EXPLORING THE ADAPTABILITY OF THE UNIT INVERSE WEIBULL DISTRIBUTION FOR MODELING DATA ON THE UNIT INTERVAL

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Abstract

This paper derives a new lifetime distribution called the unit inverse Weibull distribution(UIWD) from inverse weibull distribution. Various statistical properties such as the survival function, hazard rate function, revised hazard rate function, cumulative hazard rate function, moments, and quartiles have been discussed. Additionally, we have explored other properties like skewness, kurtosis, order statistics, and the quantile function. Various methods of estimation, including maximum likelihood, moments, percentiles, and the Cramer-Von Mises, have been discussed. Simulation studies were conducted to assess the accuracy and precision of the parameters. Comparative analyses were performed to highlight the effectiveness and utility of the proposed model in comparison to other existing models, using two real-life applications. Finally, real life data analysis reveals that derived distribution can provide a better fit than several well-known distributions.

Keywords: Unit inverse Weibull distribution, statistical properties, estimation, simulation study

1. Introduction

Over the last couple of decades, multiple authors have introduced a range of fresh methodologies for creating novel sets of distributions. This has significantly expanded the potential for accurately modeling real-world data across a variety of fields. This concept of devising new models and families has garnered notable attention recently, often termed as "parameter addition" and "parameter induction." The core objective of these efforts is to formulate models capable of effectively capturing real-life phenomena by utilizing available data sets from diverse domains.

In applied statistics, a prevalent hurdle involves addressing uncertain occurrences that exist within the confined range of (0,1). Consider instances from the real world, where measurements frequently involve proportions, fractions representing certain attributes, scores obtained from aptitude assessments, assorted indices, rates, and various other data points that inherently lie within the interval (0,1). In such scenarios, continuous distributions characterized within the domain of (0,1) prove indispensable for the probabilistic representation of these phenomena. The distribution that holds sway over the unit interval finds application across numerous sectors, encompassing economics and biology.

Distributions that find definition over the (0,1) range are conventionally harnessed to model random variables that are inherently limited within the confines of (0,1), such as percentages and proportions. The Beta distribution, renowned for its convenience and utility across a plethora of statistical domains, is a standard choice for tackling such scenarios. Nevertheless, there are situations where the Beta distribution might fall short in adequately elucidating the data, thus prompting the quest for alternative distributions defined within the unit interval.

Various distributions defined on the unit interval have been proposed in the literature, including Topp-Leone [1], Johnson SB [2], unit Gamma [3], Kumaraswamy [4], Arcsine [5], unit Logistic [6], generalized Beta type I [7], Simplex [8], standard two-sided Power[9], Mc Arcsine [10], Log-Lindley [11], two-sided generalized Kumaraswamy [12], and Log-Xgamma [13].

More recently, researchers have proposed new families of transformed distributions on the unit interval. Examples include the unit Birnbaum-Saunders [14], unit Lindley [15], unit inverse Gaussian [16], unit Gompertz [17], unit improved second-degree Lindley [18], Log-weighted exponential [19] , Logit Slash [20], and unit generalized Half Normal [21] distributions.

In this study, we propose a probability distribution called the Unit Inverse Weibull Distribution (UIWD) specifically designed for modeling data on the interval (0,1). The UIWD is derived from a type transformation of the Inverse Weibull Distribution (IWD), and we provide various methods of estimation approach for estimating its parameters.

The motivation behind the development of the UIWD arises from several factors. Firstly, the UIWD exhibits simple and closed-form expressions for its distribution function and quantile function. Moreover, it demonstrates superior fitting performance compared to other commonly used distributions on the unit interval. The UIWD allows for the derivation of various statistical properties, and we utilize the various estimators method to estimate its parameters. Through simulation studies, we assess the accuracy and precision of different estimators and compare the UIWD model with existing models to showcase its utility and effectiveness. Overall, this study aims to contribute to the understanding and application of the UIWD as a flexible probability distribution for modeling data on the unit interval (0,1).

2. The Unit Inverse Weibull Distribution: Derivation of Pdf and Cdf

The two-parameter Weibull distribution defines the probability density function (PDF) of a random variable U as

$$f(u;\alpha,\beta) = \alpha\beta u^{\beta-1}e^{-\alpha u^{\beta}}, \quad u > 0; \alpha,\beta > 0. \tag{1}$$

To explore a related distribution, we introduce a transformation by defining $V = 1/U$. Consequently, V follows the two-parameter Inverse Weibull Distribution (IWD), and its PDF is given by

$$f(v;\alpha,\beta) = \alpha\beta v^{-\beta-1}e^{-\alpha v^{-\beta}}, v > 0; \alpha,\beta > 0. \tag{2}$$

Now, we propose the Unit Inverse Weibull Distribution (UIWD) by introducing a further transformation, $X = 1/1 + V$. As a result, X follows the UIWD, and its PDF is expressed as

$$f(x;\alpha,\beta) = \alpha\beta\left(\frac{1}{x^2}\right)\left(\frac{1}{x}-1\right)^{-\beta-1}e^{-\alpha\left(\frac{1}{x}-1\right)^{-\beta}}, 0 < x < 1; \alpha,\beta > 0. \tag{3}$$

Here, α and β serve as shape parameters in the UIWD, with the constraint that $\alpha > 0$ and $\beta > 0$.

