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SOME INFERENTIAL ASPECTS ON THREE-STATION TANDEM QUEUE

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Abstract

Considered is a three-station tandem queue with service times at stations 1, 2, and 3 are exponentially distributed with customers arriving according to the Poisson process at station 1. Given that the stationary distribution is the product of three independent geometric distributions with the intensity parameters, maximum likelihood estimators and Bayes estimators of the intensity parameters based on the number of customers present at different time periods are obtained. Furthermore, the minimal posterior risk and minimum Bayes risk of the estimators are computed. Also, a simulation study is conducted to evaluate the performance of the estimators obtained.

Keywords: Three-station tandem queue, Classical inference, Bayesian inference, MCMC sampling

1. Introduction

Most works on queuing models are restricted to deriving the formulations for transient or stationary (steady state) solutions and do not take into account the related statistical inference issues. Some of the crucial tools to understanding any random phenomenon using stochastic models are classical inference and Bayesian inference. The past has not paid much attention to the analysis of queuing systems in all these directions. Standard parametric models are highly suitable whenever the systems are completely observable in terms of their fundamental random components, such as inter-arrival times and service times.

Estimation of the parameters associated with the queueing models are integral part of queuing theory. Frequently, previous experiments or analyses of the inter-arrival time or service time data have revealed some information about the parameters of the distributions of inter-arrival time or service time. The Bayesian approach offers the framework for formally integrating prior knowledge with the facts currently available.

Here are some of the queueing system research that have been done in the past where the estimate of queueing parameters was done using both classical and Bayesian methods. Inter-arrival and service times were used as the observed data in an empirical Bayesian framework by [9] to estimate the parameters for various queueing systems. Based on the number of customers present at various sampling time points, [5] computed an maximum likelihood estimator (MLE) and Bayes estimator of traffic intensity in an M/M/1 queueing model. Regarding tandem queues with dependent service time structures, [2] studied statistical inferential aspects. Using the classical inference method, they modelled tandem queues and estimated the parameters. The statistical analysis of a tandem queue with blocking was then undertaken by [3] and focused on a two station tandem queue. Again, [1] investigated the Bayesian inference for a two station tandem queue, calculated the traffic intensities for the two stations, and determined the confidence

interval of the estimators. In the M/M/1 queue with bivariate priors, Bayes estimation has been studied by [6]. Then [4] performed a simulation research applying the Markov Chain Monte Carlo (MCMC) approach including the Metropolis-Hastings (M-H) algorithm and explored the Bayesian inference of the Markovian queuing model with two heterogeneous servers.
This paper attempts a detailed study of a three station tandem queue with customers arriving according to the Poisson process, with rate λ for service at station 1 and service times at station 1, station 2, and station 3 being exponentially distributed with service rates μ_1, μ_2 and μ_3 respectively. The maximum likelihood and Bayes estimators of the intensity parameters ρ_1, ρ_2 and ρ_3 are computed using the number of customers present at various sampled time points under the assumption that the stationary distribution is the product of three independent geometric distributions with parameters ρ_1, ρ_2 and ρ_3 accordingly. Additionally, the minimal Bayes risk of the estimators and the minimum posterior risk related to Bayes estimators are derived.
This paper is structured as follows: Section 1 discussed an introduction to tandem queues as well as some early research in this area. Section 2 explored the model, the system description, and the inferential aspects of the model. Section 3 looked at the estimated number of customers in the system and its implications. Section 4 examined the model using simulation. Finally, Section 5 contains the paper's conclusions.

