PERFORMANCE ANALYSIS OF BULK ARRIVAL GENERAL SERVICE QUEUE WITH FEEDBACK, IMPATIENT CUSTOMERS AND SECOND OPTIONAL SERVICE

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Abstract

This paper analyzes the steady state behavior of batch arrival non-Markovian service queue with feedback, balking, reneging, and second optional service (SOS). The steady-state probabilities are computed using the probability generating function. After completing the first essential service (FES), if a customer is unsatisfied with it, he may choose to rejoin the system (feedback), opt for the SOS, or depart from the system with specific probabilities. Once a customer arrives, he decides immediately to join the queue or refuses to join (balking). Furthermore, after joining the queue if a customer does not get service within a specific time, may become impatient, and decide to leave the line without getting any service (reneging). Reneging time follows exponential distribution while service time (FES and SOS) follow general distribution. Also, the cost model was presented to determine the optimal service rates to minimize the expected cost. Finally, various performance measures and numerical illustrations are provided.

Keywords: Batch arrival; Steady State; Non Markovian; Feedback; Balking; Reneging; First essential service, Second optional service; Queue

I. Introduction

In queueing theory, items may arrive in batches. Known as batch arrival queueing models. A perfect example of such models is a digital communication system as [1] studied batch arrival queue systems with breakdown and repairs in which the services are performed in two different stages. At the end of each second phase of service, the server takes a compulsory vacation. The service times of the two stages follow general distributions. The expected number of units in the system has been obtained using the probability generating function. In [2] the probability generating functions have been used to study the transient and the steady state behavior of a batch arrival system and batch service with SOS. The service time distribution of both FES and SOS are exponential. [3] analyzed the steady state of $M^X/G/1$ queue with a retrial and two stages of heterogeneous services with admission, feedback, and general retrial time. The arrivals join with dependent admission due to the server state. The supplementary variable approach has been used to derive the stationary equations, the generating functions of the number of customers in the system and the orbit, and the mean queue size in the system and the orbit. Prominent research papers on the batch arrival queues can be found in [4], [5], [6], [7], [8], [9] and the references therein.

Many authors have studied customer behavior in the queueing system whereby some customers, upon arrival, decide to join the queue or refuse to join the queue. This situation is referred to as balking. The other situation is reneging where a customer upon joining the queue and waiting a specific period of time without getting service, may get impatient and may leave the queue. These two terminologies of balking and reneging are referred to as impatience behavior. [10] analyzed a single server queue model with impatience where the customers lose patience if the wait is more than the threshold they fixed. Later in [11] a study on batch arrival queue system with vacation and breakdown is done. The server provides two stages of service one by one in succession, and the customer may renege during breakdown or vacation period. Recently [12] studied batch arrival queueing system with balking, three types of heterogeneous service, and vacation. The impatient customers are assumed to balk during the period when the server is activated on the system or when the server is on vacation. Many related studies on balking are found in [13], [14], [15], [16], [17], [18], etc.

Several researchers have studied queueing systems with feedback, such as [19] investigated a batch arrival system with two-phase heterogeneous service, breakdown, and compulsory server vacation. After a customer completes two stages of services and if feels unsatisfied with the service, then he may join the tail of the queue as a feedback customer for receiving another service with a certain probability otherwise he leave the system. Later, [20] studied an M/G/1 with feedback and vacation. They consider the service times as independent and identically distributed with different rates when the customer is served with feedback or without feedback. Recently, in [21] the authors have investigated an $M/M^b/1$ with SOS and feedback. The customers are served in batches with batch size of maximum capacity b. After customers complete FES, if they are unsatisfied, they will rejoin the queue and retake the service; otherwise, they opt for SOS or leave the system. Other studies on feedback are found in [22], [23], [24], [25], [26], etc.

In queueing literature, we found studies on batch arrival non-Markovian queue systems, which include some assumptions such as feedback, balking, and reneging. The queue systems with balking, reneging, and feedback have many applications in our lives. For example, inventory and production, call centers, computer networks, etc.Therefore, adding SOS to the model which includes feedback, balking, and reneging will make the model more adaptable, and motivates us to explore its behavior under a steady state environment. We use the probability generating function to obtain the steady-state probabilities. Some important performance measures are obtained. Also, some interesting special cases were discussed. The cost analysis is derived by using the method of Quasi-Newton method. Finally, some numerical results are presented in the form of tables and graphs to show the effect of parameters on the performance measures.

This paper is structured as follows: description of the model and governing equations are presented in Section 2. In Section 3, we study the steady-state solution. Some performance measures are obtained in Section 4. In Section 5, we discuss some particular cases. Cost analysis and numerical illustrations are presented in Section 6. Finally, Section 7 concludes our paper.

II. Model Description and Mathematical Formulation

In this paper, we study an $M^X/G/1$ queue with SOS, balking, reneging and feedback. A brief description of the model is presented in the following lines:

- Customers arrive in bathes of the random size, say X , say X, in a compound Poisson process with probability $P(X = j) = c_j$, so that $\lambda c_j dt$ is the probability of first order that j $(j = 1, 2, ...)$ customers (units) arrives at the system during a short interval of time $(t,\ t + dt]$. Further, $\sum_{j=1}^{\infty} c_j = 1$, $0 \le c_j \le 1$ for all j, where $\lambda > 0$ is the mean arrival rate of batches.
- The first-come, first-served (FCFS) discipline of service is followed.
- The service time for FES and SOS are assumed to follow general arbitrary distribution with distribution functions $F(x)$ and $H(x)$ and the density functions are $f(x)$ and $h(x)$,

respectively. Let $\mu(x)dx, \beta(x)dx$ be the conditional probabilities of the completion of FES and SOS, respectively during the interval $(x, x + dx]$ with elapsed service time x, so that

$$\mu(x) = \frac{f(x)}{1 - F(x)} \quad \text{and} \quad f(s) = \mu(s)e^{-\int_0^s \mu(x)dx},$$
$$\beta(x) = \frac{h(x)}{1 - H(x)} \quad \text{and} \quad h(v) = \beta(v)e^{-\int_0^v \beta(x)dx}.$$

- When a customer arrives, he/she joins the line with probability b or refuses to join the line (balking) with probability $1 - b$.
- We assume that customers may leave the system after joining the queue without getting any service (renege) during FES and SOS and the reneging times is assume to follow exponential distribution with parameter α.
- After completion of FES, a customer may join the SOS with probability r_0 or depart from the system with probability r_1 or rejoin the system (feedback) if not satisfied with FES with probability r_2 where $r_0 + r_1 + r_2 = 1$.
- All various stochastic processes included in the system are mutually independent .

