ON SOME PROPERTIES AND APPLICATIONS OF THE TYPE II HALF -LOGISTIC EXPONENTIATED FRECHET DISTRIBUTION

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Abstract

As the dimensions of available data for analysis continues to grow rapidly, it becomes imperative to develop new probability distributions that can more accurately represent various phenomena. In this research paper, we introduce a novel continuous probability distribution known as the Type II Half-Logistic Exponentiated Frechet Distribution, characterized by four positive parameters. This distribution expands upon the traditional Frechet distribution by introducing two additional parameters. We derive a significant density representation for this distribution. Furthermore, we delve into several statistical and mathematical properties associated with the Type II Half-Logistic Exponentiated Frechet distribution. This includes explicit expressions for key metrics such as the quantile function, probability weighted moments, moments, moments generating function, reliability function, hazard function, and order statistics. To estimate the model parameters effectively, we employ a maximum likelihood estimation technique and present the results of a simulation study. Our research underscores the superiority of this new distribution by applying it to two real-world datasets. Notably, the findings demonstrate that the Type II Half-Logistic Exponentiated Frechet distribution outperforms other considered distributions in fitting the two real datasets.

Keywords: Type II Half-Logistic Exponentiated-G, Frechet distribution, Moments function, Reliability function, Maximum likelihood, Order Statistics.

1. Introduction

Many types of univariate continuous distributions exist, but research in various fields, including engineering, environmental science, finance, and medicine, has shown that real-world data often does not follow the classical distributions. To address this issue, extended forms of these distributions have been developed to provide more flexibility in data modeling. The Frechet distribution, also called the type II extreme value distribution, plays a vital role in extreme value theory and has numerous applications. There have been several modifications and enhancements to the Frechet distribution proposed in the statistical literature to further improve its usefulness. In recent times, various extensions of the Frechet distribution have been introduced by several researchers in the academic literature. Nadarajah and Kotz [16] were the pioneers of the exponentiated Frechet distribution, while Nadarajah and Gupta [17] introduced the beta Frechet distribution. Mahmoud and Mandouh [12] put forth the transmuted Frechet distribution, Da Silva *et al.,* [7] defined the gamma extended Frechet distribution, Krishna *et al.,* [10] introduced the Marshall-Olkin Frechet distribution, and Mead and Abd-Eltawab [13] introduced the Kumaraswamy Frechet distribution. Elbatal *et al.,* [8] conducted a study on the transmuted exponentiated Frechet distribution, Afify *et al.,* [1] investigated the transmuted Marshall-Olkin Frechet distribution, Afify *et al.,* [3] proposed the Kumaraswamy Marshall-Olkin Frechet distribution, Afify *et al.,* [2] explored the Weibull Frechet distribution, Tablada and Cordeiro [20] defined the modified Frechet distribution, and Mead *et al.,* [15] introduced the beta exponential Frechet distribution.

In a recent study, Bello *et al.,* [4] proposed a new distribution family called the Type II Half-Logistic Exponentiated-G (TIIHLEt-G). This distribution family is defined by two positive shape parameters, denoted by λ and α, and can be applied to any arbitrary cumulative distribution function (cdf) $H\left(x,\vartheta\right)$. The cumulative distribution function (cdf) and the probability density function for TIIHLEt-G are detailed as follows:

$$F_{TIIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta})=\frac{2H^{\alpha\lambda}(x;\boldsymbol{\beta})}{\left[1+H^{\alpha\lambda}(x;\boldsymbol{\beta})\right]},\ x>0,\ \ \lambda,\alpha>0 \tag{1}$$

and

$$f_{TIIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta})=\frac{2\lambda\alpha h(x;\boldsymbol{\beta})H^{\alpha-1}(x;\boldsymbol{\beta})\left[H^{\alpha(\lambda-1)}(x;\boldsymbol{\beta})\right]}{\left[1+H^{\alpha\lambda}(x;\boldsymbol{\beta})\right]^{2}},x>0,\ \lambda,\alpha>0 \tag{2}$$

The cdf and pdf of the Frechet distribution are given as

$$H(x;\theta,\delta)=e^{-(\frac{\theta}{x})^{\delta}}\ ,\quad x>0,\theta,\delta>0\,, \tag{3}$$

$$h(x;\theta,\delta)=\delta\theta^{\delta}x^{-\delta-1}e^{-(\frac{\theta}{x})^{\delta}}\ ,\ x>0,\theta,\delta>0 \tag{4}$$

The most important goal of this paper is to enhance the flexibility of a statistical model by extending the conventional two-parameter Frechet distribution. This novel model is referred to as the Type II Half Logistic Exponentiated Frechet (TIIHLEtF) distribution. The structure of this paper is organised as follows: In Section 2, we introduce and define the TIIHLEtF distribution. Section 3 presents valuable representations for the TIIHLEtF distribution. Section 4 focuses on deriving statistical properties such as probability-weighted moments, ordinary moments, moments-generating function, quartile function, reliability function, hazard function, and order statistics. In Section 5, we estimate the parameters of the new model using the maximum likelihood estimation (MLE) approach. To demonstrate the efficiency and consistency of MLE, we conducted a simulation study in Section 6. In Section 7, we apply the new model to two real datasets to illustrate its practical utility. Finally, Section 8 provides a conclusion for the paper.