2. System description and steady state probability

Consider a simplified one channel queuing system consisting of three service stations as in the figure 1. A customer that arrives for servicing must pass through station 1, station 2 and station 3

Figure 1: *System configuration*

before finishing the service. The model's underlying assumptions are as follows:

1. Arrivals occur according to the Poisson distribution with mean rate λ at station 1.
2. Service times at station 1, station 2 and station 3 are exponentially distributed with service rates μ_1, μ_2 and μ_3 respectively.
3. A queue of infinite size is allowed in front of station 1 and station 2 but at most one customer is permitted to wait between station 2 and station 3.
4. Each station is either free or busy.
5. If a customer in station i, $i = 1, 2$ completes their service before station $(i+1), i = 1, 2$ becomes free, then it is said that station $i, i = 1, 2$ is blocked.

Let $p_{n_1,n_2,n_3}(t)$ be the probability that there are n_1 customers in station 1, n_2 customers in station 2 and n_3 customers in station 3 at time t (in queue or in system). In the steady state it can be shown that,

$$p_{n_1,n_2,n_3}(t) = \rho_1^{n_1}(1-\rho_1)\rho_2^{n_2}(1-\rho_2)\rho_3^{n_3}(1-\rho_3), \quad n_1,\ n_2 = 0,1,2,3,... \ \&\ n_3 = 0,1,$$

where, $\rho_i = \frac{\lambda}{\mu_i}$, $i = 1, 2, \&\ 3$ and steady state results exist provided $\rho_i < 1$.

2.1. Classical Inference

The likelihood function of the number of customers present at n different time points $t_1, t_2, t_3, ..., t_n$ is given by

$$l((\rho_1, \rho_2, \rho_3)|((x_1, y_1, z_1), ..., (x_n, y_n, z_n))) = \rho_1^{\sum_{i=1}^n x_i}(1-\rho_1)^n \rho_2^{\sum_{i=1}^n y_i}(1-\rho_2)^n \rho_3^{\sum_{i=1}^n z_i}(1-\rho_3)^n. \quad (1)$$

Taking logarithms and differentiating the log-likelihood function of (1) with respect to ρ_1, ρ_2 and ρ_3 and equating to zero, we get the MLEs of ρ_1, ρ_2, ρ_3 and are given by

$$\hat{\rho_1} = \frac{\sum_{i=1}^n x_i}{n + \sum_{i=1}^n x_i}, \hat{\rho_2} = \frac{\sum_{i=1}^n y_i}{n + \sum_{i=1}^n y_i} \text{ and } \hat{\rho_3} = \frac{\sum_{i=1}^n z_i}{n + \sum_{i=1}^n z_i}.$$

In other words,

$$\hat{\rho_1} = \frac{T_1}{n + T_1}, \hat{\rho_2} = \frac{T_2}{n + T_2} \text{ and } \hat{\rho_3} = \frac{T_3}{n + T_3},$$

where,

$$T_1 = \sum_{i=1}^n x_i \sim NB(n, 1-\rho_1), T_2 = \sum_{i=1}^n y_i \sim NB(n, 1-\rho_2) \text{ and } T_3 = \sum_{i=1}^n z_i \sim NB(n, 1-\rho_3)$$

and T_1, T_2 and T_3 are independent (see, [8]). Clearly, the probability mass functions (pmfs) of T_1, T_2 and T_3 are given by

$$\begin{aligned}
P[T_1 = t_1] &= \binom{t_1 + n - 1}{n-1}(1-\rho_1)^n \rho_1^{t_1}, \\
P[T_2 = t_2] &= \binom{t_2 + n - 1}{n-1}(1-\rho_2)^n \rho_2^{t_2} \text{ and} \\
P[T_3 = t_3] &= \binom{t_3 + n - 1}{n-1}(1-\rho_3)^n \rho_3^{t_3},
\end{aligned}$$

where, $t_1 = 0, 1, 2, .., t_2 = 0, 1, 2, ..$ and $t_3 = 0, 1, 2, ...$ It can be shown that

$$E(T_1) = \frac{n^2 \rho_1}{1-\rho_1},\ E(T_2) = \frac{n^2 \rho_2}{1-\rho_2} \text{ and } E(T_3) = \frac{n^2 \rho_3}{1-\rho_3}.$$

