RT&A, No 4 (76)
Volume 18, December 2023

On MIXTURE OF BURR XII AND NAKAGAMI DISTRIBUTIONS: PROPERTIES AND APPLICATIONS

Hemani Sharma and Parmil Kumar

•

Department of Statistics, University of Jammu, J&K.
hemanisharma124@gmail.com, parmil@yahoo.com

Abstract

The Burr XII and Nakagami distributions hold significant importance in both lifetime distribution and wealth distribution analyses. The Burr XII distribution serves as a valuable tool for understanding the distribution of wealth and wages within specific societies, while the Nakagami distribution finds its application in the realm of communication engineering. The incorporation of finite mixture distributions, aimed at accounting for unobserved variations, has gained substantial traction, particularly in the estimation of dynamic discrete choice models. This research delves into the fundamental characteristics of the mixture Burr XII and Nakagami distributions. The study introduces parameter estimation techniques and explores various aspects, including the cumulative distribution function, hazard rate, failure rate, inverse hazard function, odd function, cumulative hazard function, r^{th} moment, moment generating function, characteristic function, moments, mean and variance, Renyi and Beta entropies, mean deviation from mean, and mean time between failures (MTBF). The paper also addresses the estimation of the mixing parameter through a Bayesian approach. To illustrate the effectiveness of the proposed model, two real-life datasets are examined.

Keywords: Nakagami distribution, Burr XII, mixture distribution, Renyi and beta entropy, MTBF

1. Introduction

Mixture distributions play a pivotal role in modeling complex phenomena where a single underlying distribution cannot fully capture the observed data. These distributions offer a flexible framework by combining multiple component distributions, each representing a distinct pattern or source of variation. In various fields, ranging from statistics and economics to engineering and biology, mixture distributions have gained prominence due to their ability to accommodate diverse data characteristics. By utilizing mixture distributions, researchers can uncover latent subpopulations, account for unobserved factors, and enhance the accuracy of modeling real-world phenomena. The exploration of mixture distributions was instigated by Pearson [14], who initially investigated the blending of two normal distributions. Following Pearson's pioneering work [14], there emerged a significant hiatus in the advancement of mixture distributions. However, this period of inactivity was later reinvigorated by subsequent scholars. Decay [7] built upon Pearson's foundation, while Hasselblad [8] delved extensively into the intricate realm of finite mixture distributions. Numerous practical scenarios involve interpreting observed data as a blend originating from two or more distinct distributions. By leveraging this notion, we have the capability to merge different statistical distributions, thereby forging a novel distribution that inherits the traits of its individual components. This approach holds true particularly when dealing with continuous underlying random variables. An essential characteristic of a mixture distribution is that its cumulative distribution function (CDF) is formed by skillfully combining the CDFs of its constituent distributions through a convex amalgamation. Similarly, the probability density function (PDF) of the mixture can be elegantly expressed as a convex blend, seamlessly incorporating the individual PDFs of its constituent parts. This versatile mathematical framework not only allows us to effectively handle intricate data but also enables the extraction of meaningful insights by synergistically harnessing the statistical attributes of diverse distributions. Assume T is a continuous random variable with the following probability density function:

$$f(t_i) = \sum_{i=1}^{k} p_j f_j(t); t > 0, k > 0 \tag{1}$$

where $\leq p_j \leq 1$, j=1,2,3...,k and$\sum_{i=1}^{k} p_i = 1$. The corresponding c.d.f. is given by

$$F(t_i) = \sum_{i=1}^{k} p_j F_j(t); t > 0, k > 0 \tag{2}$$

where k is the number of components, the parameters $p_1, p_2, ..., p_n$are called mixing parameters, where p_i, represent the probability that a given observation comes from population $''i''$ with density $f_i(.)$, and $f_1(.), f_2(.), ..., f_3(.)$, are the component densities of the mixture. When the number of components k=2, a two component mixture and can be written as:

$$f(t) = p_1 f_1(t) + p_2 f_2(t) \tag{3}$$

Different research studies have explored various mixture distribution models and their applications. For example, Zaman and colleagues [19] introduced a mixture of chi-square distributions using Poisson elements. Chen and co-authors [4] discussed likelihood inference in finite mixture models, explaining different scenarios for these models. Martin and collaborators [11] utilized a mixture of chi-square distributions to describe the distribution of light in imaging, and they also introduced this distribution to model unstructured light distribution. Daniyel et al. [6] proposed a mixture of Burr II and Weibull distributions and examined their properties. Jaspers and co-workers [10] used a Bayesian approach to estimate mixing weights for multivariate mixture models. Nasiri and Azarian [13] explored estimating mixing proportions for a mixture of two chi-square distributions using various methods and compared their performance. Singh and colleagues [16] introduced a generalized distribution for modeling lifetime data and discussed its statistical properties with real-world applications. Daghestani and team [5] proposed a mixture of Lindley and Weibull distributions, investigated their statistical properties, and estimated their unknown parameters. In this context, this study focuses on the utilization of mixture distributions, specifically exploring the properties and applications of the mixture Burr XII and Nakagami distributions. By delving into their mathematical foundations, parameter estimation methods, and various statistical measures, we aim to provide a comprehensive understanding of how mixture distributions can enhance our ability to capture the intricate nuances present in real-world datasets.

The three-parameter Burr XII distribution was introduced by Titterington et al. [17]. For values of x greater than zero, its cumulative distribution function and probability density function are as follows:

$$F(x; q, d, b) = 1 - \left[1 + (\frac{x}{q})^b\right]^{-d}; d, b, q > 0 \tag{4}$$

And

$$f(x; q, d, b) = bdq^{-b}x^{b-1}\left[1 + (\frac{x}{q})^b\right]^{-d-1}; d, b, q > 0 \tag{5}$$

respectively, where d and b are the shape parameters and q is the scale parameter. It is frequently employed to model income data.

Nakagami [12] introduced the Nakagami distribution, also referred to as the Nakagami-m distribution, to describe radio signal fading. It involves a shape parameter denoted as 'm.' If a random variable 'X' conforms to the Nakagami distribution with a shape parameter $\alpha > 0.5$ and a scale parameter $\lambda > 0$, its probability density function (p.d.f) can be expressed in the following manner:

$$f(x;\alpha,\lambda) = \frac{2}{\Gamma(\alpha)}\left(\frac{\alpha}{\lambda}\right)^{\alpha} x^{2\alpha-1}\exp\left(\frac{-\alpha}{\lambda}x^2\right); x > 0, \alpha \geq 0.5, \lambda > 0 \tag{6}$$

The corresponding commulative distribution function is given by

$$F(x) = \frac{1}{\Gamma(\alpha)}\Gamma\big(\frac{\alpha}{\lambda}x^2, \alpha\big); x > 0, \alpha \geq 0.5, \lambda > 0 \tag{7}$$

where $\Gamma(x,a) = \int_0^x t^{a-1}e^{-t}dt$ is the lower incomplete gamma function.

This particular distribution has found applications spanning a diverse range of fields, encompassing hydrology, the transmission of multimedia data across networks, investigations into medical imaging, and the modeling of high-frequency seismogram envelopes.

The primary objective of this research paper is to introduce and explore a novel distribution, which emerges as a fusion of the Burr XII and Nakagami distributions, termed as the Mixture of Burr II and Nakagami Distribution (MBND). The study delves into the statistical properties inherent to this mixture distribution, in addition to the techniques for parameter estimation through Maximum Likelihood Estimation. Furthermore, the paper extends its investigation to include the estimation of the mixing parameter using a Bayesian approach. To validate the efficacy of the proposed model, real-world data is employed, allowing for a comparative assessment against other optimally fitting models.

2. Probability Density Function of MBND

The PDF of the mixture of Burr XII Nakagami distribution (MBND) has the following form

$$f(x;q,d,b,\alpha,\lambda) = p_1 f_1(x;q,d,b) + p_2 f_2(x;\alpha,\lambda) \tag{8}$$

Here, with p_1 and p_2 representing the mixing proportions where their sum equals one ($p_1 + p_2 = 1$), $f_1(x;q,d,b)$ signifies the PDF of the Burr XII distribution, and $f_2(x;\alpha,\lambda)$ denotes the PDF of the Nakagami distribution. Combining these individual probability densities results in the mixture density as follows:

$$f(x;q,d,b,\alpha,\lambda) = p_1 bdq^{-b}x^{b-1}\big[1+\big(\frac{x}{q}\big)^b\big]^{-d-1} + p_2\frac{2}{\Gamma(\alpha)}\left(\frac{\alpha}{\lambda}\right)^{\alpha}x^{2\alpha-1}\exp\left(\frac{-\alpha}{\lambda}x^2\right) \tag{9}$$

where $x > 0$ and $d, b, q >, \lambda > 0, \alpha \geq 0.5$, $p_1 + p_2 = 1$.

3. Cumulative Probability distribution of MBND

The cumulative distribution function is given by

$$F(x;q,d,b,\alpha,\lambda) = p_1 F_1(x;q,d,b) + p_2 F_2(x;\alpha,\lambda) \tag{10}$$

Here, with p_1 and p_2 representing the mixing proportions where their sum equals one ($p_1 + p_2 = 1$), $f_1(x;q,d,b)$ signifies CDF of the Burr XII distribution, and $f_2(x;\alpha,\lambda)$ denotes the CDF of the Nakagami distribution. Combining these individual probability densities results in the mixture density as follows:

$$F(x;q,d,b,\alpha,\lambda) = p_1\left(1-\big[1+\big(\frac{x}{q}\big)^b\big]^{-d}\right) + p_2\left(\frac{1}{\Gamma(\alpha)}\Gamma\big(\frac{\alpha}{\lambda}x^2,\alpha\big)\right) \tag{11}$$

where $x > 0$, $d, b, q >, \lambda > 0, \alpha \geq 0.5$, $p_1 + p_2 = 1$.

