Exponentiated Discrete Hypo Exponential Distribution and its Generalizations

Krishnakumari.K[1] and Dais George[2]

[1]Research Scholar, St.Thomas College, Palai
(SAS SNDP YOGAM College, Konni), Kerala, India.
[2]Catholicate College, Pathanamthitta, Kerala, India.
krishnavidyadharan@gmail.com, daissaji@rediffmail.com

Abstract

Generalizations of standard probability distributions is a thought-provoking concept in statistical literature and was inspired by many researchers in recent days. This is because the addition of parameters may increase the flexibility of the new models. Now a days various generalization techniques are available in literature. In this work, we proposed a generalization of discrete hypo exponential distribution and studied its various properties. A real data analysis is carried out and check the flexibility of the new model by comparing it with other standard distributions. Two generalizations of the proposed distribution are introduced.

Keywords: Discrete hypo exponential distribution, Estimation, Generalization, Moments, Stress-strength analysis.

1. Introduction

Over the last few decades, there has been growing interest in adding supplementary parameters to the baseline distributions to broaden generalized families of distributions. The addition of parameters may increase the flexibility of the new models. So generalization of the standard distributions are attracted by many researchers and are prominent in recent days. In literature there exists various generalization techniques and for a detailed review, see Tahir and Nadarajah[32]. These techniques resulted in the generalizations of various standard distributions. For details see, Gupta and Kundu [14], Eugene et.al [11], Zografos and Balakrishnan [33], Gomez-Deniz [13], Mahmoudi and Zakerzadeh [19], Cordeiro and Castro [4], Nadarajah [24], Nadarajah et.al [26], Cordeiro et al. [5], Ristic and Balakrishnan [30], Lemonte et.al [17], Liyanage and Pararai [18], Merovci and Elbatal [22], Merovci and Sharma [23], Nadarajah and Bakar [25], Ahmad and Ghazal [1], Sulami [2] etc. Recently exponentiated family of distributions due to Lehman [16] has got special attention and various standard distributions were generalized. The most prominent distribution introduced in the 20^{th} century is the exponentiated exponential distribution and inspired by this many existing distributions were generalized and for details see, Pal et al. [28], Nekoukhou and Bidram [27], Morshedy et al. [8], El-Bassiouny et al. [7], Morshedy et al.[10], Morshedy et al. [9], Mashhadzadeh and Mirmostafaee [21] and Baharith and Alamoudi [3].
The layout of this article is in this way. In Section 2, we introduced exponentiated discrete hypo exponential distribution and studied its various properties. In Section 3 the parameters of the distribution is done through non linear maximization method. To evaluate the performance of the nlm estimator a simulation study is done in Section 4. A real data analysis is done in Section 5. In Section 6 some generalizations of the proposed distribution are introduced. Some concluding remarks are recorded in Section 7.

2. Exponentiated Discrete Hypo Exponential Distribution

Consider the discrete hypo exponential (DHE) distribution having model parameters ϕ_1, $\phi_2 > 0$, $\phi_1 \neq \phi_2$ with the distribution function

$$F(x;\phi_1,\phi_2) = \frac{\phi_2}{\phi_2-\phi_1}(1-e^{-\phi_1 x}) - \frac{\phi_1}{\phi_2-\phi_1}(1-e^{-\phi_2 x}) \tag{1}$$

By inserting (1) into the resilience parameter family of distributions, the distribution function of the resulting distribution is given by

$$\begin{aligned} G(x;\phi_1,\phi_2,\alpha) &= [F(x;\phi_1,\phi_2)]^{\alpha} \\ &= \frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}} \end{aligned} \tag{2}$$

where

$$V(x;\phi_1,\phi_2,\alpha) = [\phi_2(1-e^{-\phi_1 x}) - \phi_1(1-e^{-\phi_2 x})]^{\alpha} \tag{3}$$

We call such a random variable X, having distribution function (2), is an exponentiated DHE distribution with parameters ϕ_1, $\phi_2 > 0$, $\phi_1 \neq \phi_2$, $\alpha > 0$ and denote it as EDHE (ϕ_1, ϕ_2, α).
The probability mass function(pmf) of EDHE distribution is given by

$$\begin{aligned} P(X=x) &= v(x;\phi_1,\phi_2,\alpha) \\ &= \frac{V(x+1;\phi_1,\phi_2,\alpha) - V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}; x = 0,1,2,.... \end{aligned} \tag{4}$$

The plots of pmf of EDHE distribution is given in Figure 1.