Also

$$Var(T_1) = \frac{n^2 \rho_1}{(1-\rho_1)^2},\ Var(T_2) = \frac{n^2 \rho_2}{(1-\rho_2)^2} \text{ and } Var(T_3) = \frac{n^2 \rho_3}{(1-\rho_3)^2}.$$

Since $\hat{\rho_1}$, $\hat{\rho_2}$ and $\hat{\rho_3}$ are one to one functions of T_1,T_2 and T_3 respectively, it is clear that $\hat{\rho_1}$, $\hat{\rho_2}$ and $\hat{\rho_3}$ assume the values $\frac{t_1}{n+t_1}$, $\frac{t_2}{n+t_2}$ and $\frac{t_3}{n+t_3}$ respectively with t_1,t_2,t_3 = $0, 1, 2, 3, \cdots$ Further, the joint pmf of $\hat{\rho_1}$, $\hat{\rho_2}$ and $\hat{\rho_3}$ is given by

$$\begin{aligned}
P[\hat{\rho_1} = u, \hat{\rho_2} = v, \hat{\rho_3} = w] &= P\left[\frac{t_1}{n+t_1} = u, \frac{t_2}{n+t_2} = v, \frac{t_3}{n+t_3} = w\right] \\
&= P\left[t_1 = \frac{nu}{1-u}\right] P\left[t_2 = \frac{nv}{1-v}\right] P\left[t_3 = \frac{nw}{1-w}\right] \\
&= \binom{\frac{nu}{1-u} + n - 1}{n-1}(1-\rho_1)^n \rho_1^{\frac{nu}{1-u}} \binom{\frac{nv}{1-v} + n - 1}{n-1}(1-\rho_2)^n \rho_2^{\frac{nv}{1-v}} \\
&\times \binom{\frac{nw}{1-w} + n - 1}{n-1}(1-\rho_3)^n \rho_3^{\frac{nw}{1-w}}.
\end{aligned}$$

In the next section, Bayes estimators of ρ_1, ρ_2 and ρ_3 and their Bayes risks are found.

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2.2. Bayesian Inference

The number of customers present at various sampled time points is used to determine the Bayes estimators of ρ_1, ρ_2 and ρ_3 as well as their Bayes risks. The natural conjugate prior density for (ρ_1, ρ_2, ρ_3) is taken to be the product of three independent Beta distributions of first kind with the parameters (m_1, n_1), (m_2, n_2) and (m_3, n_3), respectively. As a result, we suppose that (ρ_1, ρ_2, ρ_3) has a prior distribution that is the product of three separate Beta distributions of the first kind, each with the parameters (m_1, n_1), (m_2, n_2) and (m_3, n_3). That is,

$$\tau(\rho|(m_1,n_1),(m_2,n_2),(m_3,n_3)) = \frac{1}{\beta(m_1,n_1)\beta(m_2,n_2)\beta(m_3,n_3)}\rho_1^{m_1-1}(1-\rho_1)^{n_1-1}\rho_2^{m_2-1} \times (1-\rho_2)^{n_2-1}\rho_3^{m_3-1}(1-\rho_3)^{n_3-1},$$

where $0 < \rho_1, \rho_2, \rho_3 < 1$, $\rho = (\rho_1, \rho_2, \rho_3)$, $m' = (m_1, n_1)$, $n' = (m_2, n_2)$ and $p' = (m_3, n_3)$.
The marginal probability density function (pdf) of $T = (T_1, T_2, T_3) = (\sum_{i=1}^n x_i, \sum_{i=1}^n y_i, \sum_{i=1}^n z_i)$, which is called the predictive pdf and is given by