2. Type II Half-Logistic Exponentiated Frechet (TIIHLEtF) Distribution

In this section, we introduce a novel model referred to as the TIIHLEtF distribution. A random variable X is considered to follow the TIIHLEtF distribution if its cumulative distribution function (cdf) is derived by substituting equation (3) into equation (1) in the following approach:

$$F_{TIIHLEtF}(x;\lambda,\alpha,\theta,\delta)=\frac{2e^{-\alpha\lambda(\frac{\theta}{x})^{\delta}}}{1+e^{-\alpha\lambda(\frac{\theta}{x})^{\delta}}}\ ,x>0,\lambda,\alpha,\theta,\delta>0 \tag{5}$$

and its corresponding pdf is

$$f_{TIIHLEtF}(x;\lambda,\alpha,\theta,\delta)=\frac{2\lambda\alpha\delta\theta^{\delta}x^{-\delta-1}e^{-(\frac{\theta}{x})^{\delta}}e^{-(\alpha-1)(\frac{\theta}{x})^{\delta}}e^{-\alpha(\lambda-1)(\frac{\theta}{x})^{\delta}}}{\left[1+e^{-\alpha\lambda(\frac{\theta}{x})^{\delta}}\right]^{2}}\ ,x>0,\lambda,\alpha,\theta,\beta>0 \tag{6}$$

where θ is a scale parameter and λ,α,δ are shape parameters.

3. Expansion of Density

In this section, we have obtained a valuable expression for the probability density function (pdf) and cumulative distribution function (cdf) of the TIIHLEtF distribution. This achievement is attributed to our utilization of the generalized binomial series given as:

$$\left(1+Z\right)^{-\beta}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta+i-1}{i}z^{i} \tag{7}$$

For $|z|<1$ and β is a positive real non integer. The density function of the TIIHLEtF distribution is derived by applying the binomial theorem from equation (7) to equation (6).

$$f_{TIIHLEtF}(x;\lambda,\alpha,\theta,\delta)=2\lambda\alpha\delta\theta^{\delta}x^{-\delta-1}e^{-(\frac{\theta}{x})^{\delta}}e^{-(\alpha-1)(\frac{\theta}{x})^{\delta}}e^{-\alpha(\lambda-1)(\frac{\theta}{x})^{\delta}}\left[1+e^{-\alpha\lambda(\frac{\theta}{x})^{\delta}}\right]^{-2}$$

Now, using the generalized binomial theorem, we can write

$$\left[1+e^{-\alpha\lambda(\frac{\theta}{x})^{\delta}}\right]^{-2}=\sum_{i=0}^{\infty}(-1)^{i}\binom{1+i}{i}\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha\lambda i}$$

Then, the pdf can be written as:

$$f_{TIIHLEtF}(x;\lambda,\alpha,\theta,\delta)=\sum_{i=0}^{\infty}\eta_{p}\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha\lambda(i+1)} \tag{8}$$

where $\eta_{p}=2\lambda\alpha\delta\theta^{\delta}x^{-\delta-1}(-1)^{i}\binom{1+i}{i}$

In addition, an expansion for the $\left[F(x;\lambda,\alpha,\theta,\delta)\right]^{h}$ is produced, with h being an integer, and the binomial expansion is worked out once more.

$$\left[F(x;\lambda,\alpha,\theta,\delta)\right]^{h}=2^{h}\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{h}\left[1+\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha\lambda}\right]^{-h}$$

$$\left[1+\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha\lambda}\right]^{-h}=\sum_{j=0}^{h}(-1)^{j}\binom{h+j-1}{j}\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha\lambda j}$$

The cdf can be written as:

$$\left[F(x;\lambda,\alpha,\theta,\delta)\right]^{h}=\sum_{j=0}^{h}\varphi_t\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha\lambda(j+h)} \tag{9}$$