Formulation of Mathematical Model

The state of the system at time t is defined by the Markov process as

$$\{(L_q(t), M(t), \varepsilon_i(t)); i = 1,2, \ t \geq 0\},$$

where $L_q(t)$ is the queue length at time t, $M(t)$ be the state of the server at time t which is given by

$$M(t) = \begin{cases} 0, & \text{the server is idle and the queue is empty at time t,} \\ 1, & \text{the server is operating FES at time } t, \\ 2, & \text{the server is operating SOS at time } t. \end{cases}$$

and $\varepsilon_i(t)$ is the elapsed service time of a batch in service ($i = 1$ for FES and $i = 2$ for SOS) at time t. The state space of the Markov process is given as follows:

$$\Omega = \{\{0,0\}U\{n,i,\varepsilon_1\}U\{n,i,\varepsilon_2\}; n \geq 0, i = 1,2.\}$$

The probabilities involved in this model are defined as

- $Q(t)$ is the probability that the system is empty and the server is in idle.
- $P_{n,i}(x,t)$ is the probability of n ($n \geq 0$) units in the queue, with one unit in the service, elapses service time is x and the server is providing FES for $i = 1$ and SOS for $i = 2$.

According to the description that is given in the previous section, the differential-difference equations are formulated as follows:

$$\frac{d}{dt}Q(t) + \lambda Q(t) = r_1 \int_0^\infty P_{0,1}(x,t)\mu(x)dx + \int_0^\infty P_{0,2}(x,t)\beta(x)dx, \tag{1}$$

$$\frac{\partial}{\partial x}P_{0,1}(x,t) + \frac{\partial}{\partial t}P_{0,1}(x,t) = -(\lambda b + \mu(x))P_{0,1}(x,t) + \alpha P_{1,1}(x,t), \tag{2}$$

$$\frac{\partial}{\partial x}P_{n,1}(x,t) + \frac{\partial}{\partial t}P_{n,1}(x,t) = -(\lambda b + \mu(x) + \alpha)P_{n,1}(x,t) + \lambda b \sum_{i=1}^{n} c_i P_{n-i,1}(x,t) + \alpha P_{n+1,1}(x,t), \quad n \geq 1, \tag{3}$$

$$\frac{\partial}{\partial x}P_{0,2}(x,t) + \frac{\partial}{\partial t}P_{0,2}(x,t) = -(\lambda b + \beta(x))P_{0,2}(x,t) + \alpha P_{1,2}(x,t), \tag{4}$$

$$\frac{\partial}{\partial x}P_{n,2}(x,t) + \frac{\partial}{\partial t}P_{n,2}(x,t) = -(\lambda b + \beta(x) + \alpha)P_{n,2}(x,t) + \lambda b \sum_{i=1}^{n} c_i P_{n-i,2}(x,t) + \alpha P_{n+1,2}(x,t), \ n \geq 1. \tag{5}$$

Equations (1)-(5) must be solved at $x = 0$ with the following boundary conditions

$$P_{n,1}(0,t) = \lambda c_{n+1}Q(t) + r_1 \int_0^\infty P_{n+1,1}(x,t)\mu(x)dx + r_2 \int_0^\infty P_{n,1}(x,t)\mu(x)dx + \int_0^\infty P_{n+1,2}(x,t)\beta(x)dx, \ n \geq 0, \tag{6}$$

$$P_{n,2}(0,t) = r_0 \int_0^\infty P_{n,1}(x,t)\mu(x)dx, \ n \geq 0. \tag{7}$$

. At steady state, i.e, as $t \to \infty$, the above probabilities are denoted by Q, $P_{n,i}(x)$ and their derivatives with respect to time t vanish.

III. Steady State Solution of the Model

Considering the model in steady state, the state equations (1) - (7) are given as follows:

$$\lambda Q = r_1 \int_0^\infty P_{0,1}(x)\mu(x)dx + \int_0^\infty P_{0,2}(x)\beta(x)dx, \tag{8}$$

$$\frac{\partial}{\partial x}P_{0,1}(x) + (\lambda b + \mu(x))P_{0,1}(x) = \alpha P_{1,1}(x), \tag{9}$$

$$\frac{\partial}{\partial x}P_{n,1}(x) + (\lambda b + \mu(x) + \alpha)P_{n,1}(x) = \lambda b \sum_{i=1}^{n} c_i P_{n-i,1}(x) + \alpha P_{n+1,1}(x), \quad n \geq 1, \tag{10}$$

$$\frac{\partial}{\partial x}P_{0,2}(x) + (\lambda b + \beta(x))P_{0,2}(x) = \alpha P_{1,2}(x), \tag{11}$$

$$\frac{\partial}{\partial x}P_{n,2}(x) + (\lambda b + \beta(x) + \alpha)P_{n,2}(x) = \lambda b \sum_{i=1}^{n} c_i P_{n-i,2}(x) + \alpha P_{n+1,2}(x) \ n \geq 1. \tag{12}$$

The boundary conditions are given by

$$P_{n,1}(0) = \lambda c_{n+1}Q + r_1 \int_0^\infty P_{n+1,1}(x)\mu(x)dx + r_2 \int_0^\infty P_{n,1}(x)\mu(x)dx + \int_0^\infty P_{n+1,2}(x)\beta(x)dx, \ n \geq 0, \tag{13}$$

$$P_{n,2}(0) = r_0 \int_0^\infty P_{n,1}(x)\mu(x)dx, \ n \geq 0. \tag{14}$$

Generating Functions of the Queue Length

The main purpose of this subsection is to solve the equations (8) - (14) using bi-variate probability generating functions (PGFs). The PGFs are defined as follows:

$$P_i(x,z) = \sum_{n=0}^{\infty} P_{n,i}(x)z^n, \quad |z| \leq 1, \ x > 0, \ i = 1,2. \tag{15}$$

$$P_i(0,z) = \sum_{n=0}^{\infty} P_{n,i}(0)z^n, \quad |z| \leq 1, \ i = 1,2. \tag{16}$$

$$C(z) = \sum_{j=1}^{\infty} c_j z^j, \quad |z| \leq 1. \tag{17}$$

lemma 1. *For $x > 0$ we have*

$$(I)\ \frac{\partial}{\partial x}P_1(x,z) + \big(\lambda b(1-C(z)) + \mu(x) + \alpha - \frac{\alpha}{z}\big)P_1(x,z) = 0, \tag{18}$$

$$(II)\ \frac{\partial}{\partial x}P_2(x,z) + \big(\lambda b(1-C(z)) + \beta(x) + \alpha - \frac{\alpha}{z}\big)P_2(x,z) = 0. \tag{19}$$

Proof. (I) Multiplying equations (9) and (10) by appropriate power z^n, summing them from $n = 0$ to $n = \infty$, and using the definition of PGFs, we get the result.
(II) Similarly, from equations (11) and (12), we get the desired result. □

lemma 2. *For $x > 0$, we have*

$$(I)\ P_1(x,z) = P_1(0,z)e^{-[\eta(z)]x - \int_0^x \mu(t)dt}, \tag{20}$$

$$(II)\ P_2(x,z) = P_2(0,z)e^{-[\eta(z)]x - \int_0^x \beta(t)dt}, \tag{21}$$

where $\eta(z) = \lambda b(1-C(z)) + \alpha - \frac{\alpha}{z}$.