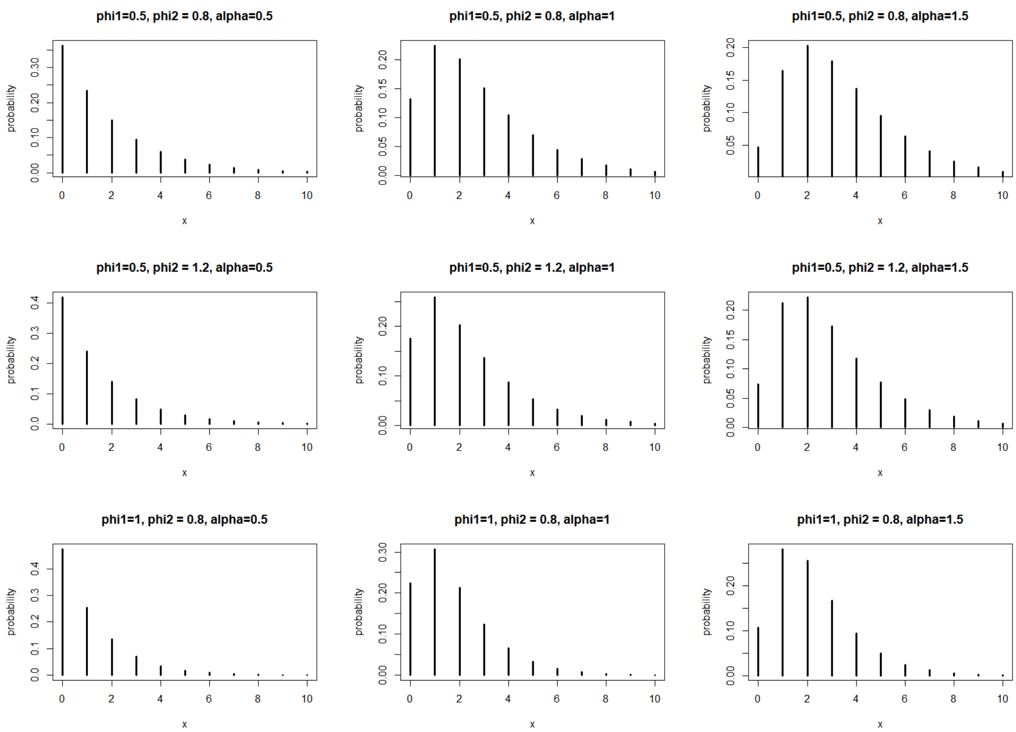

Figure 1: *Plots of pmf of EDHE distribution*

From Figure 1 it is understood that the EDHE distribution is unimodel. Since every log-concave density is unimodel, it is also inferred that EDHE distribution is log-concave.

2.1. Reliability characteristics

$$\text{Survival function}, S(x) = 1 - \frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^\alpha}\ ; x = 0,1,2..., \quad \text{and}$$

$$\text{hazard rate}, r(x) = \frac{V(x+1;\phi_1,\phi_2,\alpha) - V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^\alpha - V(x;\phi_1,\phi_2,\alpha)}$$

The plots of hazard rate of EDHE distribution is given in Figure 2.

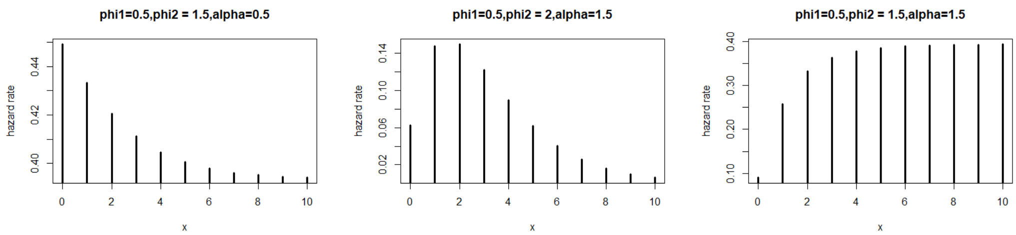

Figure 2: *Plots of hazard rate of EDHE distribution.*

From Figure 2, it is evident that for various model parameters, the hazard rate functions can be decreasing, increasing and increasing-decreasing, which makes the EDHE distribution more flexible and can model different types of data sets such as count data, failure time data etc.

2.2. Moments

Let X $\sim$ EDHE(ϕ_1,ϕ_2,α), then for $n \geq 1$,

$$E(X^n) = \sum_{x=0}^{\infty} x^n \left[\frac{V(x+1;\phi_1,\phi_2,\alpha) - V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^\alpha}\right].$$

In particular

$$E(X) = \sum_{x=0}^{\infty} x \left[\frac{V(x+1;\phi_1,\phi_2,\alpha) - V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^\alpha}\right].$$

and

$$E(X^2) = \sum_{x=0}^{\infty} x^2 \left[\frac{V(x+1;\phi_1,\phi_2,\alpha) - V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^\alpha}\right].$$

The expression for V(X) can be obtained using he relation

$$V(X) = E(X^2) - [E(X)]^2$$

2.3. Infinite Divisibility

According to Steutel and Van Harn [31], a necessary condition for infinite divisibility of a discrete distribution P_y is that $P_0 > 0$. For EDHE distribution this condition is satisfied for all values of the parameters. Hence it is infinitely divisible.

2.4. Theorem

If X follows an exponentiated hypo exponential distribution with parameters ϕ_1 , ϕ_2 and α then the random variable W=[X] follows a exponentiated discrete hypo exponential distribution with parameters ϕ_1, ϕ_2 and α .

Proof:
Consider w=0, 1, 2,... then using Lemma1 of Krishna and Pundir (2009), we have

$$\begin{aligned} P(W \geq w) &= P([X] \geq w) \\ &= P(X \geq w) \\ &= 1 - \left[\frac{\phi_2(1-e^{-\phi_1 x}) - \phi_1(1-e^{-\phi_2 x})}{\phi_2 - \phi_1}\right]^{\alpha} \end{aligned}$$

which is the survival function of EDHE distribution and hence the theorem.

2.5. Order Statistics

Order statistics are sample values placed in ascending order. It is a very useful concept in statistical sciences. It has a far reaching applications especially in modeling auctions, car races, insurance policies and estimating parameters of distributions etc.
Let $X_{(1:n)} \leq X_{(2:n)} \leq X_{(3:n)} \leq ... \leq X_{(n:n)}$ represents the order statistics obtained from the i.i.d. EDHE(ϕ_1, ϕ_2, α) distribution of size n. Then probability mass function of first order statistics is given by

$$f_{X_{(1:n)}}(x) = \left[1 - \frac{V(x-1;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\right]^n - \left[1 - \frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\right]^n$$

and the distribution function is

$$F_{X_{(1:n)}}(x) = 1 - \left[1 - \frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\right]^n.$$

The probability mass function of n^{th} order statistics is given by

$$f_{X_{(n:n)}}(x) = \left[\frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\right]^n - \left[\frac{V(x-1;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\right]^n$$

and the distribution function is

$$F_{X_{(n:n)}}(x) = \left[\frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\right]^n$$

where $V(x;\phi_1,\phi_2,\alpha)$ is given by (3).