where $\varphi_t=2^{h}(-1)^{j}\binom{h+j-1}{j}$

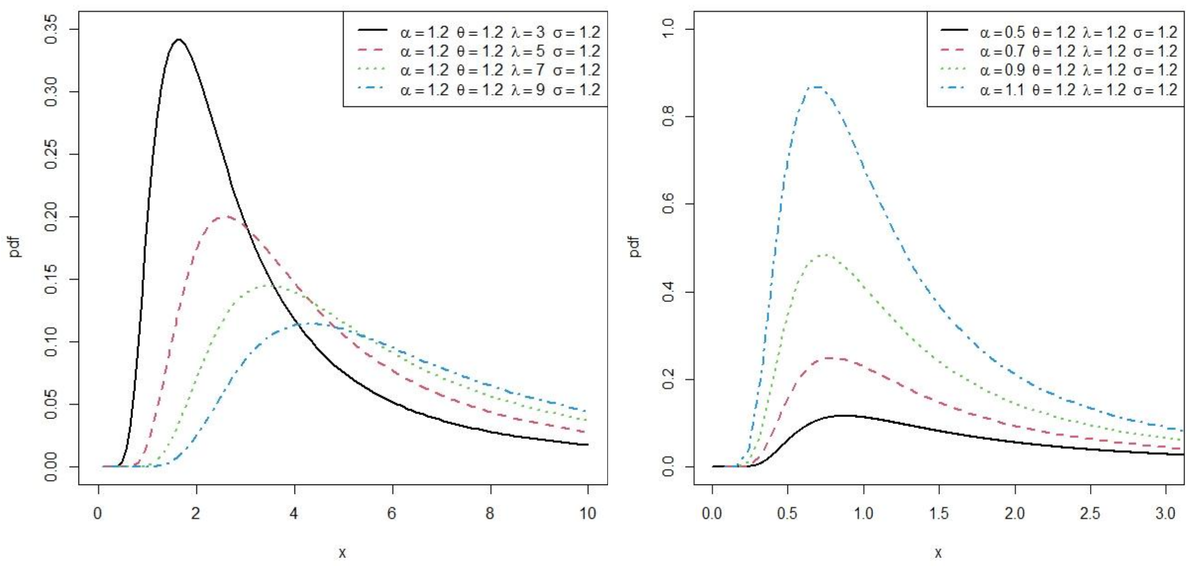

Figure 1: Plots of Pdf of TIIHLEtF distribution for different values of parameters.

4. Statistical Properties

In this section, we derived some statistical properties of the new of distribution.

4.1. Probability weighted moments

Greenwood et al. [10] introduced a concept known as probability weighted moments (PWMs). This technique is employed to create estimators in the inverse form for both distribution parameters and quantiles. The notations used for probability weighted moments is $\tau_{r,s}$, and these moments can be computed for a random variable X by utilizing the relationship outlined below.

$$\tau_{r,s}=E\left[X^{r}F(X)^{s}\right]=\int_{-\infty}^{\infty}x^{r}f(x)(F(x))^{s}dx \tag{10}$$

The PWMs for the TIIHLEtF distribution are obtained by inserting equations (8) and (9) into (10), and then replacing h with s in the following manner.

$$\tau_{r,s}=\sum_{i=0}^{\infty}\sum_{j=0}^{h}\eta_p\varphi_t\int_0^{\infty}x^{r}\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha\lambda(i+1+j+s)}dx \tag{11}$$

Consider the integral

$$\int_0^\infty x^r \left[e^{-(\frac{\theta}{x})^\delta} \right]^{\alpha\lambda(i+1+j+s)} dx$$

Let $y = \alpha\lambda(i+1+j+s)(\frac{\theta}{x})^\delta \Rightarrow x = \left[\frac{\alpha\lambda(i+1+j+s)\theta^\delta}{y} \right]^{\frac{1}{\delta}}; dx = \frac{dyx^{\delta-1}}{\delta\theta^\delta \alpha\lambda(i+1+j+s)}$

Then

$$\int_0^\infty \left[\frac{\alpha\lambda(i+1+j+s)\theta^\delta}{y} \right]^{\frac{r}{\delta}} e^{-y} \frac{dyx^{\delta-1}}{\delta\theta^\delta \alpha\lambda(i+1+j+s)}$$

$$\int_0^\infty y^{-\frac{r}{\delta}} e^{-y} dy = \Gamma(1-\frac{r}{\delta})$$

Hence, the PWMs of TIIHLEtF can be expressed in the following manner.

$$\tau_{r,s} = \sum_{i=0}^{\infty}\sum_{j=0}^{s} (\alpha\lambda)^{\frac{r}{\delta}} \theta^r (i+1+j+s)^{\frac{r}{\delta}-1} \eta_p \varphi_t \Gamma(1-\frac{r}{\delta}) \tag{12}$$

Now,

$$\varphi_t = 2^s (-1)^j \binom{s+j-1}{j}$$

and

$$\eta_p = 2(-1)^i \binom{1+i}{i}$$

4.2. Moments

As moments play a crucial role in statistical analysis, particularly in practical applications, we proceed to derived the rth moment for the newly introduced distribution.

$$\mu_r^{'} = E(x^r) = \int_0^\infty x^r f(x) dx \tag{13}$$

By using the expansion of the pdf in equation (8), we have

$$E(X^r) = \sum_{i=0}^{\infty} \eta_p \int_0^\infty x^r \left[e^{-(\frac{\theta}{x})^\delta} \right]^{\alpha\lambda(i+1)} dx \tag{14}$$

Consider the integral

$$\int_0^\infty x^r \left[e^{-(\frac{\theta}{x})^\delta} \right]^{\alpha\lambda(i+1)} dx$$

Let $w=\alpha\lambda(i+1)(\frac{\theta}{x})^{\delta} \Rightarrow x=\left[\frac{\alpha\lambda(i+1)\theta^{\delta}}{w}\right]^{\frac{1}{\delta}}; dx=\frac{dwx^{\delta-1}}{\alpha\lambda(i+1)\theta^{\delta}\delta}$