The cumulative distribution function (CDF) of the Unit Inverse Weibull Distribution (UIWD) is defined as follows:

$$F(x|\alpha,\beta) = P(X \leq x) = 1 - e^{-\alpha(1/x-1)^{-\beta}}, \quad 0 < x < 1, \quad \alpha,\beta > 0. \tag{4}$$

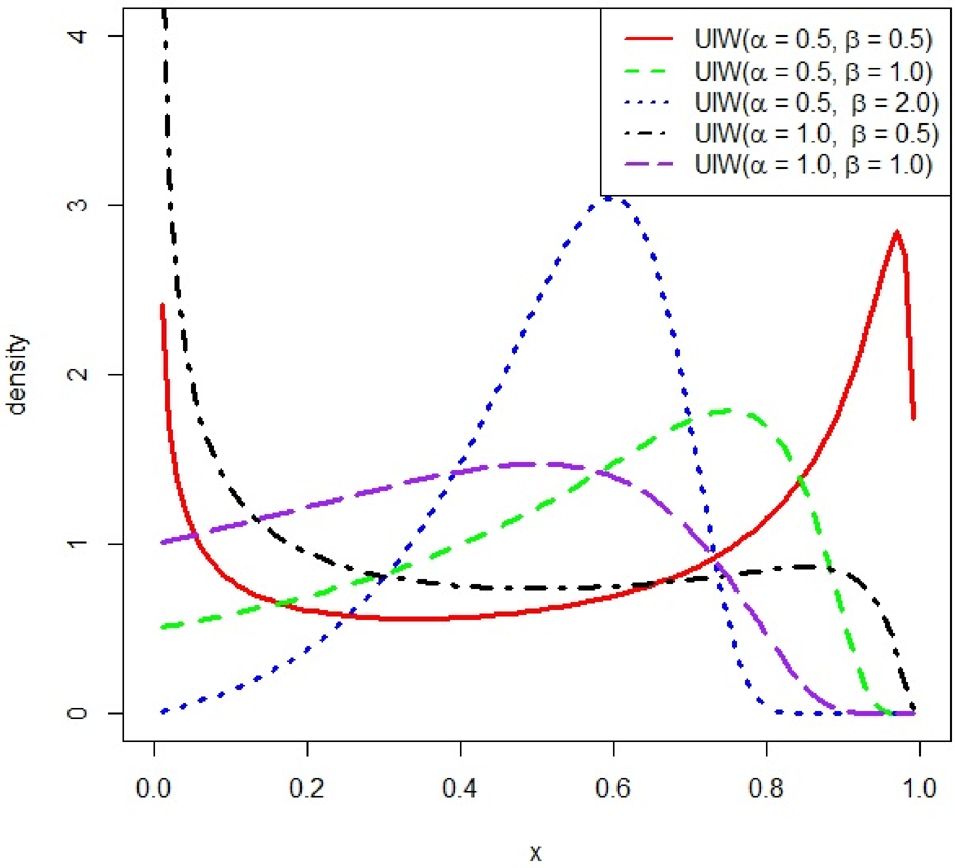

Figure 1: *Pdf of UIWD with different values of alpha and beta*

3. Reliability Properties

3.1. Survival Function

The survival function of the Unit Inverse Weibull Distribution (UIWD) is given by:

$$S(x|\alpha,\beta) = 1 - F(x) = e^{-\alpha(1/x-1)^{-\beta}}, \quad 0 < x < 1, \quad \alpha, \beta > 0. \tag{5}$$

Here the parameters α and β of the UIWD control the shape and scale of the survival function.

The survival function of the Unit Inverse Weibull Distribution (UIWD) exhibits the following characteristics:

1.Monotonic Decrease: The survival function is a monotonically decreasing function. As the value of x increases within the (0, 1) interval, the probability of the random variable exceeding x decreases. This is evident from Figure 3.

2.Asymptotic Behavior: As x approaches 0, the survival function approaches 1. This indicates that the probability of the random variable being greater than a value close to 0 tends to 1. In other words, the UIWD has a high probability of taking on values very close to 0.

3. As x approaches 1, the survival function approaches 0. This implies that the probability of the random variable exceeding a value close to 1 tends to 0. Consequently, the UIWD has a low probability of taking on values very close to 1.

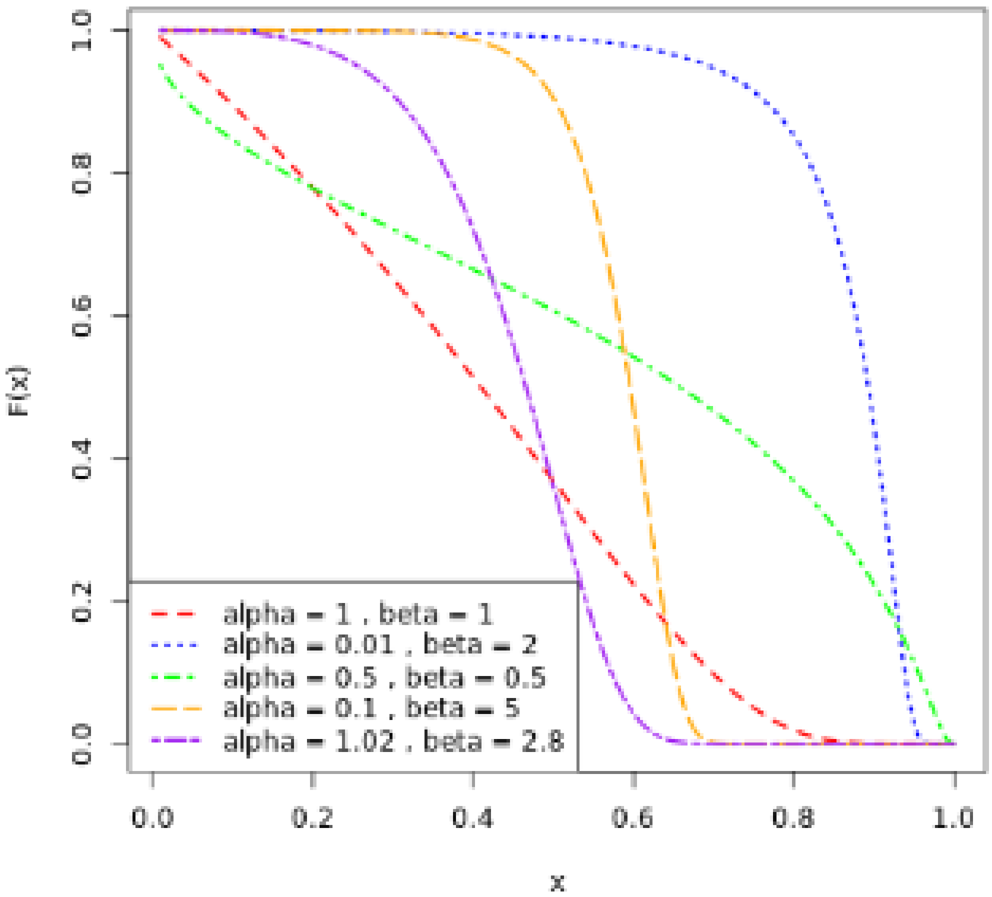

Figure 2: *Distribution function of UIWD with different values of alpha and beta*

