BAYESIAN ESTIMATION OF TOPP-LEONE LINDLEY (TLL) DISTRIBUTION PARAMETERS UNDER DIFFERENT LOSS FUNCTIONS USING LINDLEY APPROXIMATION

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Abstract

In this study, we present the Bayesian estimates of the unknown parameters of the Topp-Leone Lindley distribution using the maximum likelihood and Bayesian methods. In this study, the Bayes theorem was adopted for obtaining the posterior distribution of the shape parameter and scale parameter of the Topp-Leone Lindley distribution assuming the Jeffreys' (non-informative) prior for the shape parameter and the Gamma (conjugate) prior for the scale parameter under three different loss functions namely: Square Error Loss Function, Linear Exponential Loss Function and Generalized Entropy Loss Function. The posterior distribution derived for both parameters are not solvable analytically, it requires a numerical approximation techniques to obtain the solution. The Lindley approximation techniques was adopted to obtain the parameters of interest. The loss function were used to derive the estimates of both parameters with an assumption that the both parameters are unknown and independent. To ascertain the accuracy of these estimators, the proposed Bayesian estimators under different loss functions are compared with the corresponding maximum likelihood estimator using a Monte Carlo simulation on the performance of these estimators according to the mean square error and BIAS based on simulated samples simulated from the Topp-Leone Lindley distribution. . It was also observed for any fixed value of the parameters, as sample size increases, the mean square errors of the Bayesian Estimates and maximum likelihood estimates decrease. Also, the maximum likelihood estimates and Bayesian estimates converge to the same value as the sample gets larger except for Generalized Entropy Loss Function.

Keywords: Bayesian estimation, Prior Distribution, Loss Functions, Lindley's Approximation, Topp-Leone Lindley distribution

1. INTRODUCTION

Topp and Leone [1] introduced a distribution with finite support whose cumulative distribution function (cdf) has a closed form-expression called the Topp-Leone (the J-Shaped) distribution. This distribution has been used to model several phenomenon representing the time until the occurrence of a particular event. Data from such studies are called the survival data or lifetime data. Nadarajah and Kotz [2] studied and disclosed the usefulness of the Topp-Leone distribution in the analysis of interval-bounded data. In their study of the mathematical properties, it was observed that the Topp-Leone distribution exhibit bathtub failure rate functions and the closed form of the moments werederived, which disclosed the wide range of its applications in reliability study. The disclosure of the important properties of the Topp-Leone distribution by Nadarajah and Kotz [2] has attracted the interest of authors which is evident in statistical literature. For instance, see the work of Ghitany et al [3], Zhou et al [4], Kotz and Seier [5], Nadarajah [6], Zghoul [7], amongst others.The cumulative frequency distribution (cdf) and probability density function (pdf) of the Topp-Leone (TL) distribution are respectively given as

$$G(t) = t^{\alpha}(2-t)^{\alpha} = [t(2-t)]^{\alpha} = \left[1-(1-t)^2\right]^{\alpha}, \qquad 0<t<1, \quad \alpha>0 \tag{1}$$

and

$$G(t) = 2\alpha(2-t)\left[1-(1-t)^2\right]^{\alpha-1} \qquad 0<t<1, \quad \alpha>0 \tag{2}$$

The TL distribution is on a unit interval support $(0,1)$; this means that it cannot be used in the analysis of survival data, which are not on a unit interval support. To overcome this setback, Al-Shomrani et al [8] presented the Topp-Leone generated family of distribution with cdf and pdf given as

$$G(x;\alpha,\Phi) = F(x;\Phi)^{\alpha}(2-F(x;\Phi))^{\alpha} = \left[1-\left(\bar{F}(x;\Phi)\right)^2\right]^{\alpha}, x>0, \ \alpha>0 \tag{3}$$

and

$$g(x;\alpha,\Phi) = 2\alpha f(x)\bar{F}(x;\Phi)\left[1-\left(\bar{F}(x;\Phi)\right)^2\right]^{\alpha-1}, \qquad x>0, \ \alpha>0 \tag{4}$$

Where $f(x;\Phi)$, $F(x;\Phi)$ and $\bar{F}(x;\Phi)$ are respectively the pdf, cdf and survival functions of the baseline distribution and Φ is the vector of parameters of the baseline distribution. Nzei and Ekhosuehi [9] used the logit of the TL-G family to presented the Topp-Leone Lindley (TL-L) distribution withthe probability density function (pdf) and the cumulative distribution function (cdf) for the Topp-Leone Lindley (TL-L) distribution respectively expressed as;

$$g(x) = \frac{2\alpha\theta^2}{\theta+1}(1+x)\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-2\theta x}\left\{1-\left[\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^2\right\}^{\alpha-1}, \ x>0, \ \alpha, \ \theta>0 \tag{5}$$

and

$$G(x) = \left\{1-\left[\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^2\right\}^{\alpha}, \ x>0, \ \alpha, \ \theta>0 \tag{6}$$

The Reliability (survival) function of the TL-L distribution is given as

$$R(x) = 1-\left\{1-\left[\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^2\right\}^{\alpha} \tag{7}$$

In addition, the corresponding hazard rate function of the TL-L distributionis expressed as

$$h(x) = \frac{\frac{2\alpha\theta^2}{\theta+1}(1+x)\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-2\theta x}\left\{1-\left[\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^2\right\}^{\alpha-1}}{1-\left\{1-\left[\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^2\right\}^{\alpha}} \tag{8}$$

The CDF, pdf and hazard rate function of the TL-L distribution are shown in Figure (1), (2) and (3) respectively for different values of the parameters α and θ.

The aim of this study is to obtainthe Bayesian estimates of the parameters α and θ for TL-L distribution under different loss functions. The Bayesian framework is considered under the square errorloss function (SELF) presented by Legendre [10] and Gauss [11], linear exponential (LINEX) loss function presented by Varian [12] and general entropy loss function (GELF) presented by Calabria and Pulcini [13] to obtain the Bayes estimators of the unknown parameters α and θ.

