CHARACTERIZATION OF SOME CONTINUOUS DISTRIBUTIONS BY CONDITIONAL VARIANCE OF RECORD VALUES

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Abstract

Characterization of a probability distribution gives a unique property enjoyed by that distribution. Various approaches are available in the literature to characterize distributions through record values. Many researchers have characterized Exponential, Pareto, and Power function distributions using moments, conditional expectation, and some other characteristics of record values. In this paper, we have characterized these three distributions through conditional variance of adjacent record values. The results have been verified using numerical computation.

Keywords: Characterization of continuous distributions, conditional variance, record values.

1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables with distribution function $(df)F(x)$ and probability density function $(pdf)f(x)$. Let $X_{U(r)}$ be the r th upper record value, then the conditional pdf of $X_{U(r+1)}$ given $X_{U(r)} = x, 1 \leq r < s$ is given by (Ahsanullah, 2004)[1]

$$f\left(X_{U(r+1)} = y \mid X_{U(r)} = x\right) = \frac{f(y)}{\bar{F}(x)} \tag{1.1}$$

where $\bar{F}(x) = P(X > x) = 1 - F(x)$.
One can transform the upper record into lower record values by replacing the original sequence of $\left(X_j\right)$ by $(-X_i, j \geq 1)$ (Ahsanullah, 2004) [1]. Let $X_{L(r)}$ be the r-th lower record value, then the conditional pdf of $X_{L(r+1)}$ given $X_{L(r)} = x, 1 \leq r < s$ is given by

$$f\left(X_{L(r+1)} \mid X_{L(r)} = x\right) = \frac{f(y)}{F(x)}. \tag{1.2}$$

The record values have been extensively studied in literature. For an excellent review, one may refer to Ahsamullah (2004) [1]. Arnold at al. (1998) [2] and Nevzorov (2001) [3] amongst others. Characterization of distributions through conditional expectations of record values have been considered, among others, by Nagaraja, H.N. and Nevzorov, V.B. (1997) [4], Franco and Ruiz(1997)

[5], Athar et al. (2003) [6], Khan et al. (2010) [7] and Faizan and Khan (2011) [8].

Beg, M.I. and Kirmani. S.N.U.A. (1978) [9] characterized exponential distribution by a weak homoscedasticity. Khan and Beg (1987) [10] extended the result of Beg and Kirmani (1978) for Weibull distribution. Khan et al. (2008) [11] characterized a general class of distribution by conditional variance of order statistics and Shah et al. (2018) [12] characterized Pareto and power function distributions by conditional variance of order statistics, In this paper we have characterized exponential, Pareto and power function distributions by conditional variance of record values.

2. Characterization Results

Theorem 2.1: Let x be a random variable with df $F(x)$ and $E\left(X^2\right) < \infty$. Then for $r < s$

$$V\left[X_{x(r+1)} \mid X_{v(r)} = x\right] = \theta^2 \tag{2.1}$$

for some $\theta > 0$ if and only if

$$\bar{F}(x) = e^{-re}; \quad x > 0. \tag{2.2}$$

Proof: First we will prove (2.2) implies (2.1). It is easy to see that from (1.1) and (2.2)

$$E\left[X_{u(r+1)} \mid X_{u(r)} = x\right] = x + \theta \tag{2.3}$$

and

$$E\left[X^2_{u(r+1)} \mid X_{u(r))} = x\right] = x^2 + 2x\theta + 2\theta^2 \tag{2.4}$$

Now, using (2.3) and (2.4), we have

$$V\left[X_{u(r+1)} \mid X_{u(r)} = x\right] = \theta^2$$

For sufficiency part, we have from (2.2)

$$\int_x^\infty y^2 \frac{f(y)}{\bar{F}(x)} dy - \left(\int_x^\infty y \frac{f(y)}{\bar{F}(x)} dy\right)^2 = \theta^2$$

$$\bar{F}(x) \int_x^\infty y^2 f(y) dy - \left(\int_x^\infty y f(y) dy\right)^2 = \theta^2 \bar{F}^2(x) \tag{2.5}$$

Differentiating (2.5) twice w.r.t. x and simplifying, we get

$$\int_x^\infty y f(y) dy = x\bar{F}(x) + \theta^2 f(x) \tag{2.6}$$

Now differentiate (2.6) again w.r.t. x, we get

$$\bar{F}(x) = -\theta^2 f'(x)$$

and hence the result.