3.2. Hazard Rate,Reversed Hazard Rate, Cumulative Hazard Rate Functions

The hazard rate function of the Unit Inverse Weibull Distribution (UIWD) is given by:

$$h(x|\alpha,\beta) = \frac{f(x)}{1-F(x)}$$

$$= \alpha\beta\left(\frac{1}{x^2}\right)\left(\frac{1}{x}-1\right)^{-\beta-1}, \quad 0 < x < 1; \quad \alpha,\beta > 0. \tag{6}$$

Hazard Rate function has the following characteristics:

1. Monotonic Increase: The hazard function is a monotonically increasing function. As the value of x increases within the (0, 1) interval, the hazard rate, which measures the instantaneous rate of occurrence of an event, increases. This is evident from Figure 4.

2. Asymptotic Behavior: As x approaches 1, the hazard rate approaches infinity. This indicates that the event becomes more likely to occur as time approaches the maximum value of 1.

The reversed hazard rate function of the Unit Inverse Weibull Distribution (UIWD) is given by:

$$h_{\text{rev}}(x|\alpha,\beta) = \frac{f(x)}{F(x)}$$

$$= \frac{\alpha\beta\left(\frac{1}{x^2}\right)\left(\frac{1}{x}-1\right)^{-\beta-1} e^{\left(-\alpha\left(\frac{1}{x}-1\right)^{-\beta}\right)}}{1-e^{\left(-\alpha\left(\frac{1}{x}-1\right)^{-\beta}\right)}}, 0 < x < 1, \quad \alpha,\beta > 0. \tag{7}$$

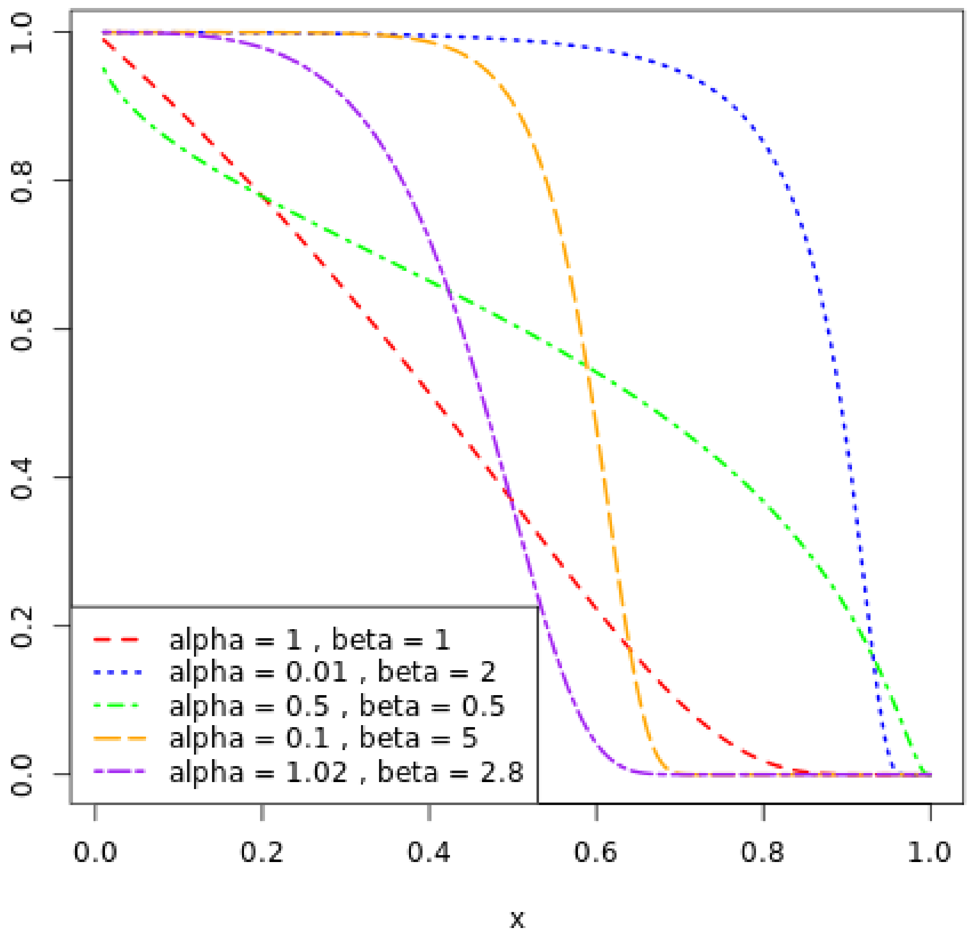

Figure 3: *Survival function of UIWD with different values of alpha and beta*

The cumulative hazard rate function of the Unit Inverse Weibull Distribution (UIWD) is given by:

$$C(x;\alpha,\beta) = -lns(x)$$

$$= \alpha\left(\frac{1}{x}-1\right)^{-\beta}, \quad 0 < x < 1, \quad \alpha,\beta > 0. \tag{8}$$

4. Moments and Related Properties

4.1. Raw Moments and central moments

The r^{th} raw moment about the origin of the random variable X is:

$$\mu_r' = \int_0^1 x^r \alpha\beta\left(\frac{1}{x^2}\right)\left(\frac{1}{x}-1\right)^{-\beta-1} e^{-\alpha\left(\frac{1}{x}-1\right)^{-\beta}} dx$$

