SOME APPLICATIONS OF TRANSMUTED LOG-UNIFORM DISTRIBUTION

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Abstract

As a generalization of the Log Uniform distribution, Transmuted Log - Uniform distribution is introduced and its properties are studied.We obtained graphical representations of its pdf, cdf, hazard rate function and survival function. We have derived statistical properties such as moments, mean deviations, and the quantile function for the Transmuted Log-Uniform distribution. We also obtained the order statistics of the new distribution. Method of maximum likelihood is used for estimating the parameters. Estimation of stress strength parameters is also done. We applied the Transmuted Log-Uniform distribution to a real data set and compared it with Transmuted Weibull distribution and Transmuted Quasi-Akash distribution. It was found that the Transmuted Log-Uniform distribution was a better fit than the Transmuted Weibull distribution and Transmuted Quasi-Akash distribution distributions based on the values of the AIC, CAIC, BIC, HQIC, the Kolmogorov-Smirnov (K-S) goodness-of-fit statistic and the p-values.

Keywords: Transmuted distribution, Transmuted Log- Uniform distribution,Stress- strength parameters

1. Introduction

The Transmuted family of distributions was first introduced by Shaw and Buckley (2007) and has since been widely used in various fields including finance, engineering, and environmental sciences. According to quadratic rank transmutation map (QRTM) technique approach, a random variable X is said to have a Transmuted distribution, if its cdf is given by,

$$D(x) = (1+\lambda)G(x) - \lambda(G(x))^2; \quad -1 \leq \lambda \leq 1 \tag{1}$$

where G(x) is the c.d.f of the base distribution.

The corresponding probability density function (p.d.f) with parameter λ is given by:

$$d(x) = g(x)(1+\lambda-2\lambda G(x)); \quad -1 \leq \lambda \leq 1 \tag{2}$$

where λ is a scale parameter.

There are different families of distributions which are useful for developing flexible compound distributions for solving real life problems. Transmuted distributions have emerged as the superior option, surpassing their standard counterparts in terms of flexibility and performance. Some of the models studied were Transmuted Exponential Lomax distribution by Abdullahi and Ieren [1], Transmuted complementary Weibull Geometric distribution by Afify [2], the Transmuted Weibull Lomax distribution by Afify [3], the Transmuted Weibull distribution by Aryaland Tsokos

[5], Transmuted Additive Weibull distribution by Elbatal and Aryal [6], Transmuted Quasi Akash distribution by Hassan [8], Transmuted Exponentiated Gamma distribution by Hussian [9], Transmuted modified Weibull distribution by Khanand King [11], Transmuted Inverse Weibull distribution by Khan [10], Transmuted Gompertz distribution by Khan [12], Transmuted Lindley distribution by Mansour [13], Transmuted Rayleigh distribution by Merovci [15], Transmuted Pareto distribution by Merovci and Puka [14], Transmuted Inverse Exponential distribution by Oguntunde and Adejumo [16] and Transmuted generalized Uniform distribution by subramanian [18] etc.

In this article we present a new generalization of Log-Uniform distribution called the Transmuted Log-Uniform distribution.

2. Transmuted Log-Uniform Distribution

A Log-Uniform distribution is a probability distribution where the logarithm of the random variable is uniformly distributed.

A random variable X is said to have the Log-Uniform distribution with parameter λ if its probability density function is defined as,

$$g(x) = \begin{cases} \frac{1}{x\left[ln(b)-ln(a)\right]}; & \text{if } , a \leq x \leq b, 0 < a < b, a, b \in R \\ 0; & \text{otherwise} \end{cases} \tag{3}$$

where a and b are the parameters of the distribution and they are location parameters that define the minimum and maximum values of the distribution on the original scale and ln is the natural Log function (the logarithm to base e).

