Analysis of $MAP/PH_1, PH_2, PH_3/1$ Queueing-Inventory System with Two Commodities

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Abstract

In this work, a single server implements a two-commodity inventory queueing system. We assume that both commodities have a finite capacity. Customers arrive by a Markovian Arrival Process, there is a need for a single item, and either or both types of commodities are required, and this requirement is modeled using certain probabilities. The lead times are exponentially distributed, and the service times have a PH distribution. We use matrix analytical techniques to investigate the queueing inventory system and adopt an (s, S)-type replenishment policy that is dependent on the type of commodity. In the steady state, the joint and individual probability distribution of the E_{system}, inventory level, and server status is obtained. A few significant performance measures are attained. Our mathematical concept is then illustrated with a few numerical examples.

Keywords: Queueing-inventory; Markovian Arrival Process; Phase-type distribution; (s, S)-type policy; Two-Commodity.

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1. Introduction

Many researchers have been interested in the study of queueing inventory systems, and proposals involving two commodities have been made. Sigman and Levi [18] presented the M/G/1 queueing-inventory model with exponentially distributed lead time under light traffic in 1992. Several models with various ordering criteria have been developed to operate such systems. Balintfy [5] and Silver, E.A., [19] both contributed to the development of the joint ordering policy. A two-commodity inventory system with zero lead time and an equal demand process was examined, according to Krishnamoorthy et al. [12] and Anbazhagan and Arivarignan [2].

Neuts [15] developed, studied, and instructed MAP in 1984. Chakravarthy [8] derived the Markovian arrival process by depicting matrix (D_0, D_1) as the guideline for the MAP at the dimension m, where D_0 governs for no arrival, where D_1 governs for arrival. The generator of the matrix Q defined by $D = D_0 + D_1$ is an irreducible stochastic matrix. A single-server inventory system using Markovian Arrival Process (MAP)-based arrivals were studied by Paul Manuel et al. [16].

Yadavalli et al. [22] considered a two-commodity stochastic inventory system with joint and individual ordering policies, Poisson arrivals and lost sales. Anbazhagan et al. [3] for their consideration of a two-commodity continuous review inventory system with substitutable items and Markovian demands. When the sum of the two commodities' on-hand inventory levels reaches a certain level s, reordering for supply is initiated. A two-commodity inventory problem was studied by Krishnamoorthy and Varghese [13] with no lead time and Markovian shifts in demand for the first, second, and both commodities.

Binitha Benny et al. [6] considered a total cost inventory system with a single server and the buffer capacity will be limited. Customers arrive through a Poisson process, and the probabilities used to determine the demand for each commodity or both commodities depend on which commodity is being purchased. Sivakumar et al. [20] investigated a total cost continuous review inventory system with a demand renewal and ordering policy, a policy combination known as ordering individual commodities and ordering both commodities jointly.

A two-commodity model with a compliment and regular working vacations is examined by Lakshmanan et al. in their study [14]. Each customer orders service at a convenient moment, and both commodities are independent of their ordering procedures. Each customer is given a finite retry orbit when the requested item is out of stock or the server is overloaded. Schwarz et al. [17] looked into a brand-new type of stochastic network that shows a product from steady-state distribution. There, integrated models for networks of service stations and inventories were constructed using stochastic networks. They assume that even though a server with associated inventory stops accepting new customers when the stock is out, lost sales are still recorded in the system.

According to Yadavalli et al. [23], the three types of demand for the two goods are comparable. They looked at a system with a phase-type distributed lead time and perishable items. A Markovian arrival process governs the occurrence of all three different kinds of demands. Each commodity's lifetime has an exponential distribution with unique properties. A continuous-time Markov chain that identified the system was used to give a stability analysis and identify individual ordering strategies. Amirthakodi [1] thought of an inventory system with one server service facility and a limited number of trial feedback customers. An inventory system with a single server, two commodities, queue-dependent services for a finite queue, and an optional retrial facility was examined by Jeganathan et al. [10].

Federgruen et al. [9] investigated a continuous review multi-item inventory system with demands generated by independent compound Poisson processes using the (S, c, s) ordering strategy. One consequence of implementing this approach is the requirement to find three optimal variables for each item. Kalpakam and Arivarignan [11] proposed a policy with fewer variables for making decisions and for an (s, S) policy generated by [11] that is appropriate for related but non-substitutable items, a single reorder level s is determined. The total cost is determined by the average inventory, a customer in queue, and reorder rates, according to Berman [7], who provided a deterministic approximation for their inventory system with a service facility.

The demand for each commodity occurs in independent Poisson processes with a variety of parameters in two-commodity retrial inventory systems with varied ordering strategies has been studied by Sivakumar [21] and Jeganathan and Anbazhagan [4]. The constant retrial policy was taken into consideration in both experiments. In other words, a signal is sent out when there are i demands in the orbit according to an exponential distribution that is independent of the orbit's number.