Substituting u = $\left(\frac{1}{x}-1\right)^{-\beta}$, we get

$$\mu_r' = \int_0^\infty \alpha \frac{1}{(1+u^{\frac{-1}{\beta}})} e^{-\alpha u}\, du$$

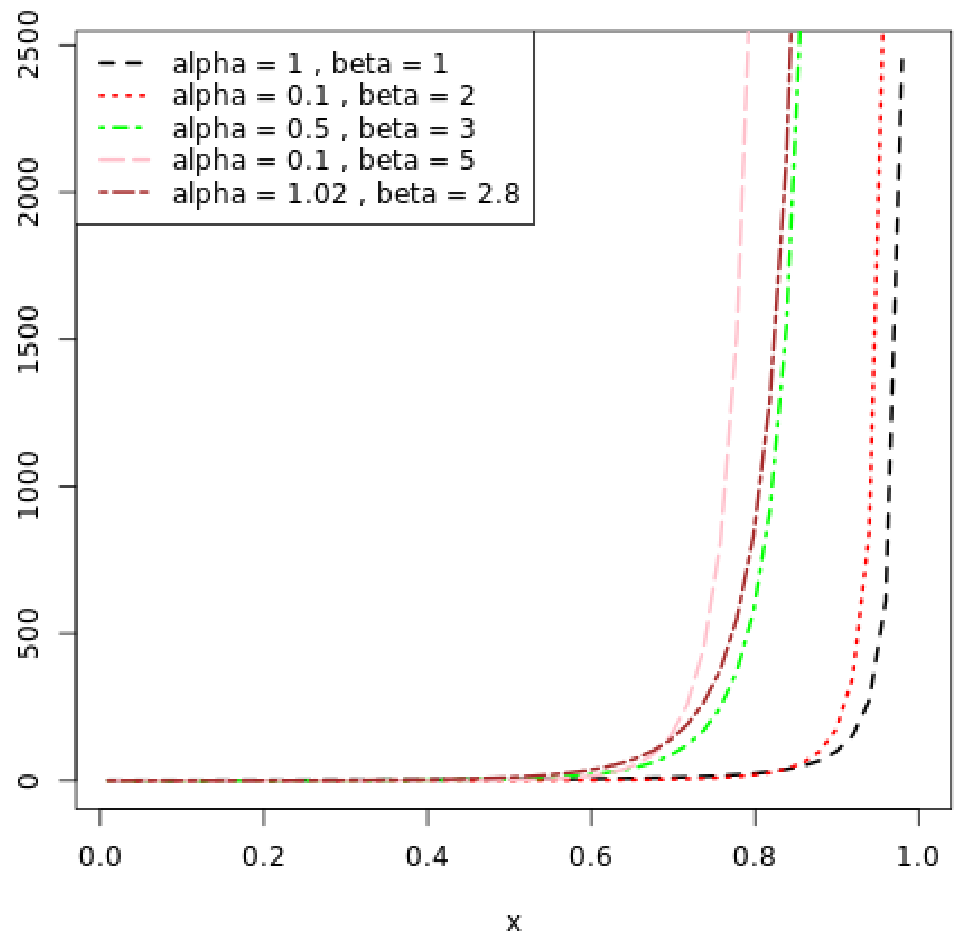

Figure 4: *Hazard Rate function of UIWD with different values of alpha and beta*

$$= 1 + \sum_{k=1}^{\infty} (-1)^k r(r+1)......(r+k-1) \frac{\Gamma_{1-\frac{k}{\beta}}}{\alpha^{\frac{-k}{\beta}}}, \beta > k. \tag{9}$$

The first four raw moments are obtained by putting r = 1, 2, 3,4 in (9) .

The central moments are obtained from the raw moments with the help of the recurrence relationship between raw moments and central moments.

$\mu_1 = 0$
$\mu_2 = \mu_2' - {\mu_1'}^2$
$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$
$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

4.2. Quartiles

Consider the function $f(x)$ of UIWD. The first quartile (Q_1) is given by:

$$\int_0^{Q_1} f(x)\, dx = \frac{1}{4}$$

which implies

$$Q_1 = \frac{1}{1 + (0.7213\alpha)^{-\beta}}. \tag{10}$$

The median (m, Q_2) is given by:

$$\int_0^m f(x)\,dx = \frac{1}{2}$$

which implies

$$m = \frac{1}{1 + (1.4423\alpha)^{-\beta}}. \tag{11}$$

The third quartile (Q_3) is given by:

$$\int_0^{Q_3} f(x)\,dx = \frac{3}{4}$$

which implies

$$Q_3 = \frac{1}{1 + (3.4761\alpha)^{-\beta}}. \tag{12}$$

4.3. Bowley's Coefficient of Skewness

The Bowley's coefficient of skewness (S_{kp}) of UIWD is given by:

