ON THE PROPERTIES AND APPLICATIONS OF TOPP-LEONE GOMPERTZ INVERSE RAYLEIGH DISTRIBUTION

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Abstract

In this study, we introduce a new four-parameter continuous probability distribution known as the Topp-Loene Gompertz Inverse Rayleigh (TLGoIRa) distribution. This novel model extends the Gompertz Inverse Rayleigh distribution. We present various mathematical properties of the distribution, including moments, moment generating functions, quantile functions, survival functions, hazard functions, reversed hazard functions, and odd functions. We also derive the distribution of order statistics, yielding both the maximum and minimum order statistics. This process of parameter estimation using the maximum likelihood estimation method is discussed. Furthermore, we present two real-life applications that illustrate the effectiveness and robustness of the TLGoIRa distribution when compared to several considered lifetime models. Our analysis reveals that the TLGoIRa distribution demonstrates superior robustness in comparison to the competing lifetime models. Additionally, the study highlights the distribution's efficacy in fitting biomedical datasets.

Keywords: Bladder cancer patients, goodness of fit, continuous probability, Gompertz Inverse Rayleigh, adequacy model.

I. Introduction

The Inverse Rayleigh distribution, originally introduced by [1], stands as a continuous probability distribution with significant utility in modeling the time until failure of a system. It emerges as a specialized variant within the broader framework of the inverse Weibull (IW) distribution, a potent tool for modeling lifetime data. The realm of statistical research has witnessed a fervent exploration of the inverse Rayleigh distribution, with scholars such as [2], [3], [4], [5], [6], [7], [8], and [9] delving into its intricacies, unraveling its nuances, and investigating its various generalizations and extensions. This study is inherently driven by a paramount objective: to fortify the existing model by incorporating additional shape parameters into the GoIRa distribution, as initially proposed by [9]. This augmentation is sought to yield a heightened degree of robustness, amplifying its aptness in fitting real-world datasets that abound within the realm of medical science. The proposed enhancement is envisaged to concomitantly bolster the goodness-of-fit of the model to such complex and diverse datasets, rendering it a more potent tool in the hands of researchers, clinicians, and decision-makers tasked with extracting meaningful insights and informed decisions from the intricate fabric of medical data. By infusing the GoIRa distribution with further shape parameters, the study seeks to attain a higher degree of flexibility, enabling the model to more accurately capture the multifaceted variability present in medical datasets. This endeavor is underscored by a firm statistical foundation, harnessing robust quantitative techniques to ensure the viability and efficacy of the proposed enhancement. The profound implications of such an augmented model resonate across medical research, aiding in predictive modeling, risk assessment, and optimal resource allocation, ultimately advancing the frontiers of knowledge and facilitating improved healthcare outcomes. The GoIRa distribution will be combined with the Toppleone family of distributions, introducing additional skewness to the GoIRa distribution. This augmentation aims to enhance the baseline distribution's capacity to accurately model datasets that demonstrate a significant degree of skewness.

II. Methods

2.1 Topp-Leone Gompertz Inverse Rayleigh (TLGoIRa) distribution

In this section a new continuous probability distribution function (pdf) known as TLGoIRa distribution is derived. Also, some plots of its pdf and hazard rate function (hrf) were plotted in order to assess the shape of the new distribution.
The cdf and pdf of the family of distribution proposed by [10] are given as:

$$F\left(x;\theta,\boldsymbol{\xi}\right)=\left[1-\left[1-G\left(x;\boldsymbol{\xi}\right)\right]^{2}\right]^{\theta} \tag{1}$$

$$f\left(x;\theta,\boldsymbol{\xi}\right)=2\theta g\left(x;\boldsymbol{\xi}\right)\left[1-G\left(x;\boldsymbol{\xi}\right)\right]\left[1-\left[1-G\left(x;\boldsymbol{\xi}\right)\right]^{2}\right]^{\theta-1} \tag{2}$$

where ξ is the vector of parameters of the baseline distribution.
where $G(x;\xi)$ is the cumulative distribution function (cdf) of the baseline distribution with vector of parameter ξ.
for $x\geq 0,\theta,\xi>0$,
where equations (1) and (2) are the cdf and pdf of the family of distributions proposed by [10].
The cdf and pdf of the GoIRa distribution are given a

$$G\left(x;\alpha,\lambda,\beta\right)=1-e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta}\right]} \tag{3}$$

$$g\left(x;\alpha,\lambda,\beta\right)=2\alpha\lambda^{2}x^{-3}e^{-\left(\frac{\lambda}{x}\right)^{2}}\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta-1}e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta}\right]} \tag{4}$$

To obtain the cdf of the new model, equation (3) is inserted into equation (1) as

$$F\left(x;\theta,\alpha,\lambda,\beta\right)=\left[1-\left[e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta}\right]}\right]^{2}\right]^{\theta} \tag{5}$$

On differentiating equation (5), the pdf of TLGoIRa distribution is obtained which is given as

$$f\left(x;\theta,\alpha,\lambda,\beta\right)=4\alpha\theta\lambda^{2}x^{-3}e^{-\left(\frac{\lambda}{x}\right)^{2}}\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta-1}e^{2\left(\frac{\alpha}{\beta}\right)\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta}\right]}\left[1-\left[e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta}\right]}\right]^{2}\right]^{\theta-1} \tag{6}$$