Proof. Integrating equations (18) and (19) in the interval $[0,x]$, we get the desired result. □

lemma 3. *For $x > 0$, we have*

$$(I)\ \int_0^{\infty} P_1(x,z)\mu(x)dx = P_1(0,z)F^*(\eta(z)). \tag{22}$$

$$(II)\ \int_0^{\infty} P_2(x,z)\beta(x)dx = P_2(0,z)H^*(\eta(z)). \tag{23}$$

where $F^*[\eta(z)]$, $H^*[\eta(z)]$ are the Laplace-Steiltjes transform (LST) of the service times $F(x)$ and $H(x)$, respectively.

$$F^*[\eta(z)] = \int_0^{\infty} e^{-(\eta(z))x}dF(x),$$

$$H^*[\eta(z)] = \int_0^{\infty} e^{-(\eta(z))x}dH(x).$$

Proof.
Multiplying equations (20) and (21) by $\mu(x)$ and $\beta(x)$, respectively and integrating with respect to x, we get the result. □

lemma 4. *The PGFs $P_i(z)$, $i = 1,2$ are given by*

$$(I)\ P_1(z) = \frac{\lambda(C(z)-1)\big[1-F^*(\eta)\big]Q}{\big[z - r_1F^*(\eta) - r_2zF^*(\eta) - r_0F^*(\eta)H^*(\eta)\big]\eta(z)}, \tag{24}$$

$$(II)\ P_2(z) = \frac{r_0\lambda(C(z)-1)F(\eta(z))\big[1-H^*(\eta(z))\big]Q}{\big[z - r_1F^*(\eta(z)) - r_2zF^*(\eta(z)) - r_0F^*(\eta(z))H^*(\eta(z))\big]\eta(z)}. \tag{25}$$

where $P_i(z) = \int_0^{\infty} P_i(x,z)dx$, $i = 1,2$.

Integrating equations (20) and (21) by parts, we get

$$P_1(z) = P_1(0,z)\left(\frac{1-F^*(\eta(z))}{\eta(z)}\right), \tag{26}$$

$$P_2(z) = P_2(0,z)\left(\frac{1-H^*(\eta(z))}{\eta(z)}\right). \tag{27}$$

Now, we have to find $P_1(0,z), P_2(0,z)$.
Multiplying equation (13) by appropriate powers of z^n, summing them from $n = 0$ to ∞, and using the definition of PGFs, we get

$$\begin{aligned} zP_1(0,z) &= \lambda C(z)Q + r_1\int_0^\infty P_1(x,z)\mu(x)dx + zr_2\int_0^\infty P_1(x,z)\mu(x)dx \\ &+ \int_0^\infty P_2(x,z)\beta(x)dx - \left[r_1\int_0^\infty P_{0,1}(x)\mu(x)dx + \int_0^\infty P_{0,2}(x)\beta(x)dx\right] \end{aligned} \tag{28}$$

Substituting equation (8) into equation (28), we get

$$\begin{aligned} zP_1(0,z) &= \lambda C(z)Q + r_1\int_0^\infty P_1(x,z)\mu(x)dx + r_2 z\int_0^\infty P_1(x,z)\mu(x)dx \\ &+ \int_0^\infty P_2(x,z)\beta(x)dx - \lambda Q. \end{aligned} \tag{29}$$

Substituting equations (22) and (23) in equation (29), we get

$$\begin{aligned} zP_1(0,z) &= \lambda(C(z)-1)Q + r_1F^*(\eta(z))P_1(0,z) + r_2zF^*(\eta(z))P_1(0,z) \\ &+ P_2(0,z)H^*(\eta(z)), \end{aligned} \tag{30}$$

Similarly, multiplying equation (14) by appropriate powers of z^n, summing them from $n = 0$ to ∞, and using the definition of PGFs, we get

$$P_2(0,z) = r_0\int_0^\infty P_1(x,z)\mu(x)dx. \tag{31}$$

Substituting equation (22) in equation (31), we obtain

$$P_2(0,z) = r_0F^*(\eta(z))P_1(0,z). \tag{32}$$

Substituting equation (32) in equation (30), we get

$$\begin{aligned} zP_1(0,z) &= \lambda(C(z)-1)Q + r_1F^*(\eta(z))P_1(0,z) + r_2zF^*(\eta(z))P_1(0,z) \\ &+ r_0F^*(\eta(z))H^*(\eta(z))P_1(0,z). \end{aligned} \tag{33}$$

After algebraic calculations, we get

$$P_1(0,z) = \frac{\lambda(C(z)-1)Q}{z - r_1F(\eta(z)) - r_2zF(\eta(z)) - r_0F(\eta(z))H(\eta(z))}. \tag{34}$$

Substituting equation (34) in equation (32), we get

$$P_2(0,z) = \frac{r_0\lambda(C(z)-1)F(\eta(z))Q}{z - r_1F(\eta(z)) - r_2zF(\eta(z)) - r_0F(\eta(z))H(\eta(z))}. \tag{35}$$

After substituting equations (34) and (35) in equations (26) and (27) respectively, and some algebraic calculations, the equations (24) and (25) are obtained.

lemma 5. *The PGF of the queue size is given by*

$$P_q(z) = \frac{\left[\lambda(C(z)-1)Q\right]\left[1-F^*(\eta(z)) + r_0F^*(\eta(z)) - r_0F^*(\eta(z))H^*(\eta(z))\right]}{\left[\lambda b(1-C(z)) + \alpha - \frac{\alpha}{z}\right]\left[z - r_1F^*(\eta(z)) - r_2zF^*(\eta(z)) - r_0F^*(\eta(z))H^*(\eta(z))\right]} \tag{36}$$