$$S_{kp} = \frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$$

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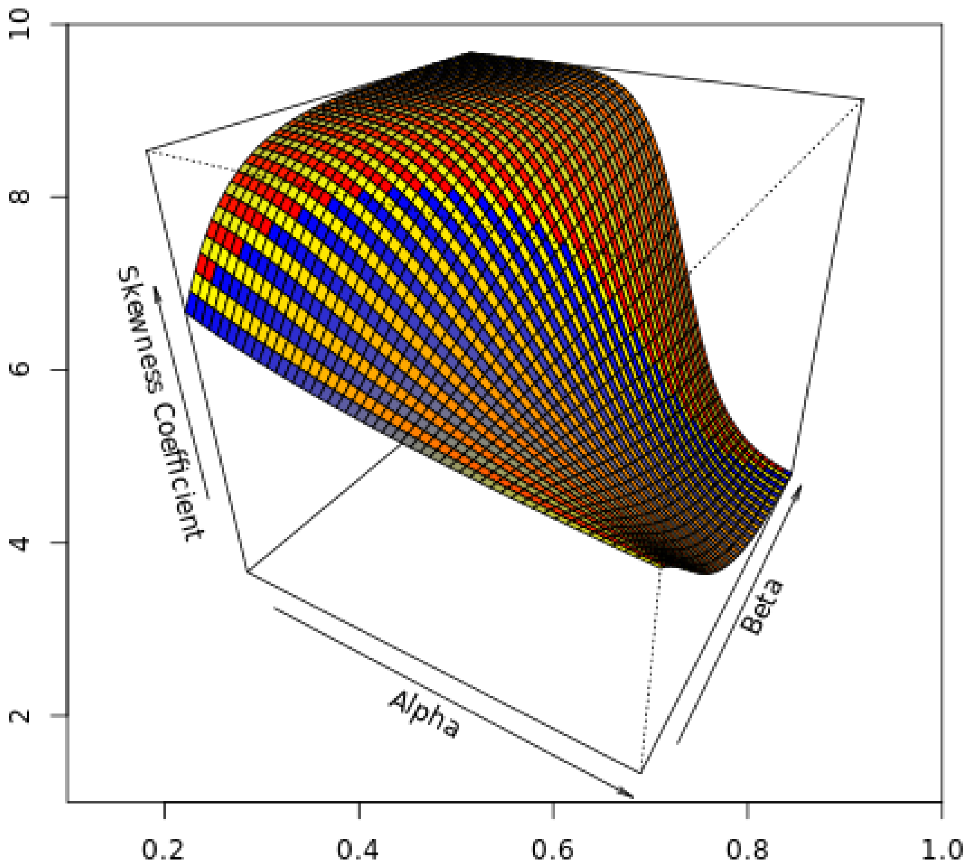

Figure 5: *skewness of UIWD*

4.4. Percentile Coefficient of Kurtosis

The percentile coefficient of kurtosis of UIWD is given by:

$$K = \frac{Q.D}{P_{90} - P_{10}}$$

where

$$Q.D = \frac{Q_3 - Q_1}{2}$$

10^{th} percentile, P_{10} is given by

$$\int_0^{P_{10}} f(x)\, dx = \frac{1}{10}$$

implies

$$P_{10} = \frac{1}{1 + (0.4343\alpha)^{-\beta}}. \tag{13}$$

90^{th} percentile, P_{90} is given by

$$\int_0^{P_{90}} f(x)\, dx = \frac{9}{10}$$

implies

$$P_{90} = \frac{1}{1 + (9.4912\alpha)^{-\beta}}. \tag{14}$$

Table 1: *Skewness, Kurtosis, Mean, and Variance for Different α and β*

α	β	Skewness	Kurtosis	Mean	Variance
0.5	0.2	0.1208	0.2563	0.5975	0.0712
0.5	1	0.1673	0.2855	0.5385	0.0164
0.5	2.6	0.3706	0.3848	0.5127	0.0028
0.5	7.8	0.8538	0.4932	0.5038	0.0004
1	0.2	0.1154	0.2571	0.3687	0.0880
1	1	0.0407	0.2970	0.4037	0.0122
1	2.6	-0.2735	0.3704	0.4496	0.0022
1	7.8	-0.8829	0.4644	0.4817	0.0003
2	0.2	0.1101	0.2579	0.1511	0.0448
2	1	-0.0716	0.2924	0.2773	0.0063
2	2.6	-0.6071	0.2304	0.3879	0.0015
2	7.8	-0.9905	0.0362	0.4597	0.0003
10	0.2	0.0981	0.2596	0.0010	0.0000
10	1	-0.2137	0.2654	0.0844	0.0003
10	2.6	-0.6969	0.1336	0.2592	0.0005
10	7.8	-0.9910	0.0096	0.4094	0.0002

4.5. Order Statistics

Let's consider a random sample of size n from the UIWD, denoted as $X_1, X_2, \ldots, X_n$. The order statistics of the UIWD are defined as $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, where $X_{(1)}$ represents the smallest observed value, $X_{(2)}$ represents the second smallest value, and so on, up to $X_{(n)}$, which represents the largest observed value.