1.1. Motivation for the proposed model

The main motivating factor for our model is the Textile scenario. Buyers usually go to a Textile shop to purchase one or more (like churidar, sarees, shirts, kurtas, and so on) items or goods. Let's say there are n various items and people are shopping for the product i with probability, p_i, $1 \leq i \leq n$. Customers shop for objects $i_1, ..., i_k$, for $2 \leq k \leq n$ with probability $p_{i_1,...,i_k}$, where $i_1, ..., i_k$ is an element of the set of integers $1, 2, ..., n$. Customers will be served only those products that are in stock of the ones requested if all of the requested different goods are not in stock. If a Buyer is unable to obtain any product, they will be disappointed. A customer has a $2^n - 1$ different possibility to shop for the products and we will concentrate on the case where $n = 2$.

1.2. Research Gap

Benny et al. [6] worked with two-commodity in the single server queueing inventory system and arrival follows the Poisson process and service follows an exponential distribution. This article examines two-commodity in the inventory with arrival following MAP and service times following Phase-type distributions. The authors handle (s, S) policy, and both individual and joint orders are obtained. In this article, we develop (s, S) policy, both individual and joint orders, and numerical implementation of 2D using Matlab software.

1.3. Viewpoint for This Work

The manuscript for this work is synchronized as follows: A brief explanation of our model is provided in Section 2. Our model's notations and matrix generation are described in Section 3. Section 4 contains our model's steady-state probability. Section 5 provides performance measures. Numerical illustrated in Section 6. The conclusion is given in Section 7.

2. Model Description

Consider a single server queueing model subject to a two-commodity. Customers arrive according to a MAP and each commodity has a single item demand. The MAP is specified by two m x m matrices (D_0, D_1), $D = D_0 + D_1$, which is an irreducible infinitesimal generator. The matrix D_0 means no arrival similarly, the matrix D_1 means arrival.

There is a need for a single unit, and either or both types of commodities are required, and this requirement is modeled using certain probabilities. The lead times are exponentially distributed, and the service times have a PH distribution. Customers may want both commodities or only one, depending on some predetermined probability. Only when services are being offered are the customers' needs disclosed. If the requested item is not available, the customer permanently exits the system. When only one of the requested items is available and both are demanded, the customer is given the one that is in stock. In the case where both commodity inventory levels are 0, customers are not allowed to join the system. However, customers join the system even when the server is operating and no more inventory is available. For the customer's needed item to be provided at the time the item is taken for service, it is planned that the items will be replenished during the current service. When a customer cannot get the commodity they need at the time of service, the customer is also lost.

When taken for service, the customer requests item I_i with probability c_i, for $i = 1, 2$ or both I_1 and I_2 with probability c_3 such that $c_1 + c_2 + c_3 = 1$. After a random period of service, the requested item is delivered to the customer. The service times for processing orders for I_1, I_2 or both $I_1 and I_2$ are PH- distribution with represented by (α_v, T_v), $1 \leq v \leq 3$. Whose matrix is order n_v with $T_v^0 + T_v e = 0$ implies that $T_v^0 = -T_v e$. Here λ is the arrival rate, which is signified as $\lambda = \pi_1 D_1 e$, where π_1 is the steady-state probability vector. The mean service rate is denoted by $\mu_v = [\alpha_v(-T_v)^{-1} e_{n_v}]^{-1}$.

For both commodities, the system has a maximum capacity of S_i items. We utilize a (s_i, S_i) replenishment strategy for the commodity I_i, where $i = 1, 2$. That is, an order is placed for just that item to raise the inventory level of commodity I_i back to S_i, i = 1, 2 at the time of replenishment, anytime it drops to s_i. For parameters, β_i, for $i = 1, 2$, the lead time has an exponential distribution.

3. The QBD process's infinitesimal generation matrix

The following notations and assumptions are used to explain our model of producing QBD processes in this section.

Notations

We will define the following notations:

- $\otimes$ -Kronecker product of two matrices of various dimensions resulting in a block matrix.
- $\oplus$ - Kronecker sum of two matrices of various dimensions resulting in a block matrix.
- I_m stand for identity matrix of m rows and m columns.
- e - A column vector of the suitable order. Each of its entries is one.
- $N(t)$ represents the total number of customers in the queue.
- $V(t)$ represents the server's status at epoch t.

$$V(t) = \begin{cases} 0, & \text{if server is idle} \\ 1, & \text{if server is busy with } I_1 \\ 2, & \text{if server is busy with } I_2 \\ 3, & \text{if server is busy with } I_1 \text{ and } I_2 \end{cases}$$

- $L_i(t)$ stands for the excess inventory level of commodity I_i, $i = 1, 2$.
- $S(t)$ stands for phases of the service.
- $M(t)$- The Markovian arrival process is considered in phases.
- Let Y=$\{Y(t) : t \geq 0\}$, where $Y(t) = \{N(t), V(t), I_1(t), I_1(t), S(t), M(t)\}$ is a $CTMC$ with state space

$$\Phi = \phi(0) \bigcup_{i=1}^{\infty} \phi(i). \tag{1}$$

where
$\phi(0) = \{(0, 0, a_1, a_2, k) : \ 0 \leq a_1 \leq S_1, \ 0 \leq a_2 \leq S_2, \ 1 \leq k \leq m\}$
$\cup\{(0, v, a_1, a_2, j_v, k) : \ 1 \leq v \leq 3, \ 0 \leq a_1 \leq S_1, \ 0 \leq a_2 \leq S_2, \ 1 \leq j_v \leq n_v, \ 1 \leq k \leq m\}$
and for $p \geq 1$,

$$\phi(p) = \{(p, v, a_1, a_2, j_v, k) : \ 1 \leq v \leq 3, \ 0 \leq a_1 \leq S_1, \ 0 \leq a_2 \leq S_2, \ 1 \leq j_v \leq n_v, \ 1 \leq k \leq m\}.$$