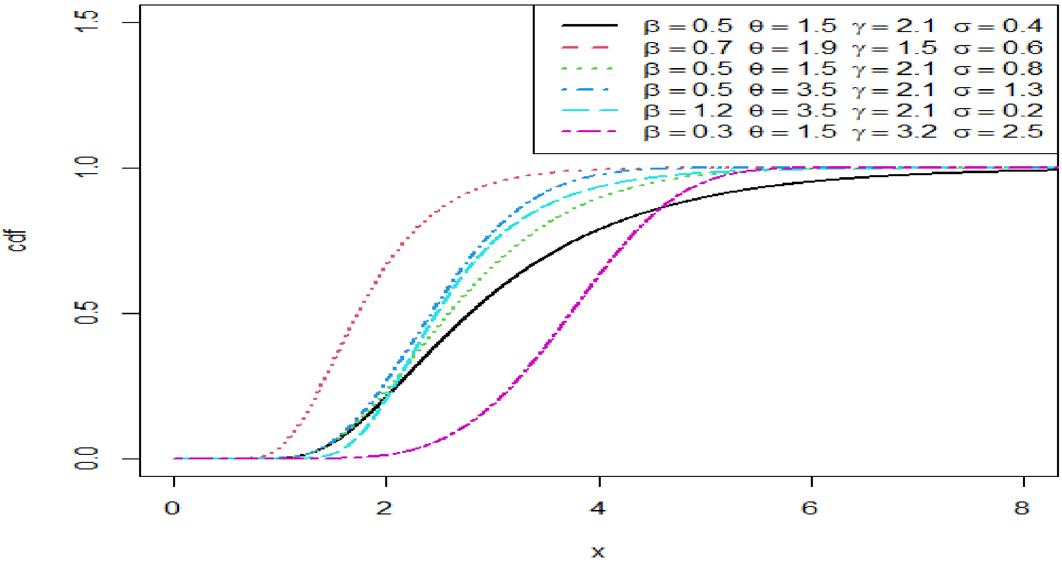

Figure 1: *Plots of cdf of TLGoIRa distribution for different parameter value*

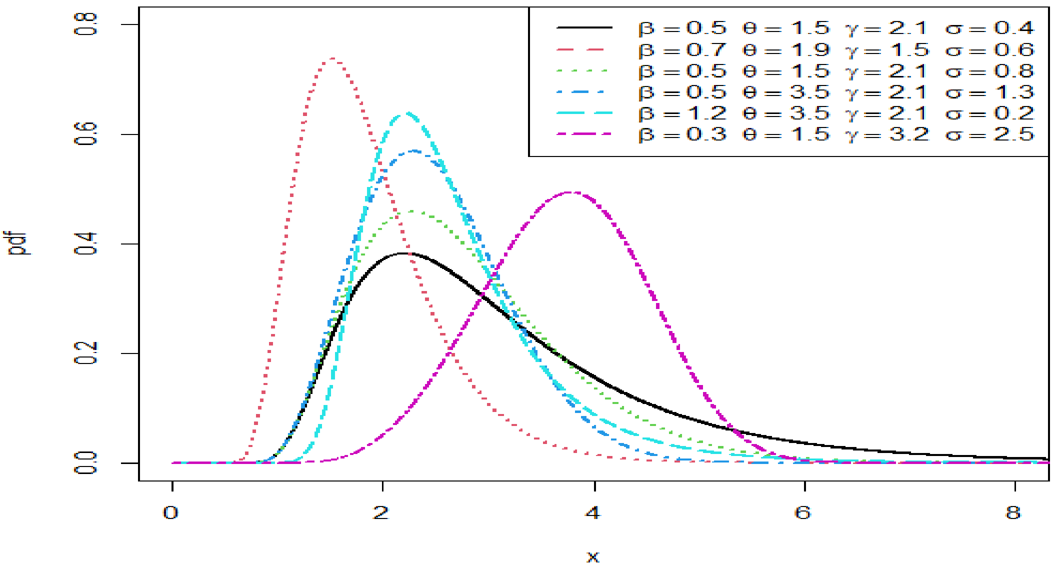

Figure 2: *Plots of pdf of TLGoIRa distribution for different parameter values*

Where $x \geq 0$, $\lambda > 0$ is the scale parameter and $\alpha, \theta, \beta > 0$ are the shape parameters respectively

2.1.1 Expansion of density

In this section the pdf in equation (6) is expanded using binomial expansion. Expanding the last term in equation (6), we have

$$e^{2\left(\frac{\alpha}{\beta}\right)\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta}\right]} = \sum_{i=0}^{\infty}\frac{\left(\frac{2\alpha}{\beta}\right)^{i}}{i!}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta}\right]^{i}$$

$$\left[1-\left[e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{-\beta}\right]}\right]^2\right]^{\theta-1}=\sum_{j=0}^{\infty}(-1)^j\binom{\theta-1}{j}\left[e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{-\beta}\right]}\right]^{2j}$$

$$\left[e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{-\beta}\right]}\right]^{2j}=\sum_{k=0}^{\infty}\frac{\left(\frac{2j\alpha}{\beta}\right)^k}{k!}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{-\beta}\right]^k$$

$$\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{-\beta}\right]^{k+1}=\sum_{m=0}^{\infty}(-1)^m\binom{k+1}{m}\left[1-e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{-\beta m}$$

$$\left[1-e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{-\beta(1+m)-1}=\sum_{w=0}^{\infty}(-1)^w\binom{-\beta(1+m)-1}{w}\left[e^{-\left(\frac{\lambda}{x}\right)^2}\right]^w$$