2.6. Entropy

The Shannon's entropy of random variable X having probability mass function P(x) is given by

$$H(X) = E(-logP(x)).$$

For EDHE distribution, H(X) is obtained as

$$H(X) = -\sum_{x=0}^{\infty}\left[\frac{V(x+1;\phi_1,\phi_2,\alpha) - V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\right] log \left[\frac{V(x+1;\phi_1,\phi_2,\alpha) - V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\right].$$

Renyi's entropy of order $\beta > 0$ $(\beta \neq 1)$ is

$$H_{\beta}(p) = \frac{1}{1-\beta} log \sum_{x=0}^{\infty}\left[\frac{V(x+1;\phi_1,\phi_2,\alpha) - V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\right]^{\beta}.$$

2.7. Stress-strength Analysis

Stress-strength analysis is a mechanism and a topic used in reliability engineering. When the probability of stress exceeding the strength of an item, that item fails. Hence the expected reliability (R) is calculated as

$$R = P(stress \leq strength) = \sum_{x=0}^{\infty} f_{stress}(x) R_{strength}(x)$$

where the strength and stress are in the positive domain. When stress $\sim$ EDHE$(\phi_1, \phi_2, \alpha_1)$ and strength$\sim$ EDHE$(\phi_3, \phi_4, \alpha_2)$, the expected reliability is

$$R = \sum_{x=0}^{\infty} \frac{V(x+1;\phi_1,\phi_2,\alpha_1) - V(x;\phi_1,\phi_2,\alpha_2)}{(\phi_2-\phi_1)^{\alpha_1}} \left[1 - \frac{V(x;\phi_3,\phi_4,\alpha_2)}{(\phi_4-\phi_3)^{\alpha_2}}\right].$$

Tables 1-4 show the numerical values of R for different values of stress-strength parameters.

Table 1: *Values of R for* $\phi_1 = 0.1$, $\phi_2 = 0.3$, $\phi_3 = 0.3$, $\phi_4 = 0.6$ *and different values of* α_1 *and* α_2

	α_2					
α_1	0.2	0.6	1	1.5	2	2.5
0.2	0.5185	0.6342	0.6962	0.7409	0.7685	0.7874
0.6	0.1707	0.2952	0.3708	0.4319	0.4735	0.5040
1	0.0735	0.1588	0.2166	0.2676	0.3050	0.3338
1.5	0.0336	0.0834	0.1207	0.1564	0.1843	0.2068
2	0.0181	0.0479	0.0718	0.0960	0.1157	0.1321

Table 2: *Values of R for* $\phi_1 = 0.3$, $\phi_2 = 0.5$, $\phi_3 = 0.6$, $\phi_4 = 0.8$ *and different values of* α_1 *and* α_2

	α_2					
α_1	0.2	0.6	1	1.5	2	2.5
0.2	0.6338	0.7240	0.7777	0.8185	0.8441	0.8617
0.6	0.2799	0.4162	0.5027	0.5733	0.6210	0.6557
1	0.1430	0.2657	0.3487	0.4210	0.4730	0.5127
1.5	0.0753	0.1690	0.2373	0.3011	0.3499	0.3890
2	0.0469	0.1169	0.1712	0.2249	0.2680	0.3037

Table 3: *Values of R for* $\phi_1 = 0.5$, $\phi_2 = 0.8$, $\phi_3 = 0.5$, $\phi_4 = 0.8$ *and different values of* α_1 *and* α_2

	α_2					
α_1	0.2	0.6	1	1.5	2	2.5
0.2	0.7372	0.8258	0.8748	0.9092	0.9293	0.9421
0.6	0.4272	0.5967	0.6952	0.7686	0.8138	0.8442
1	0.2720	0.4598	0.5747	0.6649	0.7235	0.7646
1.5	0.1759	0.3550	0.4713	0.5685	0.6352	0.6838
2	0.1277	0.2886	0.3991	0.4962	0..5658	0.6183

Table 4: *Values of R for α_1=$\alpha_2 = 0.6$, and different values of ϕ_1, ϕ_2, ϕ_3, ϕ_4*

$\phi_2 = \phi_4$

$\phi_1 = \phi_3$	0.6	0.7	0.8	1	1.5
0.1	0.4940	0.4974	0.50000	0.5037	0.5091
0.2	0.5437	0.5470	0.54972	0.5540	0.5611
0.3	0.5627	0.5668	0.5704	0.5761	0.5863
0.4	0.5751	0.5803	0.5848	0.5923	0.6058
0.5	0.5850	0.5912	0.5967	0.6059	0.6226

From tables 1-3 it is clear that for fixed values of ϕ_1, ϕ_2, ϕ_3, ϕ_4 and α_1 reliability increases as α_2 tends to infinity. But the reliability decreases with α_1 tends to infinity for fixed values of ϕ_1, ϕ_2, ϕ_3, ϕ_4 and α_2 . Table 4 shows that reliability increases with increasing values of ϕ_1, ϕ_3 for fixed values of ϕ_2, ϕ_4, α_1 and α_2. Also for fixed values of ϕ_1, ϕ_3, α_1 and α_2, reliability increases with increasing values of ϕ_2 , ϕ_4.

