THE SABUR DISTRIBUTION: PROPERTIES AND APPLICATION RELATED TO ENGINEERING DATA

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Aijaz Ahmad[1*], Afaq Ahmad[2] and Aafaq A. Rather[3]

[1]Department of Mathematics, Bhagwant University, Ajmer, India.
[2]Department of mathematical science, IUST, Awantipora, Kashmir, India.
[3]Symbiosis Statistical Institute, Symbiosis International (Deemed University), Pune, India.

ahmadaijaz4488@gmail.com[1*] , aderaafaq@gmail.com[2] , aafaq7741@gmail.com[3]

Abstract

This paper introduces a novel probability distribution called the Sabur distribution (SD), characterized by two parameters. It offers a comprehensive analysis of this distribution, encompassing various properties such as moments, moment-generating functions, deviations from the mean and median, mode and median, Bonferroni and Lorenz curves, Renyi entropy, order statistics, hazard rate functions, and mean residual functions. Furthermore, the paper delves into the graphical representation of the probability density function, cumulative distribution function and hazard rate function to provide a visual understanding of their behavior. The distribution's parameters are estimated using the well-known method of maximum likelihood estimation. The paper also showcases the practical applicability of the Sabur distribution through real-world examples, underscoring its performance and relevance in various scenarios.

Keywords: Moments, moment generating function, reliability measures, mean deviations, maximum likelihood function.

Subject classification: 60E05, 62E15.

1. Introduction

Statistical distributions hold great importance in fields such as biomedicine, engineering, economics, and various scientific domains. Two widely recognized distributions, namely the exponential distribution and the gamma distribution, are often used as lifetime distributions for analyzing statistical data. Among these, the exponential distribution stands out due to its singular parameter and several intriguing statistical properties, notably its memory less property and constant hazard rate characteristic. In the realm of statistics, numerous extensions of these distributions have been developed to enhance their flexibility and applicability. One notable contribution to this literature is attributed to Lindley in [10]. He introduced a one-parameter lifetime distribution characterized by the following probability density function:

$$f(y,\beta)=\frac{\beta^2}{(1+\beta)}(1+y)e^{-\beta y}\quad ;y>0\,,\beta>0$$

In recent years, researchers have made significant advancements in the study of the Lindley distribution and have proposed various one- and two-parameter distributions to model complex datasets effectively. A notable contribution was made by Ghitney et al. [8], who conducted an extensive study on the Lindley distribution. They demonstrated that the Lindley distribution outperforms the exponential distribution when applied to modelling waiting times before bank customer service. Additionally, they highlighted that the contours of the hazard rate function for the Lindley distribution show an increasing trend, while the mean residual life function is a decreasing function of the random variable. Zakerzadeh and Dolati [16] and Nadarajah et al. [12] extended the Lindley distribution by introducing new parameters and evaluating the performance of these extended distributions using various datasets. Over the years, several authors have made contributions to modify the Lindley distribution. For instance, Merovci [11] introduced the transmuted Lindley distribution and discussed its various properties. Sharma et al. [14, 15] introduced the inverse of the Lindley distribution and examined its unique characteristics. Shanker et al. [13] developed a novel lifetime distribution called the Akash distribution, which demonstrated superior performance compared to both the exponential and Lindley distributions. Ahmad et al. [1] introduced the transmuted inverse Lindley distributions and conducted analyses of their properties, Ahmad et al [2, 3], Bhaumik, D. K. et al. [5], Flaih, A et al. [6]. Each of these distributions comes with its own set of advantages and limitations when applied to analyzing complex data.

In this paper, the authors aim to introduce a new two-parameter distribution that offers greater flexibility and improved results compared to existing distributions. The probability density function of this newly established two-parameter distribution is as follows

$$f(y,\alpha,\beta) = \frac{\beta^2}{\alpha\beta+\beta^2+1}\left(\alpha+\beta+\frac{\beta}{2}y^2\right) e^{-\beta y} \quad ; y > 0, \alpha, \beta > 0 \tag{2.1}$$

The proposed distribution is named as Sabur distribution which is a combination of two distributions, Exponential distribution having scale parameter β and gamma distribution having shape parameter 3 with scale parameter β.With combining proportion as $\frac{\beta(\alpha+\beta)}{\alpha\beta+\beta^2+1}$

$$f(y,\alpha,\beta) = \pi\phi_1(y,\beta) + (1-\pi)\phi_2(y,\beta)$$

Where $\pi = \frac{\beta(\alpha+\beta)}{\alpha\beta+\beta^2+1}$

$$\phi_1(y,\beta) = \beta e^{-\beta y}, \qquad \phi_2(y,3,\beta) = \frac{\beta^3}{2}y^2 e^{-\beta y}$$

The cumulative distribution function of (1.1) is given as

$$F(y,\alpha,\beta) = 1-\left[1+\frac{\beta^2y^2+2\beta y}{2(\alpha\beta+\beta^2+1)}\right]e^{-\beta y} \; ; y > 0, \alpha, \beta > 0 \tag{2.1}$$