On substituting all the expansions into equation (6), we have

$$f(x)=4\alpha\theta\lambda^2\sum_{i,j,k,m,w=0}^{\infty}\frac{\left(\frac{2\alpha}{\beta}\right)^i}{i!}\frac{\left(\frac{2j\alpha}{\beta}\right)^k}{k!}(-1)^{j+m+w}\binom{\theta-1}{j}\binom{k+1}{m}\binom{-\beta(1+m)-1}{w}x^{-3}\left[e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{w+1} \tag{7}$$

Now

$$f(x;\theta,\alpha,\lambda,\beta)=\sum_{i,j,k,m,w=0}^{\infty}\gamma_q x^{-3}\left[e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{w+1} \tag{8}$$

where

$$\gamma_q=4\alpha\theta\lambda^2\frac{\left(\frac{2\alpha}{\beta}\right)^i}{i!}\frac{\left(\frac{2j\alpha}{\beta}\right)^k}{k!}(-1)^{j+m+w}\binom{\theta-1}{j}\binom{k+1}{m}\binom{-\beta(1+m)-1}{w}$$

Equation (7) is the expansion of equation (6) which will be used to derive some of the properties of the distribution.
Also, equation (5) is expanded as

$$\left[F(x;\theta,\alpha,\lambda,\beta)\right]^h=\left[1-\left[e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{-\beta}\right]}\right]^2\right]^{\theta h}$$

$$\left[1-\left[e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta}\right]}\right]^{2}\right]^{\theta h}=\sum_{z=0}^{h}(-1)^{z}\binom{\theta h}{z}\left[e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta}\right]}\right]^{2z}$$

$$\left[e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta}\right]}\right]^{2z}=\sum_{t=0}^{\infty}\frac{\left(\frac{2z\alpha}{\beta}\right)^{t}}{t!}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta}\right]^{t}$$

$$\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta}\right]^{t}=\sum_{d=0}^{\infty}(-1)^{d}\binom{t}{d}\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta d}$$

$$\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta d}=\sum_{p=0}^{\infty}(-1)^{p}\binom{-\beta d}{p}\left[e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{p}$$

Substituting all the expansions into equation (5), we have

$$\left[F(x;\theta,\alpha,\lambda,\beta)\right]^{h}=\sum_{z,d,p=0}^{\infty}\sum_{t=0}^{h}\frac{\left(\frac{2z\alpha}{\beta}\right)^{t}}{t!}(-1)^{z+d+p}\binom{t}{d}\binom{-\beta d}{p}\binom{\theta h}{z}\left[e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{p} \tag{9}$$

Now

$$\left[F(x;\theta,\alpha,\lambda,\beta)\right]^{h}=\sum_{t=0}^{h}\zeta_{c}\left[e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{p} \tag{10}$$

where

$$\zeta_{c}=\sum_{z,d,p=0}^{\infty}\frac{\left(\frac{2z\alpha}{\beta}\right)^{t}}{t!}(-1)^{z+d+p}\binom{t}{d}\binom{-\beta d}{p}\binom{\theta h}{z}$$

Equation (9) is the expansion of equation (5) which will be used to derive some of the properties of the distribution.

2.1.2 Properties of the TLGoIRa distribution

In this section, some of the mathematical and statistical properties of TLGoIRa distribution such as the quantile function, moments, moment generating function, reliability measure, odds function, reversed hazard function and order statistics are derived.

2.1.2.1 Moments

$$E\left(X^{r}\right)=\int_{0}^{\infty}x^{r}f(x)\,dx \tag{11}$$

$$=\sum_{i,j,k,m,w=0}^{\infty}\gamma_{q}\int_{0}^{\infty}x^{r-3}\left[e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{w+1}dx$$

$$\int_0^\infty x^{r-3}\left[e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{w+1} dx = \frac{\lambda^r \Gamma\left(1-\frac{r}{2}\right)}{(w+1)^{1-\frac{r}{2}}}$$

$$E\left(X^r\right) = \sum_{i,j,k,m,w=0}^{\infty} \gamma_q \frac{\lambda^r \Gamma\left(1-\frac{r}{2}\right)}{(w+1)^{1-\frac{r}{2}}} \tag{12}$$

Equation (12) is the moments of TLGoIRa distribution. To obtain mean, we set *r* = *1* in equation (12).