3.1. The Infinitesimal Generator Matrix

The infinitesimal generator matrix of the Markov chain is given by:

$$Q = \begin{bmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & 0 & \dots \\ B_{10} & A_1 & A_0 & 0 & 0 & 0 & 0 & \dots \\ B_{20} & A_2 & A_1 & A_0 & 0 & 0 & 0 & \dots \\ B_{30} & A_3 & A_2 & A_1 & A_0 & 0 & 0 & \dots \\ B_{40} & A_4 & A_3 & A_2 & A_1 & A_0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \dots \end{bmatrix} \tag{2}$$

The following describes Markov chain transitions and the corresponding rates:
The matrix B_{00} governs,

- $(0, v, a_1, a_2, j_v, k) \rightarrow (0, 0, a_1, a_2, k)$ with rate $T_v{}^0 \otimes I_m$ for $1 \leq v \leq 3$, $0 \leq a_1 \leq S_1$, $0 \leq a_2 \leq S_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$,

- $(0,0,a_1,a_2,k) \to (0,0,S_1,a_2,k)$ with rate $\beta_1 I_m$ for $0 \leq v \leq 3$, $0 \leq a_1 \leq s_1$, $0 \leq a_2 \leq S_2$, $1 \leq k \leq m$,
- $(0,v,a_1,a_2,j_v,k) \to (0,v,S_1,a_2,j_v,k)$ with rate $\beta_1 I_{n_v m}$ for $1 \leq v \leq 3$, $0 \leq a_1 \leq s_1$, $0 \leq a_2 \leq S_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$,
- $(0,0,a_1,a_2,k) \to (0,0,a_1,S_2,k)$ with rate $\beta_2 I_m$ for $0 \leq v \leq 3$, $0 \leq a_1 \leq S_1$, $0 \leq a_2 \leq s_2$, $1 \leq k \leq m$,
- $(0,v,a_1,a_2,j_v,k) \to (0,v,a_1,S_2,j_v,k)$ with rate $\beta_2 I_{n_v m}$ for $1 \leq v \leq 3$, $0 \leq a_1 \leq S_1$, $0 \leq a_2 \leq s_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$,
- $(0,0,0,a_2,k) \to (0,2,0,a_2-1,j_2,k)$ with rate $\alpha_2 \otimes (c_2+c_3)D_1$ for $1 \leq a_2 \leq S_2$,$1 \leq j_2 \leq n_2$, $1 \leq k \leq m$,
- $(0,0,a_1,0,k) \to (0,1,a_1-1,0,j_1,k)$ with rate $\alpha_1 \otimes (c_1+c_3)D_1$ for $1 \leq a_1 \leq S_1$, $1 \leq j_1 \leq n_1$, $1 \leq k \leq m$,
- $(0,0,a_1,a_2,k) \to (0,1,a_1-1,a_2,j_1,k)$ with rate $\alpha_1 \otimes c_1 D_1$ for $1 \leq a_1 \leq S_1$, $1 \leq a_2 \leq S_2$, $1 \leq j_1 \leq n_1$, $1 \leq k \leq m$,
- $(0,0,a_1,a_2,k) \to (0,2,a_1,a_2-1,j_2,k)$ with rate $\alpha_2 \otimes c_2 D_1$ for $1 \leq a_1 \leq S_1$, $1 \leq a_2 \leq S_2$, $1 \leq j_2 \leq n_2$, $1 \leq k \leq m$,
- $(0,0,a_1,a_2,k) \to (0,3,a_1-1,a_2-1,j_3,k)$ with rate $\alpha_3 \otimes c_3 D_1$ for $1 \leq a_1 \leq S_1$, $1 \leq a_2 \leq S_2$, $1 \leq j_3 \leq n_3$, $1 \leq k \leq m$.