2. THE MAXIMUM LIKELIHOOD ESTIMATION (MLE)

Let x_i, $i = 1, 2, 3, \cdots, n$ be a random sample from the TL-L distribution, then the maximum likelihood function of (5) denoted by $L(x, \delta)$ is defined as:

$$L(x;\alpha,\theta) = \prod_{i=1}^{n}\left\{\frac{2\alpha\theta^2}{\theta+1}(1+x)\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-2\theta x}\left[1-\left(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\right]^{\alpha-1}\right\} \tag{9}$$

and the log-likelihood function denoted by $\ell_n(x,\alpha,\theta)$ is given as

$$\ell_n(x,\alpha,\theta) = n\ln\left(2\alpha\theta^2\right) - n\ln(1+\theta) - 2\theta\sum_{i=1}^{n}x + \sum_{i=1}^{n}\ln(1+x) + \sum_{i=1}^{n}\ln\left(\frac{\theta+1+\theta x}{\theta+1}\right)$$
$$+(\alpha-1)\sum_{i=1}^{n}\ln\left[1-\left(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\right] \tag{10}$$

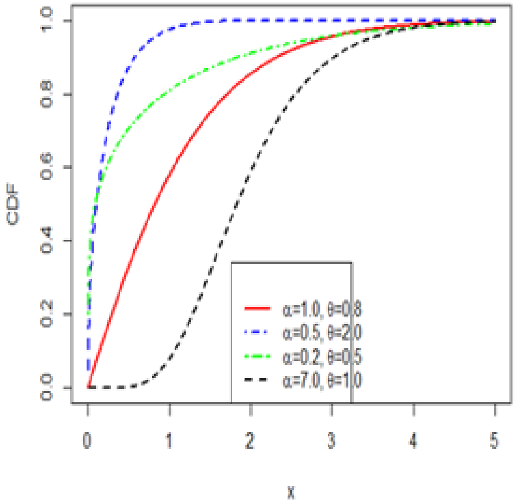

Figure 1: *The CDF of TL-L Distribution*

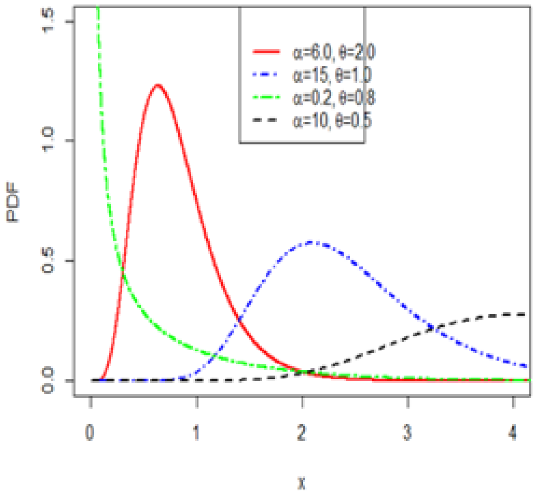

Figure 2: *The PDF of TL-L Distribution*

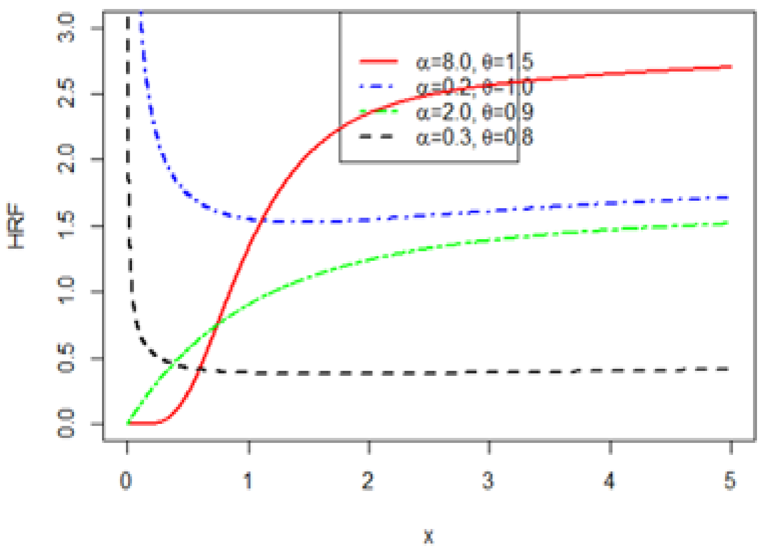

Figure 3: *The HRF of TL-L Distribution*

To obtain the MLEs of the TL-L, we solve the equations of the partial derivatives of the log-likelihood function with respect to the parameters $\frac{\partial \ell n}{\partial \alpha} = 0$ and $\frac{\partial \ell n}{\partial \theta} = 0$. These partial derivatives with respect to the parameters α and θ are:

$$\frac{\partial \ell n}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln\left[1 - \left(\frac{\theta + 1 + \theta x}{\theta + 1} e^{-\theta x}\right)^2\right] \tag{11}$$

$$\frac{\partial \ell n}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{\theta+1} - 2\sum_{i=1}^{n} x + \frac{1}{\theta+1}\sum_{i=1}^{n}\frac{x}{\theta+1+\theta x} + 2\theta(\alpha-1)\sum_{i=1}^{n}\frac{(\theta+1+\theta x)e^{-2\theta x}}{(\theta+1)\left[(\theta+1)^2 - (\theta+1+\theta x)e^{-2\theta x}\right]} \tag{12}$$

The solution of $\frac{\partial \ell n}{\partial \alpha} = 0$, is the MLE of $\hat{\alpha}$ which given as

$$\hat{\alpha} = \frac{-n}{\sum_{i=1}^{n} \ln\left[1 - \left(\frac{\theta + 1 + \theta x}{\theta + 1} e^{-\theta x}\right)^2\right]} \tag{13}$$

By replacing α (12) in $\frac{\partial \ell n}{\partial \theta} = 0$ with estimate in (13), we have expression in terms of the parameter θ as

$$\frac{2n}{\theta} - \frac{n}{\theta+1} - 2\sum_{i=1}^{n} x + \frac{1}{\theta+1}\sum_{i=1}^{n}\frac{x}{\theta+1+\theta x} + 2\theta\left(\frac{-n}{\sum_{i=1}^{n} \ln\left[1 - \left(\frac{\theta + 1 + \theta x}{\theta + 1} e^{-\theta x}\right)^2\right]} - 1\right)\sum_{i=1}^{n}\frac{(\theta+1+\theta x)e^{-2\theta x}}{(\theta+1)\left[(\theta+1)^2 - (\theta+1+\theta x)e^{-2\theta x}\right]} = 0 \tag{14}$$

Obviously, (14) is a complex equation, which cannot be solved analytically. Hence, solving (12) and (13) simultaneously to obtain the maximum likelihood estimates of α and θ requires iterative approach such as Newton-Raphson iterative scheme as presented by Obisesan et al [14] and Bakari et al [15] amongst others. This Newton-Raphson method can be performed with R-Software package.