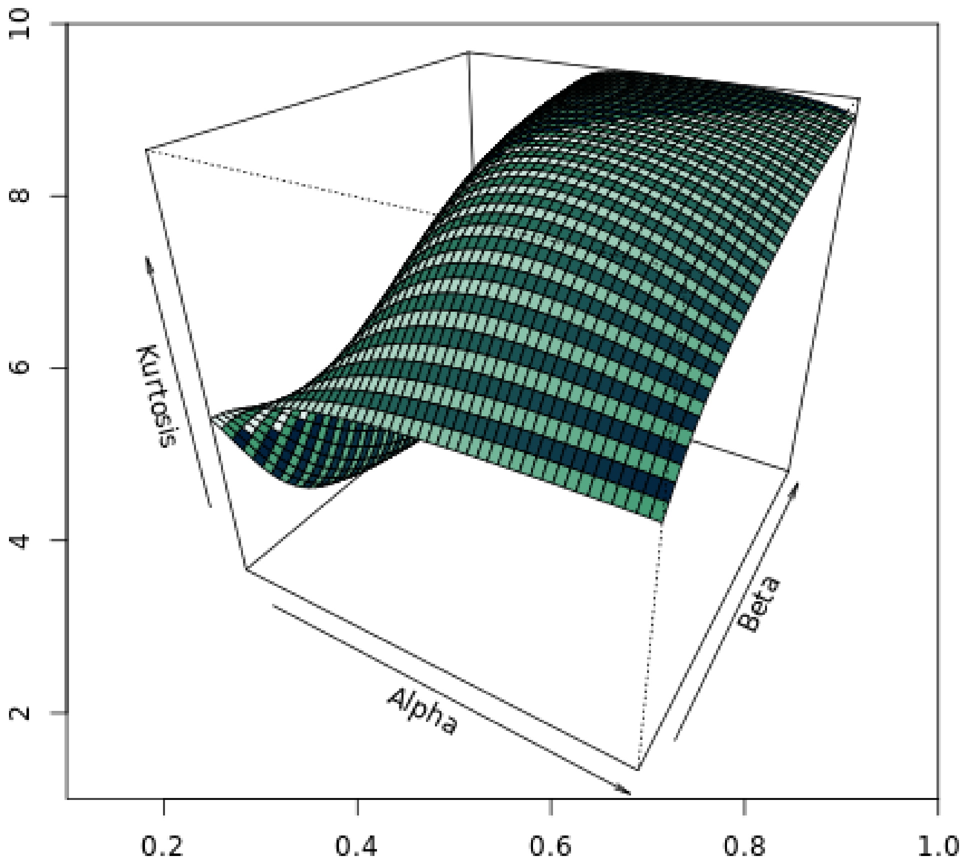

Figure 6: *Kurtosis of UIWD*

The r^{th} order statistics is given by:

$$f_{r:n}(x|\alpha,\beta) = C_{r:n}F(x|\alpha,\beta)^{r-1}(1-F(x|\alpha,\beta))^{n-r}f(x|\alpha,\beta)$$

$$= C_{r:n}[1-e^{-\alpha(1/x-1)^{-\beta}}]^{r-1}\alpha\beta\left(\frac{1}{x^2}\right)\left(\frac{1}{x}-1\right)^{-\beta-1}e^{-\alpha\left(\frac{1}{x}-1\right)^{-\beta(n-r+1)}}, \tag{15}$$

where,

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!};0<x<1, \quad \alpha,\beta>0.$$

4.6. Quantile Function

The quantile function of a distribution gives the inverse mapping of the cumulative distribution function (CDF), allowing you to find the value corresponding to a given probability.Let's denote the quantile function as Q(p), where p is the probability for which we want to find the corresponding value. Then quantile function of UIWD is given by:

$$p = F(x)$$

Then,

$$Q(p) = \frac{1}{1+\left(-\frac{\ln(1-p)}{\alpha}\right)^{\frac{-1}{\beta}}}. \tag{16}$$

5. Simulation study

In this section, we present a Monte Carlo simulation study conducted for the purpose of evaluation of the finite-sample behavior of the maximum likelihood estimates of the UIWD. Generate $n = 50, 100, 200, 400$ and 800 as sample size and considered 1000 replications for each sample size. To simulate n observations from the proposed distribution, we implemented the following algorithm:

1. Generate n random numbers from the uniform distribution $U(0, 1)$, denoted as U_i, where $i = 1, 2, \ldots, n$.
2. For each i, solve the equation $G(y_i)$ by finding the inverse of the cumulative distribution function:

$$y_i = G^{-1}(U_i) = \frac{1}{1 - \left(\frac{-\ln(1-U_i)}{\alpha}\right)^{\frac{1}{\beta}}}$$

.

Where $G^{-1}(.)$ is the quantile function of UIWD.

This algorithm allows us to generate a sample of n observations that follow the desired distribution.

Table 2: *Estimates, Bias, and MSE for Different n*

n	Estimates	Bias	MSE
50	$\hat{\alpha}$: 3.0766	-0.3234	0.1046
	$\hat{\beta}$: 1.1433	-0.0567	0.0092
100	$\hat{\alpha}$: 3.5584	0.1584	0.0551
	$\hat{\beta}$: 1.2334	0.0334	0.0071
250	$\hat{\alpha}$: 3.6005	0.2005	0.0402
	$\hat{\beta}$: 1.2855	0.0855	0.0053
500	$\hat{\alpha}$: 3.6432	0.2432	0.0292
	$\hat{\beta}$: 1.2977	0.0977	0.0036
1000	$\hat{\alpha}$: 3.4555	0.0555	0.0031
	$\hat{\beta}$: 1.2490	0.0490	0.0024

6. Estimation

6.1. Maximum Likelihood Estimator

Let's denote the sample as $\{x_1, x_2, \ldots, x_n\}$ taken from the population which follows UIWD and the parameters are α and β.

The likelihood function,

$$L(\alpha, \beta) = \prod_{i=1}^{n} \alpha\beta \left(\frac{1}{x_i^2}\right) \left(\frac{1}{x_i} - 1\right)^{-\beta-1} e^{-\alpha\left(\frac{1}{x_i}-1\right)^{-\beta}}.$$

Taking the natural logarithm of both sides, we get:

$$\ln L(\alpha, \beta) = \sum_{i=1}^{n} \ln(\alpha\beta) - 2\ln(x_i) - (\beta + 1)\ln\left(\frac{1}{x_i} - 1\right) - \alpha\left(\frac{1}{x_i} - 1\right)^{-\beta}.$$

Now, let's differentiate $\ln(L(\alpha,\beta))$ with respect to α and β and set the derivatives to zero:

$$\frac{\partial}{\partial\alpha}\ln(L(\alpha,\beta)) = 0$$

and

$$\frac{\partial}{\partial\beta}\ln(L(\alpha,\beta)) = 0$$

Taking the derivative with respect to α:

$$\frac{\partial}{\partial\alpha}\ln(L(\alpha,\beta)) = \sum_{i=1}^{n}\left[\frac{1}{\alpha} - (x_i-1)^{-\beta}\right].$$

Setting it to zero:

$$\sum_{i=1}^{n}\left[\frac{1}{\alpha} - (x_i-1)^{-\beta}\right] = 0.$$

Rearranging, we get:

$$\frac{1}{\alpha} = \sum_{i=1}^{n}(x_i-1)^{-\beta}.$$

Solving for α:

$$\hat{\alpha} = \frac{1}{\sum_{i=1}^{n}(x_i-1)^{-\beta}}. \tag{17}$$

Now, taking the derivative with respect to β:

$$\frac{\partial}{\partial\beta}\ln(L(\alpha,\beta)) = \sum_{i=1}^{n}\left[\ln(x_i-1) + \alpha(x_i-1)^{-\beta}\ln(x_i-1) - \alpha(x_i-1)^{-\beta-1}(x_i-1)\right].$$

Setting it to zero:

$$\sum_{i=1}^{n}\left[\ln(x_i-1) + \alpha(x_i-1)^{-\beta}\ln(x_i-1) - \alpha(x_i-1)^{-\beta-1}(x_i-1)\right] = 0. \tag{18}$$

It is non linear function and it can be found numerically.