2.1.2.2 Moment generating function (mgf)

$$M_{(x)}(t) = \int_0^\infty e^{tx} f(x) dx \tag{13}$$

the series expansion for e^{tx} is given as

$$e^{tx} = \sum_{v=0}^{\infty} \frac{(tx)^v}{v!} \tag{14}$$

$$M_{(x)}(t) = \sum_{v=0}^{\infty} \frac{t^v}{v!} \sum_{i,j,k,m,w=0}^{\infty} \gamma_q \frac{\lambda^v \Gamma\left(1-\frac{v}{2}\right)}{(w+1)^{1-\frac{v}{2}}} \tag{15}$$

2.1.2.3 Quantile function

Quantile function has a significant position in probability theory and it is the inverse of the cdf. The quantile function is obtained using

$$Q(u) = F^{-1}(u) \tag{16}$$

Using the inverse of equation (5), we have the quantile function of TLGoIRa distribution given as

$$x = Q(u) = \frac{\lambda}{\sqrt{-\log\left[1-\left[1-\left[\frac{\beta}{\alpha}\log\left[1-U^{\frac{1}{\theta}}\right]\right]^{\frac{1}{2}}\right]\right]^{\frac{-1}{\beta}}}} \tag{17}$$

The median is obtained by setting *u* = *0.5* in equation (17) given as

$$x_{median} = Q(0.5) = \frac{\lambda}{\sqrt{-\log\left[1-\left[1-\left[\frac{\beta}{\alpha}\log\left[1-0.5^{\frac{1}{\theta}}\right]\right]^{\frac{1}{2}}\right]\right]^{\frac{-1}{\beta}}}} \tag{18}$$

2.1.2.4 Hazard function

Hazard function is given as

$$\tau(x;\alpha,\beta,\lambda,\theta) = \frac{f(x;\alpha,\beta,\lambda,\theta)}{R(x;\alpha,\beta,\lambda,\theta)} \tag{19}$$

The hazard function of the TLGoIRa distribution is given as

$$\tau(x)=\frac{4\alpha\theta\lambda^{2}x^{-3}e^{-\left(\frac{\lambda}{x}\right)^{2}}\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta-1}e^{2\left(\frac{\alpha}{\beta}\right)\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta}\right]}\left[1-\left[e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta}\right]}\right]^{2}\right]^{\theta-1}}{1-\left[1-\left[e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta}\right]}\right]^{2}\right]^{\theta}} \tag{20}$$

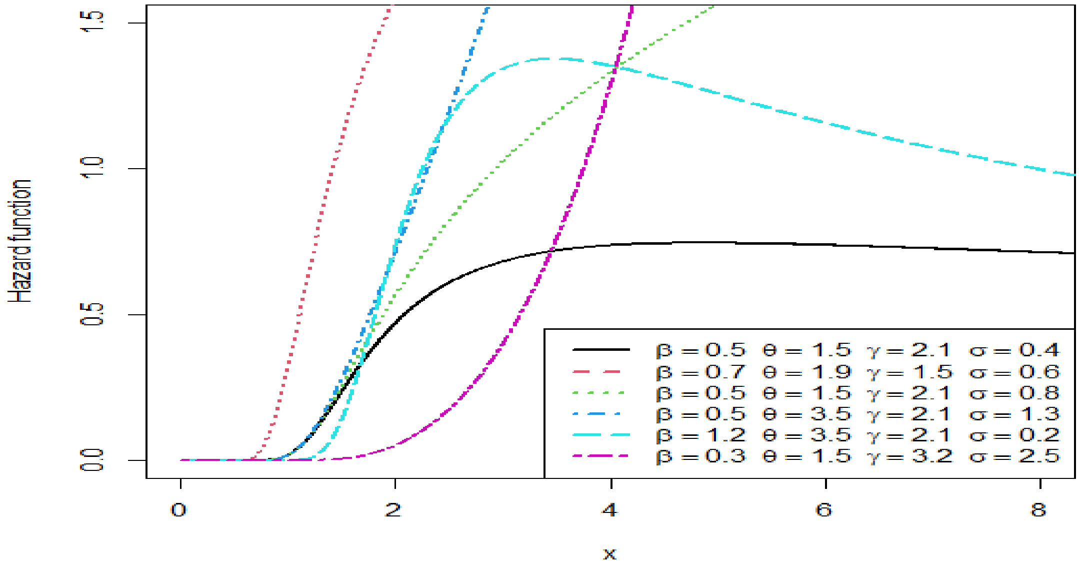

Figure 3: *Plots of hazard function of the TLGoIRa distribution for different parameter values*

2.1.2.5 Survival function

The reliability function is also known as survival function, which is the probability of an item not failing prior to some time. It can be defined as

$$R(x;\alpha,\beta,\lambda,\theta)=1-F(x;\alpha,\beta,\lambda,\theta) \tag{21}$$

The survival function of the TLGoIRa distribution is given as

$$\mathrm{R}(x)=1-\left[1-\left[e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right]^{-\beta}\right]}\right]^{2}\right]^{\theta} \tag{22}$$

