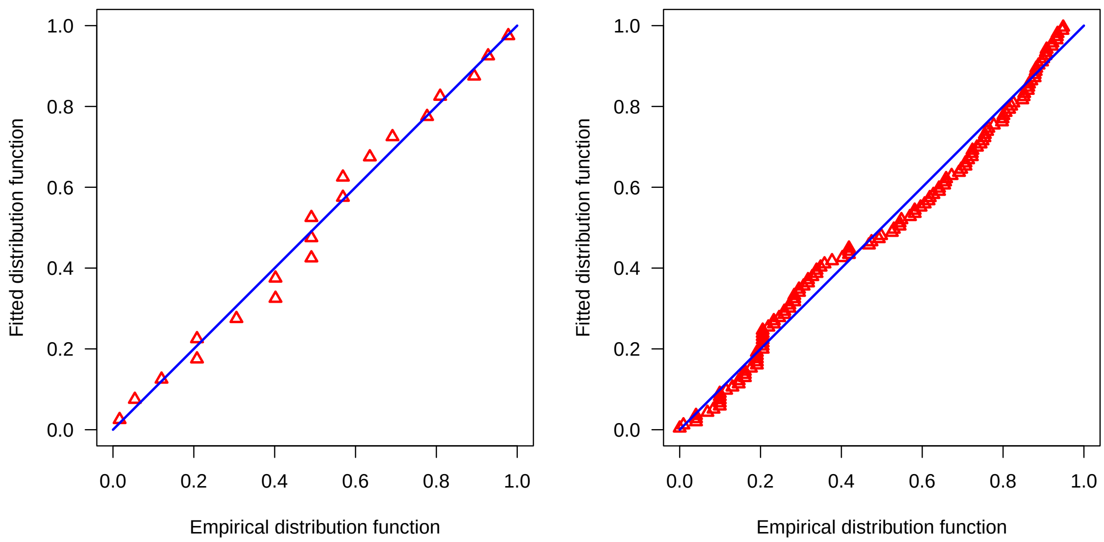

Figure 3: *Graphs of PP plot of $\pi - PHLIW$ distribution (left for dataset-I, right for dataset-II)*

Table 6: *Fitted statistics for the data set-I*

Model	-2logL	AIC	HQIC	KS	p(KS)	CVM	p(CVM)	AD	p(AD)
π-PHLIW	30.8149	34.8149	35.2036	0.1020	0.9854	0.0259	0.9895	0.1515	0.9986
LW	38.6683	44.6683	45.2514	0.1811	0.5282	0.1456	0.4060	0.8604	0.4380
APE	43.9186	47.9186	48.3074	0.2350	0.2195	0.2907	0.1433	1.6494	0.1449
Weibull	41.1728	45.1728	45.5616	0.1849	0.5009	0.1834	0.3037	1.0834	0.3155
APTW	38.0798	44.0798	44.6629	0.1557	0.7176	0.1082	0.5505	0.7223	0.5386
NAPTW	36.6835	42.6835	43.2666	0.1586	0.6955	0.0957	0.6109	0.6204	0.6268

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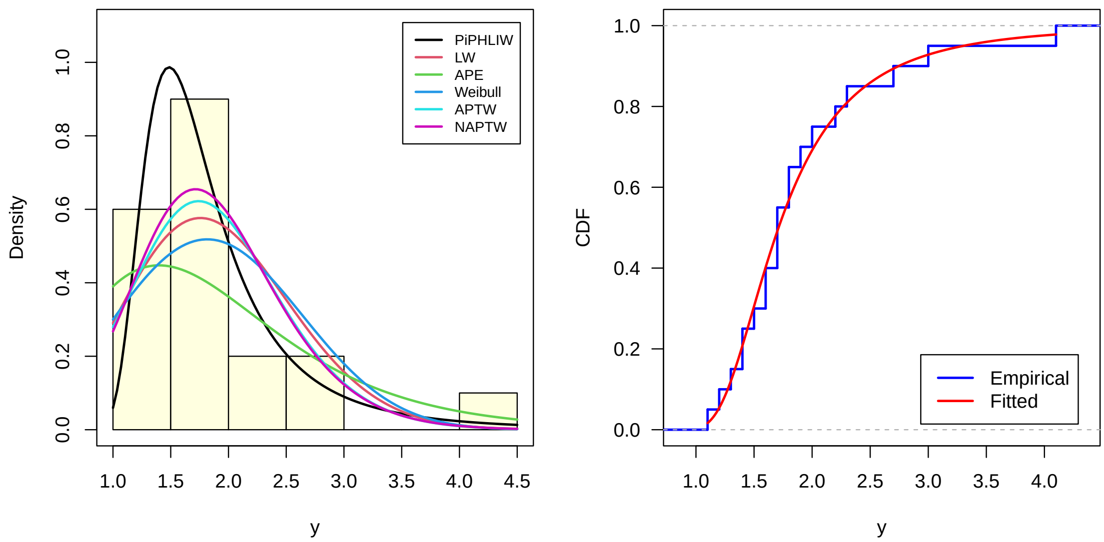