Proof. Let us suppose the PGF of the queue size irrespective of the state of the system be given by

$$P_q(z) = P_1(z) + P_2(z) \tag{37}$$

Substituting equations (24) and (25) in equation (37), we get the result. □

lemma 6. *Based on the previous results, we have*

$$Q = \frac{(-\lambda bE(X) + \alpha)\left[1 - r_2 + (-\lambda bE(X) + \alpha)[E(S) + r_0E(V)]\right]}{-[-\lambda E(X)(1-b) - \alpha](-\lambda bE(X) + \alpha)[E(S) + r_0E(V)] + (-\lambda bE(X) + \alpha)\left[1 - r_2\right]}, \tag{38}$$

Proof.
To obtain Q, we have to use the normalizing condition

$$P_q(1) + Q = 1. \tag{39}$$

Now, clearly $z = 1$ brings P_q in equation (39) to indeterminate $(\frac{0}{0})$ form. Therefore using L'Hospital's rule, we obtain

$$P_q(1) = \lim_{z\to 1} P_q(z) = \frac{\lambda C'(1)(-\lambda bC'(1) + \alpha)\left[F^{*\prime}(0) + r_0H^{*\prime}(0)\right]Q}{(-\lambda bC'(1) + \alpha)\left[1 - r_2 + (-\lambda bC'(1) + \alpha)F^{*\prime}(0) + r_0[(-\lambda bC'(1) + \alpha)H^{*\prime}(0)]\right]}. \tag{40}$$

Substituting $C(1) = 1$, $C'(1) = E(X)$, $F^*(0) = 1$, $F^{*\prime}(0) = -E(S)$, $H^*(0) = 1$, $H^{*\prime}(0) = -E(V)$ in (39), we get

$$P_q(1) = \frac{-\lambda E(X)(-\lambda bE(x) + \alpha)\left[E(S) + r_0E(V)\right]Q}{(-\lambda bE(X) + \alpha)\left[1 - r_2 - (-\lambda bE(X) + \alpha)[E(S) + r_0E(V)]\right]}. \tag{41}$$

where $E(S)$ and $E(V)$ are the mean service times for FES and SOS, respectively. $E(X)$ is the mean batch size of the arriving units.
Substituting the equation (41) in (39), the equation (38) is derived. □

IV. Performance Measures

In this section, using the PGF of the queue size distribution that we obtained in previous section, we get the mean queue size and the waiting time of a customer in the queue. Let L_q be the mean queue size which is define as following

$$L_q = \lim_{z\to 1} \frac{d}{dz} P_q(z), \tag{42}$$

where $P_q(z)$ denote the PGF of the queue size. Taking the limit of derivative of $P_q(z)$ at $z = 1$ brings equation (41) to indeterminate $(\frac{0}{0})$ form. Then using L'Hospital's rule and carrying out the derivatives at $z = 1$, we obtain

$$L_q = \frac{M''(1)N'''(1) - N''(1)M'''(1)}{3(M''(1))^2}. \tag{43}$$

Let us derive the second the third derivatives at z=1 with some algebra calculations, we get

$$\begin{aligned}
N(z) &= \big[\lambda(C(z)-1)Q\big]\big[1-F^*(\eta(z))+r_0F^*(\eta(z))-r_0F^*(\eta(z))H^*(\eta(z))\big],\\
N'(1) &= 0\\
N''(1) &= -2\lambda E(X)(-\lambda bE(X)+\alpha)[E(S)+r_0E(V)]Q,\\
N'''(1) &= -3\lambda E(X(X-1))(-\lambda bE(X)+\alpha)\big[E(S)+r_0E(V)\big]Q\\
&\quad -3\lambda E(X)\bigg[(-\lambda bE(X(X-1))+2\alpha)E(S)+2(-\lambda bE(X)+\alpha)^2E(S^2)\\
&\quad +2r_0(-\lambda bE(X)+\alpha)^2(E(S))(E(V))\\
&\quad +r_0\big[-(\lambda bE(X(X-1))+2\alpha)E(V)+2(-\lambda bE(X)+\alpha)^2E(V^2)\big]\bigg]Q,\\
M(z) &= \big[\lambda b(1-C(z))+\alpha-\frac{\alpha}{z}\big]\big[z-r_1F^*(\eta(z))-r_2zF^*(\eta(z))-r_0F^*(\eta(z))H^*(\eta(z))\big],\\
M'(1) &= 0,\\
M''(1) &= 2[-\lambda bE(X)+\alpha][1-r_2-(-\lambda bE(X)+\alpha)(E(S)+r_0E(V))],\\
M'''(1) &= -3[\lambda bE(X(X-1))+2\alpha][1-r_2-(-\lambda bE(X)+\alpha)[E(S)+r_0E(V)]]\\
&\quad -3\Big(-\lambda bE(X)+\alpha\Big)\Big[-(\lambda bE(X(X-1))+2\alpha)E(S)+2(-\lambda bE(X)+\alpha)^2E(S^2)]\\
&\quad +2r_2(-\lambda bE(X)+\alpha)(E(S))+2r_0(-\lambda bE(X)+\alpha)^2(E(S)E(V))\\
&\quad +r_0\big[-(\lambda bE(X(X-1))+2\alpha)E(V)+2(-\lambda bE(X)+\alpha)^2E(V^2)\big]\Big].
\end{aligned}$$

where $K''(1)=E(X(X-1))$ is the second factorial moment of the batch size of the arriving units, $E(S^2)$ and $E(V^2)$ are the second moment of the service time for FES and SOS, respectively. Now substituting N'', N''', M'', M''' in (43) we obtain L_q in closed form
Let W_q is the mean of waiting time of a customer in the queue. Using Little's formula we have

$$W_q = \frac{L_q}{\lambda bE(X)}. \tag{44}$$

V. Particular Cases

In this Section, we derive some particular cases from the main results obtained in this paper.
Case 1:

(1) We assume that the service time (FES and SOS) are following exponential distribution. Here, we take

$$E(S)=\frac{1}{\mu}\quad,\qquad E(S^2)=\frac{2}{(\mu)^2}$$
$$E(V)=\frac{1}{\beta}\quad,\qquad E(V^2)=\frac{2}{(\beta)^2}$$

(2) We assume that the service time (FES and SOS) are following hyper-exponential distribution. Here, we take

$$E(S)=\frac{p}{\mu_1}+\frac{1-p}{\mu_2},\quad E(S^2)=2\left(\frac{p}{(\mu_1)^2}+\frac{1-p}{(\mu_2)^2}\right)$$
$$E(V)=\frac{p}{\beta_1}+\frac{1-p}{\beta_2},\quad E(V^2)=2\left(\frac{p}{(\beta_1)^2}+\frac{1-p}{(\beta_2)^2}\right)$$