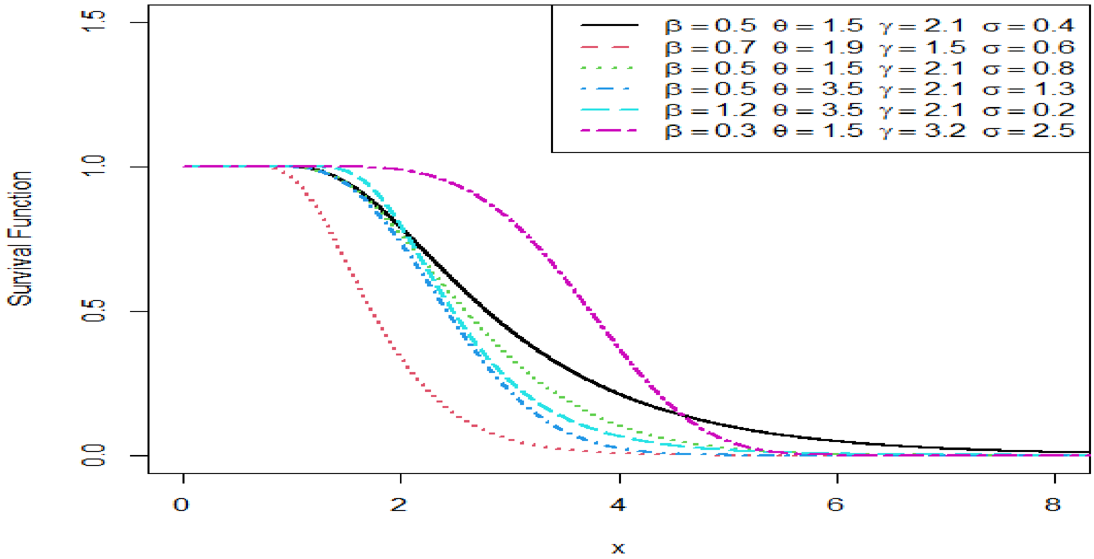

Figure 4: *Plots of survival function of the TLGoIRa distribution for different parameter values*

2.1.2.6 Reversed hazard function

Reversed hazard function of a random variable x is given as

$$\Re(x;\alpha,\beta,\lambda,\theta)=\frac{f(x;\alpha,\beta,\lambda,\theta)}{F(x;\alpha,\beta,\lambda,\theta)} \tag{23}$$

The reverse hazard rate function of the TLGoIRa distribution is given as

$$\Re(x)=\frac{4\alpha\theta\lambda^2x^{-3}e^{-\left(\frac{\lambda}{x}\right)^2}\left[1-e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{-\beta-1}e^{2\left(\frac{\alpha}{\beta}\right)\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{-\beta}\right]}\left[1-\left[e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{-\beta}\right]}\right]^2\right]^{\theta-1}}{\left[1-\left[e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{-\beta}\right]}\right]^2\right]^{\theta}} \tag{24}$$

2.1.2.7 Odds function

The odds function of the TLGoIRa distribution is given as

$$O(x;\alpha,\beta,\lambda,\theta)=\frac{\left[1-\left[e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{-\beta}\right]}\right]^2\right]^{\theta}}{1-\left[1-\left[e^{\frac{\alpha}{\beta}\left[1-\left[1-e^{-\left(\frac{\lambda}{x}\right)^2}\right]^{-\beta}\right]}\right]^2\right]^{\theta}} \tag{25}$$

2.2 Order Statistics

Let $X_1, X_2, ..., X_n$ be n independent random variable from the TLGoIRa distributions and let $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ be their corresponding order statistic. Let $F_{r:n}(x)$ and $f_{r:n}(x)$, $r = 1, 2, 3, ...n$ denote the cdf and pdf of the r^{th} order statistics $X_{r:n}$ respectively. The pdf of the r^{th} order statistics of $X_{r:n}$ is given as

$$f_{r:n}(x) = \frac{f(x)}{B(r, n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} F(x)^{v+r-1} \tag{26}$$

The pdf of r^{th} order statistic for distribution is obtained also by replacing h with v+r-1 in cdf expansion. We have

$$f_{r:n}(x) = \gamma_q \zeta_c \frac{1}{B(r, n-r+1)} \sum_{v=0}^{n-r} \sum_{i,j,k,m,w=0}^{\infty} \sum_{t=0}^{v+r-1} x^{-3} \left[e^{-\left(\frac{\lambda}{x}\right)^2} \right]^{w+p+1} \tag{27}$$

where

$$\gamma_q = 4\alpha\lambda^2 \frac{\left(\frac{2\alpha}{\beta}\right)^i}{i!} \frac{\left(\frac{2j\alpha}{\beta}\right)^k}{k!} (-1)^{j+m+w} \binom{\theta-1}{j} \binom{k+1}{m} \binom{-\beta(1+m)-1}{w}$$

and

$$\zeta_c = \sum_{z,d,p=0}^{\infty} \frac{\left(\frac{2z\alpha}{\beta}\right)^t}{t!} (-1)^{z+d+p} \binom{t}{d} \binom{-\beta d}{p} \binom{\theta(v+r-1)}{z}$$

The pdf of minimum order statistic of the distribution is obtained by setting r=1 in equation (27).

$$f_{1:n}(x) = \gamma_q \zeta_c n \sum_{v=0}^{n-1} \sum_{i,j,k,m,w=0}^{\infty} \sum_{t=0}^{v} x^{-3} \left[e^{-\left(\frac{\lambda}{x}\right)^2} \right]^{w+p+1} \tag{28}$$

Now ,

$$\zeta_c = \sum_{z,d,p=0}^{\infty} \frac{\left(\frac{2z\alpha}{\beta}\right)^t}{t!} (-1)^{z+d+p} \binom{t}{d} \binom{-\beta d}{p} \binom{\theta v}{z}$$