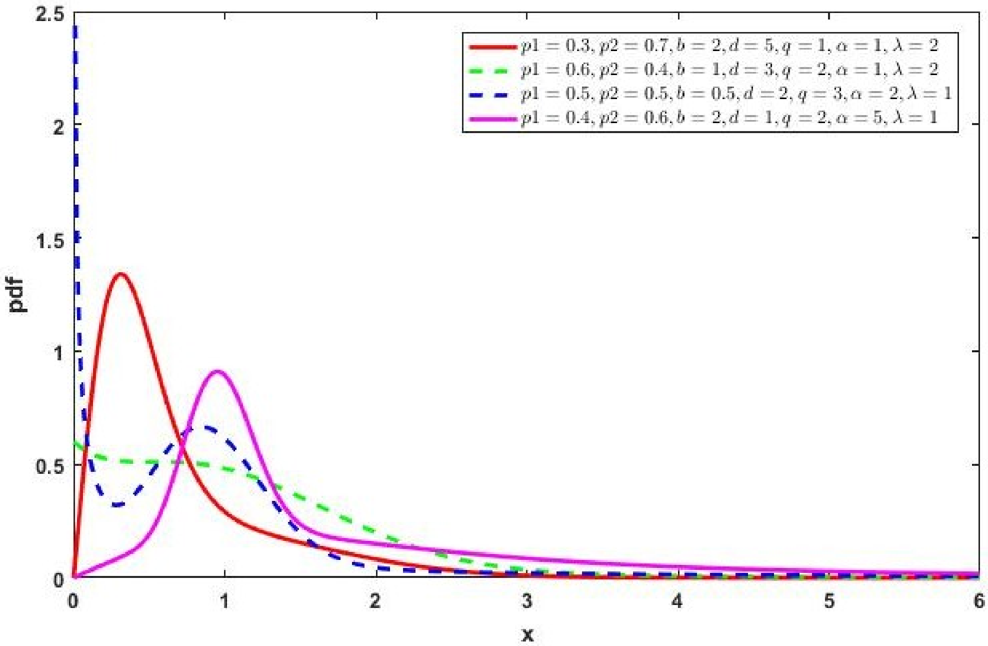


Figure 1: *Plot of the PDF for the MBND*

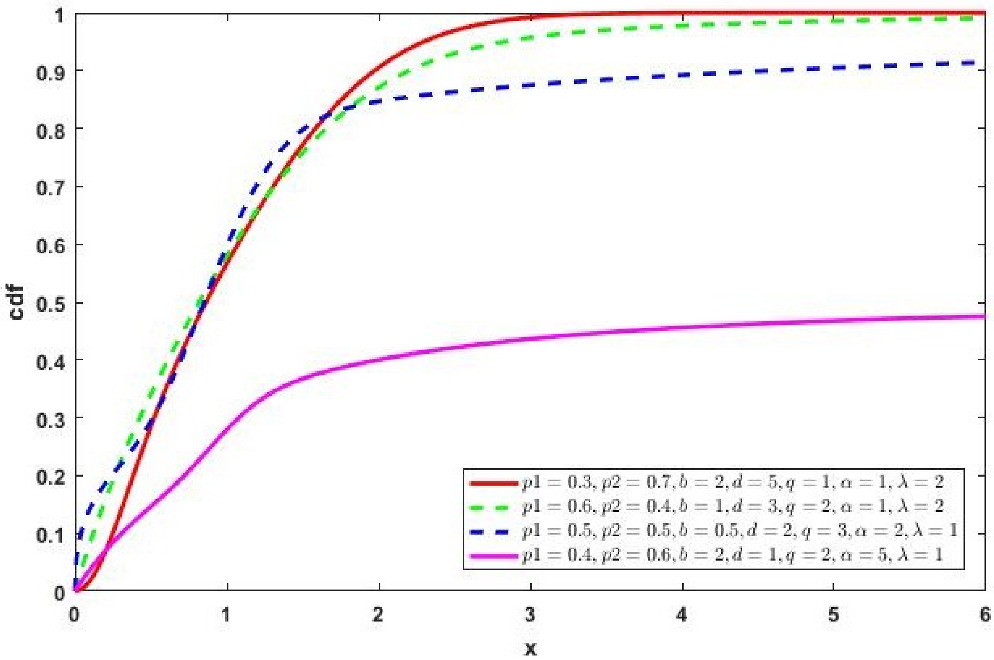


Figure 2: *Plot of the CDF for the MBND*

4. Properties of Bunami distribution

4.1. Area under the curve

Both the Burr XII and Nakagami distributions constitute complete probability density functions individually, their amalgamation as a mixture will similarly result in a comprehensive probability density function.

4.2. Reliability function

The reliability function, also called the survival function, describes a feature of a random variable related to how a system might stop working within a certain period.It is defined as the probability that the system will keep working beyond a particular point in time. Mathematically,

$$R(x) = 1 - F(x) \tag{12}$$

By inserting equation (11) into equation (12), the reliability or survival function for the MBND can be expressed in the following manner

$$R(x) = 1 - \left(p_1 \left(1 - [1 + (\frac{x}{q})^b]^{-d} \right) + p_2 \left(\frac{1}{\Gamma(\alpha)} \Gamma(\frac{\alpha}{\lambda} x^2, \alpha) \right) \right) \tag{13}$$

where $x > 0$, $d, b, q >, \lambda > 0, \alpha \geq 0.5$, $p_1 + p_2 = 1$.

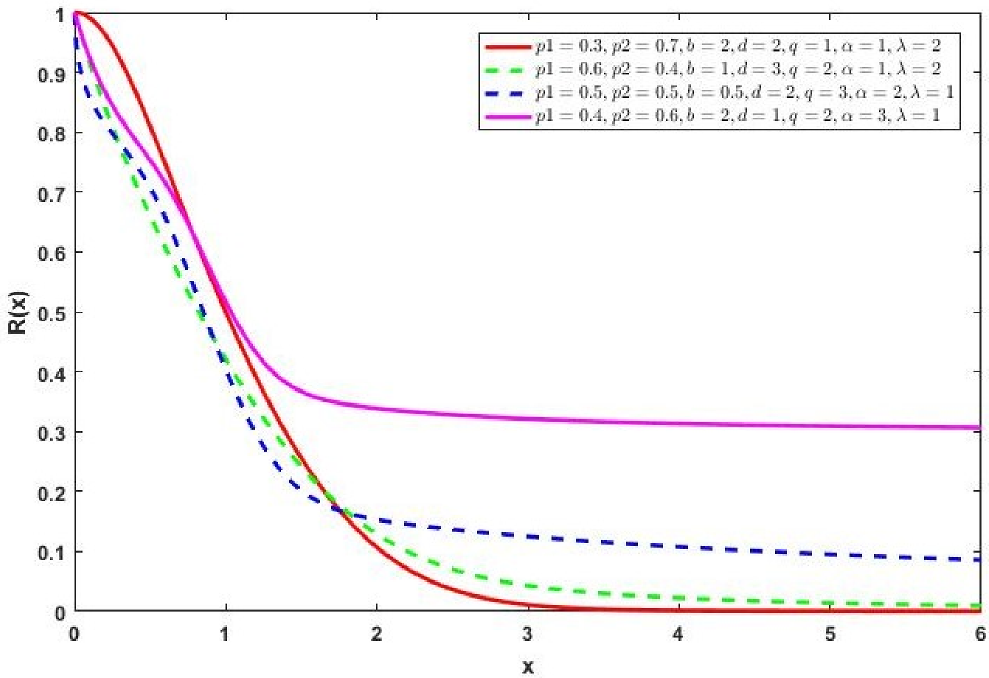


Figure 3: *Plot of the Reliability function for the MBND*

4.3. Hazard Function

The hazard function is mathematically defined as the quotient of the probability density function divided by the reliability function, and its expression takes the following form:

$$h(x) = \frac{f(x)}{R(x)} \tag{14}$$

By substituting equations (9) and (13) into equation (14), the hazard rate for the MBND can be derived, leading to the subsequent definition.

$$h(x) = \frac{p_1 bdq^{-b}x^{b-1}\left[1+\left(\frac{x}{q}\right)^b\right]^{-d-1} + p_2\frac{2}{\Gamma(\alpha)}\left(\frac{\alpha}{\lambda}\right)^{\alpha} x^{2\alpha-1}\exp\left(\frac{-\alpha}{\lambda}x^2\right)}{1-\left(p_1\left(1-\left[1+\left(\frac{x}{q}\right)^b\right]^{-d}\right)+p_2\left(\frac{1}{\Gamma(\alpha)}\Gamma\left(\frac{\alpha}{\lambda}x^2,\alpha\right)\right)\right)} \tag{15}$$

where $x > 0$, $d, b, q >, \lambda > 0, \alpha \geq 0.5$, $p_1 + p_2 = 1$.