The corresponding Cumulative distribution function (c.d.f.) is,

$$G(x) = \begin{cases} \frac{ln(x)-ln(a)}{ln(b)-ln(a)}; & \text{if } , a \leq x \leq b, 0 < a < b, a, b \in R \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

The cdf of a Transmuted Log-Uniform distribution,

$$F(x) = \begin{cases} (1+\lambda)\frac{ln(\frac{x}{a})}{ln(\frac{b}{a})} - (\lambda)\left[\frac{ln(\frac{x}{a})}{ln(\frac{b}{a})}\right]^2; & \text{if } |\lambda| \leq 1, a \leq x \leq b, \\ & 0 < a < b, \quad a, b \in R \\ 0; & \text{otherwise.} \end{cases} \tag{5}$$

The pdf of Transmuted Log-Uniform distribution is

$$f(x) = \begin{cases} \frac{(1+\lambda)}{(x)ln(\frac{b}{a})} - \frac{(2\lambda)ln(\frac{x}{a})}{(x)(ln(\frac{b}{a}))^2}; & \text{if } |\lambda| \leq 1, a \leq x \leq b, \\ & 0 < a < b, \quad a, b \in R \\ 0; & \text{otherwise} \end{cases} \tag{6}$$

The survival function of Transmuted Log-Uniform distribution is given by:

$$S(x) = \frac{(ln(\frac{b}{a}))^2 - (1+\lambda)ln(\frac{x}{a})ln(\frac{b}{a}) - (\lambda)(ln(\frac{x}{a}))^2}{(ln(\frac{b}{a}))^2} \tag{7}$$