(3) We assume that the service time (FES and SOS) are following Erlang-k distribution. Here, we take

$$E(S) = \frac{1}{\mu} \quad , \quad E(S^2) = \frac{k+1}{k(\mu)^2}$$
$$E(V) = \frac{1}{\beta} \quad , \quad E(V^2) = \frac{k+1}{k(\beta)^2}$$

Case 2: we assume the costumer may not renege during FES or SOS i.e ($\alpha = 0$), the model reduces to $M^X/G/1$ queueing system with balking, feedback and SOS.
Using this assumption in the main result of the paper, we get

$$P_q(z) = \frac{(-Q)\big[1 - F^*(\eta(z)) + r_0F^*(\eta(z)) - r_0F^*(\eta(z))H^*(\eta(z))\big]}{b\big[z - r_1F^*(\eta(z)) - r_2zF^*(\eta(z)) - r_0F^*(\eta(z))H^*(\eta(z))\big]}.$$
$$Q = \frac{b\big[1 - r_2 - \lambda bE(X)E(S) - r_0\lambda bE(X)E(V)\big]}{(1-b)(\lambda bE(X))\big[E(S) + r_0E(V)\big] + b(1-r_2)},$$
$$L_q = \lim_{z\to 1}\frac{d}{dz}P_q(z) = \frac{M'(1)N''(1) - N'(1)M''(1)}{2(M'(1))^2},$$

where N', N'', M', M'' are given in the flowing equations:

$$N'(1) = (Q)(-\lambda bE(X) + \alpha)\big[E(S) + r_0E(V)\big],$$
$$\begin{aligned} N''(1) = (Q)\Big[&(-\lambda bE(X(X-1)) + 2\alpha)E(S) + 2(-\lambda bE(X) + \alpha)^2E(S^2) \\ &+ 2r_0(-\lambda bE(X) + \alpha)^2(E(S))(E(V)) \\ &+ r_0\big[-(\lambda bE(X(X-1)) + 2\alpha)E(V) + 2(-\lambda bE(X) + \alpha)^2E(V^2)\big]\Big], \end{aligned}$$
$$M'(1) = b[1 - r_2 - (-\lambda bE(X) + \alpha)E(S) - r_0(-\lambda bE(X) + \alpha)E(V)],$$
$$\begin{aligned} M''(1) = -b\Big[&2r_2(-\lambda bE(X) + \alpha)E(S)(-\lambda bE(X(X-1)) + 2\alpha)E(S) + 2(-\lambda bE(X) + \alpha)^2E(S^2) \\ &+ 2r_0(-\lambda bE(X) + \alpha)^2(E(S))(E(V)) \\ &+ r_0\big[-(\lambda bE(X(X-1)) + 2\alpha)E(V) + 2(-\lambda bE(X) + \alpha)^2E(V^2)\big]\Big]. \end{aligned}$$

Case 3: Consider $r_0 = 0$ (no SOS), $b = 1$ (no balking), $\alpha = 1$ (no reneging) a feedback model in $M^X/G/1$ queue is obtained.

$$Q = \frac{1 - r_2 - \lambda E(X)E(S)}{1 - r_2},$$
$$L_q = \lim_{z\to 1}\frac{d}{dz}P_q(z) = \frac{M'(1)N''(1) - N'(1)M''(1)}{2(M''(1))^2},$$

where N', N'', M', M'' is given in the flowing equations:

$$N'(1) = -[(-\lambda E(X) + \alpha)E(S)]Q$$
$$N'''(1) = -[(\lambda E(X(X-1)) + 2\alpha)E(S) + 2(-\lambda E(X) + \alpha)^2E(S^2)]Q$$
$$M'(1) = [1 - r_2 - (-\lambda E(X) + \alpha)E(S)],$$
$$M''(1) = \big[-((\lambda E(X(X-1)) + 2\alpha)E(S) + 2(-\lambda E(X) + \alpha)^2E(S^2)) - 2r_2(-\lambda E(X) + \alpha)E(S)\big].$$

We note that this result agrees as special case with the result of $M^X/G/1$ queue with feedback and optional server vacations (see [4])

VI. Numerical Results and Discussion

In this section, Some numerical illustrations with discussion based on Q, L_q and W_q are provided with the purpose to illustrate the effect of the parameters ($\lambda,\ \mu,\ \beta,\ b,\ r_0,\ r_1, r_2$) on Q, L_q and W_q.

In Table 1, we show the impact of the probability of feedback (r_2) and the probability of join SOS (r_0) on the L_q. For the fixed probability of the departure (r_1), as r_2 increases and (r_0) decreases, the situation leads to an increase in L_q. This indicating that more customers feel unsatisfied and decide to rejoin the queue. We take; the service time (FES and SOS) follow Exponential distribution and $\lambda = 2,\ \mu = 5,\ \beta = 4,\ \alpha = 1, r_0 = 0.1,\ r_2 = 0.5,\ b = 0.10,\ E(X) = 1,\ E(X(X-1)) = 0$.
Also, we show in (Table 2) the impact of the mean arrival rate of batches λ and mean of reneging α on the (L_q). We observe that L_q decreases as mean reneging α increases .Thus more customers leave the the queue. For the fixed mean reneging (α), as λ increases L_q increases. We take; the service times (FES and SOS) to follow exponential distribution and $r_0 = 0.6,\ r_2 = 0.2,\ \mu = 4,\ \beta = 3, b = 0.20, E(X) = 1, E(X(X-1)) = 0$.
We show in (Table 3) the effect of batch arrival rate λ on Q and L_q when the service times (FES and SOS) are following general distribution (exponential , Erlang-κ , hyper -exponential). We observe that server's idle time Q decreases and the L_q increases as batch arrival rate λ increases. Here, when the service times (FES and SOS) to follow exponential distribution we take; $\mu = 5,\ \beta = 3,\ r_0 = 0.5,\ r_2 = 0.3,\ \alpha = 1,\ b = 0.25,\ E(X) = 1,\ E(X(X-1)) = 0$ and when they follow Erlang-κ we take $\kappa = 5,\ \mu = 5,\ \beta = 3,\ r_0 = 0.5,\ r_2 = 0.3,\ \alpha = 1,\ b = 0.25,\ E(X) = 1,\ E(X(X-1)) = 0$, and when they follow hyper-exponential $p = 0.5,\ \mu_1 = 5,\ \mu_2 = 4,\ \beta_1 = 3,\ \beta_2 = 2,,\ r_0 = 0.5,\ r_2 = 0.3,\ \alpha = 1,\ b = 0.10,\ E(X) = 1,\ E(X(X-1)) = 0$.
In Figure 1, we show the effect of batch arrival rate λ on L_q in different joining probability b. We observe that L_q increases as λ or b increases. We take; the service times (FES and SOS) to follow exponential distribution and $r_0 = 0.5,\ r_2 = 0.3,\ \mu = 5,\ \beta = 4,\ \alpha = 1,\ E(X) = 1,\ E(X(X-1) = 0$.
Also in figures 2, and 3, we show the effect of the service rate (FES and SOS) on L_q in different joining probability b. We observe that L_q decreases when the FES rate and SOS rate increase as we expected. Further, we notice that as b increases, the L_q increases i.e. additional customers joining the queue.
We take; the service times (FES and SOS) to follow exponential distribution and $\beta = 4, \alpha = 1,\ r_0 = 0.5,\ r_2 = 0.3,\ b = 0.10,\ E(X) = 1,\ E(X(X-1) = 0$, in Figure 2 and $\mu = 5, \alpha = 1,\ r_0 = 0.5,\ r_2 = 0.3,\ b = 0.10,\ E(X) = 1,\ E(X(X-1) = 0$, in Figure 3