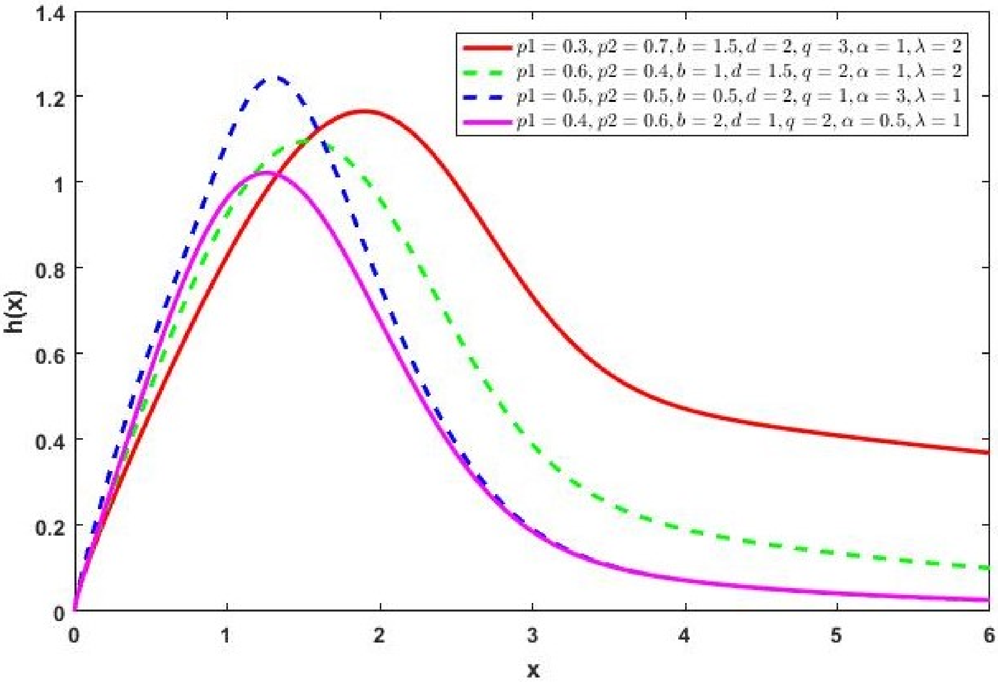


Figure 4: *Plot of the Hazard function of the MBND*

4.4. Cumulative Hazard Function

The cumulative hazard function can be defined as

$$\Lambda(x) = -\log R(x) \tag{16}$$

By substituting equation (13) into equation (16), the cumulative hazard function for the MBND can be derived, yielding the subsequent expression.

$$\Lambda(x) = -\log\left[1-\left(p_1\left(1-\left[1+\left(\frac{x}{q}\right)^b\right]^{-d}\right)+p_2\left(\frac{1}{\Gamma(\alpha)}\Gamma\left(\frac{\alpha}{\lambda}x^2,\alpha\right)\right)\right)\right] \tag{17}$$

where $x > 0$, $d, b, q >, \lambda > 0, \alpha \geq 0.5$, $p_1 + p_2 = 1$.

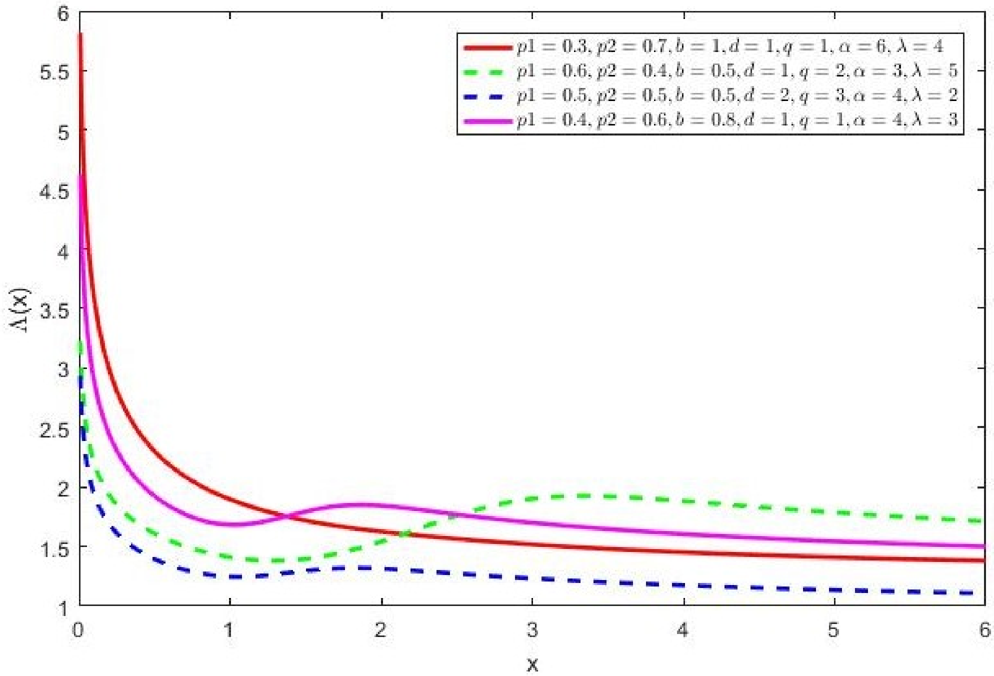


Figure 5: *Plot of Cumulative Hazard function for the MBND*

4.5. Reversed Hazard Rate

The reversed hazard rate can be conceptually understood as the result of dividing the probability density function by the cumulative distribution function, as demonstrated by the following formulation:

$$r(x) = \frac{f(x)}{F(x)} \tag{18}$$

The reversed hazard rate corresponding to the MBND can be derived by substituting equations (9) and (11) into equation (18), resulting in the following expression.

$$r(x) = \frac{p_1 bdq^{-b}x^{b-1}\left[1+\left(\frac{x}{q}\right)^b\right]^{-d-1} + p_2\frac{2}{\Gamma(\alpha)}\left(\frac{\alpha}{\lambda}\right)^{\alpha} x^{2\alpha-1}\exp\left(\frac{-\alpha}{\lambda}x^2\right)}{p_1\left(1-\left[1+\left(\frac{x}{q}\right)^b\right]^{-d}\right) + p_2\left(\frac{1}{\Gamma(\alpha)}\Gamma\left(\frac{\alpha}{\lambda}x^2,\alpha\right)\right)} \tag{19}$$

where $x > 0$, $d, b, q >, \lambda > 0, \alpha \geq 0.5$, $p_1 + p_2 = 1$.

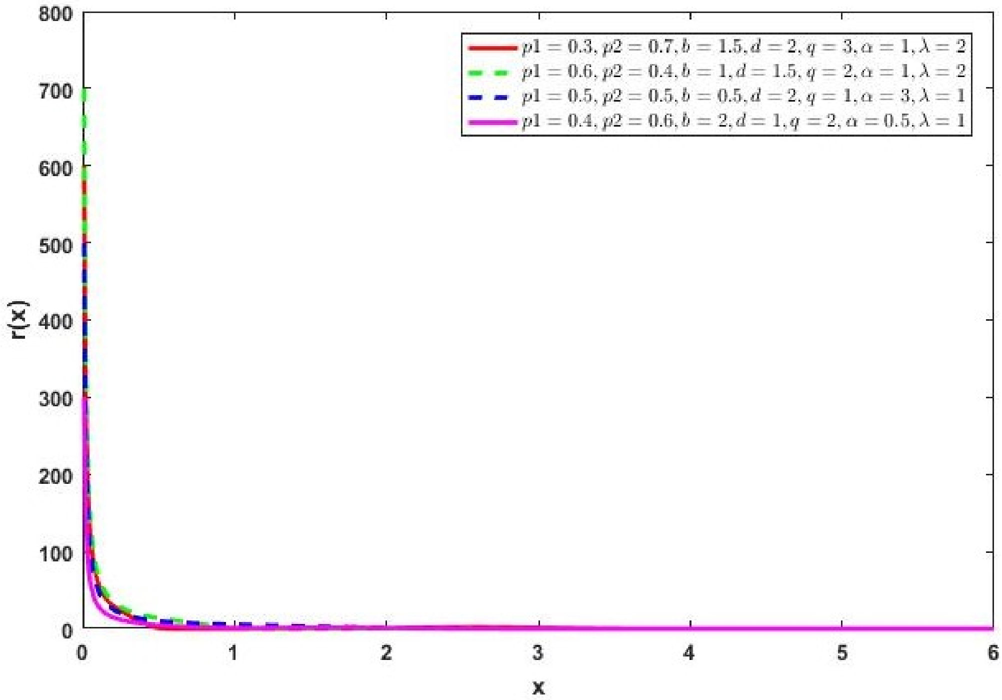


Figure 6: *Plot of the Reversed Hazard function for the MBND*

4.6. Odds Function

The odds function, represented as O(x), is characterized as the proportion of the cumulative distribution function to the reliability function, and it is expressed through the following equation:

$$O(x) = \frac{F(x)}{R(x)} \tag{20}$$

The odds function corresponding to the MBND can be derived by substituting equations (11) and (13) into equation (20), resulting in the following expression.

$$O(x) = \frac{p_1\left(1 - \left[1 + \left(\frac{x}{q}\right)^b\right]^{-d}\right) + p_2\left(\frac{1}{\Gamma(\alpha)}\Gamma\left(\frac{\alpha}{\lambda}x^2, \alpha\right)\right)}{1 - \left(p_1\left(1 - \left[1 + \left(\frac{x}{q}\right)^b\right]^{-d}\right) + p_2\left(\frac{1}{\Gamma(\alpha)}\Gamma\left(\frac{\alpha}{\lambda}x^2, \alpha\right)\right)\right)} \tag{21}$$

where $x > 0$, $d, b, q >, \lambda > 0, \alpha \geq 0.5$, $p_1 + p_2 = 1$.