Theorem 2.2: Let X be a random variable with $dfF(x)$ and $E\left(X^2\right) < \infty$. Then, for some $r < s$ and $0 < p < 1$, we have

$$V\left[X_{u(r+1)} \mid X_{u(r)} = x\right] = \frac{p}{(p-2)(p-1)^2} x^2 \tag{2.7}$$

if and only if

$$\bar{F}(x) = \left(\frac{\alpha}{x}\right)^p; \quad \alpha \le x < \infty. \tag{2.8}$$

Proof: First we will prove (2.8) implies (2.7). By using (1.1) and (2.8), it is easy to show that

$$E\left[X_{u(r+1)} \mid X_{u(r)} = x\right] = \frac{p}{p-1}x$$

and

$$E\left[X^2_{u(r+1)} \mid X_{u(r)} = x\right] = \frac{p}{p-2}x^2$$

which gives

$$V\left[X_{u(r+1)} \mid X_{u(r)} = x\right] = \frac{p}{(p-2)(p-1)^2}x^2.$$

Now, to prove (2.7) implies (2.8), we have using (1.1) and (2.7)

$$\bar{F}(x)\int_x^\infty y^2 f(y)dy - \left(\int_x^\infty y f(y)dy\right)^2 = cx^2\bar{F}^2(x) \tag{2.9}$$

where

$$c = \frac{p}{(p-2)(p-1)^2}.$$

Differentiating (2.9) twice w.r.t. x and simplifying, we get

$$\int_x^\infty y^2 f(y)dy - 2x\int_x^\infty y f(y)dy = (2c-1)x^2\bar{F}(x) - 2cx\frac{\bar{F}^2(x)}{f(x)}. \tag{2.10}$$

Now, after differentiating (2.10) w.r.t. x, we get

$$\int_x^\infty y f(y)dy = cx^2 f(x) - (4c-1)x\bar{F}(x) + c\frac{\bar{F}^2(x)}{f'(x)} - cx\frac{\bar{F}^2(x)f'(x)}{f^2(x)}. \tag{2.11}$$

Again differentiating (2.11), we get

$$\begin{aligned}&2x\frac{\bar{F}(x)f'^2(x)}{f^3(x)} - 2\frac{\bar{F}(x)f'(x)}{f^2(x)} - x\frac{\bar{F}(x)f''(x)}{f'^2(x)} + 2x\frac{f'(x)}{f(x)}\\ &+ x^2\frac{f'(x)}{\bar{F}(x)} + 6x\frac{f(x)}{\bar{F}(x)} - 6 + \frac{1}{c} = 0.\end{aligned}$$

Let $\frac{\bar{F}'(x)}{\bar{F}(x)} = y = y(x)$ bearing in mind that $f(x) = F'(x)$, $f'(x) = F''(x)$, $f''(x) = F'''(x)$, $\frac{\bar{F}''(x)}{\bar{F}(x)} = y' + y^2$, $\frac{\bar{F}'''(x)}{\bar{F}(x)} = y'' + 3yy' + y^3$, we get

$$x\frac{y'' + 3yy' + y^3}{y^2} - 2x\frac{(y'+y^2)^2}{y^3} - \left(x^2 - \frac{2}{y^2} - \frac{2x}{y}\right)\left(y' + y^2\right) - 6xy + p^2 - 4p - \frac{2}{p} - 1 = 0. \tag{2.12}$$

There exists a unique solution of the differential equation (2.12) that satisfies the prescribed initial conditions
that $y'(a) = -\frac{p}{a^2}$
and

$$y(a) = \frac{p}{a}$$

where a is any finite point in the support of F. Thus by the existence and uniqueness theorem (Boyce and Diprima, 2012) [13], we get

$$\frac{\bar{F}'(x)}{\bar{F}(x)} = y = -\frac{p}{x}$$

which implies that

$$\bar{F}(x) = (Ax)^{-p}; \quad \alpha \leq x < \infty.$$

where A is a constant to be determined and hence the Theorem.