$$\begin{aligned} f^*(t) &= \int_0^1\int_0^1\int_0^1 f(t_1,t_2,t_3;\rho_1,\rho_2,\rho_3).\tau(\rho|(m',n',p'))d\rho_1.d\rho_2.d\rho_3 \\ &= \int_0^1\int_0^1\int_0^1 P[T_1=t_1].P[T_2=t_2].P[T_3=t_3]\tau(\rho|(m',n',p'))d\rho_1.d\rho_2.d\rho_3 \\ &= \frac{\beta(t_1+m_1,n+n_1).\beta(t_2+m_2,n+n_2).\beta(t_3+m_3,n+n_3)}{\beta(m_1,n_1).\beta(m_2,n_2).\beta(m_3,n_3)}\Pi_{i=1}^3\binom{t_i+n-1}{n-1}. \end{aligned}$$

Hence the posterior distribution of $\rho = (\rho_1, \rho_2, \rho_3)$ is given by

$$\begin{aligned} q(\rho|(x,y,z)) &= \frac{f(t_1,t_2,t_3;\rho)\tau(\rho|(m',n',p'))}{\int_0^1 f(t_1,t_2,t_3;\rho)\tau(\rho|(m',n',p'))d\rho} \\ &= \frac{1}{\beta(t_1+m_1,n+n_1)}\rho_1^{(t_1+m_1)-1}(1-\rho_1)^{(n+n_1)-1} \\ &\times \frac{1}{\beta(t_2+m_2,n+n_2)}\rho_2^{(t_2+m_2)-1}(1-\rho_2)^{(n+n_2)-1} \\ &\times \frac{1}{\beta(t_3+m_3,n+n_3)}\rho_3^{(t_3+m_3)-1}(1-\rho_3)^{(n+n_3)-1}, \quad 0<\rho_1,\rho_2,\rho_3<1. \end{aligned}$$

It should be pointed out that the posterior distribution of $\rho = (\rho_1, \rho_2, \rho_3)$ is the result of the pdfs of three independent Beta distributions of first-kind with the parameters $(t_1 + m_1, n + n_1)$, $(t_2 + m_2, n + n_2)$ and $(t_3 + m_3, n + n_3)$, respectively. Therefore, under the squared error loss, the Bayes estimator of $\rho = (\rho_1, \rho_2, \rho_3)$ is given by

$$\begin{aligned} E[\rho|(x,y,z)] &= \int_0^1\int_0^1\int_0^1 \rho_1.\rho_2.\rho_3.q(\rho|(x,y,z))d\rho \\ &= \frac{t_1+m_1}{t_1+m_1+n+n_1}\frac{t_2+m_2}{t_2+m_2+n+n_2}\frac{t_3+m_3}{t_3+m_3+n+n_3}. \end{aligned}$$

Furthermore, the minimum posterior risk related to this Bayes estimator is provided by

$$V_p[\hat{\rho}^B|(x,y,z)] = diag(E[\hat{\rho_1}-\rho_1]^2, E[\hat{\rho_2}-\rho_2]^2, E[\hat{\rho_3}-\rho_3]^2),$$

where

$$\begin{aligned} E[\hat{\rho_1}-\rho_1]^2 &= \int_0^1\int_0^1\int_0^1 [\hat{\rho_1}-\rho_1]^2 q(\rho|(x,y,z))d\rho_1 d\rho_2 d\rho_3 \\ &= \frac{[n_1(n_1+1)+n]t_1^2 + n(n-2m_1n_1)t_1 + [m_1(m_1+1)n^2]}{(n+t_1)^2(t_1+m_1+n+n_1)(t_1+m_1+n+n_1+1)}, \end{aligned}$$

$$E[\hat{\rho_2} - \rho_2]^2 = \frac{[n_2(n_2+1)+n]t_2^2 + n(n-2m_2n_2)t_2 + [m_2(m_2+1)n^2]}{(n+t_2)^2(t_2+m_2+n+n_2)(t_2+m_2+n+n_2+1)} \text{ and}$$
$$E[\hat{\rho_3} - \rho_3]^2 = \frac{[n_3(n_3+1)+n]t_3^2 + n(n-2m_3n_3)t_3 + [m_3(m_3+1)n^2]}{(n+t_3)^2(t_3+m_3+n+n_3)(t_3+m_3+n+n_3+1)}.$$