The matrix $B_{(p+1)0}$, $p \geq 1$, governs

- $(p,v,0,0,j_v,k) \to (0,0,0,0,k)$ with rate ${T_v}^0 \otimes I_m$ for $1 \leq v \leq 3$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$,
- $(p,v,0,a_2,j_v,k) \to (0,0,0,a_2,k)$ with rate ${T_v}^0 {c_1}^p \otimes I_m$ for $1 \leq v \leq 3$, $1 \leq a_2 \leq S_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$,
- $(p,v,0,a_2,j_v,k) \to (0,2,0,a_2-1,j_2,k)$ with rate ${T_v}^0 {c_1}^{p-1}(c_2+c_3)\alpha_2 \otimes I_m$ for $1 \leq v \leq 3$, $1 \leq a_2 \leq S_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$,
- $(p,v,a_1,0,j_v,k) \to (0,0,a_1,0,k)$ with rate ${T_v}^0 {c_2}^p \otimes I_m$ for $1 \leq v \leq 3$, $1 \leq a_1 \leq S_1$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$,
- $(p,v,a_1,0,j_v,k) \to (0,2,a_1-1,0,j_1,k)$ with rate ${T_v}^0 {c_2}^{p-1}(c_1+c_3)\alpha_1 \otimes I_m$ for $1 \leq v \leq 3$, $1 \leq a_1 \leq S_1$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$,
- $(1,v,a_1,a_2,j_v,k) \to (0,1,a_1-1,a_2,j_1,k)$ with rate ${T_v}^0 c_1 \alpha_1 \otimes I_m$ for $1 \leq v \leq 3$, $1 \leq a_1 \leq S_1$, $1 \leq a_2 \leq S_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$,
- $(1,v,a_1,a_2,j_v,k) \to (0,2,a_1,a_2-1,j_2,k)$ with rate ${T_v}^0 c_2 \alpha_2 \otimes I_m$ for $1 \leq v \leq 3$, $1 \leq a_1 \leq S_1$, $1 \leq a_2 \leq S_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$,
- $(1,v,a_1,a_2,j_v,k) \to (0,3,a_1-1,a_2-1,j_3,k)$ with rate ${T_v}^0 c_3 \alpha_3 \otimes I_m$ for $1 \leq v \leq 3$, $1 \leq a_1 \leq S_1$, $1 \leq a_2 \leq S_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$.

The matrix A_1, $p \geq 1$, governs

- $(p,v,a_1,a_2,j_v,k) \to (p,v,S_1,a_2,j_v,k)$ with rate $\beta_1 I_{n_v m}$ for $1 \leq v \leq 3$, $0 \leq a_1 \leq s_1$, $0 \leq a_2 \leq S_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$,
- $(p,v,a_1,a_2,j_v,k) \to (p,v,a_1,S_2,j_v,k)$ with rate $\beta_2 I_{n_v m}$ for $1 \leq v \leq 3$, $0 \leq a_1 \leq S_1$, $0 \leq a_2 \leq s_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$.

The matrix A_{l+1}, $1 \leq l \leq p-1$, $p \geq 3$, governs

- $(p, v, 0, a_2, j_v, k) \rightarrow (p-l, 2, 0, a_2-1, j_2, k)$ with rate $T_v{}^0 c_1{}^{l-1}(c_2+c_3)\alpha_2 \otimes I_m$ for $1 \leq v \leq 3$, $1 \leq a_2 \leq S_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$,
- $(p, v, a_1, 0, j_v, k) \rightarrow (p-l, 1, a_1-1, 0, j_1, k)$ with rate $T_v{}^0 c_2{}^{l-1}(c_1+c_3)\alpha_1 \otimes I_m$ for $1 \leq v \leq 3$, $1 \leq a_2 \leq S_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$,
- $(p, v, a_1, a_2, j_v, k) \rightarrow (p-1, 1, a_1-1, a_2, j_1, k)$ with rate $T_v{}^0 c_1 \alpha_1 \otimes I_m$ for $1 \leq v \leq 3$, $1 \leq a_1 \leq S_1$, $1 \leq a_2 \leq S_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$,
- $(p, v, a_1, a_2, j_v, k) \rightarrow (p-1, 2, a_1, a_2-1, j_2, k)$ with rate $T_v{}^0 c_2 \alpha_2 \otimes I_m$ for $1 \leq v \leq 3$, $1 \leq a_1 \leq S_1$, $1 \leq a_2 \leq S_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$,
- $(p, v, a_1, a_2, j_v, k) \rightarrow (p-1, 3, a_1-1, a_2-1, j_3, k)$ with rate $T_v{}^0 c_3 \alpha_3 \otimes I_m$ for $1 \leq v \leq 3$, $1 \leq a_1 \leq S_1$, $1 \leq a_2 \leq S_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$.

The matrix B_{01}, governs

- $(0, v, a_1, a_2, j_v, k) \rightarrow (1, v, a_1, a_2, j_v, k)$ with rate $I_{n_v} \otimes D_1$ for $1 \leq v \leq 3$, $0 \leq a_1 \leq S_1$, $0 \leq a_2 \leq S_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$.

The matrix A_0, $p \geq 1$, governs

- $(p, v, a_1, a_2, j_v, k) \rightarrow (p+1, v, a_1, a_2, j_v, k)$ with rate $I_{n_v} \otimes D_1$ for $1 \leq v \leq 3$, $0 \leq a_1 \leq S_1$, $0 \leq a_2 \leq S_2$, $1 \leq j_v \leq n_v$, $1 \leq k \leq m$.