Also, the pdf of maximum order statistic of the distribution is obtained by setting r = n in equation (27)

$$f_{n:n}(x) = \gamma_q \zeta_c n \sum_{i,j,k,m,w=0}^{\infty} \sum_{t=0}^{v+n-1} x^{-3} \left[e^{-\left(\frac{\lambda}{x}\right)^2} \right]^{w+p+1} \tag{29}$$

where

$$\zeta_c = \sum_{z,d,p=0}^{\infty} \frac{\left(\frac{2z\alpha}{\beta}\right)^t}{t!} (-1)^{z+d+p} \binom{t}{d} \binom{-\beta d}{p} \binom{\theta(v+n-1)}{z}$$

2.3 Estimation method

The method of maximum likelihood estimation (MLE) is used in this section to estimate the parameters of the TLGoIRa distribution. For a random sample, $X_1, X_2, ..., X_n$ of size n from the TLGoIRa$(\alpha,\beta,\theta,\lambda)$, the log-likelihood function $\mathrm{L}(\alpha,\beta,\theta,\lambda)$ of (6) is given as

$$\log L = n\log(4) + n\log(\theta) + n\log(\alpha) + 2n\log(\lambda) - 3\sum_{i=1}^{n}\log(x_i) - \lambda^2\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^2$$
$$-(\beta+1)\sum_{i=1}^{n}\log\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right] + \frac{\alpha}{\beta}\sum_{i=1}^{n}\left[1-\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right]^{-\beta}\right] + (\theta-1)\sum_{i=1}^{n}\log\left[1-\left[e^{\left(\frac{\alpha}{\beta}\right)\left[1-\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right]^{-\beta}\right]}\right]^2\right] \quad (30)$$

Differentiating the log-likelihood with respect to $\lambda,\alpha,\theta,\beta$ and equating the result to zero, we have

$$\frac{\partial \log l}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log\left[1-\left[e^{-\left(\frac{\alpha}{\beta}\right)\left[1-\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right]^{-\beta}\right]}\right]^2\right] = 0 \quad (31)$$

$$\frac{\partial \log l}{\partial \alpha} = \frac{n}{\alpha} + \frac{2}{\beta}\sum_{i=1}^{n}\left[1-\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right]^{-\beta}\right] + 2(\theta-1)\sum_{i=1}^{n}\frac{e^{\left(\frac{\alpha}{\beta}\right)\left[1-\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right]^{-\beta}\right]} e^{\left(\frac{1}{\beta}\right)\left[1-\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right]^{-\beta}\right]}}{\left[1-e^{\left(\frac{\beta}{\sigma}\right)\left[1-\left[1-e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{-\sigma}\right]}\right]} = 0 \quad (32)$$

$$\frac{\partial \log l}{\partial \lambda} = \frac{2n}{\lambda} - 2\lambda\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^2 - (\beta+1)\sum_{i=1}^{n}\frac{\lambda e^{-\left(\frac{\lambda}{x}\right)^2}}{x^2\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right]} + 4\alpha\lambda\sum_{i=1}^{n}e^{-\left(\frac{\lambda}{x_i}\right)^2}\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right]^{-\beta-1}$$
$$+2\beta(\theta-1)\sum_{i=1}^{n}\frac{e^{\left(\frac{\alpha}{\beta}\right)} e^{-\left(\frac{\lambda}{x_i}\right)^2} e^{\left(\frac{\alpha}{\beta}\right)\left[1-\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right]^{-\beta}\right]}\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right]^{-\beta-1}}{\left[1-\left[e^{\left(\frac{\alpha}{\beta}\right)\left[1-\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right]^{-\beta}\right]}\right]\right]^2} = 0 \quad (33)$$

$$\frac{\partial \log l}{\partial \beta} = -\sum_{i=1}^{n} \log\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right] e^{\left(\frac{\alpha}{\beta}\right)\left[1-\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right]^{-\beta}\right]} - \frac{2\alpha}{\beta^2}\sum_{i=1}^{n}\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right]^{-\beta} e^{\left(\frac{\alpha}{\beta^2}\right)\left[1-\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right]^{-\beta}\right]} \log\left[1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right] e^{\left(\frac{\alpha}{\beta}\right)\left[1-e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{-\beta}} + 2(\theta-1)\sum_{i=1}^{n}\frac{\log\left[1-e^{-\left(\frac{\lambda}{x}\right)^2}\right]}{\left[1-e^{-\left(\frac{\beta}{\sigma}\right)\left[1-\left[1-e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{-\sigma}\right]}\right]} = 0 \quad (34)$$

Now, equations (31), (32), (33) and (34) do not have a simple analytical form and are therefore not tractable. As a result, we have to resort to non-linear estimation of the parameters using iterative method.

III. Results

3.1 Applications

In this section, we present two applications of the TLGoIRa distribution using different datasets from the biomedical field. These applications are intended to demonstrate the flexibility of the distribution in modeling real-life datasets. The data are fitted to the TLGoIRa distribution, as well as four other comparator distributions: Gompertz Inverse Rayleigh (GoIRa) distribution, Generalized Gompertz (GGo) distribution, Exponentiated Exponential (EtEx) distribution, and Inverse Rayleigh (IRa) distribution. This fitting process is carried out to test the new distribution's flexibility against these comparators. We utilized the Adequacy Model package within the R software to perform the analysis and produce the results. To evaluate the performance of the distribution, we employed the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC). These criteria were used to compare the performance of the TLGoIRa distribution with other existing distributions that align with the baseline distribution in terms of providing a good parametric fit to the dataset.

$$AIC = -2ll + 2k \quad (30)$$

$$BIC = -2ll + k\log(n) \quad (31)$$

The model selection is carried out using the AIC and the BIC. Where ll denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters, and n is the sample size from the data. The model with minimum value of AIC and BIC is chosen as the best model to fit the data set.