Table 1: *The impact of r_0 and r_2 on Q, L_q and W_q.*

r_2	r_0	Q	ρ	Lq	Wq
0.1	0.5	0.640884	0.359116	0.0296485	0.148243
0.2	0.4	0.634146	0.365854	0.0304878	0.152439
0.3	0.3	0.625850	0.374150	0.0312408	0.156204
0.4	0.2	0.615385	0.384615	0.0317308	0.158654
0.5	0.1	0.601770	0.398230	0.0315591	0.157795

Table 2: *Impact of λ and α on Q, L_q and W_q.*

λ	α	Q	ρ	L_q	W_q
$\lambda = 1.0$	$\alpha = 1$	0.720497	0.279503	0.0373921	0.186960
	$\alpha = 2$	0.781553	0.218447	0.0321112	0.160556
	$\alpha = 3$	0.820717	0.179283	0.0276886	0.138443
$\lambda = 1.5$	$\alpha = 1$	0.543147	0.456853	0.0977276	0.244319
	$\alpha = 2$	0.628099	0.371901	0.0759388	0.189847
	$\alpha = 3$	0.686411	0.313589	0.0623758	0.155940
$\lambda = 2.0$	$\alpha = 1$	0.420601	0.579399	0.1789000	0.298166
	$\alpha = 2$	0.514388	0.485612	0.1304270	0.217379
	$\alpha = 3$	0.582043	0.417957	0.1035010	0.172502

Table 3: *The impact of batch arrival rate λ on Q and L_q in General distribution service time and repair time.*

	exponential		*Erlang − κ*		*hyper − exponential*	
λ	Q	Lq	Q	Lq	Q	Lq
1.0	0.726708	0.0384350	0.726708	0.0537750	0.702857	0.0447747
1.5	0.628169	0.0701825	0.628169	0.0884330	0.598972	0.0830000
2.0	0.546392	0.1097060	0.546392	0.128442	0.514019	0.1311680
2.5	0.477435	0.1571710	0.477435	0.174245	0.443255	0.1897080
3.0	0.418502	0.2131230	0.418502	0.226523	0.383399	0.2596380

Figure 1: *The effect of batch arrival rate (λ) on (L_q) in different joining probability b*

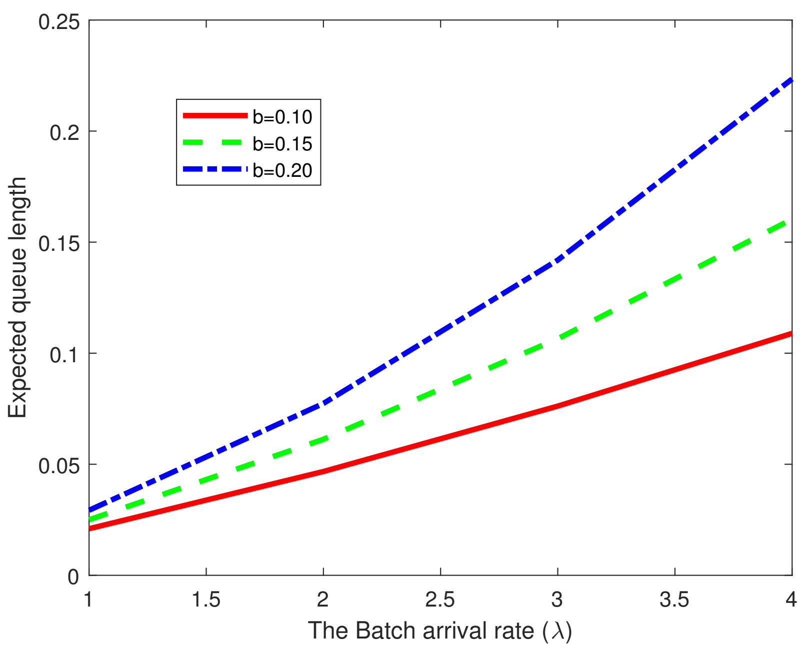


Figure 2: *The effect of the FES rate (μ) on (L_q) in different joining probability b*