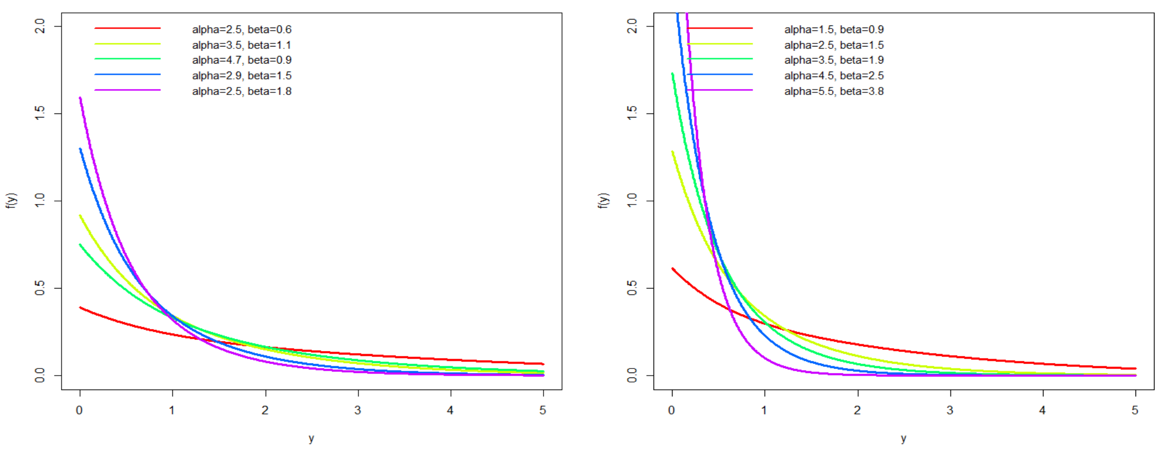

Fig 1: *The graph of p.d.f of SD for different values of parameters.*

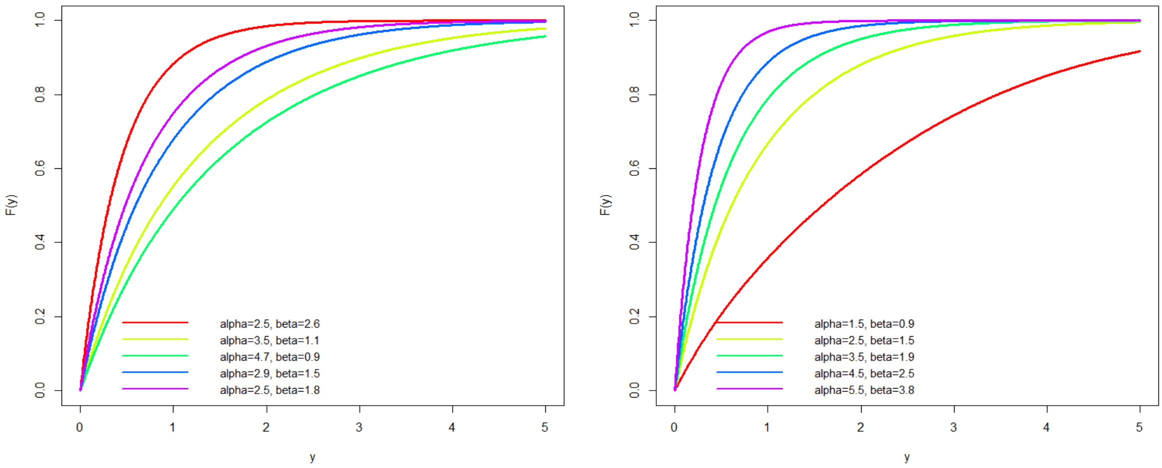

Fig 2: *The graph of c.d.f of SD for different values of parameters.*

2. Statistical Properties

In this section different properties of the Sabur distribution has been discussed such as moments, moment generating function, mode and median.

2.1 Moments of Sabur Distribution

Let us consider Y be a random variable follows the Sabur distribution then the r^{th} moment of the distribution denoted by μ_r' is given as

$$\mu_r' = E(Y^r) = \int_0^\infty y^r f(y,\alpha,\beta)dy$$

$$= \int_0^\infty y^r \frac{\beta^2}{\alpha\beta+\beta^2+1}\left(\alpha+\beta+\frac{\beta}{2}y^2\right)e^{-\beta y}\ dy$$

$$= \frac{\beta^2}{\alpha\beta+\beta^2+1}\int_0^\infty \left\{(\alpha+\beta)y^r + \frac{\beta}{2}\ y^{r+2}\right\}e^{-\beta y}\ dy$$

$$= \frac{\beta^2}{\alpha\beta+\beta^2+1}\left[\frac{(\alpha+\beta)\Gamma(r+1)}{\beta^{r+1}} + \frac{\Gamma(r+3)}{2\beta^{r+2}}\right]$$

Substituting $r = 1,2,3,4$, we obtain first four moments of the distribution about origin.