Theorem 2.3: Let X be a random variable with $df F(x)$ and $E\left(X^2\right) < \infty$. Then for $r < s$

$$V\left[X_{L(r+1)} \mid X_{L(r)} = x\right] = \frac{p}{(p+2)(p+1)^2}x^2$$

if and only if

$$F(x) = \left(\frac{x}{\beta}\right)^p; \quad 0 \leq x < \beta < \infty. \tag{2.13}$$

Proof: This can be proved on lines of Theorem 2.2

Table 1: *Verification of the characterization results in case of Exponential distribution.*

θ	X	$L.H.S.$	$R.H.S.$	$\|L.H.S. - R.H.S.\|$	$\|\frac{L.H.S.-R.H.S.}{R.H.S}\|$
1.5	0.4	2.2499	2.25	0.0001	0.00005
2.5	0.8	6.2497	6.25	0.0003	0.00005
4.5	1.6	20.2504	20.25	0.0004	0.00002
5.5	2.0	30.2454	30.25	0.0046	0.00015
6.5	2.4	42.2430	42.25	0.0070	0.00017
7.5	2.8	56.2436	56.25	0.0064	0.00011
8.5	3.2	72.2603	72.25	0.0103	0.00014
9.5	3.6	90.2302	90.25	0.0198	0.00022
10.5	4.0	110.2485	110.25	0.0015	0.00001
11.5	4.4	132.2380	132.25	0.0120	0.00009
12.5	4.8	156.1910	156.25	0.0590	0.00038
13.5	5.2	182.2197	182.25	0.0303	0.00017
14.5	5.6	210.2026	210.25	0.0474	0.00023
15.5	6.0	240.2801	240.25	0.0301	0.00013

Table 2: *Verification of the characterization results in case of Pareto distribution.*

α	p	X	$L.H.S.$	$R.H.S.$	$\|L.H.S. - R.H.S.\|$	$\|\frac{L.H.S.-R.H.S.}{R.H.S}\|$
0.3	3	0.2156	0.0347	0.0348	0.0001	0.0029
0.7	4	0.5797	0.0745	0.0747	0.0002	0.0027
1.1	5	0.8523	0.0756	0.0757	0.0001	0.0013
1.5	6	1.1692	0.0838	0.0820	0.0018	0.0220
1.9	7	1.4536	0.0699	0.0822	0.0123	0.1496
2.3	8	1.7510	0.0836	0.0834	0.0151	0.1530
2.7	9	2.0642	0.0943	0.0856	0.0087	0.1016
3.1	10	2.3171	0.0829	0.0812	0.0017	0.0209
3.5	11	2.6390	0.0850	0.0851	0.0001	0.0011
3.9	12	2.9604	0.0878	0.0869	0.0009	0.0103
4.3	13	3.2612	0.0847	0.0873	0.0026	0.0002
4.7	14	3.5998	0.0899	0.0895	0.0004	0.0045
5.1	15	3.8431	0.0862	0.0869	0.0007	0.0080
5.5	16	4.2212	0.0871	0.0905	0.0034	0.0376
5.9	17	4.5505	0.0917	0.1147	0.0230	0.2005

Conclusions:

This paper introduces a study of the Exponential, Pareto, and Power function distributions, showcasing their characterizations based on the conditional variance of adjacent record values. The validity of our findings has been confirmed through some numerical computation.

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