4. Analysis of Steady-State

The nonsingularity of B_{00} and A_1 is need for Q to be irreducible. Consider the matrix $A = \sum_{l=0}^{\infty} A_l$. Let the unique stationary distribution of A be ψ. Under the condition (Neuts [15]),

$$\psi A_0 e < \sum_{l=2}^{\infty}(l-1)\psi A_l e,$$

an irreducible Markov chain with generator Q possesses a unique stationary solution vector $Y = (y_0, y_1, y_2, \dots)$ satisfying

$$YQ = 0, \quad Ye = 1.$$

Partitioning Y as $Y = (y_0, y_1, y_2, \dots)$ where

$y_0 = (y_0(v, a_1, a_2, j_v, k) : \ 0 \leq v \leq 3, \ 0 \leq a_1 \leq S_1, \ 0 \leq a_2 \leq S_2, \ 1 \leq j_v \leq n_v, \ 1 \leq k \leq m)$,

$y_p = (y_p(v, a_1, a_2, j_v, k) : \ 1 \leq v \leq 3, \ 0 \leq a_1 \leq S_1, \ 0 \leq a_2 \leq S_2, \ 1 \leq j_v \leq n_v, \ 1 \leq k \leq m), \ for \ p \geq 1$,

where y_0 is of dimension $1 \times (S_1+1)(S_2+1)m + (S_1+1)(S_2+1)n_1 m + (S_1+1)(S_2+1)n_2 m + (S_1+1)(S_2+1)n_3 m$ and y_p for $p \geq 1$, is of dimension $1 \times (S_1+1)(S_2+1)n_1 m + (S_1+1)(S_2+1)n_2 m + (S_1+1)(S_2+1)n_3 m$. Then Y is obtained from

$$y_p = y_1 R^{p-1}, \quad p \geq 2$$

where R is the minimal nonnegative solution of the matrix equation $\sum_{j=0}^{\infty} Y^j A_j = 0$.

The boundary equations are given by

$$\sum_{p=0}^{\infty} y_p B_{p0} = 0$$

$$y_0 B_{00} + \sum_{p=1}^{\infty} y_p A_p = 0$$

The normalizing condition $Ye = 1$ gives

$$y_0 e + y_1 [I-R]^{-1} e = 1$$

R matrix is obtained using the algorithm:

$$R(0) = 0$$

$$R(p+1) = -A_0 A_1{}^{-1} - R^2(p) A_2 A_1{}^{-1} - R^3(p) A_3 A_1{}^{-1} - \dots, \quad p \geq 0$$

5. Performance Measure

- Expected number of customers in the system, $E_N = \sum_{p=1}^{\infty} py_p$
- Expected number of customers demanding I_1 alone, $E_{I_1} = c_1 E_N$
- Expected number of customers demanding I_2 alone, $E_{I_2} = c_2 E_N$
- Expected number of customers demanding both I_1 and I_2, $E_{I_{12}} = c_3 E_N$
- Expected rate of replenishment for item I_1,

$$E_{RI_1} = \beta_1 \{ \sum_{a_1=0}^{s_1} \sum_{a_2=0}^{S_2} \sum_{k=1}^{m} y_0(0, a_1, a_2, k) + \sum_{p=0}^{\infty} \sum_{v=1}^{3} \sum_{a_1=0}^{s_1} \sum_{a_2=0}^{S_2} \sum_{j_v=1}^{n_v} \sum_{k=1}^{m} y_p(v, a_1, a_2, n_v, k) \}$$

- Expected rate of replenishment for item I_2,

$$E_{RI_2} = \beta_2 \{ \sum_{a_1=0}^{S_1} \sum_{a_2=0}^{s_2} \sum_{k=1}^{m} y_0(0, a_1, a_2, k) + \sum_{p=0}^{\infty} \sum_{v=1}^{3} \sum_{a_1=0}^{S_1} \sum_{a_2=0}^{s_2} \sum_{j_v=1}^{n_v} \sum_{k=1}^{m} y_p(v, a_1, a_2, n_v, k) \}$$

- Expected reorder rate of commodity I_1,

$$E_{R_1} = \mu_1 \sum_{p=0}^{\infty} \sum_{a_2=0}^{S_2} \sum_{j_1=1}^{n_1} \sum_{k=1}^{m} y_p(1, s_1 + 1, a_2, n_1, k)$$

- Expected reorder rate of commodity I_2,

$$E_{R_2} = \mu_2 \sum_{p=0}^{\infty} \sum_{a_1=0}^{S_1} \sum_{j_2=1}^{n_2} \sum_{k=1}^{m} y_p(2, a_1, s_2 + 1, n_2, k)$$

- Expected reorder rate of commodity I_1 and I_2,

$$E_{R_{12}} = \mu_3 \sum_{p=0}^{\infty} \sum_{j_3=1}^{n_3} \sum_{k=1}^{m} y_p(3, s_1 + 1, s_2 + 1, n_3, k)$$

6. Numerical Implementation

In this section, we examine the outcome of our system using numerical and graphical representations. The three different MAP representations are distinct with the following variance and correlation structures and their mean values are 1.

Arrival in Erlang of order 2(ERL-A):

$$D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix} D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

Arrival in Exponential(EXP-A):

$$D_0 = [-1] D_1 = [1]$$

Arrival in Hyper exponential(HYP-EXP-A):

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix} D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}$$

Let us consider PH-distributions for the service process as follows:
ERL-S (Service in Erlang of order 2):

$$\alpha_1 = \alpha_2 = \alpha_3 = [1, 0] \quad T_1 = T_2 = T_3 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

EXP-S(Service in Exponential):

$$\alpha_1 = \alpha_2 = \alpha_3 = [1] \quad T_1 = T_2 = T_3 = [-1]$$

HYP-EXP-S(Service in Hyper exponential):

$$\alpha_1 = \alpha_2 = \alpha_3 = [0.8, 0.2] \quad T_1 = T_2 = T_3 = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}$$

6.1. Illustration

In the following tables 1, 2 and 3, we have examined the impact of the arrival rate λ on the expected system size. Fix $S_1 = 8$, $S_2 = 10$, $s_1 = 2$, $s_2 = 3$, $\mu_1 = 2$, $\mu_2 = 3$, $\mu_3 = 4$, $\beta_1 = 2$, $\beta_2 = 3$, $c_1 = 0.1$, $c_2 = 0.1$, $c_3 = 0.8$.