3. BAYESIAN ESTIMATION (BE)

The main belief of Bayesian statistics that distinguishes it from the classical statistics is that it consider the parameter(s) of the given model to be random variables with prior distribution denoted by $\pi(\Phi)$ .In this Section, we discuss the Bayesian estimates for the parameters of the TL-L distribution using the Jeffreys' (non-nformative) prior for α and the Gamma (conjugate) prior for θ under some loss functions namely; squared error loss function (SELF), linear exponential loss function (LINEX) andgeneral entropy loss function (GELF). We discuss these loss functions and the priorsbriefly as follows:

3.1 The Square Error Loss Function (SELF)

The square error loss function, which is the simplest and the most commonly used symmetric loss function in the literature by authors, see Rastogi and Merovci [16] and Sangeeta et al [17] amongst others. It isdefined as

$$L_{SELF}(\hat{\Phi}, \Phi) = (\hat{\Phi} - \Phi)^2 \tag{15}$$

The Bayesian estimate under SELF is $\hat{\Phi}_{BSELF} = E_{\Phi}(\Phi \mid \underline{x})$.

This is the expectation considered with regard to the posterior density. SELF assigns the same magnitude of error to both over estimation and under estimation because of its symmetric nature, which is not always true in many practical scenario Kaur et al. [18].

3.2 The Linear Exponential Loss Function (LINEX)

Varian [23] presented an asymmetric loss function defined as

$$L_{LINEX}\left(\hat{\Phi},\Phi\right) = e^{m(\hat{\Phi}-\Phi)} - m(\hat{\Phi}-\Phi) - 1 \tag{16}$$

Where $m \neq 0$ is the shape parameter of the LINEX loss function. Zellner[19] studied the properties of this loss function and showed that for $m > 0$, over estimation is more costly than under estimation. When $m < 0$, the loss function increases almost exponentially for $d < 0$ and almost linearly for $d > 0$, where $d = \hat{\Phi} - \Phi$. The Bayesian estimate under the LINEX loss function is given as

$$\hat{\Phi}_{BLINEX} = -\frac{1}{m}\ln\left[E_{\Phi}\left(e^{-m\Phi} \mid \underline{x}\right)\right] \tag{17}$$

3.3 The General Entropy Loss Function (GELF)

The general entropy loss function (GELF) was proposed by Calabria and Pulcini [13] as an alternative to the modified LINEX loss function and it is defined as

$$L_{GELF}\left(\hat{\Phi},\Phi\right) = \left(\frac{\hat{\Phi}}{\Phi}\right)^{k} - k\ln\left(\frac{\hat{\Phi}}{\Phi}\right) - 1 \tag{18}$$

Where $k \neq 0$ and it determines the shape of the loss function. When $k < 0$, it shows there is more of under estimation than over estimation. On the other hand, when $k > 0$ shows more of over estimation than under estimation. The Bayes estimate of Φ under the general entropy loss function is given as

$$\hat{\Phi}_{BGELF} = \left[E_{\Phi}\left(\Phi^{-k} \mid \underline{x}\right)\right]^{-\frac{1}{k}} \tag{19}$$

It is important to note that for $k = -1$, $\hat{\Phi}_{GELF} = \hat{\Phi}_{SELF}$ i.e. the general entropy loss function reduces to the square error loss function at $k = -1$.

3.4 Prior Distributions:

The choice of prior distribution for an unknown parameter(s) is an important part of Bayesian statistics. For the Bayes estimate of the parameters α and θ, we consider the Jeffreys' (non-informative) prior for α and the Gamma (conjugate) prior for θ. Then the prior distributions are defined below as:

$$\pi_1(\Phi) \propto \sqrt{I(\Phi)} \tag{20}$$

Where $I(\Phi) = -E\left\{\frac{\partial^2 \ell n}{\partial \Phi^2}\right\}$ which is the Fisher's Information. For the TL-L distribution, the Jeffreys' prior of α is defined as

$$\pi_1(\alpha) = \frac{1}{\alpha} \qquad \alpha > 0 \tag{21}$$

and

$$\pi_2(\theta) = \frac{q^p}{\Gamma(p)} \theta^{p-1} e^{-q\theta} \qquad \theta > 0,\ p > 0,\ q > 0 \tag{22}$$

The joint prior distribution of the parameters α and θ is defined as a combination of the priors as

$$\pi(\alpha, \theta) = \frac{q^p}{\alpha\, \Gamma(p)} \theta^{p-1} e^{-q\theta} \tag{23}$$

3.5 Posterior Distribution

The posterior distribution function of an unknown probability distribution parameter Φ is the formula used to compute the conditional probability density of the distribution parameter Φ given the data $X = x$ through the Bayes formula defined as

$$P(\Phi \mid \underline{x}) = \frac{L(x \mid \Phi)\pi(\Phi)}{\int_{\Theta} L(x \mid \Phi)\pi(\Phi) d\Phi} \tag{24}$$

Where the prior distribution of the unknown parameter is $\pi(\Phi)$, $L(x \mid \Phi)$ is the likelihood function of the density of X and Φ is vector of the unknown parameter. Then the posterior distribution of the TL-L distribution parameters α and θ is obtained by substituting (9) and (23) into (24) to be

$$P(\alpha, \theta \mid \underline{x}) = \frac{\frac{2^n \alpha^{n-1} \theta^{2n}}{(\theta+1)^n \Gamma(p)} \prod_{i=1}^{n} (1+x)\left(\frac{\theta+1+\theta x}{\theta+1}\right)\left\{1-\left[\frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}\right]^2\right\}^{\alpha-1} e^{-\theta(2x+q)}}{\int_0^\infty \int_0^\infty \frac{2^n \alpha^{n-1} \theta^{2n}}{(\theta+1)^n \Gamma(p)} \prod_{i=1}^{n} (1+x)\left(\frac{\theta+1+\theta x}{\theta+1}\right)\left\{1-\left[\frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}\right]^2\right\}^{\alpha-1} e^{-\theta(2x+q)} d\alpha\, d\theta} \tag{25}$$

Obviously, the posterior distribution in (25) for the estimation of TL-L parameters, α and θ is in a rational form which cannot be reduced to a closed form, making tedious to evaluate the posterior distribution in order to obtain the Baye's estimators. However, one can used the approach developed by Lindley [20], to approximate these Bayes estimators.