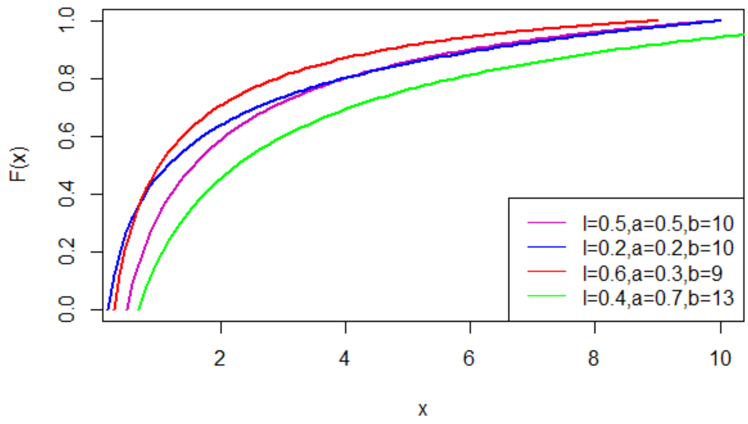

Figure 1: *Plot of cdf of the Transmuted Log-Uniform distribution*

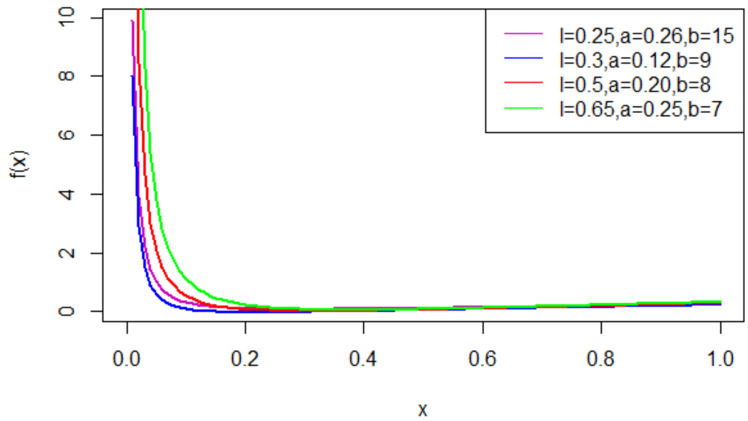

Figure 2: *Plot of pdf of the Transmuted Log-Uniform distribution*

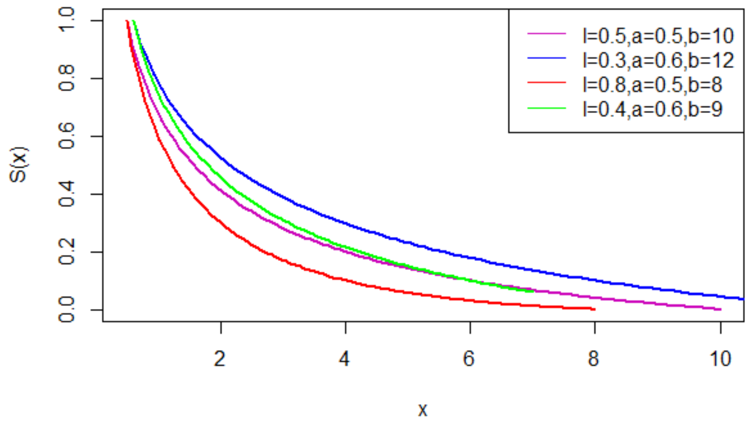

Figure 3: *Plot of survival function of the Transmuted Log-Uniform distribution*

The failure rate function or hazard function of Transmuted Log-Uniform distribution is:

$$h(x) = \frac{(1+\lambda)ln(\frac{b}{a}) - (2\lambda)ln(\frac{x}{a})}{x[(ln(\frac{b}{a}))^2 - (1+\lambda)ln(\frac{x}{a})ln(\frac{b}{a}) - (\lambda)(ln(\frac{x}{a}))^2]} \tag{8}$$

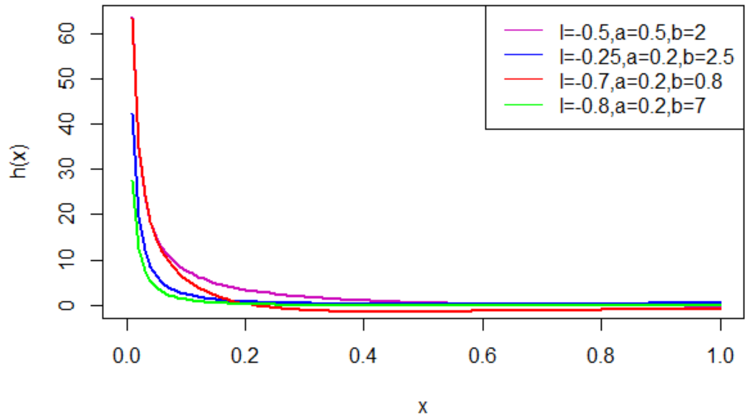

Figure 4: *Plot of hazard rate function of the Transmuted Log-Uniform distribution*

- Special Case:
  If we put$\lambda = 0$, then Transmuted Log-Uniform distribution reduces to Log-Uniform distribution.

3. Statistical properties

3.1. Moments

Let X is a random variable following Transmuted Log-Uniform distribution with parameters a,b,λ and then the r^{th} moment for a given probability distribution is given by:

$$\mu_r' = \int_a^b x^r \left[\frac{(1+\lambda)}{x \ln \frac{b}{a}} - \frac{2\lambda \ln \frac{x}{a}}{x(\ln \frac{b}{a})^2} \right] dx$$

$$E(X^r) = \mu_r' = \frac{(1+\lambda)(b^r - a^r)}{r \ln(\frac{b}{a})} - \frac{2\lambda a^r}{(\ln(\frac{b}{a}))^2} \left[(\frac{b}{a}) \ln(\frac{b}{a}) - (\frac{b}{ar}) + (\frac{1}{r^2}) \right] \tag{9}$$

Mean of the Transmuted Log-Uniform distribution is obtained as:
$\mu_1' = \frac{(1+\lambda)(b-a)}{ln(\frac{b}{a})} - \frac{2\lambda a}{(ln(\frac{b}{a}))^2} \left[(\frac{b}{a})ln(\frac{b}{a}) - (\frac{b}{a}) + 1 \right]$.