$$\mu_1' = \frac{\alpha\beta+\beta^2+3}{\beta(\alpha\beta+\beta^2+1)}\ ,\mu_2' = \frac{2(\alpha\beta+\beta^2+6)}{\beta^2(\alpha\beta+\beta^2+1)}$$

$$\mu_3' = \frac{6(\alpha\beta+\beta^2+10)}{\beta^3(\alpha\beta+\beta^2+1)}\ ,\mu_4' = \frac{24(\alpha\beta+\beta^2+15)}{\beta^4(\alpha\beta+\beta^2+1)}$$

Therefore, the mean and variance of Sabur distribution is given as

$$\mu = E(Y) = \frac{\alpha\beta+\beta^2+3}{\beta(\alpha\beta+\beta^2+1)}$$

The central moments of Sabur distribution can be obtained by using above raw moments

$$\mu_2 = \sigma^2 = \mu_2' - (\mu_1')^2 = \frac{\alpha^2\beta^2+2\alpha\beta^3+8\alpha\beta+\beta^4+8\beta^2+3}{\beta^2(\alpha\beta+\beta^2+1)^2}$$

$$\mu_3 = \frac{2\beta^6+60\beta^4+6\alpha\beta^5+4\alpha^2\beta^4+2\alpha(\alpha^2+\alpha+59)\beta^3+2(30\alpha^2+39)\beta^2+78\alpha\beta+36}{\beta^3(\alpha\beta+\beta^2+1)^3}$$

$$\mu_4 = \frac{\left\{\begin{matrix} 9\beta^8+222\beta^6-24\alpha\beta^8-12\alpha\beta^7(4\alpha+3)-6\alpha^2\beta^6(4\alpha-13) \\ -6\alpha\beta^5(2\alpha^2-33\alpha-95)+\alpha^2\beta^4(2\alpha^2+24\alpha+894)+\alpha\beta^3(534\alpha^2-144\alpha+852) \\ +\beta^2(702\alpha+690)+812\alpha\beta+495 \end{matrix}\right\}}{\beta^4(\alpha\beta+\beta^2+1)^4}$$

Coefficient of variation (C.V)= $\frac{\sigma}{\mu_1'} = \frac{\sqrt{\alpha^2\beta^2+2\alpha\beta^3+8\alpha\beta+\beta^4+8\beta^2+3}}{\beta(\alpha\beta+\beta^2+1)(\alpha\beta+\beta^2+3)}$

Coefficient of skewness ($\sqrt{\beta_1}$)= $\frac{\mu_3}{(\mu_2)^{3/2}} = \frac{2\beta^6+60\beta^4+6\alpha\beta^5+4\alpha^2\beta^4+2\alpha(\alpha^2+\alpha+59)\beta^3+2(30\alpha^2+39)\beta^2+78\alpha\beta+36}{(\alpha^2\beta^2+2\alpha\beta^3+8\alpha\beta+\beta^4+8\beta^2+3)^{3/2}}$

Coefficient of kurtosis (β_2) $= \frac{\mu_4}{(\mu_2)^2} = \frac{\left\{\begin{matrix} 9\beta^8+222\beta^6-24\alpha\beta^8-12\alpha\beta^7(4\alpha+3)-6\alpha^2\beta^6(4\alpha-13) \\ -6\alpha\beta^5(2\alpha^2-33\alpha-95)+\alpha^2\beta^4(2\alpha^2+24\alpha+894)+\alpha\beta^3(534\alpha^2-144\alpha+852) \\ +\beta^2(702\alpha+690)+812\alpha\beta+495 \end{matrix}\right\}}{(\alpha^2\beta^2+2\alpha\beta^3+8\alpha\beta+\beta^4+8\beta^2+3)}$

Index of dispersion(γ) $= \frac{\sigma^2}{\mu_1'} = \frac{\alpha^2\beta^2+2\alpha\beta^3+8\alpha\beta+\beta^4+8\beta^2+3}{\beta(\alpha\beta+\beta^2+1)(\alpha\beta+\beta^2+3)}$

2.2. Moment Generating Function of Sabur Distribution

Let us consider Y be a random variable follows the Sabur distribution then moment generating function of the distribution denoted by $M_Y(t)$ is given as

$$\begin{aligned} M_Y(t) = E(e^{ty}) &= \int_0^\infty e^{ty} f(y,\alpha,\beta)dy \\ &= \frac{\beta^2}{\alpha\beta+\beta^2+1}\int_0^\infty \left(\alpha+\beta+\frac{\beta}{2}y^2\right) e^{-(\beta-t)y}dy \\ &= \frac{\beta^2}{\alpha\beta+\beta^2+1}\left\{\frac{(\alpha+\beta)}{(\beta-t)}+\frac{\beta}{(\beta-t)^3}\right\} \\ &= \frac{1}{\alpha\beta+\beta^2+1}\left\{(\alpha\beta+\beta^2)\sum_{k=0}^{\infty}\left(\frac{t}{\beta}\right)^k+\sum_{k=0}^{\infty}\binom{k+2}{k}\left(\frac{t}{\beta}\right)^k\right\} \\ &= \sum_{k=0}^{\infty}\frac{2(\alpha\beta+\beta^2)+(k+1)(k+2)}{2(\alpha\beta+\beta^2+1)}\left(\frac{t}{\beta}\right)^k \end{aligned}$$