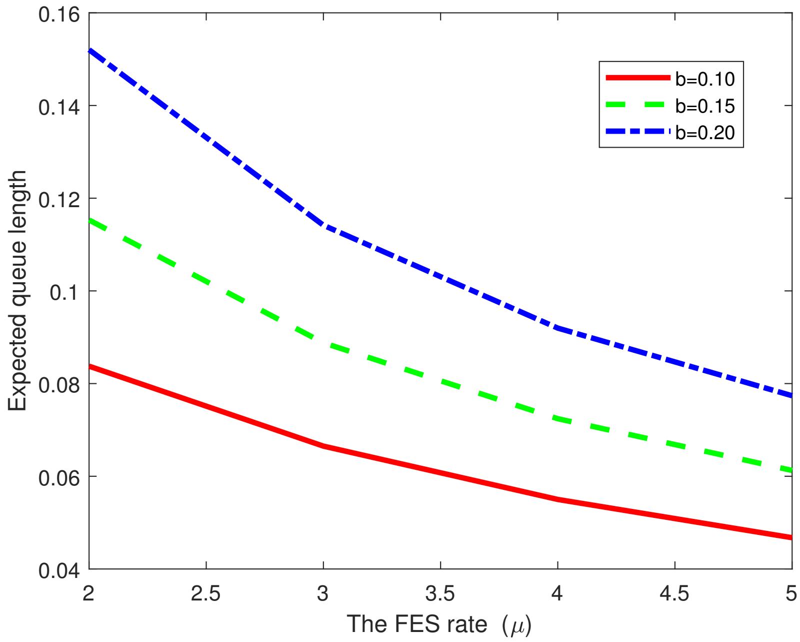
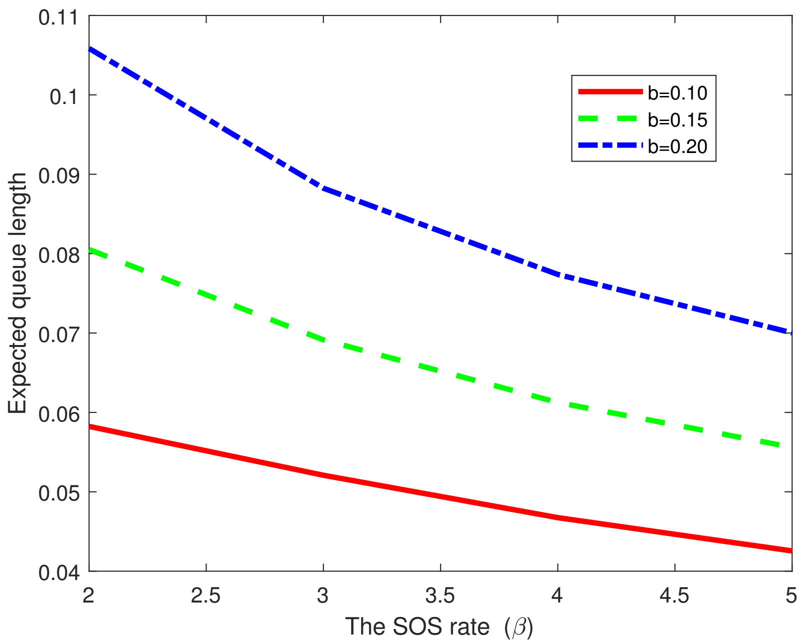


Figure 3: *The effect of SOS rate (β) on (L_q) in different joining probability b*



The Cost Model

To achieve the optimal service rate in FES and SOS with a minimum expected cost function, we have developed the expected cost function per unit time as :

$$f(\mu, \beta) = CL + C_1\mu + C_2\beta + C_r\alpha, \tag{45}$$

where :

- C = cost per unite time per customer present in the queue.
- C_1 = cost per unite time during FES.
- C_2 = cost per unite time during SOS.
- C_r = cost per unite time when the customer renege.

The cost minimization problem $f(\mu, \beta)$ can be presented mathematically as

$$f(\mu^*, \beta^*) = \underset{s.t\mu,\beta>0}{\text{Minimize}} f(\mu, \beta). \tag{46}$$

We use the Quasi- Newton method to search for (μ, β) until the minimum of $f(\mu, \beta)$ is obtained. For details of Quasi- Newton method, one may refer Lewis and Overton [27].

Table 4: *Impact of r_0 and r_2 on the expected cost*

r_0	r_2	μ^*	β^*	$f(\mu^*,\ \beta^*)$
	r_2 = 0.20	1.41917	0.917929	51.2880
r_0= 0.2	r_2 = 0.40	1.74319	1.05108	60.1129
	r_2 = 0.60	2.34484	1.29993	76.0659
	r_2 = 0.20	1.44822	1.07991	54.1650
r_0= 0.2	r_2 = 0.40	1.78370	1.24700	63.5951
	r_2 = 0.60	2.40812	1.56344	80.7229
	r_2 = 0.20	1.47416	1.21697	56.5886
r_0= 0.3	r_2 = 0.40	1.81941	1.41499	66.5526
	r_2 = 0.60	2.46299	1.79318	84.7184

From Table 4, we notice that for fixed r_0, $(\mu^*,\ \beta^*)$ and $f(\mu^*,\ \beta^*)$ increase with the increase of r_2. This is because many customers have not satisfied with the service and repeat the service, leading to high-cost implications.
Similarly, for fixed r_2, as r_0 increases, we observe that both $(\mu^*,\ \beta^*)$ and $f(\mu^*,\ \beta^*)$ increase . This is due to the fact that as r_0 increases, customers tend to enter SOS service, thereby increasing the service rate, which in turn results in an increase of cost. We take the service times (FES and SOS) to follow exponential distribution and $\lambda = 2,\ \mu = 2,\ \beta = 1,\ \alpha = 0.1,\ b = 0.2,\ E(X) = 1,\ E(X(X-1)) = 0$.

Table 5: *Impact of α and b on the expected cost*

α	b	μ^*	β^*	$f(\mu^*,\ \beta^*)$
	b = 0.20	1.47416	1.21697	56.5886
α= 0.10	b = 0.25	1.76159	1.45299	64.9474
	b = 0.30	2.03931	1.67967	72.8668
	b = 0.20	1.39146	1.14110	56.955
α= 0.15	b = 0.25	1.67757	1.37735	65.2815
	b = 0.30	1.95431	1.60419	73.1769
	b = 0.20	1.30875	1.06509	57.3181
α= 0.20	b = 0.25	1.59351	1.30166	65.6134
	b = 0.30	1.86928	1.52869	73.4854

Table 5 shows the impact of reneging rate α on the minimum expected cost function $f(\mu^*, \beta^*)$for different values of joining probability b. In this table, we observe that the optimal service rates (μ^*, β^*) and expected cost $f(\mu^*, \beta^*)$ increase as both α and b increase. Particularly, For fixed b as α increases, customers departure from the queue which leads to decrease the service rates μ^* , β^* and increase cost, so that to balance the system profitability.We take; the service times (FES and SOS) to follow exponential distribution and ($\lambda = 2,\ \mu = 2,\ \beta = 1,\ r_0 = 0.4,\ r_2 = 02,\ E(X) = 1,\ E(X(X-1)) = 0.$)

VII. Conclusion

In this paper, we analyzed the steady state behavior of a single server batch arrival non -Markovian batch service queue with a second optional service, balking, reneging and feedback using the supplementary variable technique to get the probability generating function of the number of customers in the system. The mean of the queue size and waiting time of a customer in the queue were obtained. Some interesting special cases were discussed. We assumed general distribution for the service time. The cost model was presented to determine the optimal service rates to minimize the expected cost. Finally, the numerical results through graphical illustrations and tables were presented.

References

[1] Rajan, B. S., Ganesan, V. and Rita, S. (2020). Batch arrival poisson queue with breakdown and repairs. *International Journal of Mathematics in Operational Research*. 17(3), 424–435. https://doi.org/10.1504/IJMOR.2020.110033.

[2] P. Vijaya Laxmi, G. Andwilile Abrahamu and E. Girija Bhavani. (2021). Performance of a Single Server Batch Queueing Model with Second Optional Service under Transient and Steady State Domain.*Reliability: Theory & Applications*. 16 (4 (65)), 226-238. https://doi:10.24412/1932-2321-2021-465-226-238.