2.8. Estimation

In this section we estimate the parameters ϕ_1, ϕ_2 and α of EDHE distribution using the method of maximum likelihood. Let us take a random sample$X_1, X_2 ... X_n$ of size n from EDHE distribution. Then the logarithm of likelihood function is

$$logL = \sum_{x=0}^{\infty} log \left[\frac{V(x+1; \phi_1, \phi_2, \alpha) - V(x; \phi_1, \phi_2, \alpha)}{(\phi_2 - \phi_1)^{\alpha}} \right]$$

The maximum likelihood estimators of ϕ_1, ϕ_2 and α are obtained by solving the equations

$$\frac{\partial logL}{\partial \phi_1} = 0, \frac{\partial logL}{\partial \phi_2} = 0, \frac{\partial logL}{\partial \alpha} = 0.$$

But these equations cannot be solved analytically. So we use Non Linear Maximization (nlm) method for estimating the parameters ϕ_1, ϕ_2 and α.

3. Simulation Study

In this section, we use Monte-Carlo simulation method to illustrate the performance of the nlm estimator of the parameters ϕ_1 and ϕ_2 and α. We generate 5000 random samples of sizes n=20, 30, 75 and 100 from the $HE(\phi_1, \phi_2)$ distribution for some true values of the parameter set (ϕ_1, ϕ_2) = (15,18), (15,21), (16,18) and (16,21). We discretize the generated data and find out 5000 estimates of ϕ_1 and ϕ_2 and α using (4)for each sample sizes. The estimate of the parameter, average bias and mean square error of the estimate (MSE) are computed and it is given in Table 5 to Table 12.

Table 5: *Values of estimates, average bias and average MSE for ϕ_1=15 and different values of, ϕ_2, α and n=20.*

ϕ_1	ϕ_2	α	$\hat{\phi_1}$	Bias(ϕ_1)	MSE(ϕ_1)	$\hat{\phi_2}$	Bias(ϕ_2)	MSE(ϕ_2)	$\hat{\alpha}$	Bias(α)	MSE(α)
15	18	0.4	14.5668	-0.4331	0.1876	17.8307	-0.1693	0.0287	0.3775	-0.0224	0.00005
		0.8	14.4120	-0.5879	0.3457	17.7375	-0.2624	0.0689	0.7133	-0.0866	0.0075
15	21	0.4	14.8996	-0.1003	0.0101	20.3668	-0.6331	0.4009	0.3370	-0.0629	0.004
		0.8	14.3655	-0.6344	0.4025	20.6269	-0.3730	0.1392	0.6763	-0.1236	0.0153

Table 6: *Values of estimates, average bias and average MSE for ϕ_1=15 and different values of ϕ_2, α and n=30.*

ϕ_1	ϕ_2	α	$\hat{\phi_1}$	Bias(ϕ_1)	MSE(ϕ_1)	$\hat{\phi_2}$	Bias(ϕ_2)	MSE(ϕ_2)	$\hat{\alpha}$	Bias(α)	MSE(α)
15	18	0.4	14.5689	-0.4310	0.1858	17.8316	-0.1683	0.0283	0.3807	-0.0192	0.00004
		0.8	14.5509	-0.4490	0.2016	17.5699	-0.4300	0.185	0.7356	-0.0643	0.0041
15	21	0.4	14.9042	-0.0957	0.0092	20.6514	-0.3485	0.1215	0.3487	-0.0512	0.0026
		0.8	14.5340	-0.4659	0.2171	20.6291	-0.3708	0.1375	0.6883	-0.1116	0.0125

Table 7: *Values of estimates, average bias and average MSE for ϕ_1=15 and different values of ϕ_2, α and n=75.*

ϕ_1	ϕ_2	α	$\hat{\phi_1}$	Bias(ϕ_1)	MSE(ϕ_1)	$\hat{\phi_2}$	Bias(ϕ_2)	MSE(ϕ_2)	$\hat{\alpha}$	Bias(α)	MSE(α)
15	18	0.4	14.6128	-0.3871	0.1499	17.8372	-0.1627	0.0265	0.4090	0.0099	0.00001
		0.8	14.4120	-0.5879	0.3457	17.7375	-0.2624	0.0689	0.7133	-0.0866	0.0075
15	21	0.4	14.9614	-0.0380	0.0014	20.8440	-0.1550	0.0241	0.3502	-0.0497	0.0024
		0.8	14.6453	-0.3546	0.1258	20.6317	-0.3683	0.1356	0.6979	-0.1020	0.0104

Table 8: *Values of estimates, average bias and average MSE for ϕ_1=15 and different values of ϕ_2, α and n=100.*

ϕ_1	ϕ_2	α	$\hat{\phi_1}$	Bias(ϕ_1)	MSE(ϕ_1)	$\hat{\phi_2}$	Bias(ϕ_2)	MSE(ϕ_2)	$\hat{\alpha}$	Bias(α)	MSE(α)
15	18	0.4	14.6154	-0.3845	0.1478	17.8376	-0.1623	0.0263	0.4089	0.0089	0.00001
		0.8	14.7504	-0.2494	0.0622	17.8349	-0.2494	0.0273	0.7988	-0.0011	0.0000
15	21	0.4	15.0249	0.0249	0.00006	20.8483	-0.1516	0.0230	0.37770	-0.0226	0.00005
		0.8	15.0320	0.0320	0.0010	20.65213	-0.3478	0.1210	0.7445	-0.0554	0.0031

Table 9: *Values of estimates, average bias and average MSE for ϕ_1=16 and different values of ϕ_2, α and n=20.*