Figure 4: *Fitted PDF (left) and fitted CDF vs empirical CDF (right) (dataset-I)*

Table 7: *Estimated parameters using MLE method for the data set-II*

Model	parameter	SE	parameter	SE	parameter	SE
π-PHLIW(β, δ)	2.7544	0.2511	0.7849	0.0530	–	–
LW(α, β, λ)	90.4554	4.5499	0.5342	0.0312	0.0498	0.0117
APE(α, λ)	0.0544	0.0426	0.0322	0.0075	–	–
Weibull(α, β)	0.1488	0.0267	0.7463	0.0489	–	–
APTW$(\alpha, \delta, \lambda)$	1.0000	0.6757	0.7404	0.0303	0.1501	0.0444
NAPTW(α, β, λ)	307.9351	272.1076	0.3584	0.0184	1.1521	0.0184

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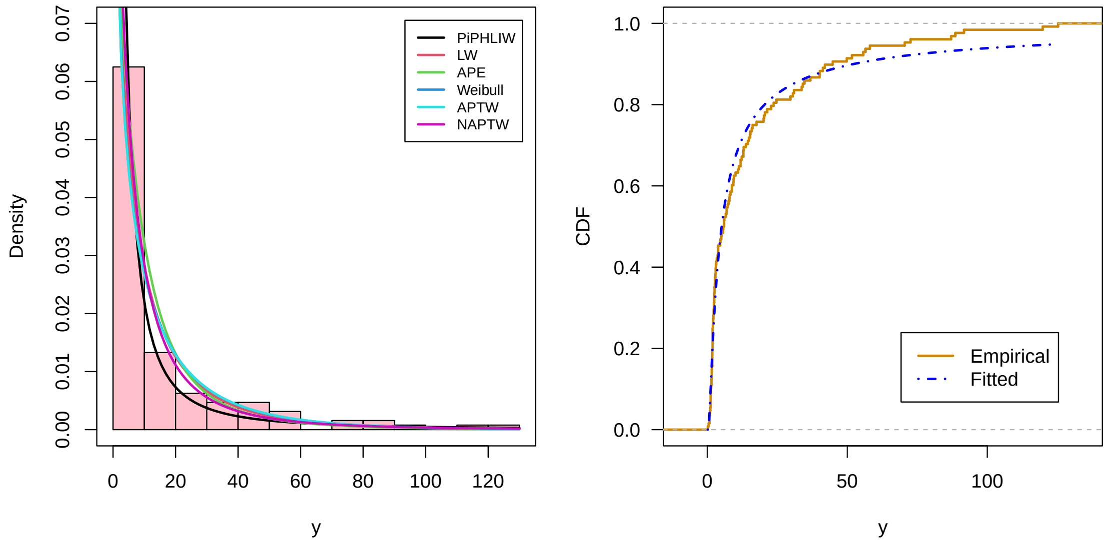

Figure 5: *Fitted PDF (left) and fitted CDF vs empirical CDF (right) (dataset-II)*

Table 8: *Fitted statistics for the data set-II*

Model	-2logL	AIC	HQIC	KS	p(KS)	CVM	p(CVM)	AD	p(AD)
π-PHLIW	921.5701	925.5701	927.8877	0.0599	0.7479	0.1416	0.4165	1.0183	0.3477
LW	933.4466	939.4466	942.923	0.1219	0.0445	0.4260	0.0616	2.6348	0.0422
APE	935.1745	939.1745	941.4921	0.1518	0.0055	0.6127	0.0207	3.6848	0.0125
Weibull	939.3848	943.3848	945.7024	0.1163	0.0628	0.4712	0.0470	2.9577	0.0288
APTW	939.4115	945.4115	948.8879	0.1160	0.0637	0.4767	0.0455	2.9670	0.0285
NAPTW	926.9602	932.9602	936.4366	0.1096	0.0926	0.2956	0.1390	1.9828	0.0940

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