Therefore, $E[V_p(\hat{\rho}^B|(x,y,z))]$ gives a minimum Bayes risk of $\hat{\rho}^B = (\hat{\rho_1}^B, \hat{\rho_2}^B, \hat{\rho_3}^B)$ with respect to the marginal distribution $h(x,y,z)$ of (x,y,z), where $(x,y,z) = (x_1,y_1,z_1),(x_2,y_2,z_2),...,(x_n,y_n,z_n)$ is derived as follows:
The marginal distribution $h(x,y,z)$ of (x,y,z) is given by

$$\begin{aligned} h(x,y,z) &= \int_0^1\int_0^1\int_0^1 L(\rho|(x,y,z)).\tau(\rho|(m',n',p'))d\rho_1.d\rho_2.d\rho_3 \\ &= \frac{\beta(m_1+t_1,n+n_1)\beta(m_2+t_2,n+n_2)\beta(m_3+t_3,n+n_3)}{\beta(m_1,n_1)\beta(m_2,n_2)\beta(m_3,n_3)} \end{aligned}$$

resulting in the minimum Bayes risk factor

$$r_{\tau,\hat{\rho}^B} = E[V_p(\hat{\rho}^B|(x,y,z))] = E[diag(E[\hat{\rho_1}-\rho_1]^2, E[\hat{\rho_2}-\rho_2]^2, E[\hat{\rho_3}-\rho_3]^2)].$$

3. Expected number of customers in the system

The expected number of customers in the system is defined by

$$\begin{aligned} L_s &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{1}(n_1+n_2+n_3)p_{n_1,n_2,n_3}(t) \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{1}(n_1+n_2+n_3)\rho_1^{n_1}(1-\rho_1)\rho_2^{n_2}(1-\rho_2)\rho_3^{n_3}(1-\rho_3) \\ &= (1-\rho_1)(1-\rho_2)(1-\rho_3)\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{1}(n_1+n_2+n_3)\rho_1^{n_1}\rho_2^{n_2}\rho_3^{n_3} \\ &= \frac{\rho_1}{1-\rho_1}+\frac{\rho_2}{1-\rho_2}+\frac{\rho_3}{1-\rho_3}. \end{aligned}$$

Therefore,

$$L_s = \lambda\left[\frac{1}{(\mu_1-\lambda)}+\frac{1}{(\mu_2-\lambda)}+\frac{1}{(\mu_3-\lambda)}\right]. \tag{2}$$

In the next section, we obtain a $100(1-\alpha)\%$ asymptotic confidence interval for the expected number of customers in the system.

3.1. Maximum Likelihood Estimator for the expected number of customers in the system

Given an exponential inter-arrival time population with the parameter λ, let $X_1, X_2, ..., X_n$ be a random sample of size n. Let $Y_{i1}, Y_{i2}, ..., Y_{in}$ represent a random sample of size n taken from a population of service times with an exponential distribution and parameter $\mu_i, i = 1,2,3$. Therefore, it is clear that

$$E[\bar{X}] = \frac{1}{\lambda}, \quad E[\bar{Y_i}] = \frac{1}{\mu_i}, \quad i = 1,2,3.$$

Here $\bar{X}$ and $\bar{Y_i}$, $i = 1,2,3$, respectively represents sample means for inter-arrival times and service times. It can be shown that $\bar{X}$ and $\bar{Y_i}$, $i = 1,2,3$ are, respectively, the MLEs of $\frac{1}{\lambda}$ and $\frac{1}{\mu_i}$, $i = 1,2,3$.