Table 1: *Arrival rate(λ) vs E_N*

ERL-S			
λ	ERL-A	EXP-A	HYP-EXP-A
1	0.038353086	0.090270325	0.197559861
1.1	0.050688987	0.113781945	0.256893077
1.2	0.065624277	0.141308387	0.329498505
1.3	0.083549669	0.173400817	0.417940725
1.4	0.104926605	0.210715786	0.525249133
1.5	0.130305758	0.254041436	0.654975645
1.6	0.160351157	0.304331913	0.81126147
1.7	0.195872159	0.36275318	0.998927098
1.8	0.23786654	0.430744844	1.223609249
1.9	0.287579538	0.51010493	1.491978161
2.0	0.346586293	0.603108157	1.812077611

Table 2: *Arrival rate(λ) vs E_N*

EXP-S			
λ	ERL-A	EXP-A	HYP-EXP-A
1	0.062592538	0.120376635	0.254416346
1.1	0.08207371	0.151739687	0.328650275
1.2	0.105500217	0.188464919	0.418427098
1.3	0.133449138	0.231291358	0.526393629
1.4	0.166604203	0.281099839	0.655616534
1.5	0.205783572	0.338948174	0.809643341
1.6	0.251976048	0.406117251	0.992584752
1.7	0.306389033	0.484172226	1.20923196
1.8	0.370513054	0.575044848	1.465227137
1.9	0.446210016	0.681145853	1.767309959
2	0.535836027	0.80552094	2.123668949

Table 3: *Arrival rate(λ) vs E_N*

HYP-EXP-S			
λ	ERL-A	EXP-A	HYP-EXP-A
1	0.249031469	0.32018599	0.571239637
1.1	0.319730773	0.403892787	0.721444007
1.2	0.403462978	0.502016937	0.897044873
1.3	0.502065778	0.616543171	1.10107116
1.4	0.617713186	0.749810152	1.336958686
1.5	0.752987938	0.904583472	1.608618699
1.6	0.910971089	1.084145482	1.920517083
1.7	1.095353193	1.292405822	2.27776399
1.8	1.310572349	1.534037132	2.686212657
1.9	1.561985236	1.814640848	3.152564574
2	1.856077719	2.140947812	3.684475578

We observe that from the above tables 1, 2 and 3:

- As arrival rate (λ) increases, the variety of arrangements of arrival and service times then the corresponding E_N also increases.
- Observe the arrival times, E_N rises more quickly in the case of $HYP - EXP - A$ and more slowly in the case of $ERL - A$. Similarly, it rises gradually in the case of $ERL - S$ and rapidly in the case of $HYP - EXP - A$.

6.2. Illustration

We have investigated the consequence of the arrival rate λ against the Expected to reorder rate of commodity I_1 (E_{R_1})in the obeying table 4, 5 and 6. Fix $S_1 = 8$, $S_2 = 10$, $s_1 = 2$, $s_2 = 3$, $\mu_1 = 2$, $\mu_2 = 3$, $\mu_3 = 4$, $\beta_1 = 2$, $\beta_2 = 3$, $c_1 = 0.1$, $c_2 = 0.1$, $c_3 = 0.8$.

Table 4: *Arrival rate(λ) vs E_{R_1}*

ERL-S			
λ	ERL-A	EXP-A	HYP-EXP-A
1.0	0.000025	0.003443	0.000174
1.1	0.000036	0.003781	0.000228
1.2	0.000050	0.004121	0.000290
1.3	0.000069	0.004461	0.000360
1.4	0.000092	0.004804	0.000437
1.5	0.000121	0.005148	0.000522
1.6	0.000155	0.005495	0.000612
1.7	0.000196	0.005846	0.000707
1.8	0.000245	0.006199	0.000806
1.9	0.000302	0.006557	0.000908
2.0	0.000367	0.006918	0.001013

Table 5: *Arrival rate(λ) vs E_{R_1}*

EXP-S			
λ	ERL-A	EXP-A	HYP-EXP-A
1.0	0.003780	0.015642	0.004010
1.1	0.004109	0.017110	0.004381
1.2	0.004435	0.018566	0.004750
1.3	0.004758	0.020010	0.005117
1.4	0.005080	0.021444	0.005481
1.5	0.005402	0.022870	0.005844
1.6	0.005723	0.024289	0.006203
1.7	0.006047	0.025702	0.006560
1.8	0.006372	0.027112	0.006914
1.9	0.006701	0.028519	0.007264
2.0	0.007034	0.029924	0.007610