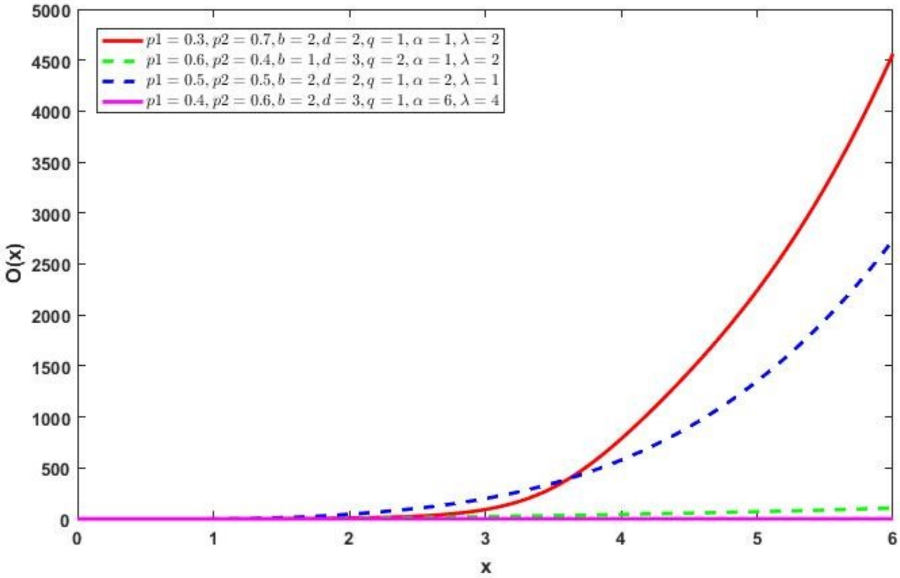


Figure 7: *Plot of the Odd Function for the MBND*

4.7. r^{th} Moment about Origin

The r^{th} moment of the real-valued function can be acquired by applying

$$\mu_r' = \int x^r f(x) dx \tag{22}$$

$$\mu_r' = \int_0^\infty x^r \left(p_1 b d q^{-b} x^{b-1} \left[1 + (\frac{x}{q})^c\right]^{-k-1} + p_2 \frac{2}{\Gamma(\alpha)} \left(\frac{\alpha}{\lambda}\right)^\alpha x^{2\alpha-1} \exp\left(\frac{-\alpha}{\lambda} x^2\right) \right) dx \tag{23}$$

$$\mu_r' = p_1 \int_0^\infty x^r b d q^{-c} x^{c-1} \left[1 + (\frac{x}{q})^b\right]^{-d-1} dx + p_2 \int_0^\infty x^r \frac{2}{\Gamma(\alpha)} \left(\frac{\alpha}{\lambda}\right)^\alpha x^{2\alpha-1} \exp\left(\frac{-\alpha}{\lambda} x^2\right) dx \tag{24}$$

$$\mu_r' = p_1 q^r \frac{\Gamma(\frac{r}{b}+1)\Gamma(k-\frac{r}{b})}{\Gamma(d)} + p_2 \frac{\Gamma(\alpha+\frac{r}{2})}{\Gamma\alpha} \left(\frac{\lambda}{\alpha}\right)^{\frac{r}{2}} \tag{25}$$

4.8. Raw moments about Origin

Putting r=1, 2, 3 and 4 in (25),first four raw moments are

$$\mu_1' = p_1 q^1 \frac{\Gamma(\frac{1}{b}+1)\Gamma(k-\frac{1}{b})}{\Gamma(d)} + p_2 \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma\alpha} \left(\frac{\lambda}{\alpha}\right)^{\frac{1}{2}} = Mean \tag{26}$$

$$\mu_2' = p_1 q^2 \frac{\Gamma(\frac{2}{b}+1)\Gamma(k-\frac{2}{b})}{\Gamma(d)} + p_2 \frac{\Gamma(\alpha+1)}{\Gamma\alpha} \left(\frac{\lambda}{\alpha}\right) \tag{27}$$

$$\mu_3' = p_1 q^3 \frac{\Gamma(\frac{3}{b}+1)\Gamma(k-\frac{3}{b})}{\Gamma(d)} p_2 \frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma\alpha} \left(\frac{\lambda}{\alpha}\right)^{\frac{3}{2}} \tag{28}$$

$$\mu_4' = p_1 q^4 \frac{\Gamma(\frac{4}{b}+1)\Gamma(k-\frac{4}{b})}{\Gamma(d)} + p_2 \frac{\Gamma(\alpha+2)}{\Gamma\alpha} \left(\frac{\lambda}{\alpha}\right)^2 \tag{29}$$

4.9. Moments about Mean

$$\mu_1 = 0 \tag{30}$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = Variance \tag{31}$$

$$\mu_2 = p_1 q^2 \frac{\Gamma(\frac{2}{b}+1)\Gamma(k-\frac{2}{b})}{\Gamma(d)} + p_2 \frac{\Gamma(\alpha+1)}{\Gamma\alpha}\left(\frac{\lambda}{\alpha}\right) - \left(p_1 q^1 \frac{\Gamma(\frac{1}{b}+1)\Gamma(k-\frac{1}{b})}{\Gamma(d)} + p_2 \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma\alpha}\left(\frac{\lambda}{\alpha}\right)^{\frac{1}{2}}\right)^2 \tag{32}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + (\mu_1')^3$$

$$\mu_3 = p_1 q^3 \frac{\Gamma(\frac{3}{b}+1)\Gamma(k-\frac{3}{b})}{\Gamma(d)} p_2 \frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma\alpha}\left(\frac{\lambda}{\alpha}\right)^{\frac{3}{2}} - 3\left[p_1 q^2 \frac{\Gamma(\frac{2}{b}+1)\Gamma(k-\frac{2}{b})}{\Gamma(d)} + p_2 \frac{\Gamma(\alpha+1)}{\Gamma\alpha}\left(\frac{\lambda}{\alpha}\right)\right]$$
$$\left[p_1 q^1 \frac{\Gamma(\frac{1}{b}+1)\Gamma(k-\frac{1}{b})}{\Gamma(d)} + p_2 \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma\alpha}\left(\frac{\lambda}{\alpha}\right)^{\frac{1}{2}}\right] + \left[p_1 q^1 \frac{\Gamma(\frac{1}{b}+1)\Gamma(k-\frac{1}{b})}{\Gamma(d)} + p_2 \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma\alpha}\left(\frac{\lambda}{\alpha}\right)^{\frac{1}{2}}\right]^3 \tag{33}$$

$$\mu_4 = \mu_3' - 4\mu_1'\mu_3' + 6(\mu_1')^2(\mu_2') - 3(\mu_1')^4$$

$$\mu_4 = p_1 q^3 \frac{\Gamma(\frac{3}{b}+1)\Gamma(k-\frac{3}{b})}{\Gamma(d)} p_2 \frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma\alpha}\left(\frac{\lambda}{\alpha}\right)^{\frac{3}{2}} - 4\left(p_1 q^1 \frac{\Gamma(\frac{1}{b}+1)\Gamma(k-\frac{1}{b})}{\Gamma(d)} + p_2 \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma\alpha}\left(\frac{\lambda}{\alpha}\right)^{\frac{1}{2}}\right)$$
$$p_1 q^3 \frac{\Gamma(\frac{3}{b}+1)\Gamma(k-\frac{3}{b})}{\Gamma(d)} p_2 \frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma\alpha}\left(\frac{\lambda}{\alpha}\right)^{\frac{3}{2}} + 6\left(p_1 q^1 \frac{\Gamma(\frac{1}{b}+1)\Gamma(k-\frac{1}{b})}{\Gamma(d)} + p_2 \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma\alpha}\left(\frac{\lambda}{\alpha}\right)^{\frac{1}{2}}\right)^2$$
$$p_1 q^2 \frac{\Gamma(\frac{2}{b}+1)\Gamma(k-\frac{2}{b})}{\Gamma(d)} + p_2 \frac{\Gamma(\alpha+1)}{\Gamma\alpha}\left(\frac{\lambda}{\alpha}\right) - 3\left(p_1 q^1 \frac{\Gamma(\frac{1}{b}+1)\Gamma(k-\frac{1}{b})}{\Gamma(d)} + p_2 \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma\alpha}\left(\frac{\lambda}{\alpha}\right)^{\frac{1}{2}}\right)^4 \tag{34}$$

4.10. Measure of Skewness and Kurtosis

Skewness refers to the measure of asymmetry exhibited by a probability distribution. It indicates the extent to which the distribution leans or deviates from being symmetric around its mean. Kurtosis helps characterize the peakedness or flatness of the distribution's curve. Skewness and kurtosis are symbolized as β_1 and β_2 respectively, and their mathematical representation is given as:

$$\beta_1 = \frac{(\mu_3)^2}{(\mu_2)^3} \tag{35}$$

Using eq.(32) and (33) in the above equation, we can get the value of skewness.

And,

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2} \tag{36}$$

Using eq.(32) and (34) in the equation (36), the expression for the β_2 can be obtained.