Then

$$\int_0^{\infty}\left[\frac{\alpha\lambda(i+1)\theta^{\delta}}{w}\right]^{\frac{r}{\delta}} e^{-w}\frac{dwx^{\delta-1}}{\alpha\lambda(i+1)\theta^{\delta}}$$

$$\int_0^{\infty} w^{-\frac{r}{\delta}} e^{-w} dw = \Gamma(1-\frac{r}{\beta})$$

The r^{th} moment for TIIHLEtF distribution can be written as follows

$$E(X^r)=\sum_{i=0}^{\infty}\eta_p\theta^r(\alpha\lambda)^{\frac{r}{\delta}}(i+1)^{\frac{r}{\delta}-1}\Gamma(1-\frac{r}{\delta}) \tag{15}$$

Now

$$\eta_p = 2(-1)^i\binom{1+i}{i}$$

The mean and variance of TIIHLEtF distribution are as follows

$$E(X)=\sum_{i=0}^{\infty}\eta_p\theta(\alpha\lambda)^{\frac{1}{\delta}}(i+1)^{\frac{1}{\delta}-1}\Gamma(1-\frac{1}{\delta}) \tag{16}$$

and

$$\operatorname{var}\left(X\right)=\sum_{i=0}^{\infty}\eta_p\theta(\alpha\lambda)^{\frac{1}{\delta}}(i+1)^{\frac{1}{\delta}-1}\Gamma(1-\frac{1}{\delta})-\left[\sum_{i=0}^{\infty}\eta_p\theta(\alpha\lambda)^{\frac{1}{\delta}}(i+1)^{\frac{1}{\delta}-1}\Gamma(1-\frac{1}{\delta})\right]^2 \tag{17}$$

4.3. Moment generating function (mgf)

The Moment Generating Function of x is given as:

$$M_x\left(t\right)=E\left(e^{tx}\right)=\int_0^{\infty}e^{tx}f(x)dx \tag{18}$$

where the expansion of $e^{tx}=\sum_{m=0}^{\infty}\frac{t^m x^m}{m!}$

The moment generating function of TIIHLEtF distribution is given by

$$M_x(t)=\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^m\eta_p\theta^m(\alpha\lambda)^{\frac{m}{\delta}}(i+1)^{\frac{m}{\delta}-1}\Gamma(1-\frac{m}{\delta})}{m!} \tag{19}$$

4.4. Reliability function

The reliability function, also referred to as the survivor function, provides the probability that an individual or patient will endure beyond certain specified duration of time. In other words, it gives the likelihood of survival beyond a particular time point. It's defined as

$$R(x;\lambda,\alpha,\theta,\delta)=\frac{1-\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha\lambda}}{1+\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha\lambda}} \tag{20}$$

4.5. Hazard function

The hazard function represents the likelihood of an event of interest happening within a relatively brief time interval is defined as follow:

$$T(x;\lambda,\alpha,\theta,\beta)=\frac{2\lambda\alpha\delta\theta^{\delta}x^{-\delta-1}e^{-(\frac{\theta}{x})^{\delta}}\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha-1}\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha(\lambda-1)}}{1-\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{2\alpha\lambda}} \tag{21}$$

4.6. Quantile Function

The quantile function plays a crucial role in generating random variables from continuous probability distributions, making it a key element in probability theory. Specifically, for a given value 'x,' the quantile function is denoted as F(x) = u, where 'u' follows a uniform distribution between 0 and 1 (U(0,1)). To simulate the TIIHLEtF distribution, one can readily achieve this by reversing equation (5), resulting in the definition of the quantile function Q(u).

$$x=Q(u)=\frac{\theta}{\left[-\log\left[\frac{U}{2-U}\right]^{\frac{1}{\alpha\lambda}}\right]^{\frac{1}{\delta}}} \tag{22}$$