3.6 Lindley's Approximation

Lindley [20] developed a method for reducing the posterior distribution in Bayesian estimation, which involves integral that can't be expressed in closed form. This method provides a simplified form of Bayesian estimator, which makes it easier to apply in practice. Several authors have used the Lindley approximation to obtain the Bayes estimate for some lifetime distribution in the literature; amongst whom are Hummara and Ahmad [21], Adegoke et al ([22], [23]), Kamran et al [24], Bashiru et al [25], etc. Lindley developed an asymptotic approximation to the ratio

$$I(X) = \frac{\int_{(\alpha,\, \theta)} Z(\alpha, \theta) e^{L(\alpha, \theta) + U(\alpha, \theta)} \partial(\alpha, \theta)}{\int_{(\alpha,\, \theta)} e^{L(\alpha, \theta) + U(\alpha, \theta)} \partial(\alpha, \theta)} \tag{26}$$

Where $Z(\alpha, \theta)$ is a function of the distribution parameter α and θ, $L(\alpha, \theta)$ is the log-likelihood function and $U(\alpha, \theta)$ is the log of the prior distribution function $\pi(\alpha, \theta)$. Therefore, $I(X)$ is evaluated as

$$I(X) = Z(\alpha,\theta) + \frac{1}{2}\left[Z_{11}\sigma_{11} + Z_{22}\sigma_{22}\right] + \left(U_1 Z_1 \sigma_{11} + U_2 Z_2 \sigma_{22}\right) + \frac{1}{2}\left[L_{111} Z_1 \sigma_{11}^2 + L_{222} Z_2 \sigma_{22}^2\right]$$
$$+ \frac{1}{2}\left[L_{122} Z_1 \sigma_{11}\sigma_{22} + L_{112} Z_2 \sigma_{11}\sigma_{22}\right] \tag{27}$$

Therefore, for an unknown parameter α, the Lindley approximation is can be expressed as

$$E[Z(\alpha \mid \underline{x})] = Z(\hat{\alpha}, \hat{\theta}) + \frac{1}{2}\left[Z_{11}\sigma_{11}\right] + \left(U_1 Z_1 \sigma_{11}\right) + \frac{1}{2}\left[L_{111} Z_1 \sigma_{11}^2 + L_{122} Z_1 \sigma_{11}\sigma_{22}\right] \tag{28}$$

Similarly, for an unknown parameter θ, the Lindley approximation is can be expressed as

$$E[Z(\theta|\underline{x})]=Z(\hat{\alpha},\hat{\theta})+\frac{1}{2}[Z_{22}\sigma_{22}]+(U_2Z_2\sigma_{22})+\frac{1}{2}[L_{222}Z_2\sigma_{22}^2+L2_{221}Z_2\sigma_{11}\sigma_{22}] \quad (29)$$

Where the elements of the Lindley approximation in (27 - 29) are as given below

$$Z_1=\frac{\partial Z(\alpha,\theta)}{\partial\alpha};\quad Z_2=\frac{\partial Z(\alpha,\theta)}{\partial\theta};\quad Z_{11}=\frac{\partial^2 Z(\alpha,\theta)}{\partial\alpha^2};\quad Z_{22}=\frac{\partial^2 Z(\alpha,\theta)}{\partial\theta^2};\quad Z_{12}=\frac{\partial^2 Z(\alpha,\theta)}{\partial\alpha\partial\theta}$$

and $Z_{21}=\frac{\partial^2 Z(\alpha,\theta)}{\partial\theta^2\partial\alpha}$.

$$P(\alpha,\theta)=\ln\pi(\alpha,\theta)=p\ln q-\ln\Gamma(p)-\ln\alpha+(p-1)\ln\theta-q\theta;$$

$$U_1=\frac{\partial P(\alpha,\theta)}{\partial\alpha}=-\frac{1}{\alpha}\quad\text{and}\quad U_2=\frac{\partial P(\alpha,\theta)}{\partial\theta}=\frac{p-1}{\theta}-q.$$

$$L_{111}=\frac{\partial^3\ln L(\alpha,\theta)}{\partial\alpha^3}=\frac{2n}{\alpha^3}$$

$$L_{222}=\frac{\partial^3\ln L(\alpha,\theta)}{\partial\theta^3}=\frac{2n}{\theta^3}-\frac{2n}{(\theta+1)^3}+\frac{1}{\theta+1}\sum_{i=1}^{n}\frac{x(1+x)^2}{(\theta+1+\theta x)^3}-\frac{2}{(\theta+1)^2}\sum_{i=1}^{n}\frac{x(1+x)}{(\theta+1+\theta x)^2}$$
$$+\frac{2}{(\theta+1)^3}\sum_{i=1}^{n}\frac{x}{(\theta+1+\theta x)}+\frac{2(\alpha+1)}{\theta+1}\sum_{i=1}^{n}\frac{CBA''-CAB''-C'A'B+C'B'A}{C^2}-\frac{2(\alpha-1)}{(\theta+1)^2}\sum_{i=1}^{n}\frac{BA'-AB'}{B^2}$$

$$L_{122}=\frac{\partial^3\ln L(\alpha,\theta)}{\partial\alpha\partial\theta^2}=\frac{2}{\theta+1}\sum_{i=1}^{n}\frac{BA'-AB'}{B^2}-\frac{2}{(\theta+1)^2}\sum_{i=1}^{n}\frac{A}{B}$$

$$L_{112}=\frac{\partial^3\ln L(\alpha,\theta)}{\partial\alpha^2\partial\theta}=0$$

$$\sigma_{11}=-\frac{1}{L_{11}}=\frac{\alpha^2}{n}\quad\text{and}$$

$$\sigma_{22}=-\frac{1}{L_{22}}=\left[\frac{n}{\theta^2}-\frac{n}{(\theta+1)^2}-\frac{1}{\theta+1}\sum_{i=1}^{n}\frac{x(1+x)}{(\theta+1+\theta x)^2}+\frac{2}{(\theta+1)^2}\sum_{i=1}^{n}\frac{x}{(\theta+1+\theta x)}-\frac{2(\alpha+1)}{(\theta+1)}\right.$$
$$\left.+\sum_{i=1}^{n}\frac{BA'-AB'}{B^2}\ \frac{2(\alpha-1)}{(\theta+1)^2}\sum_{i=1}^{n}\frac{A}{B}\right]^{-1}$$