3.2. Quantile function

The Quantile function of Transmuted Log-Uniform distribution is obtained by inverting distribution function.

$$p = (1+\lambda)\frac{ln(\frac{x}{a})}{ln(\frac{b}{a})} - \lambda \left[\frac{ln(\frac{x}{a})}{ln(\frac{b}{a})} \right]^2$$

$$x = a\exp\left(\frac{\left[(1+\lambda)+\sqrt{\lambda^2+2\lambda+1-4\lambda p}\right]\ln\frac{b}{a}}{2\lambda}\right) \tag{10}$$

The median, 2^{nd} quartile is obtained by putting $p = \frac{1}{2}$ in (10)

$$x = a\exp\left(\frac{\left[(1+\lambda)+\sqrt{\lambda^2+1}\right]\ln\frac{b}{a}}{2\lambda}\right) \tag{11}$$

3.3. Mean deviation

Let X follows Transmuted Log-Uniform distribution with mean μ and median M.

- Mean Deviation from the Mean is given by:

$$\delta_1(x) = \int_a^b |x-\mu| f(x)dx \ = 2\mu(F(\mu)-1)+2T(\mu) \tag{12}$$

where μ is the mean of the distribution and

$$T(\mu) = \int_\mu^b x f(x)dx$$

$$T(\mu) = \int_\mu^b x\left[\frac{(1+\lambda)}{(x)ln(\frac{b}{a})} - \frac{(2\lambda)ln(\frac{x}{a})}{(x)(ln(\frac{b}{a}))^2}\right]dx$$

$$T(\mu) = \frac{(1+\lambda)(b-\mu)}{ln(\frac{b}{a})} - \frac{(2\lambda)}{(ln(\frac{b}{a}))^2}\left[b\left(\ln\frac{b}{a}-1\right)-\mu\left(\ln\frac{\mu}{a}-1\right)\right] \tag{13}$$

- Similarly, the Mean Deviation about Median is:

$$\delta_2(x) = \int_a^b |x-M| f(x)dx \ = 2T(M)-\mu \tag{14}$$

where M is the median of the distribution and μ is the mean of the distribution and

$$T(M) = \int_M^b x f(x)dx$$

$$T(M) = \frac{(1+\lambda)(b-M)}{ln(\frac{b}{a})} - \frac{(2\lambda)}{(ln(\frac{b}{a}))^2}\left[b\left(\ln\frac{b}{a}-1\right)-M\left(\ln\frac{M}{a}-1\right)\right] \tag{15}$$

The mean deviations about mean is obtained by substituting the mean, cdf and $T(\mu)$ in (12). The mean deviations about median is obtained by substituting the mean, cdf and $T(M)$ in (14).

3.4. Order Statistics

Let $X_{(1)}$,$X_{(2)}$,$X_{(3)}$,...,$X_{(n)}$ denote the order statistics of a random sample X_1,X_2,X_3,...,X_n drawn from the continuous distribution with pdf $f_X(x)$ and cdf $F_X(x)$, then the pdf of r^{th} order statistics $X_{(r)}$ is given by:

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{(r-1)}[1-F(x)]^{(n-r)} \tag{16}$$

Using the equations (5) and (6) the probability density function of r^{th} order statistics $X_{(r)}$ of Transmuted Log-Uniform distribution is given by:

$$f_{x_{(r)}}(x,a,b,\lambda) = \frac{n!}{(r-1)!(n-r)!}\left[\frac{(1+\lambda)}{x\ln(\frac{b}{a})} - \frac{(2\lambda)\ln(\frac{x}{a})}{x(\ln(\frac{b}{a}))^2}\right]\left[(1+\lambda)\frac{\ln(\frac{x}{a})}{\ln(\frac{b}{a})} - \lambda\left[\frac{\ln(\frac{x}{a})}{\ln(\frac{b}{a})}\right]^2\right]^{(r-1)}$$
$$\left[1-((1+\lambda)\frac{\ln(\frac{x}{a})}{\ln(\frac{b}{a})} - (\lambda)\left[\frac{\ln(\frac{x}{a})}{\ln(\frac{b}{a})}\right]^2)\right]^{(n-r)} \tag{17}$$