4.11. Moment Generating Function

The moment generating function is a mathematical function that can be defined as follows: $M_x(t) = E(e^{tx})$, provided it exists.

where $E(e^{tx}) = \int e^{tx} f(x)dx$ and is given by

$$E(e^{tx}) = \int_0^\infty e^{tx}\left(p_1 bdq^{-b}x^{b-1}\left[1+(\frac{x}{q})^b\right]^{-d-1} + p_2\frac{2}{\Gamma(\alpha)}\left(\frac{\alpha}{\lambda}\right)^\alpha x^{2\alpha-1}\exp\left(\frac{-\alpha}{\lambda}x^2\right)\right)dx$$

$$E(e^{tx}) = p_1\int_0^\infty e^{tx}bdq^{-b}x^{b-1}\left[1+(\frac{x}{q})^b\right]^{-d-1}dx + p_2\int_0^\infty e^{tx}\frac{2}{\Gamma(\alpha)}\left(\frac{\alpha}{\lambda}\right)^\alpha x^{2\alpha-1}\exp\left(\frac{-\alpha}{\lambda}x^2\right)dx \tag{37}$$

$$E(e^{tx}) = p_1\sum_{j=0}^{\infty}\frac{(qt)^j}{j!}\frac{\Gamma(\frac{j}{b}+1)\Gamma(k-\frac{j}{b})}{\Gamma(d)} + p_2\left[1-t\sqrt{\frac{\lambda}{\alpha}}\right]^{-\alpha} \tag{38}$$

4.12. Characteristic Function

The characteristic function is derived from the moment generating function by substituting 't' with 'it', and its expression is given as follows:

$$\phi_x(t) = E(e^{itx}) \tag{39}$$

$$E(e^{itx}) = p_1\sum_{j=0}^{\infty}\frac{(qit)^j}{j!}\frac{\Gamma(\frac{j}{b}+1)\Gamma(k-\frac{j}{b})}{\Gamma(d)} + p_2\left[1-it\sqrt{\frac{\lambda}{\alpha}}\right]^{-\alpha} \tag{40}$$

4.13. Mean deviation about Mean

The mean deviation represents the average of the absolute differences between data points and the mean of a dataset. It offers a way to estimate the level of variability in a population. The mean deviation about mean can be obtained as follows:

$$\delta_1(x) = \int_0^\infty |x-\mu| f(x)dx \tag{41}$$

where $\mu = E(x)$. It can be calculated by the following relationship

$$\delta_1(x) = \int_0^\mu (\mu - x)f(x)dx + 2\int_\mu^\infty (\mu-x)f(x)dx$$

$$\delta_1(x) = 2\int_0^\mu (\mu-x)f(x)dx$$

$$\delta_1(x) = 2\left[\mu F(\mu) + \int_0^\mu xf(x)dx\right] \tag{42}$$

$$\delta_1(x) = 2\Bigg[\left(p_1 q^1\frac{\Gamma(\frac{1}{b}+1)\Gamma(k-\frac{1}{b})}{\Gamma(d)} + p_2\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma\alpha}\left(\frac{\lambda}{\alpha}\right)^{\frac{1}{2}}\right)\left(p_1\left(1-\left[1+(\frac{x}{q})^b\right]^{-d}\right) + p_2\left(\frac{1}{\Gamma(\alpha)}\Gamma(\frac{\alpha}{\lambda}\mu^2,\alpha)\right)\right)$$
$$-\left(p_1 bd\sum_{j=0}^{\infty}(-1)^j\binom{d+1}{j}\frac{\left(\frac{\mu}{q}\right)^{\frac{j+1}{b}+1}}{j+\frac{1}{b}+1} + p_2\frac{1}{\Gamma(\alpha)}\sqrt{\frac{\lambda}{\alpha}}\sum_{i=0}^{\infty}\frac{(-i)}{i!}\frac{(\frac{\alpha}{\lambda}\mu^2)^{i+\alpha-\frac{1}{2}}}{i+\alpha+\frac{1}{2}}\right)\Bigg] \tag{43}$$

4.14. Renyi Entropy

The quantification of uncertainty inherent in random variable X can be computed using Renyi entropy [15]. This can be articulated using the subsequent equation:

$$\delta_r(x) = \frac{1}{1-r}\log\left(\int f^r(x)dx\right) \tag{44}$$

where

$$f^r(x) = \left[p_1 bdq^{-b}x^{b-1}\left[1+(\frac{x}{q})^b\right]^{-d-1} + p_2\frac{2}{\Gamma(\alpha)}\left(\frac{\alpha}{\lambda}\right)^{\alpha}x^{2\alpha-1}\exp\left(\frac{-\alpha}{\lambda}x^2\right)\right]^r \tag{45}$$

By using binomial expansion

$$(a+b)^n = \sum_{k=0}^{\infty}\binom{n}{k}a^{n-k}b^k$$

The integral will take the form

$$= \sum_{k=0}^{\infty}\binom{r}{n}\int_0^{\infty}\left[p_1 bdq^{-b}x^{b-1}\left[1+(\frac{x}{q})^b\right]^{-d-1}\right]^{r-n}\left[p_2\frac{2}{\Gamma(\alpha)}\left(\frac{\alpha}{\lambda}\right)^{\alpha}x^{2\alpha-1}\exp\left(\frac{-\alpha}{\lambda}x^2\right)\right]^n dx$$

Due to their independence, the integral can be treated as the multiplication of the two integrals, leading to the resulting expression for the Renyi entropy as:

$$\delta_r(x) = \frac{1}{1-r}\log\left[\frac{(p_1 bdq^{-b})^{r-n}q^{(b-1)(r-n)+1}}{b}\frac{\Gamma\left(\frac{(r-n)(b-1)+1}{b}\right)\Gamma\left(\frac{brd-bnd+r-n-1}{b}\right)}{\Gamma\left(\frac{(r-n)(b-1)+1}{b}+\frac{brd-bnd+r-n-1}{b}\right)}\right.$$
$$\left.+p_2\frac{2^{n-1}}{(\Gamma(\alpha))^n}\left(\frac{\alpha}{\lambda}\right)^{1-\frac{n}{2}}\left(\frac{1}{n}\right)^{\left(\alpha n-\frac{n}{2}+1\right)}\Gamma(\alpha n-\frac{n}{2}+\frac{1}{2})\right] \tag{46}$$

4.15. β-Entropy

The β-entropy is formulated as an extension of the Shannon entropy with a single parameter. This concept was initially introduced by Havrda and Charvat [9], and subsequently adapted for use in physical contexts by Tsallis [18]. The β-entropy can be defined as:

$$H_{\overline{\beta}} = \frac{1}{\overline{\beta}-1}\left[1-\int_0^{\infty}f^{\overline{\beta}}(x)\right]dx, for\overline{\beta}\neq 1. \tag{47}$$

β-entropy for the MBND is given by

$$H_{\overline{\beta}} = \frac{1}{\overline{\beta}-1}\int_0^{\infty}\left[p_1 bdq^{-b}x^{b-1}\left[1+(\frac{x}{q})^b\right]^{-d-1} + p_2\frac{2}{\Gamma(\alpha)}\left(\frac{\alpha}{\lambda}\right)^{\alpha}x^{2\alpha-1}\exp\left(\frac{-\alpha}{\lambda}x^2\right)\right]^{\overline{\beta}}dx, for\overline{\beta}\neq 1. \tag{48}$$

By using binomial expansion

$$(a+b)^n = \sum_{k=0}^{\infty}\binom{n}{k}a^{n-k}b^k$$

The integral will take the form

$$= \sum_{n=0}^{\overline{\beta}}\overline{\beta}n\int_0^{\infty}\left[p_1 bdq^{-b}x^{b-1}\left[1+(\frac{x}{q})^b\right]^{-d-1}\right]^{\overline{\beta}-n}\left[p_2\frac{2}{\Gamma(\alpha)}\left(\frac{\alpha}{\lambda}\right)^{\alpha}x^{2\alpha-1}\exp\left(\frac{-\alpha}{\lambda}x^2\right)\right]^n dx$$

Due to their independence, the integral can be treated as the multiplication of the two integrals, leading to the resulting expression for the β-entropy as:

$$H_{\overline{\beta}} = \frac{1}{1-\overline{\beta}}\left[\frac{(p_1 bdq^{-b})^{\overline{\beta}-n} q^{(b-1)(\overline{\beta}-n)+1}}{b} \frac{\Gamma\left(\frac{(\overline{\beta}-n)(b-1)+1}{b}\right)\Gamma\left(\frac{b\overline{\beta}d-bnd+\overline{\beta}-n-1}{b}\right)}{\Gamma\left(\frac{(\overline{\beta}-n)(b-1)+1}{b}+\frac{b\overline{\beta}d-bnd+\overline{\beta}-n-1}{b}\right)} \right.$$
$$\left. + p_2 \frac{2^{n-1}}{(\Gamma(\alpha))^n}\left(\frac{\alpha}{\lambda}\right)^{1-\frac{n}{2}}\left(\frac{1}{n}\right)^{\left(\alpha n-\frac{n}{2}+1\right)}\Gamma\left(\alpha n-\frac{n}{2}+\frac{1}{2}\right)\right] \tag{49}$$

4.16. Mean Time between failures

The Mean Time Between Failures, often abbreviated as MTBF, serves as an indicator of a product's or component's reliability. This metric represents the average interval between successive failures in a process. Its calculation involves determining:

$$MTBF = \int_0^\infty tf(t)dt \tag{50}$$

It can be calculated as

$$MTBF = \int_0^\infty t\left(p_1 bdq^{-b}t^{b-1}\left[1+(\frac{t}{q})^b\right]^{-d-1} + p_2\frac{2}{\Gamma(\alpha)}\left(\frac{\alpha}{\lambda}\right)^\alpha t^{2\alpha-1}\exp\left(\frac{-\alpha}{\lambda}t^2\right)\right)dt \tag{51}$$

$$= p_1\int_0^\infty tbdq^{-b}t^{b-1}\left[1+(\frac{t}{q})^b\right]^{-d-1}dx + p_2\int_0^\infty x^r\frac{2}{\Gamma(\alpha)}\left(\frac{\alpha}{\lambda}\right)^\alpha t^{2\alpha-1}\exp\left(\frac{-\alpha}{\lambda}t^2\right)dt \tag{52}$$

$$MTSF = p_1 q\frac{\Gamma(\frac{1}{b}+1)\Gamma(k-\frac{1}{b})}{\Gamma(d)} + p_2\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma\alpha}\left(\frac{\lambda}{\alpha}\right)^{\frac{1}{2}} \tag{53}$$

4.17. Bonferoni and Lorenz curve

The Bonferroni Curve, introduced by Bonferroni [3], serves as a method for analyzing income inequality. It finds utility across various domains, including the examination of economic factors like income and poverty, as well as applications in reliability, demographics, insurance, and medicine.