These estimates of α and β obtained through maximizing the log-likelihood function are the maximum likelihood estimators for the given model.

6.2. Method of moments

Let $m_r{}'$ and $\mu_r{}'$ represent the sample and population raw moments, respectively, for a given data set. These moments are defined for different orders of r, where r ranges from 1 to n.

The r^{th} raw moment about the origin of the random variable X is given by:

$$\mu_r' = 1 + \sum_{k=1}^{\infty}(-1)^k r(r+1)\dots(r+k-1)\frac{\Gamma_{1-\frac{k}{\beta}}}{\alpha^{\frac{-k}{\beta}}}, \quad \beta > k.$$

The r^{th} sample raw moment, denoted as $m_r^{'}$, is calculated as:

$$m_r^{'} = \frac{1}{n}\sum_{i=1}^{n}x_i^r, \quad r = 1,2,3,\dots,n.$$

where x_i represents the individual data points in the data set.

By equating $m_r{}'$ to $\mu_r{}'$, we can establish the following set of equations:

$$m_1^{'} = \mu_1^{'}, \quad m_2^{'} = \mu_2^{'}, \quad m_3^{'} = \mu_3^{'}, \quad \dots, \quad m_n^{'} = \mu_n^{'}.$$

Solving these equations will allow us to determine the values of α and β.

6.3. Percentile estimation method

consider 25^{th} percentile (P_{25}) and 75^{th} percentile (P_{75}) of UIWD.

Then,

$$P_{25} = \frac{1}{1 + (0.7213\alpha)^{-\beta}}$$

and

$$P_{75} = \frac{1}{1 + (3.4761\alpha)^{-\beta}}$$

solving these two equations, we get,

$$\hat{\alpha} = -0.1419 + e^{\frac{\log\left(\frac{1}{P_{25}}-1\right)}{-1.4641\log\left(\frac{(1-p_{25})p_{75}}{(1-p_{75})p_{25}}\right)}} \tag{19}$$

and

$$\hat{\beta} = 1.4641\log\left(\frac{(1-p_{25})p_{75}}{(1-p_{75})p_{25}}\right). \tag{20}$$

6.4. Cramer-von mises method

The Cramer-von Mises estimation equation is used to estimate the parameters of a distribution by minimizing the Cramer-von Mises objective function. For the Unit Inverse Weibull Distribution (UIWD), the Cramer-von Mises estimation equation can be defined as follows:

$$C(\alpha, \beta) = \frac{1}{12n} + \sum\left[F(t_i|\alpha, \beta) - \frac{2i-1}{2n}\right]^2.$$

In this equation, $C(\alpha, \beta)$ represents the Cramer-von Mises objective function, α and β are the parameters of the UIWD that need to be estimated, n is the number of data points, t_i represents the observed data points, and $F(t_i|\alpha, \beta)$ is the cumulative distribution function (CDF) of the UIWD for each data point.

To estimate the parameters α and β, the Cramer-von Mises objective function is minimized by finding the values of α and β that result in the smallest value of $C(\alpha, \beta)$. This can be done using partial differentiation of α and β. Equating these to zero, we get normal equations. since these equations are non-linear, we can use iterative method to find the estimates.

7. Applications

Data set 1: Here, we apply real data to show how adaptable and applicable the suggested distribution is in comparison to a variety of other well-known distributions on the unit interval. The data set is used for fitting is COVID-19 of Britain: This data set covered a period of 47 days, from 1 May 2021 to 17 June 2021 [22]. The following information is created using daily new deaths (DNDs), daily cumulative cases (DCCs), and daily cumulative deaths (DCDs): 0.0023, 0.0023, 0.0023, 0.0046, 0.0065, 0.0067, 0.0069, 0.0069, 0.0091, 0.0093, 0.0093, 0.0093, 0.0111, 0.0115, 0.0116, 0.0116, 0.0119, 0.0133, 0.0136, 0.0138, 0.0138, 0.0159, 0.0161, 0.0162, 0.0162, 0.0162, 0.0163, 0.0180, 0.0187, 0.0202, 0.0207, 0.0208, 0.0225, 0.0230, 0.0230, 0.0239, 0.0245, 0.0251, 0.0255, 0.0255, 0.0271, 0.0275, 0.0295, 0.0297, 0.0300, 0.0302, 0.0312, 0.0314, 0.0326

Data set 2:It is the recovery rates of COVID-19 patients in Spain from 3 March to 7 May 2020 [23] .The data set are: 0.6670, 0.5000, 0.5000, 0.4286, 0.7500, 0.6531, 0.5161, 0.7895, 0.7689, 0.6873, 0.5200, 0.7251, 0.6375, 0.6078, 0.6289, 0.5712, 0.5923, 0.6061, 0.5924, 0.5921, 0.5592, 0.5954, 0.6164, 0.6455, 0.6725, 0.6838, 0.6850, 0.6947, 0.7210, 0.7315, 0.7412, 0.7508, 0.7519, 0.7547, 0.7645, 0.7715, 0.7759, 0.7807, 0.7838, 0.7847, 0.7871, 0.7902, 0.7934, 0.7913, 0.7962, 0.7971, 0.7977, 0.8007, 0.8038, 0.8289, 0.8322, 0.8354, 0.8371, 0.8387, 0.8456, 0.8490,0.8535, 0.8547, 0.8564, 0.8580, 0.8604, 0.8628, 0.6586, 0.7070, 0.7963, 0.8516.