2.3. Mode and Median of Sabur Distribution

The value or number in a data set, which are occurring repeatedly may be termed as mode while median is the middle value or number in a data set arranged in ascending order.
Taking logarithm to the pdf of Sabur distribution,we get

$$\log f(y,\alpha,\beta) = 2\log\beta - \log(\alpha\beta+\beta^2+1) + \log\left(\alpha+\beta+\frac{\beta}{2}y^2\right) - \beta y$$

Differentiate w.r.t y ,we get

$$\frac{\partial \log f(y,\alpha,\beta)}{\partial y} = \frac{\beta y}{(\alpha\beta+\beta^2+1)} - \beta$$

Equating $\frac{\partial \log f(y,\alpha,\beta)}{\partial y} = 0$, we get

$$\frac{\beta y}{(\alpha\beta+\beta^2+1)} - \beta = 0 \Rightarrow y = \frac{1\pm\sqrt{1-2\beta(\alpha+\beta)}}{\beta}$$

$$M_0 = y_0 = \frac{1\pm\sqrt{1-2\beta(\alpha+\beta)}}{\beta}$$

Using the empirical formula, we obtain median as

$$\begin{aligned} Median &= \frac{1}{3}M_0 + \frac{2}{3}\mu \\ &= \frac{1\pm\sqrt{1-2\beta(\alpha+\beta)}}{3\beta} + \frac{2(\alpha\beta+\beta^2+3)}{3\beta(\alpha\beta+\beta^2+1)} \end{aligned}$$

3. Reliability Measures

Suppose Y be a continuous random variable with cdf (y) , $y \geq 0$ .then its reliability function which is also called survival function is defined as

$$S(y) = p_r(Y > y) = \int_0^\infty f(y)\,dy = 1 - F(y)$$

Therefore, the survival function is given

$$S(y,\alpha,\beta) = 1 - F(y,\alpha,\beta) = \left[1 + \frac{\beta^2 y^2 + 2\beta y}{2(\alpha\beta + \beta^2 + 1)}\right] e^{-\beta y} \quad (3.1)$$

The hazard function of a random variable y is given as

$$H(y,\alpha,\beta) = \frac{f(y,\alpha,\beta)}{S(y,\alpha,\beta)} \quad (3.2)$$

Using equation (1.1) and equation (3.1) in (3.2), we get

$$H(y,\alpha,\beta) = \frac{2\beta^2\left(\alpha + \beta + \frac{\beta}{2}y^2\right)}{[\beta^2 y^2 + 2\beta y + 2(\alpha\beta + \beta^2 + 1)]}$$

Also, the reverse hazard function denoted as $h_r(y,\alpha,\beta)$ can be obtained as

$$h_r(y,\alpha,\beta) = \frac{f(y,\alpha,\beta)}{F(y,\alpha,\beta)} \quad (3.3)$$

Using (1.1) and (1.2) in equation (3.3), we get

$$h_r(y,\alpha) = \frac{2\beta^2\left(\alpha + \beta + \frac{\beta}{2}y^2\right)e^{-\beta y}}{2(\alpha\beta + \beta^2 + 1) - [\beta^2 y^2 + 2\beta y + 2(\alpha\beta + \beta^2 + 1)]e^{-\beta y}}$$

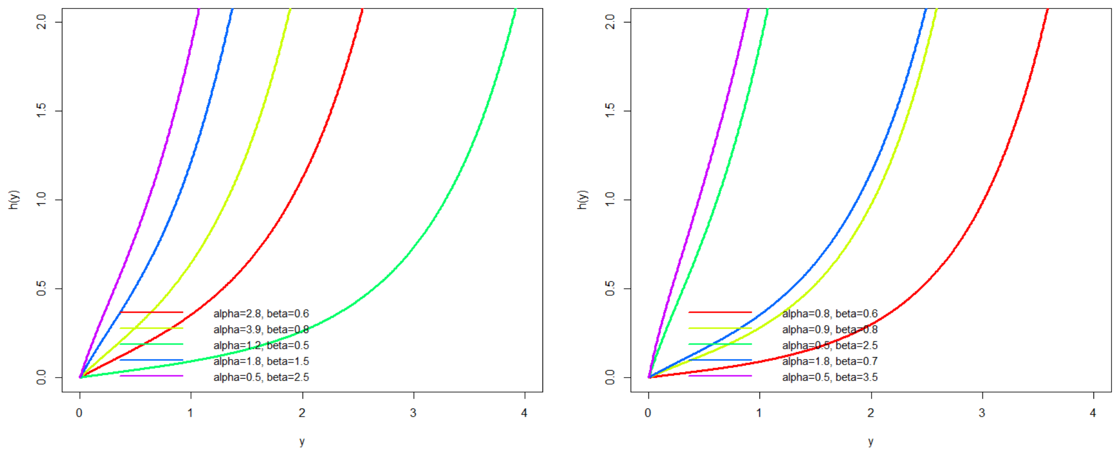

Fig 3: *The graph of hazard rate function of SD for different values of parameters.*