ϕ_1	ϕ_2	α	$\hat{\phi_1}$	Bias(ϕ_1)	MSE(ϕ_1)	$\hat{\phi_2}$	Bias(ϕ_2)	MSE(ϕ_2)	$\hat{\alpha}$	Bias(α)	MSE(α)
16	18	0.4	15.4568	-0.5431	0.2950	17.8664	-0.1335	0.0178	0.3409	-0.0590	0.0035
		0.8	15.6653	-0.3346	0.1120	17.4611	-0.5388	0.2903	0.6452	-0.1547	0.0240
16	21	0.4	15.1495	-0.8504	0.7233	20.4267	-0.5732	0.3286	0.3003	-0.0996	0.0099
		0.8	15.7852	-0.2147	0.0461	20.4893	-0.5101	0.2603	0.6690	-0.1309	0.0172

Table 10: *Values of estimates, average bias and average MSE for ϕ_1=16 and different values of ϕ_2, α and n=30.*

ϕ_1	ϕ_2	α	$\hat{\phi_1}$	Bias(ϕ_1)	MSE(ϕ_1)	$\hat{\phi_2}$	Bias(ϕ_2)	MSE(ϕ_2)	$\hat{\alpha}$	Bias(α)	MSE(α)
16	18	0.4	15.4593	-0.5406	0.2923	17.8662	-0.1337	0.0179	0.3452	-0.0547	0.0030
		0.8	15.8042	-0.1957	0.0383	17.4674	-0.5325	0.2836	0.6708	-0.1291	0.0167
16	21	0.4	15.1507	-0.8492	0.7212	20.4282	-0.5717	0.3269	0.3008	-0.0991	0.0098
		0.8	15.8184	-0.1815	0.0330	20.4918	-0.5081	0.2583	0.7341	-0.0658	0.0043

Table 11: *Values of estimates, average bias and average MSE for ϕ_1=16 and different values of ϕ_2, α and n=75.*

ϕ_1	ϕ_2	α	$\hat{\phi_1}$	Bias(ϕ_1)	MSE(ϕ_1)	$\hat{\phi_2}$	Bias(ϕ_2)	MSE(ϕ_2)	$\hat{\alpha}$	Bias(α)	MSE(α)
16	18	0.4	15.4822	-0.5177	0.2681	17.8708	-0.1291	0.0167	0.3791	-0.0208	0.00004
		0.8	15.8212	-0.1787	0.0319	17.5036	-0.4963	0.2463	0.7450	-0.0549	0.0030
16	21	0.4	15.1631	-0.8368	0.7002	20.4289	-0.5710	0.3261	0.3171	-0.08281	0.0069
		0.8	15.9054	-0.0945	0.0089	20.5039	-0.4961	0.2461	0.7946	0.0053	0.0012

Table 12: *Values of estimates, average bias and average MSE for ϕ_1=16 and different values of ϕ_2, α and n=100.*

ϕ_1	ϕ_2	α	$\hat{\phi_1}$	Bias(ϕ_1)	MSE(ϕ_1)	$\hat{\phi_2}$	Bias(ϕ_2)	MSE(ϕ_2)	$\hat{\alpha}$	Bias(α)	MSE(α)
16	18	0.4	15.4951	-0.5048	0.2548	17.8731	-0.1268	0.0161	0.3989	-0.0010	0.0000
		0.8	15.9119	-0.0880	0.0078	17.9695	-0.0304	0.00009	0.7942	-0.0057	0.0000
16	21	0.4	15.1645	-0.8354	0.6979	20.4290	-0.5710	0.3261	0.3183	-0.0816	0.0067
		0.8	15.9066	-0.0933	0.0087	20.5044	-0.4956	0.2456	0.8010	0.0010	0.0000

From tables 5-12, it is clear that as sample size increases, the average bias and average MSE becomes very small for different choices of the values of the parameters. This indicates the consistency of the estimators.

4. Real Data Analysis

For studying the efficiency of EDHE distribution we consider the data set used by Krishna and Pundir [15] and it represents the total number of carious teeth among the four deciduous molars in a sample of 100 children 10 and 11 years of old. The data are given in Table 13.

Table 13: *Observed data*

X	0	1	2	3	4
f	64	17	10	6	3

Figure 3 shows the observed data.

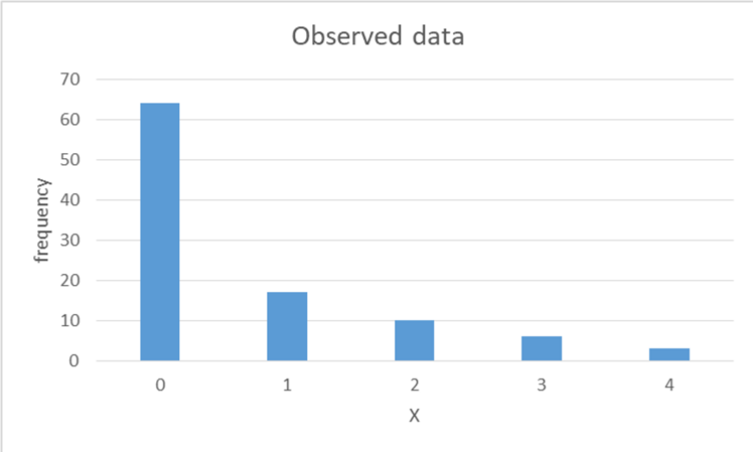

Figure 3: *Observed data.*

We fit the EDHE distribution using the empirical data set and the embeded figure is given in Figure 4.