Table 6: *Arrival rate(λ) vs E_{R_1}*

HYP-EXP-S			
λ	ERL-A	EXP-A	HYP-EXP-A
1	0.000059	0.002861	0.000272
1.1	0.000087	0.003140	0.000356
1.2	0.000124	0.003421	0.000453
1.3	0.000170	0.003705	0.000561
1.4	0.000228	0.003993	0.000679
1.5	0.000297	0.004286	0.000808
1.6	0.000378	0.004585	0.000947
1.7	0.000473	0.004888	0.001093
1.8	0.000583	0.005197	0.001247
1.9	0.000707	0.005512	0.001408
2	0.000845	0.005832	0.001573

We observe that from the above table 4, 5 and 6:

- As arrival rate (λ) increases, the variety of arrangements of arrival and service times then the corresponding E_{R_1} also increases.
- Observe the arrival times, E_{R_1} rises faster in the case of $EXP - A$ and more gradually in the case of $HYP - EXP - A$. Comparably, it rises gradually in the case of $HYP - EXP - S$ and significantly in the case of EXP-S.

6.3. Illustration

In the 2D image, the influence of arrival rate(λ) on the expected number of customers demanding both I_1 and I_{12} has been examined. Fix $S_1 = 8$, $S_2 = 10$, $s_1 = 2$, $s_2 = 3$, $\mu_1 = 2$, $\mu_2 = 3$, $\mu_3 = 4$, $\beta_1 = 2$, $\beta_2 = 3$, $c_1 = 0.1$, $c_2 = 0.1$, $c_3 = 0.8$ so that the stability condition is satisfied.

From Figures 1 to 9,

- we can visualize that as the arrival rate (λ) maximizes, both the value of E_{I_1} and $E_{I_{12}}$ maximizes.
- Furthermore, the rate of an increase of E_{I_1} and $E_{I_{12}}$ for $HYP - EXP - A$ is rapid and slow for $ERL - A$. It is also faster for $HYP - EXP - S$ and shorter for $ERL - S$.