The mean residual function denoted by$m(y)$, and is defined as

$$m(y) = \frac{1}{1 - F(y)} \int_y^\infty 1 - F(z)\,dz$$

Therefore, the mean residual function of Sabur distribution is given by

$$m(y) = \frac{\beta^2 y^2 + 4y\beta + 2(\alpha\beta + \beta^2 + 3)}{\beta[\beta^2 y^2 + 2y\beta + 2(\alpha\beta + \beta^2 + 1)]}$$

We observe that $H(0) = f(0) = \frac{\alpha\beta^2}{\alpha\beta + \beta^2 + 1}$ and $m(0) = \mu = \frac{\alpha\beta + \beta^2 + 3}{\beta(\alpha\beta + \beta^2 + 1)}$

4. Order Statistics of Sabur Distribution

Let us consider $Y_1, Y_2 \ldots Y_n$ be random sample of sample size n from sabur distribution with pdf (1.1) and cdf (1.2).Then the pdf of k^{th} order statistics is given by

$$f_{Y_{(k)}}(y) = \frac{n!}{(k-1)!\,(n-k)!}\,[F(y)]^{k-1}\,[1-F(y)]^{n-k}\,f(y) \quad ,k = 1,2,3,\ldots,n \quad (4.1)$$

Now substituting the equation (1.1) and (1.2) in equation (4.1), we obtain the k^{th} order statistics as

$$f_{Y_{(k)}}(y) = \frac{n!\,\beta^2\left(\alpha+\beta+\frac{\beta}{2}y^2\right)e^{-\beta y}}{(k-1)!\,(n-k)!\,\left(\alpha\beta+\beta^2+1\right)}\left\{1-\left[1+\frac{\beta^2y^2+2\beta y}{2(\alpha\beta+\beta^2+1)}\right]e^{-\beta y}\right\}^{k-1}$$
$$\left\{\left[1+\frac{\beta^2y^2+2\beta y}{2(\alpha\beta+\beta^2+1)}\right]e^{-\beta y}\right\}^{n-k} \quad (4.2)$$

The pdf of first order statistics Y_1 is given as

$$f_{Y_{(1)}}(y) = \frac{n\beta^2\left(\alpha+\beta+\frac{\beta}{2}y^2\right)e^{-\beta y}}{(\alpha\beta+\beta^2+1)}\left\{\left[1+\frac{\beta^2y^2+2\beta y}{2(\alpha\beta+\beta^2+1)}\right]e^{-\beta y}\right\}^{n-1}$$

And the pdf of n^{th} order statistics Y_n is given as

$$f_{Y_{(n)}}(y) = \frac{n\beta^2\left(\alpha+\beta+\frac{\beta}{2}y^2\right)e^{-\beta y}}{(\alpha\beta+\beta^2+1)}\left\{1-\left[1+\frac{\beta^2y^2+2\beta y}{2(\alpha\beta+\beta^2+1)}\right]e^{-\beta y}\right\}^{n-1}$$

5. Renyi Entropy

If Y is a continuous random variable having probability density function $f(y,\alpha,\beta)$, then Renyi entropy is defined as

$$T_R(\delta) = \frac{1}{1-\delta}\,log\left\{\int_0^\infty f^{\delta}(y)dy\right\}$$