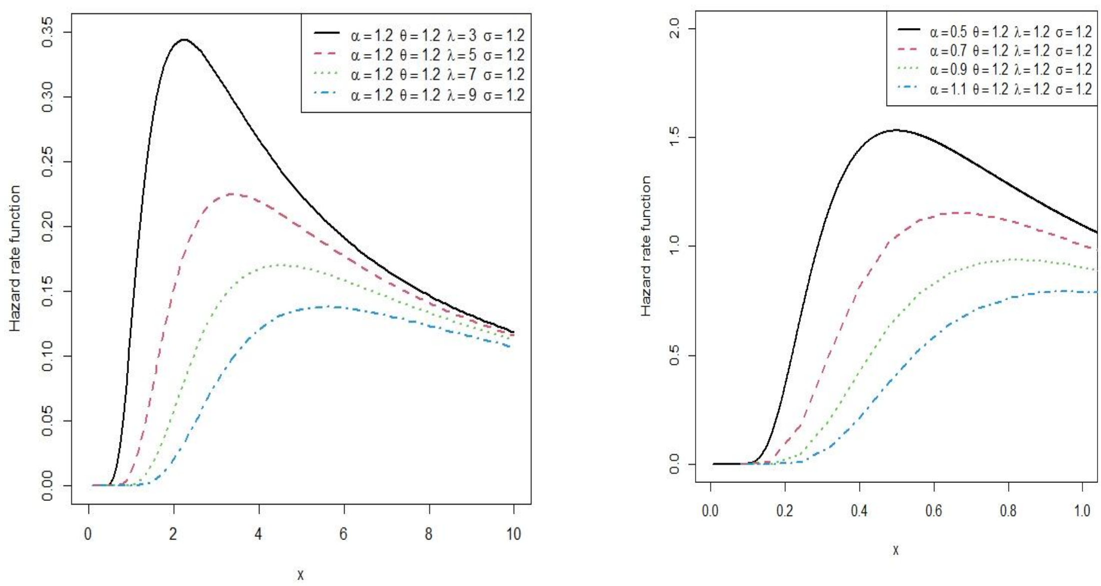

Figure 2: Plots of hazard of the TIIHLEtF distribution for different valves of parameters.

4.7. Order Statistics

Order statistics are widely applied in various statistical fields, including reliability and life testing. Consider a set of n independent and identically distributed random variables represented as X_1, X_2, ..., X_n, each following a continuous distribution function F(x). If these random variables are drawn from the TIIHLEtF distribution, we can denote the cumulative distribution function (cdf) as Fr:n(x) and the probability density function (pdf) as $f_{r:n}(x)$ for the r^{th} order statistic, where r ranges from 1 to n. In a study by David [1979], the probability density function of $X_{r:n}$ was provided.

$$f_{r:n}(x) = \frac{f(x)}{B(r,n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} F(x)^{v+r-1} \tag{23}$$

By substituting equation (8) and equation (9) into equation (23), also replacing h with v+r-1 in equation (9). We have

$$f_{r:n}(x;\lambda,\alpha,\theta,\delta) = 2^{(r+v)} \lambda\alpha\delta\theta^{\delta} x^{-\delta-1} \frac{1}{B(r,n-r+1)} \sum_{v=0}^{n-r}\sum_{i=0}^{\infty}\sum_{j=0}^{r+v-1} (-1)^{i+j+v} \binom{n-r}{v} \binom{1+i}{i} \binom{r+v+j-2}{j} \left[e^{-(\frac{\theta}{x})^{\delta}} \right]^{\alpha\lambda(i+j+r+v)} \tag{24}$$

The equation above is called the r^{th} order statistics for the TIIHLEtF distribution.
Let r = n, then the probability density function of the maximum order statistics of TIIHLEtF distribution is

$$f_{n:n}(x;\lambda,\alpha,\theta,\delta) = 2^{(n+v)} n\lambda\alpha\delta\theta^{\delta} x^{-\delta-1} \sum_{i=0}^{\infty}\sum_{j=0}^{n+v-1} (-1)^{i+j+v} \binom{1+i}{i} \binom{n+v+j-2}{j} \left[e^{-(\frac{\theta}{x})^{\delta}} \right]^{\alpha\lambda(i+j+n+v)} \tag{25}$$

Also, let r = 1, then the probability density function of the minimum order statistics of TIIHLEtF distribution is

$$f_{1:n}(x;\lambda,\alpha,\theta,\delta) = 2^{(v+1)} n\lambda\alpha\delta\theta^{\delta} x^{-\delta-1} \sum_{v=0}^{n-1}\sum_{i=0}^{\infty}\sum_{j=0}^{v} (-1)^{i+j+v} \binom{n-1}{v} \binom{1+i}{i} \binom{v+j-1}{j} \left[e^{-(\frac{\theta}{x})^{\delta}} \right]^{\alpha\lambda(i+j+1+v)} \tag{26}$$

5. Parameter Estimation

In this research paper, we investigate the application of the maximum likelihood technique to estimate the unknown parameters of the TIIHLEtF distribution when dealing with complete data. Maximum likelihood estimates (MLEs) possess advantageous characteristics that can be utilized to establish confidence intervals and provide straightforward approximations that perform well with finite data samples. In the realm of distribution theory, these approximations for MLEs can be conveniently managed, either through analytical or numerical methods. Consider a random sample of size n, denoted as x_1, x_2, x_3,...,x_n, drawn from the TIIHLEtF distribution. Then, the likelihood function, based on the observed sample, for the parameter vector $(\lambda,\alpha,\theta,\delta)^T$ is defined as follows.

$$\log L = n\log(2) + n\log(\lambda) + n\log(\alpha) + n\log(\delta) + n\delta\log(\theta) - (\delta+1)\sum_{i=1}^{n}\log(x_i)$$
$$-\alpha\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\delta} - \alpha(\lambda-1)\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\delta} - 2\sum_{i=1}^{n}\log\left[1+\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha\lambda}\right] \tag{27}$$