Data set 1 has been used by [11] and represents the sum of skin folds in 202 athletes collected at the Australian Institute of Sports as.

28.0, 98, 89.0, 68.9, 69.9, 109.0, 52.3, 52.8, 46.7, 82.7, 42.3, 109.1, 96.8, 98.3, 103.6, 110.2, 98.1, 57.0, 43.1, 71.1, 29.7, 96.3, 102.8, 80.3, 122.1, 71.3, 200.8, 80.6, 65.3, 78.0, 65.9, 38.9, 56.5, 104.6, 74.9, 90.4, 54.6, 131.9, 68.3, 52.0, 40.8, 34.3, 44.8, 105.7, 126.4, 83.0, 106.9, 88.2, 33.8, 47.6, 42.7, 41.5, 34.6, 30.9, 100.7, 80.3, 91.0, 156.6, 95.4, 43.5, 61.9, 35.2, 50.9, 31.8, 44.0, 56.8, 75.2, 76.2,101.1, 47.5, 46.2, 38.2, 49.2, 49.6, 34.5, 37.5, 75.9, 87.2, 52.6, 126.4, 55.6, 73.9, 43.5, 61.8, 88.9, 31.0, 37.6,52.8, 97.9, 111.1, 114.0, 62.9, 36.8, 56.8, 46.5, 48.3, 32.6, 31.7, 47.8, 75.1, 110.7, 70.0, 52.5, 67, 41.6, 34.8, 61.8, 31.5, 36.6, 76.0, 65.1, 74.7, 77.0, 62.6, 41.1, 58.9, 60.2, 43.0, 32.6, 48, 61.2, 171.1, 113.5, 148.9, 49.9, 59.4, 44.5, 48.1, 61.1, 31.0, 41.9, 75.6, 76.8, 99.8, 80.1, 57.9, 48.4, 41.8, 44.5, 43.8, 33.7, 30.9, 43.3, 117.8, 80.3, 156.6, 109.6, 50.0, 33.7, 54.0, 54.2, 30.3, 52.8, 49.5, 90.2, 109.5, 115.9, 98.5, 54.6, 50.9, 44.7, 41.8, 38.0, 43.2, 70.0, 97.2, 123.6, 181.7, 136.3, 42.3, 40.5, 64.9, 34.1, 55.7, 113.5, 75.7, 99.9, 91.2, 71.6, 103.6, 46.1, 51.2, 43.8, 30.5, 37.5, 96.9, 57.7, 125.9, 49.0, 143.5, 102.8, 46.3, 54.4, 58.3, 34.0, 112.5, 49.3, 67.2, 56.5, 47.6, 60.4, 34.9.

Data set 2 has been used by [12] and it represents the remission times (in months) of a random sample

of one hundred and twenty-eight (128) bladder cancer patients.
0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Table 1: *The models' MLEs and performance requirements based on data set 1*

Models	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	ll	AIC	BIC
TLGoIR	0.8992	0.2125	121.0713	1.2501	-949.7464	1907.4930	1920.7260
GoIR	0.0031	-	0.0000	0.8601	-987.5204	1981.0410	1990.9660
GGo	-0.0052	15.4031	-	0.0597	-956.0865	1918.1730	1928.9200
EtEx	0.0406	8.5786	-	-	-958.0065	1920.0130	1926.6300
IR	52.6054	-	-	-	-966.4625	1934.9250	1938.2330