where $\delta > 0$ and $\delta \neq 1$

Thus, the Renyi entropy for Sabur distribution (1.1) , is given as

$$T_R(\delta) = \frac{1}{1-\delta}\,log\left\{\int_0^\infty\left[\frac{\beta^2}{\alpha\beta+\beta^2+1}\left(\alpha+\beta+\frac{\beta}{2}y^2\right)e^{-\beta y}\right]^{\delta}dy\right\}$$
$$= \frac{1}{1-\delta}\,log\left\{\frac{\beta^{2\delta}(\alpha+\beta)^{\delta}}{(\alpha\beta+\beta^2+1)^{\delta}}\int_0^\infty\left(1+\frac{\beta}{2(\alpha+\beta)}y^2\right)^{\delta}e^{-\beta\delta y}dy\right\}$$
$$= \frac{1}{1-\delta}\,log\left\{\frac{\beta^{2\delta}(\alpha+\beta)^{\delta}}{(\alpha\beta+\beta^2+1)^{\delta}}\int_0^\infty\sum_{r=0}^{\infty}\binom{\delta}{r}\left(\frac{\beta}{2(\alpha+\beta)}y^2\right)^{r}e^{-\beta\delta y}dy\right\}$$
$$= \frac{1}{1-\delta}\,log\left\{\sum_{r=0}^{\infty}\binom{\delta}{r}\frac{\beta^{2\delta+r}(\alpha+\beta)^{\delta-r}}{2^r(\alpha\beta+\beta^2+1)^{\delta}}\int_0^\infty y^{2r}e^{-\beta\delta y}dy\right\}$$
$$= \frac{1}{1-\delta}\,log\left\{\sum_{r=0}^{\infty}\binom{\delta}{r}\frac{\beta^{2\delta+r}\,(\alpha+\beta)^{\delta-r}}{2^r(\alpha\beta+\beta^2+1)^{\delta}}\,\frac{\Gamma(2r+1)}{(\beta\delta)^{2r+1}}\right\}$$
$$= \frac{1}{1-\delta}\,log\left\{\sum_{r=0}^{\infty}\binom{\delta}{r}\frac{\beta^{2\delta-(r+1)}\,(\alpha+\beta)^{\delta-r}}{2^{r-1}(\alpha\beta+\beta^2+1)^{\delta}}\,\frac{\mathrm{r}\,\Gamma(2r)}{\delta^{2r+1}}\right\}$$

6. Mean Deviation from Mean of Sabur Distribution

The quantity of scattering in a population is evidently measured to some extent by the totality of the deviations. Let Y be a random variable from Sabur distribution with mean μ then the mean deviation from mean is defined as.

$$D(\mu) = E(|Y-\mu|) = \int_0^\infty |Y-\mu|\,f(y)\,dy$$

$$= \int_0^{\mu} (\mu - y)\, f(y)\, dy + \int_{\mu}^{\infty} (y - \mu)\, f(y) dy$$
$$= \mu \int_0^{\mu} f(y)\, dy - \int_0^{\mu} y\, f(y)\, dy + \int_{\mu}^{\infty} yf(y)\, dy - \int_{\mu}^{\infty} \mu f(y)\, dy$$
$$= \mu F(\mu) - \int_0^{\mu} y\, f(y)\, dy - \mu[1 - F(\mu)] + \int_{\mu}^{\infty} yf(y)\, dx$$
$$= 2\mu F(\mu) - 2 \int_0^{\mu} yf(y)\, dy \tag{6.1}$$

Now

$$\int_0^{\mu} yf(y)\, dy = \frac{\beta^2}{\alpha\beta + \beta^2 + 1} \int_0^{\mu} y \left(\alpha + \beta + \frac{\beta}{2} y^2\right) e^{-\beta y}\, dy$$

After solving the integral, we get

$$\int_0^{\mu} yf(y)\, dy = \mu - \left\{ \frac{\mu^3\beta^3 + 3\mu^2\beta^2 + 6\mu\beta + \beta(\alpha + \beta)(\mu\beta + 1) + 6}{2\beta(\alpha\beta + \beta^2 + 1)} \right\} e^{-\beta\mu} \tag{6.2}$$

Now substituting equation (6.2) in equation (6.1), we get

$$D(\mu) = \frac{\{\mu^2\beta^2 + 6\mu\beta + \beta^2(1 - \mu\alpha - \mu\beta) + \beta(\alpha - 2\mu) + 6\}}{\beta(\alpha\beta + \beta^2 + 1)}\, e^{-\beta\mu}$$

7. Mean Deviation from Median of Sabur Distribution

Let Y be a random variable from Sabur distribution with median M then the mean deviation from median is defined as.

$$D(M) = E(|Y - M|) = \int_0^{M} (M - y)\, dy + \int_{M}^{\infty} (y - M)\, dy$$
$$= MF(M) - \int_0^{M} yf(x)\, dy - M[1 - F(M)] + \int_{M}^{\infty} yf(y) dy$$
$$= \mu - 2 \int_0^{M} yf(y)\, dy \tag{7.1}$$

Now

$$\int_0^{M} yf(y)\, dy = \frac{\beta^2}{\alpha\beta + \beta^2 + 1} \int_0^{M} y \left(\alpha + \beta + \frac{\beta}{2} y^2\right) e^{-\beta y}\, dy$$

After solving the integral, we get

$$\int_0^{M} yf(y)\, dy = \mu - \left\{ \frac{M^3\beta^3 + 3M^2\beta^2 + 6M\beta + \beta(M + \beta)(M\beta + 1) + 6}{2\beta(\alpha\beta + \beta^2 + 1)} \right\} e^{-\beta M} \tag{7.2}$$

Now substituting equation (7.2) in equation (7.1), we get

$$D(M) = \left\{ \frac{M^3\beta^3 + 3M^2\beta^2 + 6M\beta + \beta(M + \beta)(M\beta + 1) + 6}{\beta(\alpha\beta + \beta^2 + 1)} \right\} e^{-\beta M} - \mu$$

8. Bonferroni and Lorenz Curves

In economics the relation between poverty and economy is well studied by using Bonferroni and Lorenz curves. Besides that these curves have been used in different fields such as reliability, insurance and biomedicine.