Where

$$A=x(\theta+1+\theta x)[(\theta+1)(\theta+1+\theta x)-1]e^{-2\theta x}$$
$$A'=x\left\{[(\theta+1)(\theta+1+\theta x)-1](1-x-2\theta x-2\theta x^2)+(\theta+1+\theta x)[(\theta+1+\theta x)+(\theta+1)(1+x)]\right\}e^{-2\theta x}$$
$$B=(\theta+1)^2-(\theta+1+\theta x)^2e^{-2\theta x}$$
$$B'=2(\theta+1)+2(\theta+1+\theta x)(\theta x^2+\theta x-1)e^{-2\theta x}$$
$$C=\left\{(\theta+1)^2-(\theta+1+\theta x)^2e^{-2\theta x}\right\}^2$$

$$C' = 2\left((\theta+1)^2 - (\theta+1+\theta x)^2 e^{-2\theta x} \right)\left\{ 2(\theta+1) - 2(\theta+1+\theta x)\left(\theta x^2\ \theta x - 1\right)e^{-2\theta x} \right\}$$

3.7 Lindley Approximation under the Different Loss Functions

In this section, we consider the Bayes estimators of the TL-L parameters α and θ are obtained assuming that both α and θ are unknown, using the prior in (23) under three different loss functions:

3.7.1 Under Squared Error Entropy Loss Function

a) For the parameter α, it can be seen from the SELF estimator that $Z(\alpha,\theta)=\alpha$, then $Z_1=1$, and $Z_2=Z_{11}=Z_{22}=0$, we have

$$\hat{\alpha}_{SELF} = E(\alpha|\underline{x}) = \alpha\left[1+\frac{\alpha}{2n}H_1 H_2\right] \tag{30}$$

Where $H_1 = \frac{2}{\theta+1}\sum_{i=1}^{n}\frac{BA'-AB'}{B^2} - \frac{2}{(\theta+1)^2}\sum_{i=1}^{n}\frac{A}{B}$ and

$$H_2 = \left[\frac{n}{\theta^2} - \frac{n}{(\theta+1)^2} - \frac{1}{\theta+1}\sum_{i=1}^{n}\frac{x(1+x)}{(\theta+1+\theta x)^2} + \frac{2}{(\theta+1)^2}\sum_{i=1}^{n}\frac{x}{(\theta+1+\theta x)} - \frac{2(\alpha+1)}{(\theta+1)} + \sum_{i=1}^{n}\frac{BA'-AB'}{B^2}\ \frac{2(\alpha-1)}{(\theta+1)^2}\sum_{i=1}^{n}\frac{A}{B}\right]^{-1}$$

b) For the parameter θ, it can be seen from the SELF estimator that $Z(\alpha,\theta)=\theta$, then $Z_2=1$, and $Z_1=Z_{11}=Z_{22}=0$, we have

$$\hat{\theta}_{SELF} = E[Z(\theta|\underline{x})] = \theta + H_2\left(\frac{p-1}{\theta} - q + \frac{1}{2}H_2 H_3\right) \tag{31}$$

Where

$$H_2 = \left[\frac{n}{\theta^2} - \frac{n}{(\theta+1)^2} - \frac{1}{\theta+1}\sum_{i=1}^{n}\frac{x(1+x)}{(\theta+1+\theta x)^2} + \frac{2}{(\theta+1)^2}\sum_{i=1}^{n}\frac{x}{(\theta+1+\theta x)} - \frac{2(\alpha+1)}{(\theta+1)} + \sum_{i=1}^{n}\frac{BA'-AB'}{B^2}\ \frac{2(\alpha-1)}{(\theta+1)^2}\sum_{i=1}^{n}\frac{A}{B}\right]^{-1}$$

and

$$H_3 = \frac{2n}{\theta^3} - \frac{2n}{(\theta+1)^3} + \frac{1}{\theta+1}\sum_{i=1}^{n}\frac{x(1+x)^2}{(\theta+1+\theta x)^3} - \frac{2}{(\theta+1)^2}\sum_{i=1}^{n}\frac{x(1+x)}{(\theta+1+\theta x)^2} + \frac{2}{(\theta+1)^3}\sum_{i=1}^{n}\frac{x}{(\theta+1+\theta x)} + \frac{2(\alpha+1)}{\theta+1}\sum_{i=1}^{n}\frac{CBA''-CAB''-C'A'B+C'B'A}{C^2} - \frac{2(\alpha-1)}{(\theta+1)^2}\sum_{i=1}^{n}\frac{BA'-AB'}{B^2}$$

3.7.2 Under LINEX Loss Function

a) For the parameter α, it can be seen from the LINEX estimator that $Z(\alpha,\theta)=e^{-m\alpha}$, then $Z_1=-me^{-m\alpha}$, $Z_{11}=m^2e^{-m\alpha}$ and $Z_2=Z_{22}=0$ we have

$$\hat{\alpha}_{LINEX}=E\left(e^{-m\alpha}\,|\,\underline{x}\right)=e^{-m\alpha}\left[1+\frac{m\alpha}{2}\left(\alpha-H_1H_2\right)\right] \tag{32}$$

Where

$$H_1=\frac{2}{\theta+1}\sum_{i=1}^{n}\frac{BA'-AB'}{B^2}-\frac{2}{(\theta+1)^2}\sum_{i=1}^{n}\frac{A}{B} \quad \text{and}$$

$$H_2=\left[\frac{n}{\theta^2}-\frac{n}{(\theta+1)^2}-\frac{1}{\theta+1}\sum_{i=1}^{n}\frac{x(1+x)}{(\theta+1+\theta x)^2}+\frac{2}{(\theta+1)^2}\sum_{i=1}^{n}\frac{x}{(\theta+1+\theta x)}-\frac{2(\alpha+1)}{(\theta+1)}\right.$$
$$\left.+\sum_{i=1}^{n}\frac{BA'-AB'}{B^2}\;\frac{2(\alpha-1)}{(\theta+1)^2}\sum_{i=1}^{n}\frac{A}{B}\right]^{-1}$$

b) For the parameter θ, it can be seen from the LINEX estimator that $Z(\alpha,\theta)=e^{-m\theta}$, then $Z_2=-me^{-m\theta}$, $Z_{22}=m^2e^{-m\theta}$ and $Z_1=Z_{11}=0$, we have

$$\hat{\theta}_{LINEX}=E\left(e^{-m\theta}\,|\,\underline{x}\right)=e^{-m\theta}\left\{1+mH_2\left[\frac{1}{2}\left(m-H_2H_3\right)+q-\frac{p-1}{\theta}\right]\right\} \tag{33}$$