On the other hand, the Lorenz curve provides a visual representation of the distribution of income or wealth. Though commonly associated with depicting economic inequality, it holds the ability to illustrate uneven distributions in diverse systems. The curvature of the Lorenz curve relative to a baseline, often represented as a straight line, signifies the extent of inequality – the greater the inequality, the farther the curve deviates from this baseline. The following expression is used to calculate the Bonferroni curve:

$$B_F[F(x)] = \frac{1}{\mu F(x)}\int_0^x uf(u)du \tag{54}$$

where

$$\int_0^x uf(u)du = \int_0^x u\left(p_1 bdq^{-b}u^{b-1}\left[1+(\frac{u}{q})^b\right]^{-d-1} + p_2\frac{2}{\Gamma(\alpha)}\left(\frac{\alpha}{\lambda}\right)^\alpha u^{2\alpha-1}\exp\left(\frac{-\alpha}{\lambda}u^2\right)\right)du \tag{55}$$

$$= \int_0^x p_1 ubdq^{-b}u^{b-1}\left[1+(\frac{u}{q})^b\right]^{-d-1}dx + \int_0^x p_2 u\frac{2}{\Gamma(\alpha)}\left(\frac{\alpha}{\lambda}\right)^\alpha u^{2\alpha-1}\exp\left(\frac{-\alpha}{\lambda}u^2\right)du \tag{56}$$

$$\int_0^x uf(u)du = p_1 dq \sum_{j=0}^{\infty} (-1)^j \binom{d+1}{j} \frac{\left(\frac{x}{q}\right)^{\frac{j+1}{b}+1}}{j+\frac{1}{b}+1} + p_2 \frac{1}{\Gamma(\alpha)} \sqrt{\frac{\lambda}{\alpha}} \sum_{i=0}^{\infty} \frac{(-i)}{i!} \frac{\left(\frac{\alpha}{\lambda}x^2\right)^{i+\alpha-\frac{1}{2}}}{i+\alpha+\frac{1}{2}} \quad (57)$$

$$B_F[F(x)] = \frac{p_1 dq \sum_{j=0}^{\infty} (-1)^j \binom{d+1}{j} \frac{\left(\frac{x}{q}\right)^{\frac{j+1}{b}+1}}{j+\frac{1}{b}+1} + p_2 \frac{1}{\Gamma(\alpha)} \sqrt{\frac{\lambda}{\alpha}} \sum_{i=0}^{\infty} \frac{(-i)}{i!} \frac{\left(\frac{\alpha}{\lambda}x^2\right)^{i+\alpha-\frac{1}{2}}}{i+\alpha+\frac{1}{2}}}{\left(p_1 q^1 \frac{\Gamma(\frac{1}{b}+1)\Gamma(k-\frac{1}{b})}{\Gamma(d)} + p_2 \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma\alpha} \left(\frac{\lambda}{\alpha}\right)^{\frac{1}{2}}\right) \left(p_1 \left(1 - \left[1 + \left(\frac{x}{q}\right)^b\right]^{-d}\right) + p_2 \left(\frac{1}{\Gamma(\alpha)} \Gamma\left(\frac{\alpha}{\lambda}\mu^2, \alpha\right)\right)\right)} \quad (58)$$

Now, Lorenz curve is given as

$$L(z) = \frac{\int_0^z xf(x)dx}{\mu} \quad (59)$$

where

$$\int_0^z xf(x)du = \int_0^z x \left[p_1 bdq^{-b} x^{b-1} \left[1 + \left(\frac{x}{q}\right)^b\right]^{-d-1} + p_2 \frac{2}{\Gamma(\alpha)} \left(\frac{\alpha}{\lambda}\right)^{\alpha} x^{2\alpha-1} \exp\left(\frac{-\alpha}{\lambda}x^2\right)\right] dx \quad (60)$$

The final expression for Lorenz curve is

$$L(z) = \frac{p_1 dq \sum_{j=0}^{\infty} (-1)^j \binom{d+1}{j} \frac{\left(\frac{z}{q}\right)^{\frac{j+1}{b}+1}}{j+\frac{1}{b}+1} + p_2 \frac{1}{\Gamma(\alpha)} \sqrt{\frac{\lambda}{\alpha}} \sum_{i=0}^{\infty} \frac{(-i)}{i!} \frac{\left(\frac{\alpha}{\lambda}z^2\right)^{i+\alpha-\frac{1}{2}}}{i+\alpha+\frac{1}{2}}}{p_1 q^1 \frac{\Gamma(\frac{1}{b}+1)\Gamma(k-\frac{1}{b})}{\Gamma(d)} + p_2 \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma\alpha} \left(\frac{\lambda}{\alpha}\right)^{\frac{1}{2}}} \quad (61)$$

5. Maximum Likelihood Estimation

Suppose $x_1, x_2, ...x_n$ be the random sample derived from the MBND having the probability density function defined in (9). Therefore, for n observations , the logarithm of the likelihood function is expressed as below

$$L(x) = \prod_{i=1}^{n} \left(p_1 bdq^{-b} x^{b-1} \left[1 + \left(\frac{x}{q}\right)^b\right]^{-d-1} + p_2 \frac{2}{\Gamma(\alpha)} \left(\frac{\alpha}{\lambda}\right)^{\alpha} x^{2\alpha-1} \exp\left(\frac{-\alpha}{\lambda}x^2\right)\right) \quad (62)$$

The log likelihood function is given as

$$logL(x) = \sum_{i=0}^{n} log\left(p_1 bdq^{-b} x^{b-1} \left[1 + \left(\frac{x}{q}\right)^b\right]^{-d-1} + (1-p_1) \frac{2}{\Gamma(\alpha)} \left(\frac{\alpha}{\lambda}\right)^{\alpha} x^{2\alpha-1} \exp\left(\frac{-\alpha}{\lambda}x^2\right)\right)$$

$$= nlog(p_1) + nlog(b) + nlog(d) + -nblog(q) + (b-1)log \sum_{i=1}^{n} x_i + (-d-1)log \sum_{i=1}^{n} \left[1 + \left(\frac{x_i}{q}\right)^b\right] + nlog(1-p_1)$$
$$+nlog\left(\frac{2}{\Gamma(\alpha)}\right) + \alpha nlog\left(\frac{\alpha}{\lambda}\right) + (2\alpha-1)log \sum_{i=1}^{n} x_i + log \sum_{i=1}^{n} \left(\frac{-\alpha}{\lambda} x_i^2\right) \quad (63)$$

The equation are in implicit form, so it can be solved using numerical iteration method, such as the Newton–Raphson method via R to obtain the estimates of b, d, q, α, λ and p.

6. Bayesian estimation for mixing parameter

To estimate the mixing parameter p_1 within the context of MBND, we can adopt the Beta distribution as a suitable prior due to the range constraint $p_1 \in (\theta, 1)$. Let's assume that the prior distribution for p_1 is represented by $\beta(\theta, 1)$.

$$\pi(p) = \theta p_1^{\theta-1} \tag{64}$$

The likelihood function is given by

$$L(x) = \prod_{i=1}^{n} \left(p_1 bdq^{-b} x^{b-1} \left[1 + (\frac{x}{q})^b\right]^{-d-1} + p_2 \frac{2}{\Gamma(\alpha)} \left(\frac{\alpha}{\lambda}\right)^{\alpha} x^{2\alpha-1} \exp\left(\frac{-\alpha}{\lambda} x^2\right)\right) \tag{65}$$

And, the posterior distribution of p_1 is obtained by

$$\pi(p|x) = \frac{L(x)\pi(p)}{\int_0^1 L(x)\pi(p)dp} \tag{66}$$

The marginal distribution is given as

$$m(x) = \int_0^1 L(x)\pi(p)dp \tag{67}$$

$$m(x) = \int_0^1 p_1^{\theta-1} \left[(p_1 bdq^{-b})^n \sum_{i=1}^{n} x_i^{c-1} \prod_{i=1}^{n} \left(1 + \left(\frac{x_i}{b}\right)^c\right)^{-k-1} + (1-p_1)^n \left(\frac{2}{\Gamma(\alpha)}\right)^n \left(\frac{\alpha}{\lambda}\right)^{\alpha n} \sum_{i=1}^{n} x_i^{2\alpha-1} \prod_{i=1}^{n} \exp\left(\frac{-\alpha}{\lambda} x_i^2\right)\right) \right] dp$$

$$m(x) = f_1^n(x; b, d, q)\left(\frac{\theta}{\theta+n}\right) + \theta f_2^n(x; \alpha, \lambda)\beta(\theta, n+1) \tag{68}$$

Eq.(66) becomes

$$\pi(p|x) = \frac{\theta p_1^{\theta-1}\left(p_1^n f_1^n(x; b, d, q) + (1-p_1)^n f_2^n(x; \alpha, \lambda)\right)}{f_1^n(x; b, d, q)\left(\frac{\theta}{\theta+n}\right) + \theta f_2^n(x; \alpha, \lambda)\beta(\theta, n+1)} \tag{69}$$

$$= \frac{\theta p_1^{n+\theta-1} f_1^n(x; b, d, q)}{f_1^n(x; b, d, q)\left(\frac{\theta}{\theta+n}\right) + \theta f_2^n(x; \alpha, \lambda)\beta(\theta, n+1)} + \frac{(1-p_1)^n \theta p_1^{\theta-1} f_2^n(x; \alpha, \lambda)}{f_1^n(x; b, d, q)\left(\frac{\theta}{\theta+n}\right) + \theta f_2^n(x; \alpha, \lambda)\beta(\theta, n+1)}$$

The Bayesian estimate of p_1 under Squared Error Loss Function (SELF) is the mean of the posterior distribution denoted as $p_{\hat{B}1}$.