The Bonferroni curve, $B(s)$is given as.

$$B(s) = \frac{1}{s\mu} \int_0^{t} y\, f(y)\, dy \tag{8.1}$$

Or

$$B(s) = \frac{1}{s\mu} \int_0^{s} F^{-1}(y) dy$$

And Lorenz curve, $L(s)$ is given as.

$$L(s) = \frac{1}{\mu}\int_0^t yf(y)\,dy \tag{8.2}$$

Or

$$L(s) = \frac{1}{\mu}\int_0^s F^{-1}(y)\,dy$$

Where $E(X) = \mu$ and $t = F^{-1}(s)$.

Now

$$\int_0^t yf(y)\,dy = \mu - \left\{\frac{t^3\beta^3 + 3t^2\beta^2 + 6t\beta + \beta(\alpha+\beta)(t\beta+1) + 6}{2\beta(\alpha\beta+\beta^2+1)}\right\}e^{-\beta t} \tag{8.3}$$

Substituting equation (8.3) in equations (8.1) and (8.2), we get

$$B(s) = \frac{1}{s}\left[1 - \left\{\frac{t^3\beta^3 + 3t^2\beta^2 + 6t\beta + \beta(\alpha+\beta)(t\beta+1) + 6}{2\beta(\alpha\beta+\beta^2+1)\mu}\right\}e^{-\beta t}\right]$$

And

$$L(s) = \left[1 - \left\{\frac{t^3\beta^3 + 3t^2\beta^2 + 6t\beta + \beta(\alpha+\beta)(t\beta+1) + 6}{2\beta(\alpha\beta+\beta^2+1)\mu}\right\}e^{-\beta t}\right]$$

9. Estimation of Parameters of Sabur Distribution

Suppose $Y_1, Y_2, Y_3, \dots Y_n$be random samples of size n from Sabur distribution. Then the likelihood function of Sabur distribution is given as.

$$l = \prod_{i=1}^{n} f(y_i, \alpha.\beta)$$

$$= \prod_{i=1}^{n}\left\{\frac{\beta^2}{\alpha\beta+\beta^2+1}\left(\alpha+\beta+\frac{\beta}{2}y^2\right)e^{-\beta y}\right\}$$

$$= \left(\frac{\beta^2}{\alpha\beta+\beta^2+1}\right)^n \prod_{i=1}^{n}\left(\alpha+\beta+\frac{\beta}{2}y_i^2\right)\; e^{-\beta\sum_{i=1}^{n} y_i}$$

The log likelihood function is given by

$$\log l = 2n\log\beta - n\log(\alpha\beta+\beta^2+1) + \sum_{i=1}^{n}\log\left(\alpha+\beta+\frac{\beta}{2}y_i^2\right) - \beta\sum_{i=1}^{n} y_i$$

Now, differentiating partially w. r. t parameters α and β respectively we get

$$\frac{\partial \log l}{\partial \alpha} = \frac{-n\beta}{\alpha\beta+\beta^2+1} + \sum_{i=1}^{n}\frac{1}{\alpha+\beta+\frac{\beta}{2}y_i^2}$$

$$\frac{\partial \log l}{\partial \beta} = \frac{2n}{\beta} - \frac{n(\alpha+2\beta)}{\alpha\beta+\beta^2+1} + \sum_{i=1}^{n}\frac{\left(1+\frac{y_i^2}{2}\right)}{\alpha+\beta+\frac{\beta}{2}y_i^2} - \sum_{i=1}^{n} y_i$$

Now solving $\frac{\partial \log l}{\partial \alpha} = 0\,, \frac{\partial \log l}{\partial \beta} = 0$, we get

$$\frac{-n\beta}{\alpha\beta+\beta^2+1} + \sum_{i=1}^{n}\frac{1}{\alpha+\beta+\frac{\beta}{2}y_i^2} = 0 \tag{9.1}$$

Also

$$\frac{2n}{\beta} - \frac{n(\alpha+2\beta)}{\alpha\beta+\beta^2+1} + \sum_{i=1}^{n}\frac{\left(1+\frac{y_i^2}{2}\right)}{\alpha+\beta+\frac{\beta}{2}y_i^2} - n\bar{y} = 0 \tag{9.2}$$

It is obvious that equations (9.1) and (9.2), are not in closed form, hence cannot be solved analytically for α and . In order to find the value of α and β it is imperative to apply iterative methods. The MLE of the parameters denoted as $\hat{\theta}(\hat{\alpha},\ \hat{\beta})$ of $\theta(\alpha,\beta)$ can be obtained by using Newton-Raphson method, bisection method, secant method etc.