Where

$$H_2=\left[\frac{n}{\theta^2}-\frac{n}{(\theta+1)^2}-\frac{1}{\theta+1}\sum_{i=1}^{n}\frac{x(1+x)}{(\theta+1+\theta x)^2}+\frac{2}{(\theta+1)^2}\sum_{i=1}^{n}\frac{x}{(\theta+1+\theta x)}-\frac{2(\alpha+1)}{(\theta+1)}\right.$$
$$\left.+\sum_{i=1}^{n}\frac{BA'-AB'}{B^2}\;\frac{2(\alpha-1)}{(\theta+1)^2}\sum_{i=1}^{n}\frac{A}{B}\right]^{-1}$$

and

$$H_3=\frac{2n}{\theta^3}-\frac{2n}{(\theta+1)^3}+\frac{1}{\theta+1}\sum_{i=1}^{n}\frac{x(1+x)^2}{(\theta+1+\theta x)^3}-\frac{2}{(\theta+1)^2}\sum_{i=1}^{n}\frac{x(1+x)}{(\theta+1+\theta x)^2}+\frac{2}{(\theta+1)^3}\sum_{i=1}^{n}\frac{x}{(\theta+1+\theta x)}$$
$$+\frac{2(\alpha+1)}{\theta+1}\sum_{i=1}^{n}\frac{CBA''-CAB''-C'A'B+C'B'A}{C^2}-\frac{2(\alpha-1)}{(\theta+1)^2}\sum_{i=1}^{n}\frac{BA'-AB'}{B^2}$$

3.7.3 Under GELF Loss Function

a) For the parameter α, it can be seen from the GELF estimator that $Z(\alpha,\theta)=\alpha^{-k}$, then $Z_1=-k\alpha^{-(k+1)}$, $Z_{11}=k(k+1)e^{-(k+2)}$ and $Z_2=Z_{22}=0$ we have

$$\hat{\alpha}_{GELF}=E\left(\alpha^{-k}\,|\,\underline{x}\right)=\alpha^{-k}\left[1+\frac{k\alpha}{2n}\left(k+1-H_1H_2\right)\right] \tag{34}$$

Where

$$H_1 = \frac{2}{\theta+1}\sum_{i=1}^{n}\frac{BA'-AB'}{B^2} - \frac{2}{(\theta+1)^2}\sum_{i=1}^{n}\frac{A}{B} \quad \text{and}$$

$$H_2 = \left[\frac{n}{\theta^2} - \frac{n}{(\theta+1)^2} - \frac{1}{\theta+1}\sum_{i=1}^{n}\frac{x(1+x)}{(\theta+1+\theta x)^2} + \frac{2}{(\theta+1)^2}\sum_{i=1}^{n}\frac{x}{(\theta+1+\theta x)} - \frac{2(\alpha+1)}{(\theta+1)} + \sum_{i=1}^{n}\frac{BA'-AB'}{B^2}\frac{2(\alpha-1)}{(\theta+1)^2}\sum_{i=1}^{n}\frac{A}{B}\right]^{-1}$$

b) For the parameter θ, it can be seen from the GELF estimator that $Z(\alpha,\theta)=\alpha^{-k}$, then $Z_2 = -k\,\alpha^{-(k+1)}$, $Z_{22} = k\,(k+1)\,\alpha^{-(k+2)}$ and $Z_1 = Z_{11} = 0$, we have

$$\hat{\theta}_{GELF} = E\left(\theta^{-k} \mid \underline{x}\right) = \theta^{-k}\left\{1 + k\theta H_2\left[\frac{1}{2}\left((k+1)\theta - H_2 H_3\right) + q - \frac{p-1}{\theta}\right]\right\} \tag{35}$$

Where

$$H_2 = \left[\frac{n}{\theta^2} - \frac{n}{(\theta+1)^2} - \frac{1}{\theta+1}\sum_{i=1}^{n}\frac{x(1+x)}{(\theta+1+\theta x)^2} + \frac{2}{(\theta+1)^2}\sum_{i=1}^{n}\frac{x}{(\theta+1+\theta x)} - \frac{2(\alpha+1)}{(\theta+1)} + \sum_{i=1}^{n}\frac{BA'-AB'}{B^2}\frac{2(\alpha-1)}{(\theta+1)^2}\sum_{i=1}^{n}\frac{A}{B}\right]^{-1}$$

and

$$H_3 = \frac{2n}{\theta^3} - \frac{2n}{(\theta+1)^3} + \frac{1}{\theta+1}\sum_{i=1}^{n}\frac{x(1+x)^2}{(\theta+1+\theta x)^3} - \frac{2}{(\theta+1)^2}\sum_{i=1}^{n}\frac{x(1+x)}{(\theta+1+\theta x)^2} + \frac{2}{(\theta+1)^3}\sum_{i=1}^{n}\frac{x}{(\theta+1+\theta x)} + \frac{2(\alpha+1)}{\theta+1}\sum_{i=1}^{n}\frac{CBA''-CAB''-C'A'B+C'B'A}{C^2} - \frac{2(\alpha-1)}{(\theta+1)^2}\sum_{i=1}^{n}\frac{BA'-AB'}{B^2}$$

4. NUMERICAL ANALYSIS

4.1 Monte Carlo Simulation Study

In this section, a Monte Carlo simulation studywas carried out with R Statistical software to compare the performance and accuracy of the proposed Bayesian estimators and their maximum likelihood estimates counterpart of TL-L distribution parameters α and θ by using mean square Errors (MSE) and the BIAS given as:

$$MSE = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\Phi} - \Phi\right)^2$$

and

$$BIAS = \frac{1}{N}\sum_{i=1}^{N}\left|\hat{\Phi} - \Phi\right|$$

Where N is the number of samples. In each simulation, we generate N=10,000 samples of size $n = 30, 50, 100, 200, 500, 1000$ from TL-L distribution for some sets of parameter values $\alpha = 0.84, 1.6, 2, 2.5$ and $\theta = 0.5, 2, 2.5$. We assume that p takes the values $p = 2, 5, 8, 10$; q takes the values $q = 1, 2, 5, 12$; m takes the values $m = 1, 6, 8, 15$ and c takes the values

$c = -0.25, -0.5, -0.65\ -0.75$. These results presented in Tables 1- 4 below showed the mean, MSE's and bias for estimating the parameters α and θ.