7. Simulation Study

In this part, a Monte Carlo simulation is performed using R for estimating the parameters of mixture of Burr XII and Nakagami distribution (MBND) . The values of n and the parameters are fixed as $n = (20, 40, 70, 100)$, $(b, d, q, \alpha, \lambda) = (1.5, 1, 1.5, 1.5, 1)$ respectively, the mixing parameter p_1 is fixed at 0.8 which is estimated using the classical and bayesian approach. The performance of the estimates was evaluated using the Mean Squared Error (MSE). The procedure is replicated 1000 times and obtained results are shown in table 1.

$$MSE = \frac{1}{N} \sum_{i=1}^{n} (\hat{\alpha} - \alpha)^2$$

Table 1: *Average values of estimates and their MSE in parentheses*

n	$\hat{b}$	$\hat{d}$	$\hat{q}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{p}_1$	$\hat{p}_{B1}$
20	1.9703	1.2695	0.8140	2.4148	0.6744	0.3404	0.4231
	(0.8963)	(0.5409)	(0.7580)	(0.7568)	(0.7631)	(0.4065)	(0.3932)
40	1.7349	0.8504	0.9948	2.3071	0.7112	0.5763	0.5892
	(0.7130)	(0.5140)	(0.4177)	(0.7200)	(0.6878)	(0.3896)	(0.3624)
70	1.6313	0.9631	1.4108	2.1647	0.8132	0.6296	0.6742
	(0.5990)	(0.3167)	(0.3139)	(0.4988)	(0.6157)	(0.2329)	(0.2004)
100	1.3119	0.9726	1.4372	1.9322	0.9542	0.7608	0.7814
	(0.2175)	(0.1400)	(0.1880)	(0.2604)	(0.2332)	(0.1498)	(0.1413)

From the table 1, it is noted that

- As the sample size increases, the MSE of the estimates comes down.
- With increase of n, the estimates comes closer to their true values, shows the consistency of the estimates.
- The mixing parameter performs better under the bayesian approach.

8. Real Data Analysis

In this section, we will use two real-world datasets to demonstrate the significance and versatility of the Mixture of Burr XII and Nakagami Distribution (MBND). We will compare the suitability of our proposed model (MBND) with other competing models, namely the Burr XII Lomax Distribution (BLD), Burr II Distribution (BD), and the Length Based Nakagami Distribution (LBND).
To conduct this comparison, we will employ various goodness-of-fit metrics, including the $-2\hat{l}$ statistic, the Akaike Information Criterion (AIC), the Corrected Akaike Information Criterion (AICc), and the Bayesian Information Criterion (BIC). These criteria aid in selecting the most suitable distribution by identifying the one with the lowest values of AIC, AICc, and BIC. Additionally, we will calculate the Kolmogorov-Smirnov (KS) distance and its associated p-value, providing a further assessment of the goodness of fit.

$$AIC = 2k - 2\hat{l} \tag{70}$$

$$BIC = kln(n) - 2\hat{l} \tag{71}$$

and

$$AICc = AIC + \frac{2k(k+1)}{n-k-1} \tag{72}$$

where $\hat{l}$ is the maximized log-likelihood, k is the number of parameters, and n is the sample size. Through this comprehensive analysis, we aim to establish the effectiveness of the MBND model in capturing the characteristics of the given datasets and to ascertain its superiority over the alternative distribution models.

Data Set I: The dataset pertains to the COVID-19 vaccination rate across 46 distinct countries situated in southern Africa. This dataset has been previously subjected to analysis by [2]. The data is as follows: 0.042, 0.205, 0.285, 0.319, 0.464, 0.550, 0.889, 0.895, 0.939, 0.986, 1.000, 1.088, 1.212, 1.244, 1.450, 1.593, 1.844, 2.039, 2.157, 2.167, 2.334, 2.440, 2.657, 3.685, 3.879, 4.493, 4.800, 4.944, 5.155, 5.674, 7.602, 10.004, 12.238, 12.520, 12.553, 13.063, 15.105, 15.229, 15.629, 15.848, 18.641, 18.940, 29.885, 58.162, 61.838, 72.286

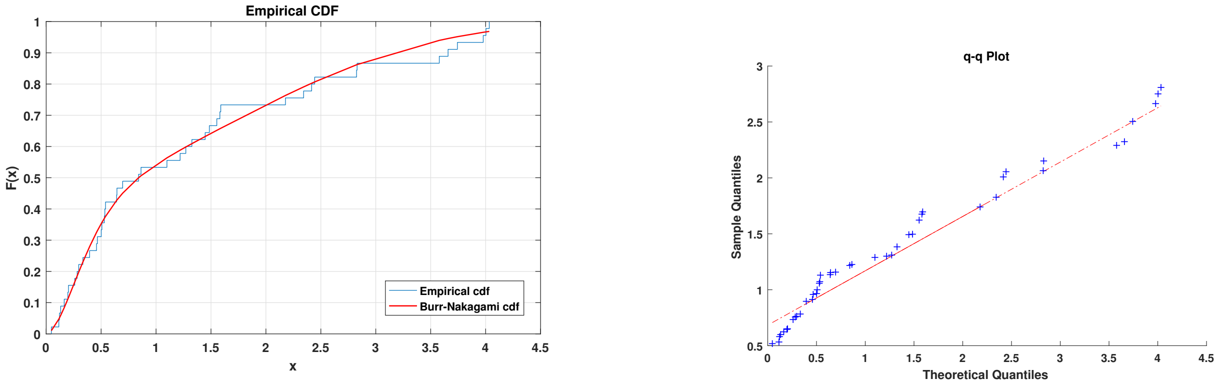


Figure 8: *(a) Ecdf plot for the dataset I (b) q-q plot for the dataset I*

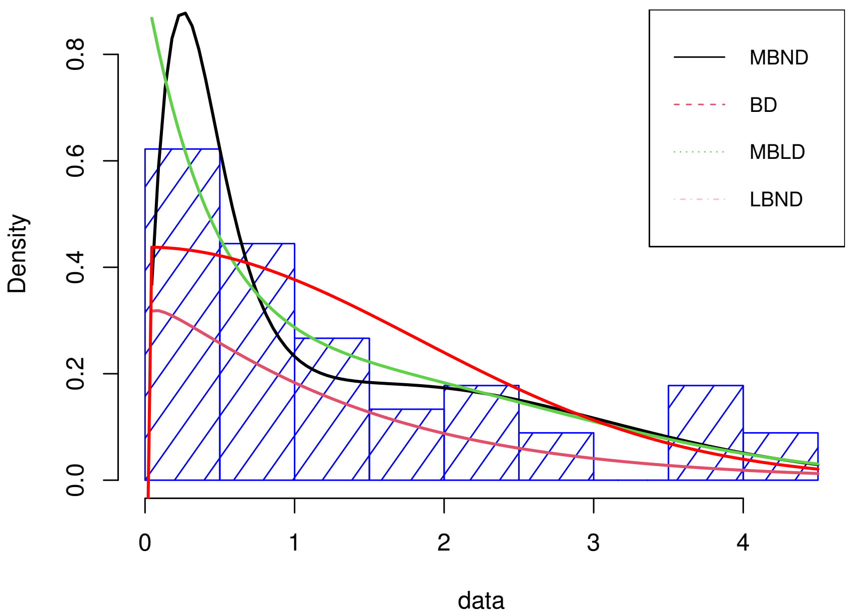


Figure 9: *Plot of the fitted densities for dataset I*

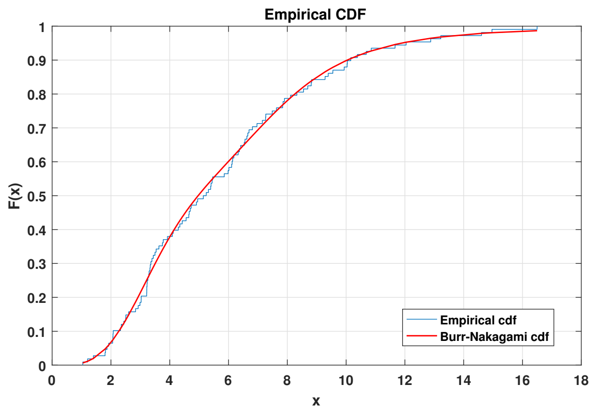
Table 2: *Maximum Likelihood Estimates of the parameters of the different models.*

Model	$\hat{b}$	$\hat{q}$	$\hat{d}$	$\hat{\alpha}$	$\hat{\lambda}$
MBND	1.7543	0.8746	3.4638	1.0266	6.3816
BLD	1.8732	4.1136	19.3219	0.03191	13.995
BD	1.0461	32.1767	28.9722	-	-
LBND	-	-	-	0.4712	1.6594

Table 3: *Goodness of fit measures for the different models.*

Model	$-2\hat{l}$	AIC	AICC	BIC	K-S	p-value
MBND	**110.3650**	**120.3650**	**121.9034**	**129.3983**	**0.0773**	**0.9313**
MBLD	114.4867	124.4867	126.0251	133.5200	0.0835	0.8864
BD	178.6642	184.6642	185.2496	190.0842	0.5233	$5.279e^{-}12$
LBND	155.8333	159.8333	160.1190	163.4466	0.1757	0.1096

Data Set II: Almongy et al. [1] analyzed this dataset to showcase the utility of an innovative extended Rayleigh distribution. This distribution was employed to model COVID-19 mortality rate data from Mexico spanning a period of 108 days, starting from March 4th and concluding on July 20th, 2020. The dataset pertains to a raw mortality rate and is characterized by its relatively unrefined nature. The data are as follows: 8.826, 6.105, 10.383, 7.267, 13.220, 6.015, 10.855, 6.122, 10.685, 10.035, 5.242, 7.630, 14.604, 7.903, 6.327, 9.391, 14.962, 4.730, 3.215, 16.498, 11.665, 9.284, 12.878, 6.656, 3.440, 5.854, 8.813, 10.043, 7.260, 5.985, 4.424, 4.344, 5.143, 9.935, 7.840, 9.550, 6.968, 6.370, 3.537, 3.286, 10.158, 8.108, 6.697, 7.151, 6.560, 2.988, 3.336, 6.814, 8.325, 7.854, 8.551, 3.228, 3.499, 3.751, 7.486, 6.625, 6.140, 4.909, 4.661, 1.867, 2.838, 5.392, 12.042, 8.696, 6.412, 3.395, 1.815, 3.327, 5.406, 6.182, 4.949, 4.089, 3.359, 2.070, 3.298, 5.317, 5.442, 4.557, 4.292, 2.500, 6.535, 4.648, 4.697, 5.459, 4.120, 3.922, 3.219, 1.402, 2.438, 3.257, 3.632, 3.233, 3.027, 2.352, 1.205, 2.077, 3.778, 3.218, 2.926, 2.601, 2.065, 1.041, 1.800, 3.029, 2.058, 2.326, 2.506, 1.923.