The components of score vector $\Delta L(\phi) = \left(\frac{\partial L(\phi)}{\partial\lambda}, \frac{\partial L(\phi)}{\partial\alpha}, \frac{\partial L(\phi)}{\partial\theta}, \frac{\partial L(\phi)}{\partial\delta}\right)^T$ are given as

$$\frac{\partial\log L}{\partial\lambda} = \frac{n}{\lambda} - \alpha\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\delta} - 2\sum_{i=1}^{n}\frac{\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha\lambda}\log\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}}{1+\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha\lambda}} = 0 \tag{28}$$

$$\frac{\partial\log L}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\delta} - (\lambda-1)\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\delta}$$
$$-2\sum_{i=1}^{n}\frac{\lambda\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha(\lambda-1)}\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}\log\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]}{1+\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha\lambda}} = 0 \tag{29}$$

$$\frac{\partial\log L}{\partial\delta} = \frac{n}{\delta} + n\log(\theta) - \sum_{i=1}^{n}x_i - \alpha\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\delta}\log\left(\frac{\theta}{x_i}\right) - \alpha(\lambda-1)\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\delta}\log\left(\frac{\theta}{x_i}\right)$$
$$-2\sum_{i=1}^{n}\frac{\lambda\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha(\lambda-1)}\alpha\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha-1}e^{-(\frac{\theta}{x_i})^{\delta}}\left(\frac{\theta}{x_i}\right)^{\delta}\log\left(\frac{\theta}{x_i}\right)}{1+\left[e^{-(\frac{\theta}{x_i})}\right]^{\alpha\lambda}} = 0 \tag{30}$$

$$\frac{\partial\log L}{\partial\theta} = \frac{n\delta}{\theta} + \alpha\delta\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\delta-1}\frac{1}{x_i} - \delta\alpha(\lambda-1)\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\delta-1}\frac{1}{x_i}$$
$$-2\sum_{i=1}^{n}\frac{\lambda\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha(\lambda-1)}\alpha\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha-1}e^{-(\frac{\theta}{x_i})^{\delta}}\left(\frac{\theta}{x_i}\right)^{\delta-1}\frac{1}{x_i}}{1+\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha\lambda}} = 0 \tag{31}$$

The MLEs are obtained by setting $\frac{\partial L(\phi)}{\partial\lambda}, \frac{\partial L(\phi)}{\partial\alpha}, \frac{\partial L(\phi)}{\partial\theta}$ and $\frac{\partial L(\phi)}{\partial\delta}$ to zero and solving these equations simultaneously. These equations cannot be solved analytically, so we have to appeal to numerical method.

6. Simulation Study

In this section, a numerical analysis will be conducted to evaluate the performance of MLE for TIIHLEtF Distribution.

Table 1: MLEs, biases and RMSE for some values of parameters

		(1,1,2.5,2.5)			(1,0.5,2,1)		
n	Parameters	Estimated Values	Bais	RMSE	Estimated Values	Bais	RMSE
20	λ	1.9103	0.9103	0.1823	1.0140	0.0140	0.0450
	α	1.0472	0.0472	0.1458	0.5247	0.0247	0.0667
	θ	2.5845	0.0845	0.1504	2.0140	0.0140	0.0434
	δ	2.5845	0.0845	1.5894	2.0140	1.0140	1.0149
50	λ	1.7149	0.7149	0.1330	1.0076	0.0076	0.0320
	α	1.0384	0.0384	0.0930	0.5058	0.0058	0.0303
	θ	2.5401	0.0401	0.1281	2.0113	0.0113	0.0392
	δ	2.5401	0.0401	1.3058	2.0113	1.0113	1.0120
100	λ	1.5193	0.5193	0.1157	1.0031	0.0031	0.0208
	α	1.0382	0.0382	0.0696	0.5008	0.0008	0.0096
	θ	2.5389	0.0389	0.1213	2.0068	0.0068	0.0294
	δ	2.5389	0.0389	0.6298	2.0068	1.0068	1.0072
250	λ	1.4246	0.4246	0.0980	1.0001	0.0001	0.0018
	α	1.0376	0.0376	0.0587	0.5000	0.0000	0.0000
	θ	2.5120	0.0120	0.1202	2.0008	0.0008	0.0102
	δ	2.5120	0.0120	0.6126	2.0008	1.0008	1.0009
500	λ	1.3032	0.3032	0.0849	1.0000	0.0000	0.0000
	α	1.0303	0.0303	0.0470	0.5000	0.0000	0.0000
	θ	2.5110	0.0110	0.1170	2.0000	0.0000	0.0000
	δ	2.5110	0.0110	0.5114	2.0000	1.0000	1.0000
1000	λ	1.1332	0.1332	0.0762	1.0000	0.0000	0.0000
	α	1.0282	0.0282	0.0396	0.5000	0.0000	0.0000
	θ	2.5032	0.0032	0.1092	2.0000	0.0000	0.0000
	δ	2.5032	0.0032	0.5036	2.0000	1.0000	1.0000

The table above shows the values of biases and RMSEs approach zero and the estimates tend to the initial (true) values as the sample increases, which indicates that the estimates are efficient and consistent.