Let $\theta_1 = \frac{1}{\mu_1}$, $\theta_2 = \frac{1}{\mu_2}$, $\theta_3 = \frac{1}{\mu_3}$ and $\theta_4 = \frac{1}{\lambda}$. Then the the expected number of customers in the system given in (2) reduces to

$$L_s = \frac{\theta_1}{(\theta_4 - \theta_1)} + \frac{\theta_2}{(\theta_4 - \theta_2)} + \frac{\theta_3}{(\theta_4 - \theta_3)}.$$

Therefore, using the invariance property of the MLE, the MLE of L_s is given by

$$\hat{L_s} = \frac{\bar{Y}_1}{\bar{X} - \bar{Y}_1} + \frac{\bar{Y}_2}{\bar{X} - \bar{Y}_2} + \frac{\bar{Y}_3}{\bar{X} - \bar{Y}_3}.$$

It should be noticed that $\hat{L_s}$ is a real valued function that is also differentiable in $\bar{Y}_1$, $\bar{Y}_2$, $\bar{Y}_3$ and $\bar{X}$.

3.2. CAN estimator for expected number of customers

By applying the multivariate central limit theorem, we have

$$\sqrt{n}\,[(\bar{Y}_1, \bar{Y}_2, \bar{Y}_3, \bar{X}) - (\theta_1, \theta_2, \theta_3, \theta_4)] \xrightarrow{d} N(0, \Sigma) \quad as \quad n \to \infty.$$

The dispersion matrix $\Sigma = ((\sigma_{ij}))$ is given by $\Sigma = diag(\theta_1^2, \theta_2^2, \theta_3^2, \theta_4^2)$. Again from [7], we have

$$\sqrt{n}(\hat{L}_s - L_s) \xrightarrow{d} N(0, \sigma^2(\theta)) \text{ as } n \to \infty,$$

where $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ and

$$\sigma^2(\theta) = \sum_{i=1}^{3} \left(\frac{\partial L_s}{\partial \theta_i}\right)^2 \sigma_{ii} = \theta_4^2 \left(\frac{\theta_1^2}{(\theta_4 - \theta_1)^4} + \frac{\theta_2^2}{(\theta_4 - \theta_2)^4} + \frac{\theta_3^2}{(\theta_4 - \theta_3)^4}\right). \tag{3}$$

Hence it is concluded that, $\hat{L}_s$ is a CAN estimator of L_s.

3.3. Confidence interval for expected number of customers

Let $\sigma^2(\hat{\theta})$ be the estimator of $\sigma^2(\theta)$ obtained by replacing θ by a consistent estimator $\hat{\theta}$, namely $\hat{\theta} = (\bar{Y}_1, \bar{Y}_2, \bar{Y}_3, \bar{X})$. Let $\hat{\sigma^2} = \sigma^2(\hat{\theta})$. Since $\sigma^2(\theta)$ is a continuous function of θ, $\hat{\sigma^2}$ is a consistent estimator of $\sigma^2(\theta)$ (see, [8]), we have

$$\hat{\sigma^2} \xrightarrow{p} \sigma^2(\theta) \text{ as } n \to \infty.$$

By Slutsky's theorem (see, [8]) ($X_n \xrightarrow{d} x$, $Y_n \xrightarrow{p} b \implies \frac{X_n}{Y_n} \xrightarrow{d} \frac{x}{b}$, $b \neq 0$), we have

$$\sqrt{n}\left(\frac{\hat{L}_s - L_s}{\hat{\sigma}}\right) \xrightarrow{d} N(0, 1) \text{ as } n \to \infty.$$

That is,

$$Pr\left[-k_{\frac{\alpha}{2}} < \sqrt{n}\left(\frac{\hat{L}_s - L_s}{\hat{\sigma}}\right) < k_{\frac{\alpha}{2}}\right] = (1 - \alpha),$$

where $k_{\frac{\alpha}{2}}$ is obtained from the standard normal table. Hence, $100(1 - \alpha)\%$ asymptotic confidence interval for L_s is given by $\left(\hat{L_s} \pm k_{\frac{\alpha}{2}} \frac{\hat{\sigma}}{\sqrt{n}}\right)$, where $\hat{\sigma}$ is obtained from the equation given in equation(3) by replacing θ_1, θ_2 and θ_3 by the corresponding MLEs $\bar{Y}_1$, $\bar{Y}_2$, $\bar{Y}_3$ and $\bar{X}$ respectively.