4. Parameter estimation

In this section, we discuss the method of maximum likelihood(ML) for the estimation of the unknown parameters a, b, λ of Transmuted Log-Uniform distribution. Let X_1,X_2,X_3,...,X_n be the random sample of size n drawn from Transmuted Log-Uniform distribution, then the likelihood function is given by:

$$L(x_i;a,b,\lambda) = \prod_{i=1}^{n}\frac{1}{x_i}\prod_{i=1}^{n}\left[\frac{1+\lambda}{ln(\frac{b}{a})} - \frac{2\lambda ln(\frac{x_i}{a})}{(ln(\frac{b}{a}))^2}\right] \tag{18}$$

The log-likelihood function is given by:

$$lnL(x_i;a,b,\lambda) = ln\left[\prod_{i=1}^{n}\frac{1}{x_i}\right] + ln\prod_{i=1}^{n}\left[\frac{1+\lambda}{ln(\frac{b}{a})} - \frac{2\lambda ln(\frac{x_i}{a})}{(ln(\frac{b}{a}))^2}\right] \tag{19}$$

Therefore, the maximum likelihood estimator of a, b and λ which maximize equation (19), must satisfy the following normal equations given by

$$\frac{\partial lnL}{\partial a} = \sum_{i=1}^{n}\frac{\ln(\frac{b}{a})(1+\lambda) - 4\lambda\ln(\frac{x_i}{a}) + 2\lambda\ln(\frac{b}{a})}{[(1+\lambda)\ln(\frac{b}{a}) - 2\lambda\ln(\frac{x_i}{a})]\ln(\frac{b}{a})} = 0 \tag{20}$$

$$\frac{\partial lnL}{\partial b} = \sum_{i=1}^{n}\frac{4\lambda\ln(\frac{x_i}{a}) - \ln(\frac{b}{a})(1+\lambda)}{[(1+\lambda)\ln(\frac{b}{a}) - 2\lambda\ln(\frac{x_i}{a})]b\ln(\frac{b}{a})} = 0 \tag{21}$$

$$\frac{\partial lnL}{\partial \lambda} = \sum_{i=1}^{n}\frac{\ln(\frac{b}{a}) - 2\ln(\frac{x_i}{a})}{(1+\lambda)\ln(\frac{b}{a}) - 2\lambda(\ln\frac{x_i}{a})} = 0 \tag{22}$$

Solving this system of equations, in a, b, β gives the MLEs of a, b, λ as $\hat{a}$,$\hat{b}$,$\hat{\lambda}$.

5. Estimation of Stress-Strength parameter

In this section, the procedure of estimating reliability of $R = P(X_2 < X_1)$ model is considered. The expression $R = P(X_2 < X_1)$ measures the reliability of a component in terms of probability and the random variables X_1 representing the stress experienced by the component does not exceed X_2 which represents the strength of the component. If stress exceeds strength, the component would fail and vice-versa.

In order to estimate the stress-strength parameter, considering two random variables X and Y with Transmuted Log-Uniform (λ_1, a, b) and Transmuted Log-Uniform (λ_2, a, b) distributions respectively. We assume that X and Y are independent random variables and the stress-strength parameter is obtained in the form:

$$R = P(Y < X) = \int_{X<Y} f(x,y)dxdy = \int_{0}^{\infty} f(x;\lambda_1,a,b)F(x;\lambda_2,a,b)dx$$

where $f(x,y)$ is the joint probability density function of random variables X and Y, having Transmuted Log-Uniform distribution so that:

$$R = \int_a^b \left[(1+\lambda_1)\left(\frac{\ln(x/a)}{\ln(b/a)}\right) - \lambda_1 \left(\frac{\ln(x/a)}{\ln(b/a)}\right)^2 \right] \left[\frac{(1+\lambda_2)}{x\ln(b/a)} - \frac{2\lambda_2 \ln(x/a)}{x(\ln(b/a))^2} \right] dx$$