From the results of the simulation study in Table 1 – 4, we summarize our observations as follows:

i. For any fixed values of the parameters α and θ, as sample size increases, the MSEs of all the estimators, both MLEs and Bayesian Estimates decrease.
ii. The values of the hyper parameters from the prior distribution have minimal effect on the posterior estimates.
iii. Generally, the terms of MSEs of the MLEs and Bayesian estimates converge to the same value as for the large sample except for GELF.

4.2 Real Data Analysis

This section present the application of TLL distribution to real data set. This data set represent 66 breaking stress of carbon fibers (in Gba) which was reported in Nicholas and Padgett [26].

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42,
2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 2.93,
3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 1.25, 4.38, 1.84,
0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88, 2.82, 2.05,
3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.35, 2.55, 2.59, 2.03, 1.61,
2.12, 3.15, 1.08, 2.56, 1.80, 2.53

Table 1: *Showing mean of ML and Bayesian estimates with corresponding MSEs and Bias for* $\theta = \alpha = 2$ *(while* $p = 2,\ q = 1,\ m = 1,\ c = -0.75$*)*

		θ				α		
n	Method	MEAN	MSE	BIAS		MEAN	MSE	BIAS
	ML	2.563265	0.317267	0.563265		2.155780	0.024267	0.155780
30	LINEX	2.563265	0.317267	0.563265		2.156686	0.024550	0.156686
	GELF	2.019669	0.000386	0.019669		1.656376	0.118120	0.343623
	SELF	2.559342	0.312864	0.559342		2.156988	0.024645	0.156988
	ML	2.485253	0.235470	0.485253		2.125277	0.015694	0.125277
50	LINEX	2.485253	0.235470	0.485253		2.130306	0.017000	0.130306
	GELF	1.965990	0.001156	0.034009		1.686560	0.098244	0.313439
	SELF	2.473202	0.223021	0.473202		2.131028	0.017197	0.131028
	ML	2.219903	0.048357	0.219903		2.067709	0.004585	0.0677091
100	LINEX	2.219903	0.048357	0.219903		2.066873	0.004473	0.066873
	GELF	1.817202	0.033414	0.182797		1.692596	0.094496	0.307403
	SELF	2.218925	0.047928	0.218925		2.066875	0.004473	0.066875
	ML	2.194559	0.037853	0.194559		1.940733	0.003512	0.059266
200	LINEX	2.194559	0.037853	0.194559		1.959422	0.001751	0.040577
	GELF	1.798195	0.040554	0.201380		1.723795	0.076289	0.276204
	SELF	2.190094	0.036136	0.190094		1.959858	0.001720	0.040141
	ML	1.926363	0.005422	0.073636		2.017662	0.000311	0.017662
500	LINEX	1.926363	0.005422	0.073636		2.017107	0.000292	0.017107
	GELF	1.634335	0.133710	0.365665		1.763728	0.055834	0.236271
	SELF	1.925588	0.005537	0.074411		2.017155	0.000294	0.017155
	ML	1.975739	0.000588	0.024260		2.010489	0.000110	0.010489
1000	LINEX	1.975739	0.000588	0.024260		2.007554	0.000057	0.007553
	GELF	1.666054	0.111519	0.333945		1.779841	0.048469	0.220158
	SELF	1.975331	0.000608	.024668		2.007565	0.000057	0.007565

Table 2: *Showing mean of ML and Bayesian estimates with corresponding MSEs and Bias for* $\theta = \alpha = 2.5$ *(while* $p = 5,\ q = 2,\ m = 8,\ k = -0.5$ *)*

		θ				α		
N	Method	MEAN	MSE	BIAS		MEAN	MSE	BIAS
	ML	3.200958	0.491343	0.700958		2.746747	0.060884	0.246747
30	LINEX	3.200958	0.491343	0.700958		2.746587	0.060805	0.246587
	GELF	0.318750	4.757852	2.181250		0.364048	4.562259	2.135951
	SELF	3.199980	0.489972	0.699980		2.746916	0.060967	0.246961
	ML	3.103670	0.364418	0.603670		2.733993	0.054753	0.233993
50	LINEX	3.103670	0.364418	0.603670		2.734001	0.054756	0.234001
	GELF	1.753758	0.556876	0.746241		0.365735	4.555087	2.134265
	SELF	3.101468	0.361764	0.601468		2.734241	0.054486	0.234241
	ML	2.775719	0.076021	0.275719		2.659853	0.025531	0.159853
100	LINEX	2.775719	0.076021	0.275719		2.659853	0.025531	0.159853
	GELF	0.362098	4.570621	2.137901		1.630771	0.755558	0.869228
	SELF	2.775484	0.075891	0.275484		2.659435	0.025420	0.159435
	ML	2.749392	0.062196	0.249392		2.463737	0.001314	0.036262
200	LINEX	2.749392	0.062196	0.249392		2.463329	0.001344	0.036670
	GELF	0.367475	4.547659	2.132524		0.405834	4.385529	2.094165
	SELF	2.748470	0.061737	0.248470		2.464147	0.001285	0.035858
	ML	2.411848	0.007770	0.088151		2.516045	0.000257	0.016045
500	LINEX	2.411848	0.007770	0.088151		2.515688	0.000246	0.015688
	GELF	0.415480	4.345220	2,084519		0.397501	4.420501	2.102498
	SELF	2.411668	0.007803	0.088338		2.515719	0.000247	0.015719
	ML	2.472058	0.000780	0.027941		2.512703	0.000161	0.012703
1000	LINEX	2.472058	0.000780	0.027941		2.512889	0.000166	0.012889
	GELF	o.404940	4.389274	2.095059		0.397943	4.418641	2.102056
	SELF	2.471967	0.000785	0.028032		2.512922	0.000166	0.012922