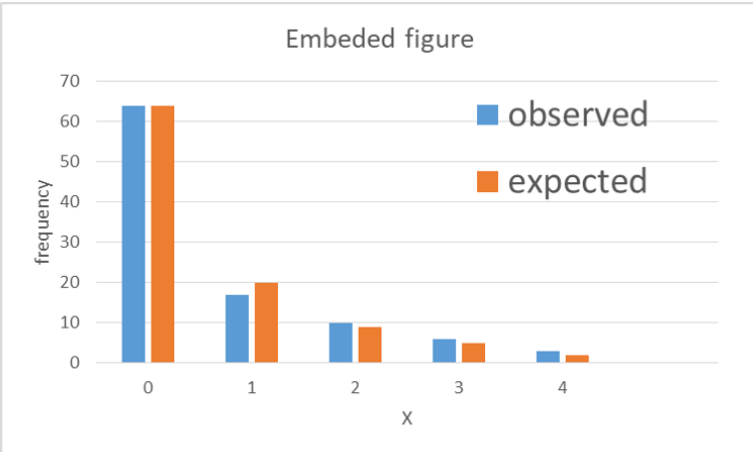

Figure 4: *Embeded figure.*

In order to assess the suitability of the proposed model, we use chi-square test of goodness of fit. Also, we compare the EDHE distribution with discrete Lindley (DL) distribution discrete Pareto (DP) distribution and the values of Log-likelihood, AIC, BIC are computed and is shown in Table 14.

Table 14: *MLE's, Chi-square value, -Log-likelihood value, AIC values, BIC values and P values for the observed data.*

Distribution fitted	estimators	Chi-square	-LL value	AIC value	BIC value	p value
DLD	$\hat{\theta} = 0.275$	6.637	113.68	229.36	229.36	0.036
DPD	$\hat{\beta} = 0.1837$	3.226	116.83	235.66	235.66	0.199
EDHED	$\hat{\phi}_1 = 0.9824779$ $\hat{\phi}_2 = 0.9824794$ $\hat{\alpha} = 0.3346$	1.2611	111.54	229.08	229.08	0.8679

From Table 14, it is inferred that the EDHE distribution is a better fit than discrete Lindley and discrete Pareto distributions.

5. Generalizations

5.1. Transmuted exponentiated discrete hypo exponential (TEDHE) distribution

Many transmuted distributions are proposed and studied in literature. For details see Rahman et al. [29] and Dey et al. [6]. In this section we present a generalization of (4) called the transmuted exponentiated discrete hypo exponential distribution. A random variable X is said to have transmuted distribution if its distribution function and probability mass functions are respectively given by

$$F(x) = G(x)[1 + \beta - \beta G(x)]; \|\beta| \leq 1 \tag{5}$$

and

$$P(X = x) = g(x)[1 + \beta - 2\beta G(x)] \tag{6}$$

where G(x), g(x) are the distribution function and probability mass function of the baseline distribution. Also if $\beta = 0$, we will get the baseline distribution. By using equations (5) and (6), the distribution function and probability mass function of the TEDHE distribution is obtained as

$$F(x;\phi_1,\phi_2,\alpha,\beta) = \frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\left[1+\beta-\beta\left(\frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\right)\right] \tag{7}$$

and

$$f(x;\phi_1,\phi_2,\alpha,\beta) = \frac{V(x+1;\phi_1,\phi_2,\alpha)-V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\left[1+\beta-2\beta\left(\frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\right)\right] \tag{8}$$

The plot of pmf of TEDHE distribution is given in Figure 5.

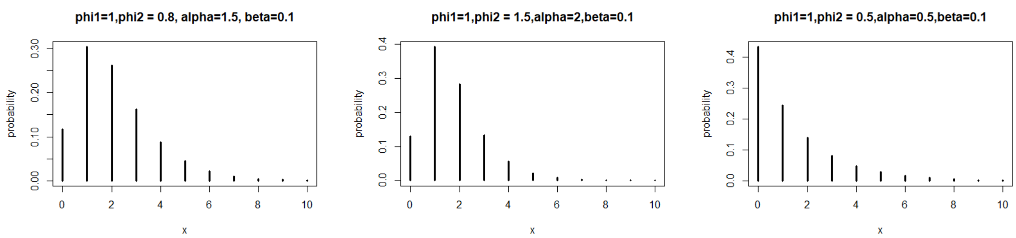

Figure 5: *Plot of pmf of TEDHE distribution.*

The survival function and hazard rate functions are given by the expressions

$$S(x) = 1 - \frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\left[1+\beta-\beta\left(\frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\right)\right]$$

and

$$h(x) = \frac{\frac{V(x+1;\phi_1,\phi_2,\alpha)-V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\left[1+\beta-2\beta\left(\frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\right)\right]}{1-\frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\left[1+\beta-\beta\left(\frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^{\alpha}}\right)\right]}$$

The hazard plots of TEDHE distribution is given in Figure 6.

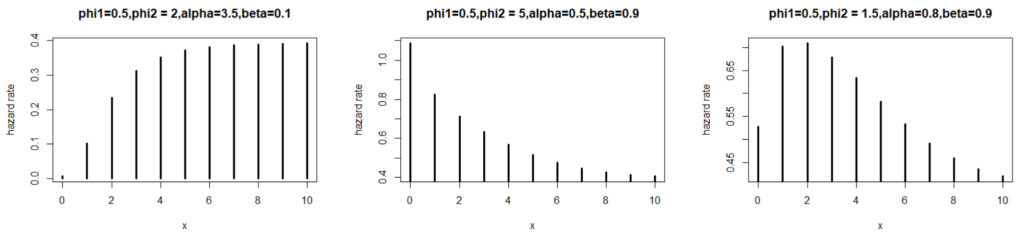

Figure 6: *Plot of hazard function of TEDHE distribution.*

From Figure 6, it is understood that for different model parameters, the hazard rate function can be decreasing, increasing and increasing-decreasing, which makes the TEDHE distribution more flexible and can model different types of data sets.