4. Computational considerations

The Bayes estimator of model parameters of three-station tandem queue with one customer being allowed to wait in the last station using an MCMC (see, [10]) simulation method as is as follows:
1. Defining the likelihood function: The likelihood function is a probability distribution that describes the probability of observing the data given the model parameters. In a queuing model it would be the probability of observing the number of customers in each station, the waiting time, and the service time given the model parameters (such as arrival rate, service rate and observation time).
2. Defining the prior distribution: The prior distribution is a probability distribution that describes the probability distribution of the model parameters before observing the data. In a queuing model it would be the probability of the arrival rate, service rate and observation time.
3. Defining the posterior distribution: The posterior distribution is the probability distribution of the model parameters given the data. It is calculated by multiplying the likelihood function and the prior distribution.
4. Specify the starting values for the MCMC chain: Choose some initial values for the model parameters that we want to estimate.
5. Run the MCMC simulation: Use an MCMC algorithm such as the M-H algorithm to generate a large number of samples from the posterior distribution.
6. Extract the samples from the MCMC chain: Retrieve the samples generated by the MCMC algorithm for each model parameters.
7. Calculate the posterior mean and standard deviation: Compute the mean and standard deviation of the samples for each model parameter. These will be the Bayes estimates of the model parameters.
8. Validate the estimates: Compare the Bayes estimates with the true values of the model parameters (if they are known) or with the estimates obtained using other methods, such as maximum likelihood estimation or method of moments.
9. Assess the convergence of the chain: Check if the chain has converged or not using methods such as trace plots, Gelman-Rubin diagnostic, or effective sample size.

4.1. Simulation

The initial values given for simulation are :
$\rho_1 = 0.3$, $\rho_2 = 0.4$, $\rho_3 = 0.7$, $m_1 = 5$, $m_2 = 6$, $m_3 = 7$, $n_1 = 10$, $n_2 = 9$, $n_3 = 8$.

Table 1: *Table 1: Table of MSE and Bias for different sample sizes.*

Sample Size	Estimates	MSE	Bias
500	0.2677	0.08643	0.17954
	0.3575	0.00374	0.07256
	0.6794	0.05953	0.09211
1000	0.2730	0.00789	0.00623
	0.3823	0.00043	0.00058
	0.6847	0.00312	0.00085
2000	0.2877	0.000036	0.00032
	0.3956	0.000023	0.000082
	0.7148	0.000016	0.00028
5000	0.3062	0.000006	0.000022
	0.4341	0.000004	0.000039
	0.7232	0.000003	0.000009

From the table 1 it is clear that when sample size increases, the Mean Square Error (MSE) and Bias are decreasing and tending to zero, indicating that the validity of the estimators obtained.

4.2. Histograms for simulation range

The histogram of the simulation range for the traffic intensities ρ_1, ρ_2 and ρ_3 is plotted. The Y axis measures the frequency and the X axis shows the range of values that the corresponding traffic intensity takes with respect to the initial value. From the figure 2, figure 3 and figure 4, it is clear that the simulation results have taken a normal curve shape.