Since the MLE of $\hat{\theta}$ follows asymptotically normal distribution which is given as

$$\sqrt{n}\left(\hat{\theta}-\theta\right) \to N\left(0, I^{-1}(\theta)\right) \tag{9.3}$$

Where $I^{-1}(\theta)$ is the limiting variance – covariance matrix of $\hat{\theta}$ and $I(\theta)$ is a 2×2 Fisher information matrix I,e

$$I(\theta) = -\frac{1}{n}\begin{bmatrix} E\left(\frac{\partial^2 \log l}{\partial^2 \alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial \alpha\, \partial \beta}\right) \\ E\left(\frac{\partial^2 \log l}{\partial \beta\, \partial \alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial^2 \beta}\right) \end{bmatrix}$$

Where

$$\frac{\partial^2 \log l}{\partial^2 \alpha} = \frac{n\beta^2}{(\alpha\beta + \beta^2 + 1)^2} - \sum_{i=1}^{n} \frac{1}{\left(\alpha + \beta + \frac{\beta}{2} y_i^2\right)^2}$$

$$\frac{\partial^2 \log l}{\partial^2 \beta} = \frac{-2n}{\beta^2} + \frac{n(\alpha^2 + 2\beta^2 + 2\alpha\beta)}{\alpha\beta + \beta^2 + 1} - \sum_{i=1}^{n} \frac{\left(1 + \frac{y_i^2}{2}\right)^2}{\left(\alpha + \beta + \frac{\beta}{2} y_i^2\right)^2}$$

$$\frac{\partial^2 \log l}{\partial \alpha \partial \beta} = \frac{\partial^2 \log l}{\partial \beta \partial \alpha} = \frac{n(\beta^2 - 1)}{\alpha\beta + \beta^2 + 1} - \sum_{i=1}^{n} \frac{\left(1 + \frac{y_i^2}{2}\right)}{\alpha + \beta + \frac{\beta}{2} y_i^2}$$

Hence the approximate $100(1-\psi)\%$ confidence interval for α and β are respectively given by

$$\hat{\alpha} \pm z_{\frac{\psi}{2}}\sqrt{I_{\alpha\alpha}^{-1}(\hat{\theta})} \;\; , \hat{\beta} \pm z_{\frac{\psi}{2}}\sqrt{I_{\beta\beta}^{-1}(\hat{\theta})}$$

Where $z_{\frac{\psi}{2}}$ is the ψ^{th} denotes percentile the standard distribution

10. Application

In this section, the importance and flexibility of the formulated distribution is illustrated by using a real life data set. And the distribution is compared with Lindley distribution (LD), Shanker Distribution (SHD), Exponential distribution (ED), inverse Lindley distribution (ILD) and Nadarajah-Haghighi distribution (HD). In order to compare the two distribution models, we consider the criteria like AIC (Akaike information criterion), CAIC (corrected Akaike information criterion) and BIC (Bayesian information criterion. The better distribution corresponds to lesser AIC, CAIC and BIC values.

Data: The data set is the strength data of glass of the aircraft window reported by Fuller *et al* [7]. The data are
18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.

From above Table 1, it has been observed that the Sabur distribution have the lesser AIC, CAIC, -logL and BIC values. Hence we can conclude that Sabur distribution leads to a better fit as compared to Lindley distribution (LD), Shanker Distribution (SHD), Exponential distribution (ED), inverse Lindley distribution (ILD) and Nadarajah-Haghighi distribution (HD)

Table 1: *MLE's, - ln L, AIC, CAIC and BIC of the fitted distributions of data sets*

Model	Parameter Estimates	S.E	-log L	AIC	CAIC	BIC
Sabur Distribution	$\boldsymbol{\alpha = 0.3501}$ $\boldsymbol{\beta = 0.0967}$	$\boldsymbol{\alpha = 0.0124}$ $\boldsymbol{\beta = 0.0080}$	**120.44**	**244.89**	**245.32**	**247.76**
LD	$\alpha = 0.0630$	$\alpha = 0.0412$	127.0	256.0	256.1	257.4
SD	$\alpha = 0.0647$	$\alpha = 0.0475$	126.15	254.3	254.5	255.8
ED	$\alpha = 0.0355$	$\alpha = 0.1507$	137.25	276.7	276.8	277.9
ILD	$\alpha = 30.153$	$\alpha = 5.2523$	137.24	276.49	276.63	277.92
NHD	$\alpha = 0.0026$ $\beta = 0.0008$	$\alpha = 9.8991$ $\beta = 3.2255$	128.59	261.19	264.06	261.62

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11. Concluding Remarks

In this paper, we introduce a new two-parameter lifetime distribution called the "Sabur distribution. We explore various mathematical properties of this distribution, including its shape, moments, hazard rate, mean residual life functions, mean deviations, and order statistics. Additionally, we derive expressions for the Bonferroni and Lorenz curves as well as the Renyi entropy measure for the proposed distribution. Furthermore, we discuss the method of maximum likelihood estimation for estimating the distribution's parameter. To demonstrate the practical utility and superiority of the Sabur distribution over existing alternatives such as the Shanker, Nadarajah-Haghighi, exponential, Lindley, and inverse Lindley distributions, we perform goodness-of-fit tests using criteria like the Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), and Bayesian Information Criterion (BIC) on real lifetime datasets.

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