On simplification we get:

$$R = \frac{(\lambda_1 - \lambda_2 + 3)}{6} \tag{23}$$

To compute the maximum likelihood estimate of R, we need to compute the maximum likelihood estimate of λ_1 and λ_2.Suppose $X_1, X_2, ..., X_n$ is random sample of size n from the Transmuted Log-Uniform (λ_1, a, b) and $Y_1, Y_2, ..., Y_m$ is an independent random sample of size m from Transmuted Log-Uniform (λ_2, a, b). Then the likelihood function of the combined random sample can be obtained as follows:

$$L = \prod_{i=1}^{n} f(x_i; \lambda_1, a, b) \prod_{i=1}^{m} f(y_i; \lambda_2, a, b)$$

$$L = \prod_{i=1}^{n} \frac{1}{x_i} \prod_{i=1}^{n} \left[\frac{1+\lambda_1}{ln(\frac{b}{a})} - \frac{2\lambda_1 ln(\frac{x_i}{a})}{(ln(\frac{b}{a}))^2} \right] \prod_{i=1}^{m} \frac{1}{x_i} \prod_{i=1}^{n} \left[\frac{1+\lambda_2}{ln(\frac{b}{a})} - \frac{2\lambda_2 ln(\frac{x_i}{a})}{(ln(\frac{b}{a}))^2} \right] \tag{24}$$

The log-likelihood function is given by:

$$lnL = ln\left[\prod_{i=1}^{n} \frac{1}{x_i} \right] + ln \prod_{i=1}^{n} \left[\frac{1+\lambda_1}{ln(\frac{b}{a})} - \frac{2\lambda_1 ln(\frac{x_i}{a})}{(ln(\frac{b}{a}))^2} \right] + ln\left[\prod_{i=1}^{m} \frac{1}{x_i} \right] + ln \prod_{i=1}^{m} \left[\frac{1+\lambda_2}{ln(\frac{b}{a})} - \frac{2\lambda_2 ln(\frac{x_i}{a})}{(ln(\frac{b}{a}))^2} \right] \tag{25}$$

The maximum likelihood estimate (MLE) of λ_1 and λ_2 can be obtained by solving the following equations:

$$\frac{\partial lnL}{\partial \lambda_1} = \sum_{i=1}^{n} \frac{\ln(\frac{b}{a}) - 2\ln(\frac{x_i}{a})}{(1+\lambda_1)\ln(\frac{b}{a}) - 2\lambda_1(\ln\frac{x_i}{a})} = 0 \tag{26}$$

$$\frac{\partial lnL}{\partial \lambda_2} = \sum_{i=1}^{m} \frac{\ln(\frac{b}{a}) - 2\ln(\frac{x_i}{a})}{(1+\lambda_2)\ln(\frac{b}{a}) - 2\lambda_2(\ln\frac{x_i}{a})} = 0 \tag{27}$$

From the equations (26) and (27), we can obtain the ML estimates $\hat{\lambda_1}$ and $\hat{\lambda_1}$. The corresponding ML estimate of R is computed from (23) by replacing λ_1 and λ_2 by their ML estimates

$$\hat{R} = \frac{(\hat{\lambda_1} - \hat{\lambda_2} + 3)}{6} \tag{28}$$

This can be used in estimation of stress-strength for the given data.

6. Simulation study and Data analysis

6.1. Simulation study

Simulation studies are an important tool for statistical research. These help researchers assess the performance of a model, understand the different properties of statistical methods and compare them. Here we take distinct combinations of parameters a, b, λ with sample size as bias and the mean square error(MSE) of the parameter estimates.