[3] Abdollahi, S., Salehi Rad and M. R. (2021). Analysis of a batch arrival retrial queue with Two-Phase Services, Feedback and admission. *Bulletin of the Iranian Mathematical Society*. 48(3), 791–804.

[4] Madan, K.C. and Al-Rawwash, M. (2005). On the $M^X/G/1$ queue with feedback and optional server vacations based on a single vacation policy.*Applied Mathematics and Computation*. 160(3), 909–919. https://doi.org/10.1016/j.amc.2003.11.037

[5] Maraghi, F. A., Madan, K.C. and Darby-Dowman, K. (2010). Batch Arrival Vacation Queue with Second Optional Service and Random System Breakdowns.*Journal of Statistical Theory and Practice*, 4(1), 137–153. https://doi.org/10.1080/15598608.2010.10411977.

[6] Bouchentouf, A.A. and Guendouzi. (2019). A. Cost optimization analysis for an$M^X/M/c$ vacation queueing system with waiting servers and impatient customers. *SeMA Journal*. 76, 309–341. https://doi.org/10.1007/s40324-018-0180-2.

[7] Khalaf, R., Madan, K.C. and Lucas, C. (2012). On an $M^X/G/1$ Queue with, Random Breakdowns, server vacations, delay Times and standby server. *International Journal of Operational Research* . 15(1), 30–46. https://doi.org/10.1504/IJOR.2012.048290.

[8] P. Vignesh, S. Srinivasan, and S. Maragatha Sundari (2019). Analysis of a non-Markovian single server batch arrival queueing system of compulsory three stages of services with fourth optional stage service. service interruptions and deterministic server vacations. *International Journal of Operational Research*, 4(1), 28-53. https://doi.org/10.1504/IJOR.2019.096937.

[9] Vignesh, P., Srinivasan, S., Sundari, S.M. and Eswar, S.K. (2023). An investigation on $M^X/G/1$ queuing model of interrupted services in the manufacturing of edible cutlery process. *International Journal of Mathematical Modelling and Numerical Optimisation*, 13(2), 173-201.

[10] Choudhury, A. (2008). newblockImpatience in Single Server Queueing Model. *American Journal of Mathematical and Management Sciences*. 28(1-2), 177–211. https://doi.org/10.1080/01966324.2008.10737723.

[11] Baruah, M., Madan, K.C. and Eldabi, T. (2013). A Two Stage Batch Arrival Queue with Reneging during Vacation and Breakdown Periods.*American Journal of Operations Research.*. 3(6), 570–580.

[12] Enogwe, Samuel, Onyeagu, Sidney and Obiora-Ilouno, Happiness. (2021). On single server batch arrival queueing system with balking, three types of heterogeneous service and Bernoulli schedule server vacation. *Mathematical Theory and Modeling*. 11(5), 40–68.

[13] Vijaya Laxmi, P., Goswami, V. and Jyothsna, K. (2013). Optimization of Balking and Reneging Queue with Vacation Interruption under N-Policy. *Journal of Optimization*, 2013, 1-9.

[14] Baruah, M., Madan, K.C. and Eldabi, T. (2013). A batch arrival queue with second optional service and reneging during vacation periods. *Investigacion Operacional*. 34(3), 244–258.

[15] Sivagnanasundararam M. and Santhanagopalan S., (2014). A Non-Markovian Multistage Batch Arrival Queue with Breakdown and Reneging.*Mathematical Problems in Engineering*. 2014. https://doi.org/10.1155/2014/519579.

[16] Singh, C.J., Jain, M. and Kumar, B. (2014). Analysis of $M^X/G/1$ queueing model with balking and vacation. *International Journal of Operational Research*. 19(2). 154–170. https://doi.org/10.1504/IJOR.2014.058952

[17] P., V. L., and George, A. A. (2020). Transient Analysis of Batch Service Queue with Second Optional Service and Reneging. *International Research Journal on Advanced Science Hub*, 2(10), 29-38. doi: 10.47392/irjash.2020.185.

[18] Cherfaoui, M., Bouchentouf, A.A. and Boualem, M. (2023). Modelling and simulation of Bernoulli feedback queue with general customers' impatience under variant vacation policy. *International Journal of Operational Research*, 46(4), 451-480.

[19] Saravanarajan, M.C. and Chandrasekaran, V.M. (2014). Analysis of $M^X/G/1$ feedback queue with two-phase service, compulsory server vacation and random breakdowns.*OPSEARCH*. 51, 235–256. https://doi.org/10.1007/s12597-013-0141-6.

[20] Shanmugasundaram, S. and Sivaram, G. (2020). M/G/1 feedback queue when server is off and on vacation.*International Journal of Applied Engineering Research.* 15(10), 1025—1028. DOI:10.37622/IJAER/15.10.2020.1025-1028

[21] P. Vijaya Laxmi, Qrewi, H. A. and George, A. A. (2022). Analysis of Markovian batch service queue with feedback and second optional service. *Reliability: Theory & Applications*. 17(2), 507–518. https://doi:10.24412/1932-2321-2022-268-507-518.

[22] Baruah, M., Madan, K.C. and Eldabi, T. (2012). Balking and Re-service in a Vacation Queue with Batch Arrival and Two Types of Heterogeneous Service.*Journal of Mathematics Research*. 4(4), 114–124.

[23] Arivudainambi, D. and Godhandaraman, P. (2012). A batch arrival retrial queue with two phases of service,feedback and K optional vacations. *Applied Mathematical Sciences (Ruse)*. 6(22), 1071-1087.

[24] Govindan, A. and Shyamala, s. (2016). Transient solution of an $M^X/G/1$ queueing model with feedback, random breakdowns, Bernoulli schedule server vacation and random setup time. *International Journal of Operational Research* . 25(2), 196–211. https://doi.org/10.1504/IJOR.2016.073956.

[25] Jain, M. and Singh, M. (2020). Transient Analysis of a Markov Queueing Model with Feedback, Discouragement and Disaster. *Int. J. Appl. Comput. Math*. 6(31), https://doi.org/10.1007/s40819-020-0777-x.

[26] Bouchentouf, A.A. and Guendouzi. (2021). A. Single Server Batch Arrival Bernoulli Feedback Queueing System with Waiting Server, K-Variant Vacations and Impatient Customers. *Operations Research Forum*, 2(1), https://doi.org/10.1007/s43069-021-00057-0.

[27] Lewis,A.S. and Overton, M.L. (2013). Nonsmooth optimazation via quasi-Newton methods. *Mathematical programming*.141, 135–163.