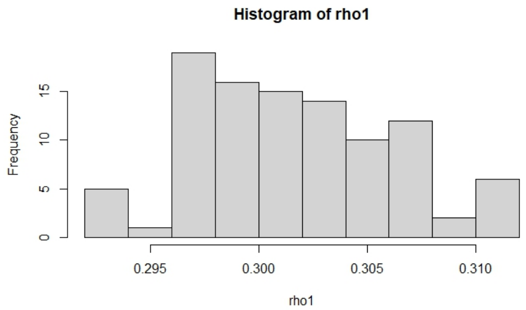

Figure 2: *Histogram 1*

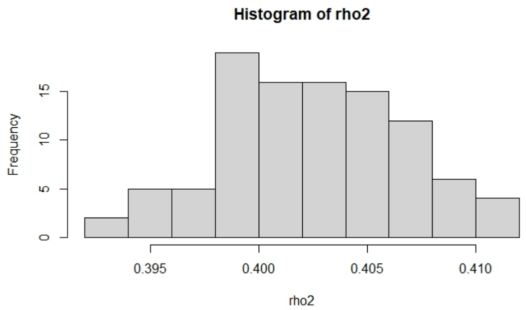

Figure 3: *Histogram 2*

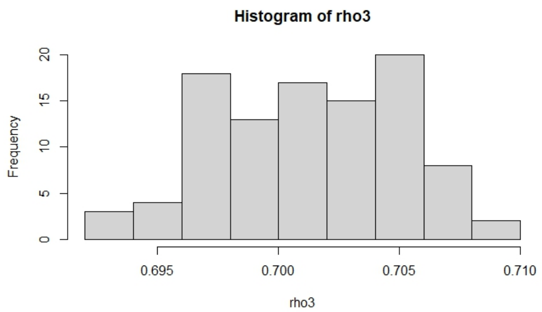

Figure 4: *Histogram 3*

5. Conclusions

In this study, we used the MLE and Bayesian techniques to estimate the traffic intensity for a three-station tandem queue where only one customer was permitted to wait between the last two stations. The Bayes estimators of ρ_1, ρ_2 and ρ_3 were obtained using the beta prior, and the minimal Bayes risk was calculated. We also estimated the expected customers for the system. Then, using Slutsky's theorem, the confidence interval for the expected number of customers was determined. A three-station tandem queue was simulated using MCMC to obtain a Bayes estimators, and the performance of the estimators are verified through a broad simulation study.

References

[1] Chandrasekhar, P. and Ambily, J. (2009). Bayesian inference for a two station tandem queue. *5th Asian Mathematical Conference, Malaysia*.

[2] Chandrasekhar, P. and Chandrasekar, B. and Yadavalli, V. S. S. (2006). Statistical inference for a tandem queue with dependent structure for service times. *In Proceedings of the Sixth IASTED International Conference on Modelling, Simulation and Optimization*, 11–13.

[3] Chandrasekhar, P. and Natarajan, R. and Yadavalli, V.S. S. (2007). Statistical analysis for a tandem queue with blocking. *In Proceedings of the Second National Conference on Management Science and Practice*.

[4] Joby, K. J. and Deepthi, V. (2020) Bayesian inference of Markovian queueing model with two heterogeneous servers. *Stochastic Modelling and Applications*, 24(1):1–14.

[5] Mukhuarjee, S. P. and Choudhury, S. (2005). Maximum likelihood and bayes estimators in m/m/1 queue. *Stochastic Modelling and Applications*, 8(2):47"55.

[6] Mukhuarjee, S. P. and Choudhury, S. (2016). Bayes estimation in m/m/1 queues with bivariate prior. *Journal of Statistics and Manegement systems*, 19:681–699.

[7] Rao, C. R. Linear Statistical inference and its applications, Wiley Eastern Pvt. Ltd., New Delhi, 1974.

[8] Rohatgi, V. K. and Saleh, A. K. An introduction to probability and statistics, John Wiley & Sons, 2015.

[9] Thiruvaiyaru, D. and Ishwar, V. B. (1992). Empirical Bayes estimation for queueing systems and networks. *Queueing Systems*, 11:179–202.

[10] Tierney, L. (1994). Markov chains for exploring posterior distributions. *The Annals of Statistics*, 22:1701–1728.