The simulation is done by using different true parameter values. The chosen true parameter values are as follows:

- $a = 7.5$,$b = 16$,$\lambda = -0.25$
  As the n increases, MSE decreases for the selected parameter values given in table 1. Moreover, the bias is close to zero as the sample size increases. Thus, as the sample size increases the estimates tend to be closer to the true parameter values.

Table 1: *Simulation study at a = 7.5, b = 16,* $\lambda = -0.25$

n	Parameter	Estimate	Bias	MSE
250	a	8.5628	1.0628	1.1296
	b	14.8466	1.1533	1.3302
	λ	0.3504	0.6004	0.3604
350	a	8.5470	1.0470	1.0962
	b	15.8661	0.9338	0.1790
	λ	0.1411	0.3912	0.1530
500	a	8.3061	0.8061	0.6499
	b	16.2075	0.2075	0.0430
	λ	0.0465	0.2965	0.0879
600	a	7.4086	0.0913	0.0083
	b	16.2010	0.2010	0.0404
	λ	-0.2993	0.0493	0.0024
750	a	7.4955	0.0044	0.0000193
	b	16.0647	0.0647	0.004182
	λ	-0.2533	0.00331	0.0000109

6.2. Data analysis

The data set given in Table 2 represents the relief times (in minutes) of twenty patients receiving an analgesic Gross and Clark (1975). We fit the Transmuted Log-Uniform distribution to a real life data set and compare the results with the Transmuted Quasi Akash distribution and Transmuted Weibull distribution.

Table 2: *Relief times of 20 patients receiving an analgesic*

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1.1	1.4	1.3	1.7	1.9
1.8	1.6	2.2	1.7	2.7
4.1	1.8	1.5	1.2	1.4
3.0	1.7	2.3	1.6	2.0

Table 3: *AIC, CAIC, BIC,and HQIC statistics of the fitted model in data set*

Distribution	AIC	CAIC	BIC	HQIC
Transmuted Log-Uniform Distribution	6.0016	8.1258	5.9790	8.1834
Transmuted Quasi Akash Distribution	49.79	51.78	50.18	50.50
Transmuted Weibull Distribution	63.3218	65.446	63.299	65.503

From the table 3, it has been observed that the Transmuted Log-Uniform Distribution possesses the lesser AIC, CAIC, BIC,and HQIC values as compared to Transmuted Quasi Akash distribution and Transmuted Weibull distribution. To check the model goodness of fit we had considered the Kolmogorov-Smirnov (K-S) test (goodness-of-fit) statistics for the relief times of patients receiving an anelgesic data.

To determine the Goodness of fit of the models, the magnitude of K-S Statistic is obtained. Since the p-value of fitted model is highest than the other distributions we have considered.Therefore the results indicate, that the proposed model performed better than other models.

7. Conclusion

In this study, we introduced a new distribution called the Transmuted Log-Uniform distribution. The distribution was generated using the Transmuted technique and the Log-Uniform distribution as the base distribution with two parameters. We obtained graphical representations of its pdf, cdf, hazard rate function and survival function. We have derived statistical properties such as moments, mean deviations, and the quantile function for the Transmuted Log-Uniform distribution. We also obtained the order statistics of the new distribution.

We used the maximum likelihood method to estimate the parameters of the distribution and the stress strength parameters. We performed a simulation study to validate the estimates of the model parameters, and it was observed that the distribution showed the least bias, with the values of mean square error decreasing as the sample size increased. Finally, we applied the Transmuted Log-Uniform distribution to a real data set and compared it with Transmuted Weibull distribution and Transmuted Quasi-Akash distribution. It was found that the new distribution was a better fit than these distributions based on the values obtained for the AIC, CAIC, BIC, HQIC, the Kolmogorov-Smirnov (K-S) goodness-of-fit statistic, and the p-values obtained for the models.

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