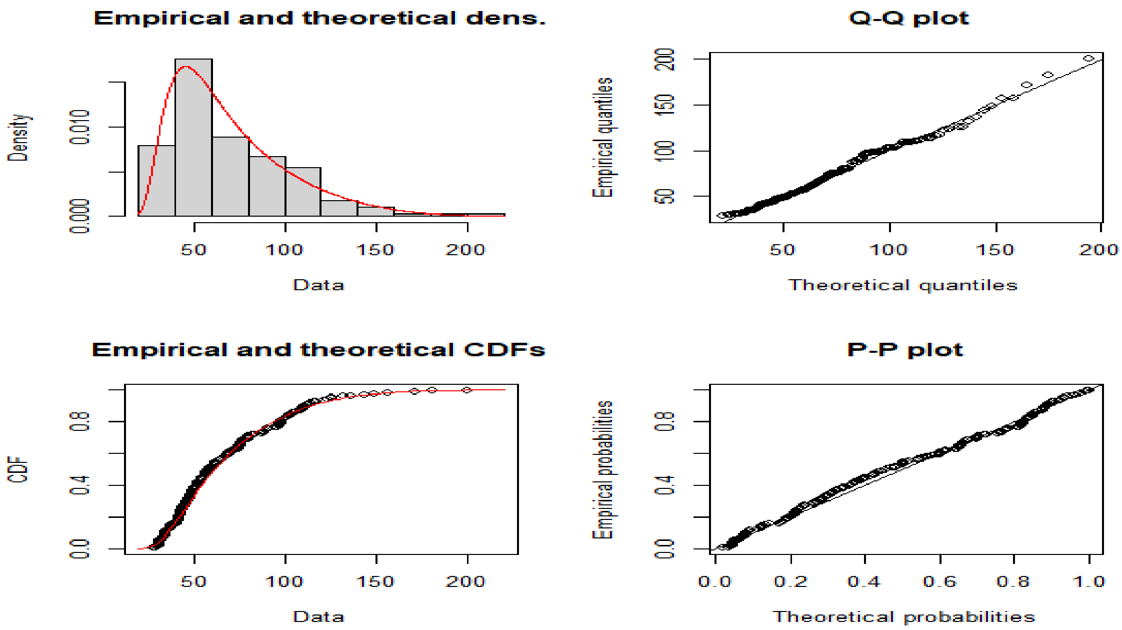

Figure 5: *Density plots for data set 1*

Table 2: *The models' MLEs and performance requirements based on data set 2*

Models	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	ll	AIC	BIC
TLGoIR	0.0002	2.6653	0.0001	0.3345	-410.6935	829.3871	834.7952
GoIR	0.0839	-	0.0041	0.5129	-413.5753	833.1505	836.1377
GGo	-0.0224	1.5034	-	0.1678	-413.1834	832.3668	835.3539
EtEx	0.1213	1.2180	-	-	-413.0776	830.1552	834.8592
IR	2.2612	-	-	-	-774.3416	1550.6830	1553.5350

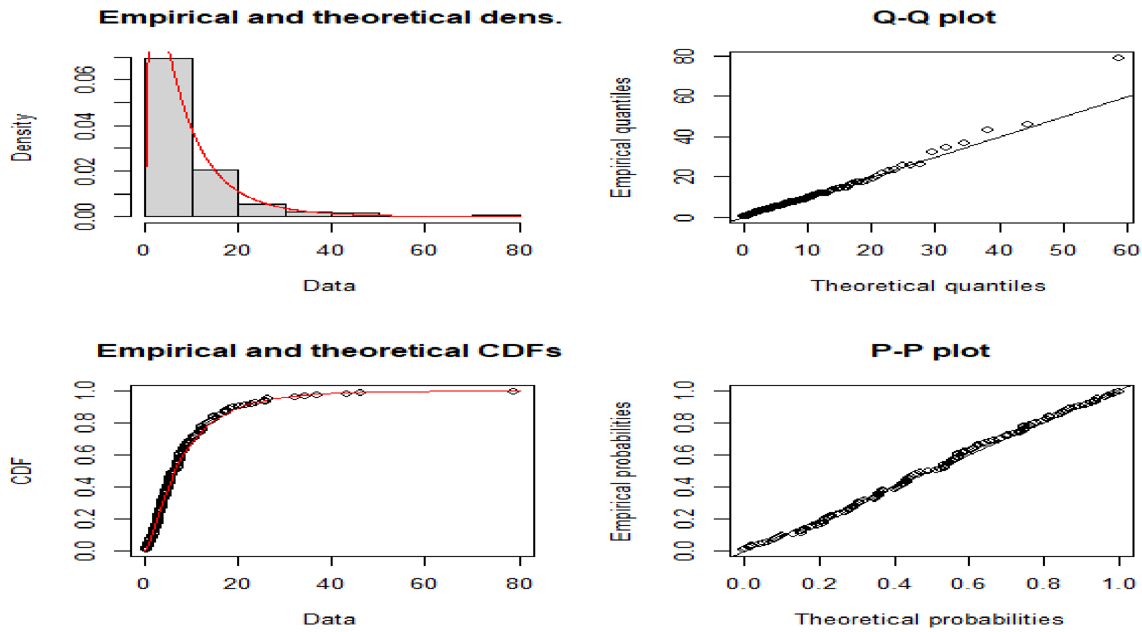

Figure 6: *Density plots for data set 2*

IV. Discussion

In this study, we have introduced and thoroughly examined a novel probability distribution, coined the Topp Leone Gompertz Inverse Rayleigh distribution. Drawing from the framework of distributions proposed by [10], we have both theoretically established its foundations and conducted practical investigations. A comprehensive exploration of the distribution's mathematical properties has been undertaken, encompassing crucial characteristics such as moments, moment generating function, quantile function, odd function, reverse hazard function, order statistics, and reliability analysis. These analyses have contributed to a robust comprehension of the distribution's behaviors and traits. The method of maximum likelihood estimation has been employed to effectively determine the distribution's parameters, lending statistical rigor to our subsequent analyses. In a bid to assess the distribution's practical utility, it has been subjected to rigorous testing using two distinct datasets from the realm of medical sciences. This assessment has entailed a comparative study against several competing distributions, including the GoIRa, Generalized Gompertz, Exponentiated Exponential, and Inverse Rayleigh distributions, all sharing common baseline distributions. The outcomes of our meticulous analyses indicate that the Topp-Leone Gompertz Inverse Rayleigh distribution stands out as the most adept candidate for fitting both datasets. The distribution's apparent superior performance among the alternatives highlights its potential to adeptly capture the underlying data-generation mechanisms. This research introduces not only a novel distribution but also offers practical insights into its capacity to effectively represent intricate real-world phenomena.

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