7. Applications to Real Data

In this section, we apply the TIIHLEtF distribution to two real datasets and perform a comparative analysis by contrasting it with fits to other distribution models. Specifically, we compare it with the Exponentiated Half-Logistic Frechet (EHLF) distribution proposed by Cordeiro *et al.,* [6], Kumaraswamy Frechet (KExF) distribution by Mead and Abd-Eltawab [14], the Gompertz Frechet (GoFr) distribution by Oguntunde *et al., [18]* , the Exponentiated Frechet (ExFr) distribution by Nadaraja and Kotz [16], and the Frechet distribution introduced by Frechet [9]. This comparison is carried out for illustrative purposes.

The EHLF distribution developed by Cordeiro *et al.* [6] has pdf defined as:

$$f(x;\alpha,\lambda,\theta,\beta)=2\alpha\lambda\theta\beta^{\theta}x^{-(\theta+1)}e^{\left[-\left(\frac{\beta}{x}\right)^{\theta}\right]}\left[1-e^{\left[-\left(\frac{\beta}{x}\right)^{\theta}\right]}\right]^{\lambda-1}\left[1-\left[1-e^{\left[-\left(\frac{\beta}{x}\right)^{\theta}\right]}\right]^{\lambda}\right]^{\alpha-1}\left[1+\left[1-e^{\left[-\left(\frac{\beta}{x}\right)^{\theta}\right]}\right]^{\lambda}\right]^{-(\alpha+1)} \quad (32)$$

The KExF distribution developed by Mead and Abd-Eltawab [14] has pdf defined as:

$$f(x;\alpha,\lambda,\theta,\beta)=\alpha\lambda\beta\theta^{\beta}x^{-\beta-1}e^{-\alpha(\frac{\theta}{x})^{\beta}}\left[1-e^{-\alpha(\frac{\theta}{x})^{\beta}}\right]^{\lambda-1} \quad (33)$$

The GoFr distribution proposed by Oguntunde *et al.*, [18] has pdf given as:

$$f(x;\alpha,\lambda,\theta,\beta)=\theta\beta\alpha^{\beta}x^{-\beta-1}e^{\left[-(\frac{\alpha}{x})^{\beta}\right]}\left[e^{\left[-(\frac{\alpha}{x})^{\beta}\right]}\right]^{\lambda-1}e^{\left[\frac{\theta}{\lambda}\left(1-\left[1-e^{\left[-(\frac{\alpha}{x})^{\beta}\right]}\right]\right)^{-\lambda}\right]} \quad (34)$$

The ExFr Distribution proposed by Nadaraja and Kotz [16] has pdf given as:

$$f(x;\alpha,\lambda,\sigma)=\alpha\lambda\sigma^{\lambda}\left[1-e^{\left(-(\frac{\sigma}{x})^{\lambda}\right)}\right]^{\alpha-1}x^{-(1+\lambda)}e^{\left(-(\frac{\sigma}{x})^{\lambda}\right)} \quad (35)$$

The Frechet distribution developed by Frechet [9] has pdf defined as:

$$f(x;\theta,\sigma)=\delta\theta^{\sigma}x^{-\sigma-1}e^{-(\frac{\theta}{x})^{\sigma}} \quad (36)$$

The two datasets utilized as illustrative examples in this application showcase the enhanced distribution flexibility and suitability of the newly proposed distribution. It also demonstrates its ability to provide the "best fit" when empirically modeling these datasets, surpassing the previously mentioned comparator distributions. All calculations were carried out using the R programming language.

Data set 1

The first dataset provided below contains information about the times at which 84 aircraft windshields experienced failures. This dataset was previously utilized in a study by Tahir *et al.*, [21].