Table 3: *Showing mean of ML and Bayesian estimates with corresponding MSEs and Bias for* $\theta = 0.5$ *and* $\alpha = 1.6$ *(while* $p = 10,\ q = 5,\ m = 15, c = -0.25$ *)*

		θ				α		
n	Method	MEAN	MSE	BIAS		MEAN	MSE	BIAS
	ML	0.482010	1.249900	1.117989		1.725412	1.501634	1.225412
30	LINEX	0.482010	1.249900	1.117989		1.725519	1.501897	1.225519
	GELF	0.833072	0.588178	0.766927		1.146106	0.417453	0.646106
	SELF	0.482010	1.249900	1.117989		1.602093	1.221410	1.102093
	ML	0.494062	1.223098	1.105937		1.700421	1.441010	1.200421
50	LINEX	0.494062	1.223098	1.105937		1.701528	1.443671	1.201528
	GELF	0.838310	0.580171	0.761689		1.141972	0.412128	0.641972
	SELF	0.494062	1.223098	1.105937		1.700645	1.441547	1.200645
	ML	0.547370	1.108029	1.052629		1.654412	1.332668	1.154412
100	LINEX	0.547370	1.108029	1.052629		1.624175	1.264541	1.124175
	GELF	0.859336	0.548582	0.74663		1.127087	0.393238	0.627087
	SELF	0.547371	1.108026	1.052628		1.553157	1.109139	1.053157
	ML	0.553372	1.095429	1.046627		1.613726	1.240386	1.113726
200	LINEX	0.553372	1.095429	1.046627		1.613730	1.240395	1.113730
	GELF	0.862086	0.544517	0.737914		1.126783	0.391974	0.626078
	SELF	0.553371	1.095430	1.046628		1.725438	1.501699	1.225438

500	ML	0.616878	0.966528	0.983122		1.607956	1.227566	1.107956
	LINEX	0.616878	0.966528	0.983122		1.607959	1.227574	1.107959
	GELF	0.888384	0.506396	0.711615		1.125163	0.391029	0.625163
	SELF	0.630569	0.939795	0.969430		1.613727	1.240389	1.113727
1000	ML	0.635364	0.903520	0.964635		1.553046	1.108906	1.053046
	LINEX	0.635364	0.903520	0.964635		1.553455	1.109767	1.053455
	GELF	0.891125	0.502502	0.708874		1.116361	0.3799015	0.616361
	SELF	0.635354	0.930540	0.964645		1.607957	1.227568	1.107957

Table 4: *Showing mean of ML and Bayesian estimates with corresponding MSEs and Bias for* $\theta = 2$ *and* $\alpha = 0.84$ *(while* $p = 5,\ q = 12,\ m = 8, c = -0.65$*)*

		θ				α		
n	Method	MEAN	MSE	BIAS		MEAN	MSE	BIAS
30	ML	2.631936	3.211034	1.791936		0.767969	1.517899	1.232030
	LINEX	2.631936	3.211034	1.791936		0.738588	1.591409	1.261411
	GELF	1.764271	0.854281	0.924271		0.820932	1.390366	1.179068
	SELF	2.408270	2.459486	1.568270		0.738978	1.592766	1.261921
50	ML	2.553476	2.936002	1.713476		0.781702	1.484247	1.218297
	LINEX	2.553476	2.936002	1.713476		0.766525	1.521766	1.233474
	GELF	1.826392	0.972970	0.986392		0.841222	1.342981	1.158778
	SELF	2.543565	2.871552	1.694565		0.766439	1.522096	1.233560
100	ML	2.231311	1.935746	1.391311		0.787585	0.469048	1.212414
	LINEX	2.231311	1.935746	1.391311		0.804772	1.429149	1.195227
	GELF	1.634268	0.630865	0.794268		0.885795	1.341462	1.114206
	SELF	2.132726	1.671155	1.292726		0.829791	1.369402	1.170208
200	ML	2.170852	1.771167	1.330852		0.8445506	1.335063	1.155449
	LINEX	2.170852	1.771167	1.330852		0.829822	1.369314	1.170177
	GELF	1.676593	0.699902	0.83659		0.866981	1.283888	1.133019
	SELF	2.216419	1.894586	1.376419		0.802809	1.433578	1.197190
500	ML	1.895829	1.114774	1.055829		0.854625	1.311883	1.145374
	LINEX	1.895829	1.114774	1.055829		0.826858	1.376269	1.173141
	GELF	1.546722	0.499456	0.706722		0.887273	1.238159	1.112726
	SELF	1.956499	1.246570	1.116499		0.831958	1.364320	1.168041
1000	ML	1.958634	1.251342	1.118634		0.855785	1.309226	1.144214
	LINEX	1.958634	1.251342	1.118634		0.902505	1.206101	1.097494
	GELF	1.512338	0.452302	0.672533		0.942452	1.118742	1.057547
	SELF	1.890693	1.103958	1.050693		0.912619	1.183110	1.087380

Table 5: *The Point Estimates of Topp-Leone Lindley distribution parameters through MLE, LINEX, GELF and SELF* $p = 6,\ q = 4,\ m = 5,\ k = -0.75$

Parameters	MLE	LINEX	GELF	SELF
θ	0.7128402	0.7128402	0.7768919	0.7128402
α	6.339166	6.339166	3.995067	6.339166

5. CONCLUSION

In estimating the parameters of probability distribution in survival analysis, Bayesian mechanism examines the nature uncertainty and provide a judicious framework for studying such problems. In this study, we considered the Bayesian Estimation (BE) for the Topp-Leone distribution parameters. The BEs were obtained using Lindley's approximation under three different loss functions, which includes Square Error Loss Function (SELF), Linear Exponential Loss Function (LINEX) and Generalized Entropy Loss Function (GELF). Monte Carlo simulation was carried out to examine the behavior of the maximum likelihood (ML) and Bayesian Estimators, which was investigated through the mean square error (MSE) and bias of the estimators. It was also observed for any fixed value of the parameters, as sample size increases, the MSEs of the Bayesian Estimates and MLEs decrease. Also, the MLEs and Bayesian estimates converge to the same value as the sample gets larger except for GELF. Generally, it was observed that the results obtained from the MLE, SELF and LINEX are more consistent than that of GELG.

Conflicts of Interest
The authors declared that there is no conflict of interest in this work.

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