5.2. Marshall-Olkin exponentiated discrete hypo exponential (MOEDHE) distribution

Marshall and Olkin [20] introduced a new method for adding a parameter $\theta(>0)$ to the baseline distribution in order to generalize it. Using this method many generalized distributions are proposed and for a detailed review see Gillariose et al. [12]. If $\overline{F}(x)$ is the survival function of a distribution, then, by Marshall-Olkin method, another survival function $\overline{G}(x)$ is obtained as

$$\overline{G}(x,\theta) = \frac{\theta\overline{F}(x)}{1-(1-\theta)\overline{F}(x)}; -\infty < X < \infty, \theta > 0.$$

The corresponding distribution function, probability mass function and hazard rate is obtained as

$$\begin{aligned} G(x,\theta) &= \frac{F(x)}{1-(1-\theta)\overline{F}(x)} \\ g(x,\theta) &= G(x,\theta) - G(x-1,\theta) \\ &= \frac{\theta f(x)}{[1-(1-\theta)\overline{F}(x)][1-(1-\theta)\overline{F}(x-1)]} \\ h(x) &= \frac{g(x)}{\overline{G}(x)}. \end{aligned}$$

where f(x) is the probability mass function corresponding to the distribution function F(x).
Using Marshall-Olkin method the survival function of MOEDHE distribution is

$$\overline{G}(x,\theta) = \frac{\theta(1-\frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^\alpha})}{1-[(1-\theta)(1-\frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^\alpha})]}.$$

The corresponding distribution function, probability mass function and hazard rate are respectively given by

$$\begin{aligned} G(x,\theta) &= \frac{\frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^\alpha}}{1-[(1-\theta)(1-\frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^\alpha})]} \\ g(x,\theta) &= \frac{\theta\frac{V(x+1;\phi_1,\phi_2,\alpha)-V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^\alpha}}{(1-[(1-\theta)(1-\frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^\alpha})])(1-[(1-\theta)(1-\frac{V(x-1;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^\alpha})])} \\ \text{and} & \\ h(x) &= \frac{\frac{V(x+1;\phi_1,\phi_2,\alpha)-V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^\alpha}}{(1-[(1-\theta)(1-\frac{V(x-1;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^\alpha})])(1-\frac{V(x;\phi_1,\phi_2,\alpha)}{(\phi_2-\phi_1)^\alpha})}. \end{aligned}$$

The plot of probability mass function of MOEDHE distribution is given in Figure 7.

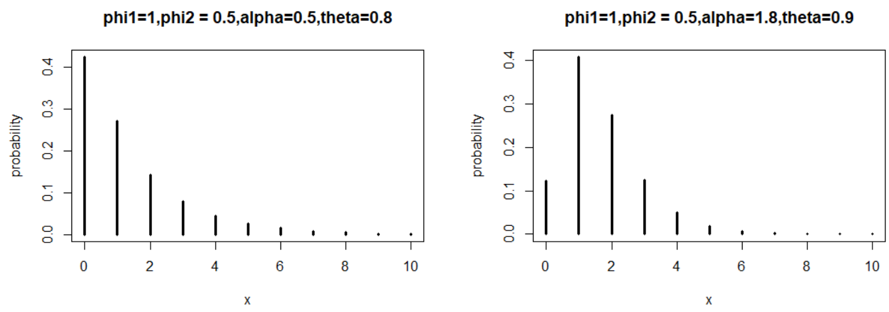

Figure 7: *Plot of pmf of MOEDHE distribution.*

The hazard plots are given in Figure 8.

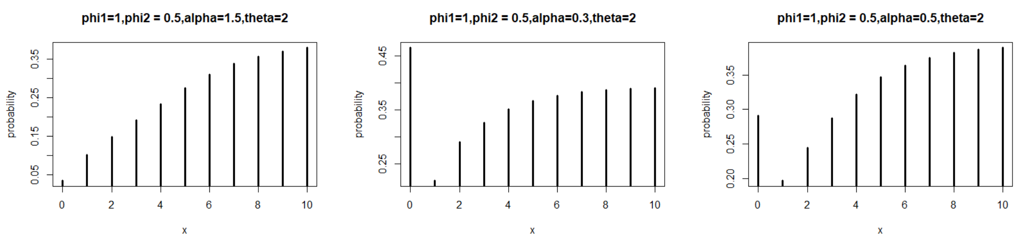

Figure 8: *Hazard plots of MOEDHE distribution.*

Figure 8 shows different shapes of hazard rate functions and so we can conclude that the MOEDHE distribution is a flexible model in modeling different types of data sets.

6. Summary

Recently, there has been thriving interest in developing new families of distributions by adding one or more additional parameters to the baseline distributions. The existence of various generalization techniques were attracted by many researchers and using one among them we proposed and studied a new distribution called exponentiated discrete hypo exponential distribution. Various distributional and structural properties of this distribution are studied. Also stress-strength analysis is carried out. To evaluate the performance of the nlm estimator, we conducted a simulation study and found that the nlm estimator is consistent. A real data application is carried out and inferred that our proposed distribution is better model than discrete Lindley and discrete Pareto distribution. Two generalizations of the proposed distribution namely transmuted exponentiated discrete hypo exponential distribution and Marshall-Olkin exponentiated discrete hypo exponential distribution are introduced.

References

[1] Ahmad, A. A. and Ghazal, M.G. M. (2020). Exponentiated additive Weibull distribution. *Reliability Engineering and System Safety*, 193, 106663.