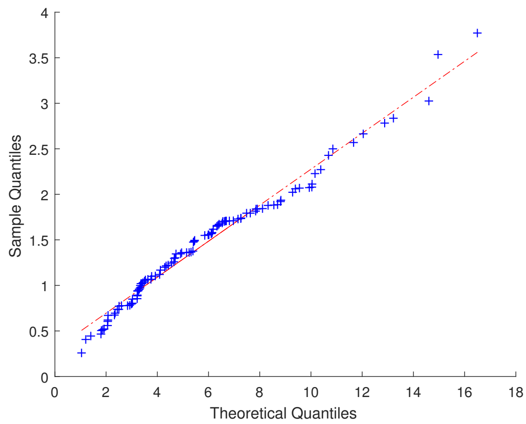


Figure 10: *(a) Ecdf plot for the dataset II (b) q-q plot for the dataset II*

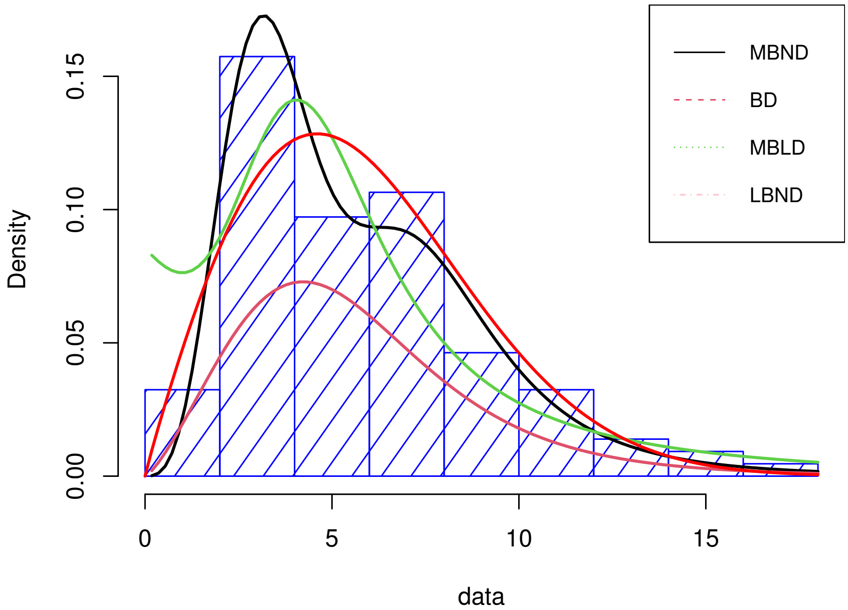


Figure 11: *Plot of the fitted densities for dataset II*

Table 4: *Maximum Likelihood Estimates of the parameters of the different models.*

Model	$\hat{b}$	$\hat{q}$	$\hat{d}$	$\hat{\alpha}$	$\hat{\lambda}$
MBND	5.2318	7.4680	0.9442	2.0159	12.4548
MBLD	3.7382	4.8573	0.9421	0.0090	18.9176
BD	2.3266	8.7808	2.6983	-	-
LBND	-	-	-	0.4719	21.1924

Table 5: *Goodness of fit measures for the different models.*

Model	$-2\hat{l}$	AIC	AICC	BIC	K-S	p-value
MBND	**530.8075**	**540.8075**	**541.3957**	**554.2181**	**0.0486**	**0.9602**
MBLD	558.2015	568.2015	568.7097	581.6121	0.1151	0.1144
BD	685.1428	691.1428	691.3735	699.1892	0.5054	$2.20e^{-}16$
LBND	538.4605	542.4605	542.5748	547.8247	0.3049	$3.676e^{-}09$

Table 2 and Table 3 provide the Maximum Likelihood Estimates (MLEs) and offer a comparison of the performance between the Mixture of Burr XII and Nakagami Distribution (MBND) and the alternative distributions for the first dataset. Similarly, Table 4 and Table 5 present the MLEs and compare the performance of MBND with other distributions for the second dataset. The findings displayed in Table 3 and 5 demonstrate that MBND achieves the lowest values for AIC, AICc, and BIC, indicating its superiority over several well-established competing models. The claim is further supported by Figures 8 and 10. Additionally, the results presented in Table 3 and 5 are corroborated by examining the Empirical Cumulative Distribution Function (ECDF) plots and Quantile-Quantile (Q-Q) plots for the MBND model in the context of both datasets, showcased in Figure 7 and 9. These visual representations lend support to the outcomes highlighted in Table 3 and 5.

9. Conclusion

In this paper, we presented a mixture of the Burr XII and Nakagami distribution (MBND). We investigate its statistical properties and demonstrate that this new mixture distribution can serve as an alternative model to some existing distributions. For MLEs, simulation schemes are developed that yield less Mean Squared Error (MSE) as sample size increases. Additionally, Bayesian estimates of the mixing parameter are computed, demonstrating their superiority over classical estimates. We utilize real-world datasets to illustrate the efficacy of the MBND model and compare it with other competing distributions. The results of this comparison reveal that the MBND model offers a better fit than the considered distributions. Through this comprehensive analysis, we establish the viability of MBND as a valuable tool for modeling and analyzing various data scenarios.

References

[1] Almetwally E. M., Aljohani H.M., Alghamdi A.S, Almongy, H. M. and Hafez, E.H. (2021). A new extended rayleigh distribution with applications of covid-19 data. *Results in Physics*, 23, 104012.

[2] Almetwally E. M., Haj Ahmad, H., Almongy, H. M. and Al-nefaie, A. H. (2022). Modeling of covid 19 vaccination rate using odd lomax inverted nadarajah- haghighi distribution. *Plos one*, 17(10), e0276181.

[3] Berferroni, C. E. Elmenti di statitica generale. Libreria Seber, Firenze., 1930.

[4] Chen, X., Ponomareva, M. and Tamer, E. Likelihood inference in some finite mixture models. *Journal of Econometrics*, 182:87–99.

[5] Daghestani, A. F., Sultan, K. S. and Al-Moisheer, A. S. (2021). Mixture of lindley and weibull distributions: Properties and estimation. *Journal of Statistics Applications and Probability*, 10(2):301—313.

[6] Daniyal, M. and Aleem, M. (2014). On the mixture of burr xii and weibull distributions. journal of statistics applications and probability, 3:251-–267.

[7] Decay, M. F. (1964). Modified poisson probability law for a point pattern more than random. (1964). *Association of American Geographers Annals*, 54:559—565.

[8] Hasselblad, V. (1968). Estimation of mixtures of distribution from exponential family. *Journal of American Statistical Association*, 64:300—304.

[9] Havrda, J. and Charvat, F. (1967). Quantification method in classification processes: concept of structural α-entropy. *Kybernetika*, 3:30-–35.

[10] Jaspers, S., Komrek, A. and Aerts, M. (2018). Bayesian estimationof multivariate normal mixtures with covariate-dependent mixing weights, with an application in antimicrobial resistance monitoring. Biometrical JournaL, 60:7—19.

[11] Martin, M., Hory, C. and Huchard, M. Chi-square law mixture for an unsupervised time frequency pattern extraction. Proceedings of the Tenth International Congress on Sound and Vibration, Stockholm, Sweden, 7-10 July, Code 62825, 2003.

[12] Nakagami, M. (1960). The m-distribution-a general formulation of intensity distribution of rapid fading. in W. C. Hoffman (ed.) *Statistical Methods in Radio Wave Propagations*, 3—36.

[13] Nasiri, P. and Azarian, A. A. (2018). Estimating the mixing proportion of mixture of two chi square distributions. *International Journal of Applied Mathematics and Information Sciences*, 12(4):753-–759.

[14] Pearson, k. (1894). Contribution to the mathematical theory of evolution. *Philosophical Transactions of the Royal Society of London*, 185:71–110, 1894.

[15] Renyi, A. On the measure of entropy and information. Proceedings of the fourth Berkely symposium on mathematical statistics and probability, 1:547–561, 1961.

[16] Singh, B. P., Singh, S. and Das, U. D. (2021). A general class of new continuous mixture distribution and applications. *J. Math. Comput. Sci.*, 11(1):585—602.

[17] Titterington, D.M., Smith, A.F.M. and Makov, U.E. Statistical Analysis of Finite Mixture Distributions. John Wiley and Sons, New York, 1985.

[18] Tsallis, C. (1968). Possible generalization of boltzmann–gibbs statistics. *Journal of Statistical Physics*, 52:479-–487.

[19] Zaman, M. R., Paul, D. N. R., Akhter, N., Howlader, M. H. and Kabir, M. S. (2006). Chi-square mixture of chi-square distributions. *Journal of Applied Sciences*, 6:243-–246.