To compare adaptability of unit inverse weibull distribution, we use the following two parameter distributions on the unit interval.

1.**Beta distribution of first kind (BD)**:

The probability density function (PDF) of the beta distribution is given by,

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$$

where $0 \leq x \leq 1$, $\alpha > 0$, $\beta > 0$, and $B(\alpha,\beta)$ is the beta function.

2.**Unit Weibull distribution(UWD)**:

The probability density function (PDF) UWD is given by,

$$f(x;\alpha,\beta) = \frac{1}{x}\alpha\beta\left(-\log x\right)^{(\beta-1)}\exp\left[-\alpha\left(-\log x\right)^{\beta}\right]$$

where $0 < x < 1$, $\alpha > 0$, and $\beta > 0$.

3. **Kumaraswamy Distribution(KD)**:

The probability density function (PDF) of the Kumaraswamy distribution is given by:

$$f(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}$$

where $0 < x < 1$, $\alpha > 0$, and $\beta > 0$.

4. **Unit Birnbaum-Saunders distribution (UBSD)**:

The probability density function (PDF) of the UBSD is given by:

$$f\left(x;\alpha,\beta\right) = \frac{1}{2x\alpha\beta\sqrt{2\pi}}\left[\left(\frac{-\beta}{\log x}\right)^{\frac{1}{2}} + \left(\frac{-\beta}{\log x}\right)^{\frac{3}{2}}\right]\exp\left\{\frac{1}{2\alpha^2}\left(\frac{\log x}{\beta} + \frac{\beta}{\log x} + 2\right)\right\}$$

where $0 < x < 1$, $\beta > 0$, and $\alpha > 0$.

5. **Unit Gompertz distribution(UGD)**:

The probability density function (PDF) of the UGD is given by:

$$f(x;\alpha,\beta) = \alpha\beta x^{-(\beta+1)}\exp\left(-\alpha\left(x^{-\beta}-1\right)\right)$$

where $0 < x < 1$, $\alpha > 0$, and $\beta > 0$.

6. **Kumaraswamy distribution(KD)**:

The pdf of KD is given by,

$$f(x) = abx^{a-1}(1-x^{a})^{b-1}, 0 < x < 1, a > 0, b > 0.$$

7. **Unit inverse weibull distribution (UIWD)**:

The pdf of UIWD is given by,

$$f(x|\alpha,\beta) = \alpha\beta\left(\frac{1}{x^2}\right)\left(\frac{1}{x}-1\right)^{-\beta-1}e^{-\alpha\left(\frac{1}{x}-1\right)^{-\beta}}$$

where $0 < x < 1$ and $\alpha,\beta > 0$.

From the table 2, we can conclude that the unit inverse weibull model has the lowest AIC, AICC, and BIC values among the listed models, indicating that it provides the best trade-off between goodness of fit and model complexity based on all three criteria. Lower values of AIC, AICC, and BIC indicate better fitting models with lower complexity.

From table 3 , we can conclude that , if the unit inverse weibull distribution model has the least values for AIC, AICC, and BIC, it indicates that the unit inverse weibull distribution model is the best-fitting model among the listed models. The lower values of these criteria suggest that the unit inverse weibull distribution model provides a better trade-off between goodness of fit and model complexity compared to the other models. Therefore, based on the provided information, the unit inverse weibull distribution model is the most favorable choice.

Table 3: *Description of Models with AIC, AICC, and BIC Values*

Model	Log Likelihood	AIC	AICC	BIC
UIWD	164.4482	-324.8965	-324.6356	-321.1128
UIGD	161.1174	-318.2347	-317.9739	-314.4511
BD	162.1896	-320.3791	-320.1182	-316.5955
UBSD	161.1062	-318.2125	-317.9516	-314.4288
UWD	152.3815	-300.763	-300.5021	-296.9793
UGD	146.5113	-289.0226	-288.7617	-285.239
KD	164.3392	-324.6785	-324.4176	-320.8948

Table 4: *Model Comparison with AIC,AICC and BIC values*

Model	Log Likelihood	AIC	AICC	BIC
UIWD	60.5479	-117.0958	-116.9053	-112.7165
UIGD	60.0268	-116.0535	-115.8631	-111.6742
BD	57.57423	-111.1486	-110.9581	-106.7692
UBSD	59.9357	-115.8715	-115.681	-111.4921
UWD	53.9658	-103.9316	-103.7411	-99.5523
UGD	46.02843	-116.0535	-115.8631	-111.6742
KD	58.8343	-113.6686	-113.4782	-109.2893

8. Conclusion

In this study, we introduced a probability distribution called the Unit Inverse Weibull Distribution (UIWD) for modeling data on the interval $(0,1)$. The UIWD was derived through a type transformation involving the Inverse Weibull Distribution (IWD). We provided the probability density function (PDF) and cumulative distribution function (CDF) of the UIWD, along with their respective mathematical expressions.We highlighted the key features and properties of the UIWD, including its simple and closed-form expressions for the distribution and quantile functions, superior fitting performance compared to other unit interval distributions, and its ability to derive various statistical properties such as the survival function, hazard rate function, revised hazard rate function, cumulative hazard rate function, moments, mode, and order statistics.

For parameter estimation, we employed different estimation methods and conducted simulation studies to assess the accuracy and precision of different estimators. The results demonstrated the effectiveness and utility of the UIWD model in capturing and analyzing data on the unit interval $(0,1)$. Comparative analyses were performed to highlight the advantages of the UIWD model over other existing models. The UIWD showed superior performance in terms of fitting data and providing a flexible framework for statistical modeling.

Overall, this research contributes to the understanding and application of the UIWD as a flexible probability distribution for modeling data on the unit interval $(0,1)$. The UIWD offers a reliable alternative to existing models and can be effectively used in various fields requiring modeling and analysis of data on the unit interval.

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