[2] Al-Sulami, D. (2020). Exponentiated Exponential Weibull Distribution. *American Journal of Applied Sciences*, **17**, 188-195.

[3] Baharith, L. A and Alamoudi, H. H. (2021). The exponentated Frechet Generator of distributions with applications. *Symmetry*, **13(4)**, 572.

[4] Cordeiro, G. M and Castro, M. (2011). A new family of generalized distributions. *J.Stat.Comput.Simul.*, **81**, 883-898.

[5] Cordeiro, G. M. ,Ortega, E. M. M and Silva, G. (2012). The Beta extended Weibull family. *J.Prob.Stat.Sci.*, **10**, 15-40.

[6] Dey, S., Kumar, D.,Anis, M. Z. and Nadarajah, S. (2021). A review of transmuted distributions. *Journal of the Indian Society for Probability and Statistics*, **22**, 47-111.

[7] El-Bassiouny, A. H., EL-Damcese, M., Abdelfattah, M. and Eliwa, M. S. (2017). Exponentiated generalized Weibull-Gompertz distribution with application in survival analysis. *Journal of Statistics Applications and Probability*, **6**, 7-16.

[8] El-Morshedy, M., El-Bassiouny, A. H. and El-Gohary, A. (2017). Exponentiated Inverse flexible Weibull extension distribution. *Journal of Statistics Applications and Probability*, **6**, 169-183.

[9] El-Morshedy, M., Eliwa, M. S., El-Gohary, A and Khalil, A. A. (2020). Bivariate exponentiated discrete Weibull distribution: Statistical Properties,estimation, Simulation and Applications. *Mathmatical Sciences* , **14(1)**.

[10] El-Morshedy, M., Eliwa, M. S. and Nagy, H. (2019). A new two-parameter exponentiated discrete Lindley distribution: Properties,estimation and Applications. *Journal of Applied Statistics* .

[11] Eugene, N., Lee, C and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and Methods*, **31**, 497-512.

[12] Gillariose, J., Tomy, L., Chesneau, C. and Jose, M. (2020). Recent developments in Marshall-Olkin distributions. *Contributions to Mathematics*,**2**, 71-75.

[13] Gomez-Deniz, E. (2010). Another Generalization of the geometric distribution. *Test 19*, 399-415

[14] Gupta, R. D and kundu, D. (1999). Generalized exponential distribution, *Aust.N.Z.J.Stat.*, **41**, 173-188.

[15] Krishna, H and Pundir, P. S. (2009). Discrete Burr and discrete Pareto distributions. *Statistical Methodology*, **6**, 177-188.

[16] Lehmann, E. L. (1953). The power of rank tests. *Ann. Math. Statistics*, **24**, 23-43.

[17] Lemonte, W., Souza, w. B and Cordeiro, G. M. (2013). The exponentiated Kumaraswamy distribution and its log-transform. *Brazillian Journal of Probability and Statistics*, **27**, 31-53.

[18] Liyange, W and Pararai, M. (2014). A generalized power lindley distribution with applications. *Asian Journal of Math. Applications and Probability*, **18**, 1-23.

[19] Mahmoudi, E and Zakerzadeh, H. (2010). Generalized Poisson-Lindley distribution. *Communications in Statistics-Theory and Methods*, **39**, 1785-1798.

[20] Marshall, A. W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. *Biometrica*, **84(3)**, 641-652.

[21] Mashhadzadeh, Z. H and Mirmostafaee, S. M. T. K. (2020). The exponentiated discrete inverse Rayleigh distribution. *Journal of Hyperstructures*, **9(1)**, 54-61.

[22] Merovci, F. and Elbatal, I. (2014). Transmuted Lindley - Geometric distribution and its applications. *Journal of Statistics Applications and Probability*, **3**, 77-91.

[23] Merovci, F. and Sharma, V. K. (2014). The Beta-Lindley distribution: Properties and Applications. *Journal of Applied Mathematics*.

[24] Nadarajah, S. (2011). The exponentiated exponential distribution:asurvey. *AStA Adv. Stat. Anal.* , **95**,219-251.

[25] Nadarajah, S. and Bakar, S. S. A. (2015). An exponentiated geometric distribution. *Applied Mathematical Modelling* , 1-10.

[26] Nadarajah, S. Bakouch, H. S. and Tahmasbi, R. (2011). A generalized Lindley distribution. *Sankhya B 73* , 331-359.

[27] Nekoukhou, V. and Bidram, H. (2015a). The exponentiated discrete Weibull distribution. *SORT 39*, 127-146.

[28] Pal, M., Ali, M. M and Woo, J. (2006). Exponentiated Weibull Distribution. *Statistica*, **66(2)**, 139-147.

[29] Rahman, M. M, Zahrani, B. L.,Shahbaz, S. H. and Shahbaz, M. Q. (2020). Transmuted probability distributions: A Review. *Pakistan Journal of Statistics and Operations Research*, **16(1)**, 83-94.

[30] Ristic, M. M and Balakrishnan, N. (2012). The gamma exponentiated exponential distribution. *J.Stat.Comput.Simul.*, **82**, 1191-1206.

[31] Steutel, F. W and Van Harn, K. Infinite divisibility of Probability Distributions on the Real Line. Vol.259, New York, Marcel Dekker, 2004.

[32] Tahir, M. H and Nadarajah, S. (2015). Parameter induction in continuous univariate distributions: Well established G-families. *Anais da Academia Brasileira de Cincias* , **87(2)**, 539-568.

[33] Zografos, K and Balakrishnan, N. (2009). On families of Beta and generalized Gamma generated distributions and associated Inference. *Statistical Methodology*, 344-362.