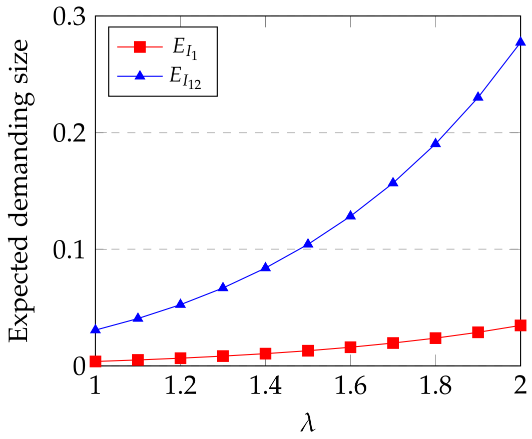

Figure 1: *Arrival rate(λ) vs both E_{I_1} and $E_{I_{12}}$ - $E_k/E_k/1$*

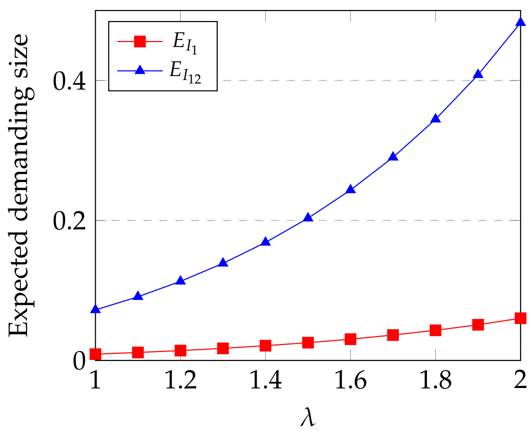

Figure 2: *Arrival rate(λ) vs both E_{I_1} and $E_{I_{12}}$ - $M/E_k/1$*

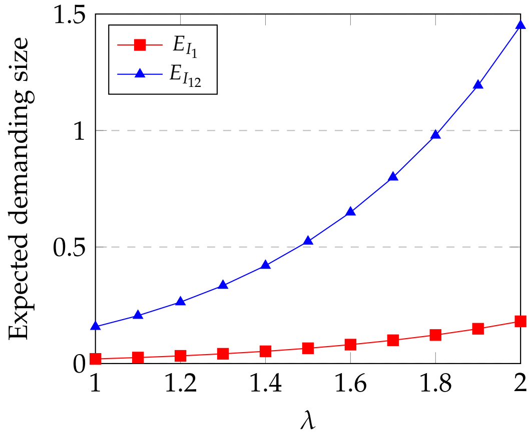

Figure 3: *Arrival rate(λ) vs both E_{I_1} and $E_{I_{12}}$ - $H_k/E_k/1$*

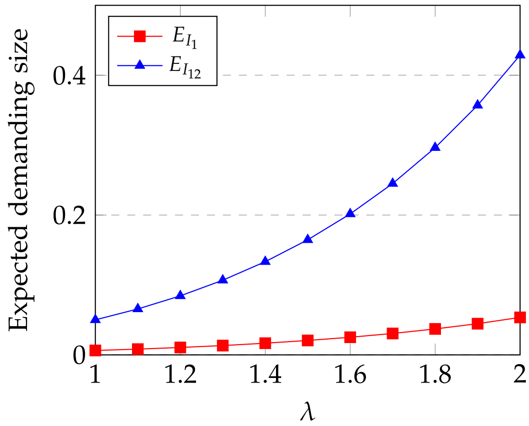

Figure 4: *Arrival rate(λ) vs both E_{I_1} and $E_{I_{12}}$ - $E_k/M/1$*

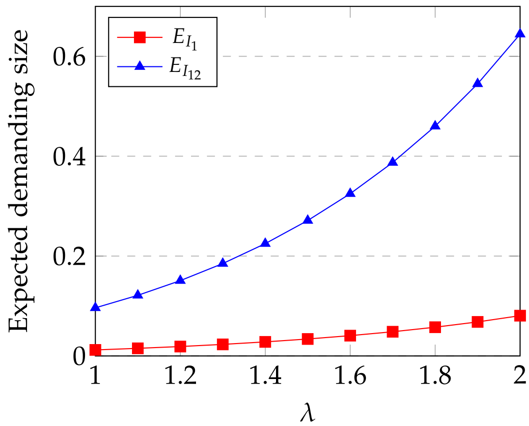

Figure 5: *Arrival rate(λ) vs both E_{I_1} and $E_{I_{12}}$ - $M/M/1$*

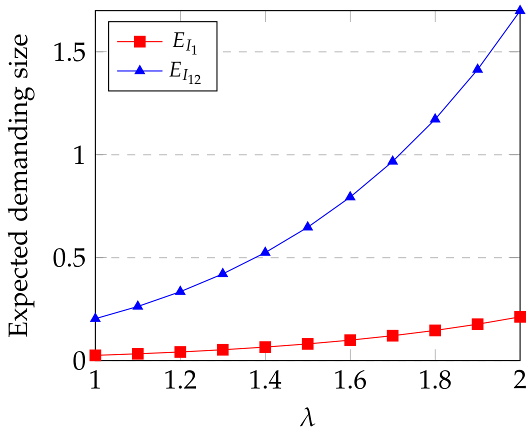

Figure 6: *Arrival rate(λ) vs both E_{I_1} and $E_{I_{12}}$ - $H_k/M/1$*

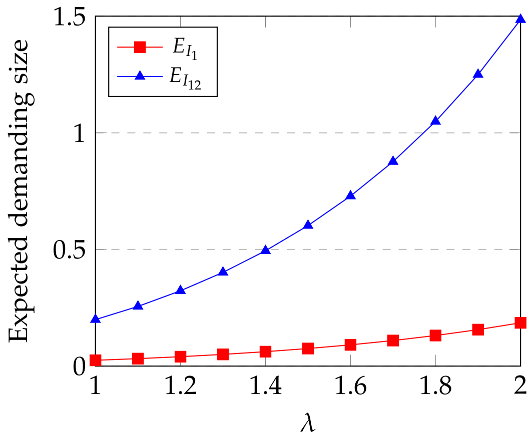

Figure 7: *Arrival rate(λ) vs both E_{I_1} and $E_{I_{12}}$ - $E_k/H_k/1$*

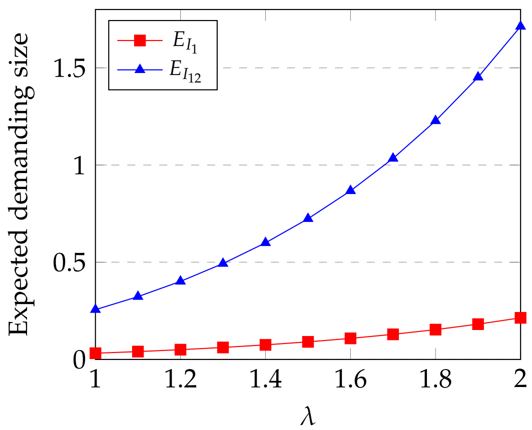

Figure 8: *Arrival rate(λ) vs both E_{I_1} and $E_{I_{12}}$ - $M/H_k/1$*

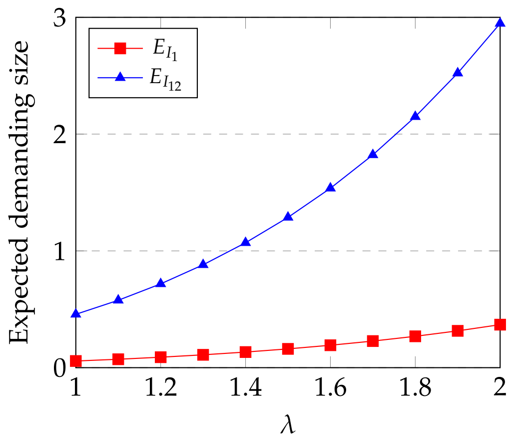

Figure 9: *Arrival rate(λ) vs both E_{I_1} and $E_{I_{12}}$ - $H_k/H_k/1$*

7. Conclusion

We looked at an inventory problem with two commodities and MAP demand arrival. When being taken for service, customers express their needs. If the requested item is unavailable, the customer is permanently removed from the system. When taken for service, if both goods are demanded, and when there is only one thing left, it is served to the customer. Depending on the type of demand, service times are distributed using a phase-type parameter. With parameter β_i for I_i, $i = 1, 2$, the lead times for each commodity are exponentially distributed. It is determined that the continuous-time Markov chain is of type GI/M/1. The stability of the system is demonstrated. Many system performance indices are developed, along with numerical examples and numerical studies.

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