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

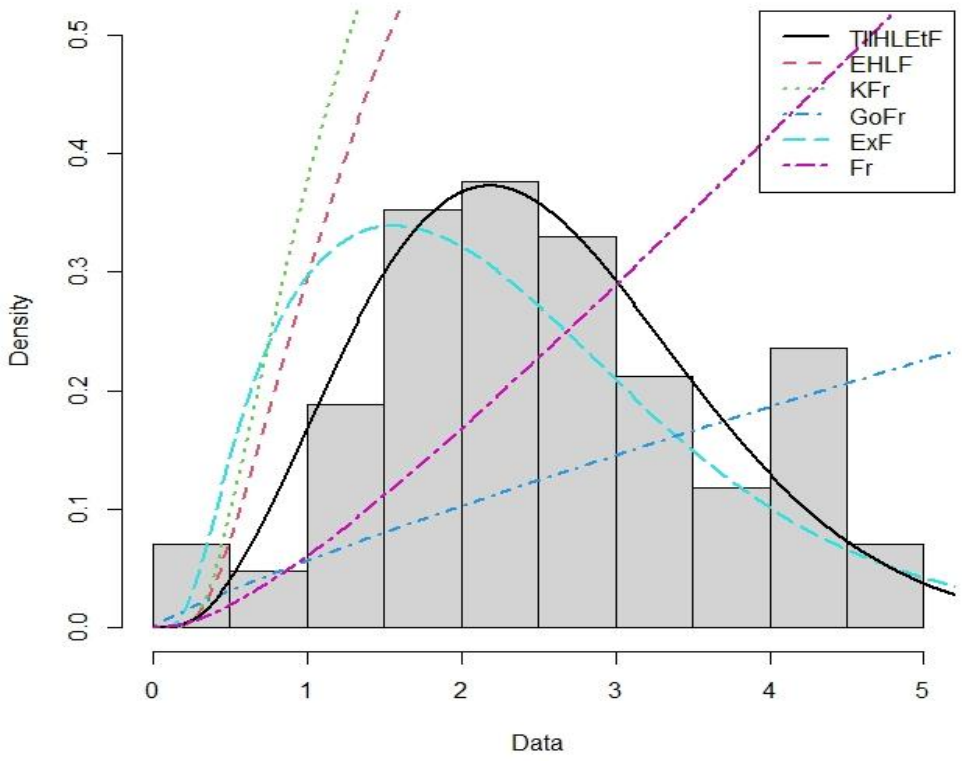

Figure 3: Fitted pdfs for the TIIHLEtF, EHLF, KFr, GoFr, ExF, and Fr distributions to the data set 1

Table 2: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 1

Distributions	α	λ	θ	δ	β	LL	AIC
TIIHLEtF	1.4258	1.4258	1.0593	0.7272	-	-60.4319	128.8638
EHLF	0.6829	22.7797	0.5953	-	16.8659	-152.1688	312.3376
KFr	13.1105	1.9176	0.1052	-	0.8131	-63.4185	134.837
GoFr	1.3750	1.5499	5.3750	-	1.3750	- 186.4972	380.9943
ExF	5.7603	0.6018		7.1979	-	-167.5459	341.0917
Fr	-	-	19.5745	0.3347	-	-146.065	296.1299

Table 2 displays the outcomes of maximum likelihood estimation for estimating the parameters of both the newly proposed distribution and five comparator distributions. Evaluating goodness of fit, the new proposed distribution exhibited the lowest AIC value, with the KFr distribution coming in a close second. A visual assessment of the fit, as shown in Figure 3, further reinforces the superiority of the proposed distribution when compared to the comparator distributions. Consequently, the newly proposed distribution is deemed the most suitable choice for modeling an aircraft windshields failure dataset from the assortment of distributions under consideration.

Data set 2

The second dataset presented below records both the instances of failure and the periods of service for a windshield. This dataset was previously employed in a study conducted by Kundu and Raqab [12].

0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483,1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881,1.262, 2.543, 5.140.

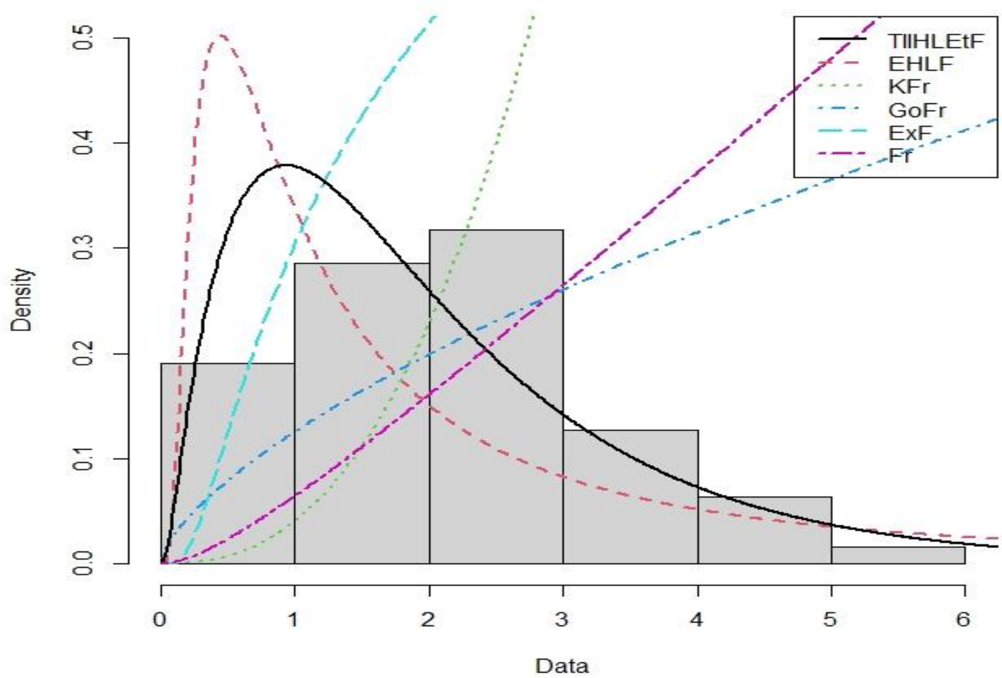

Figure 4: Fitted pdfs for the TIIHLEtF, EHLF, KFr, GoFr, ExF, and Fr distributions to the data set 2

Table 3: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 2

Distributions	α	λ	θ	δ	β	LL	AIC
TIIHLEtF	0.8728	2.1344	0.7247	0.6787	-	- 75.4283	158.8566
EHLF	1.3995	5.6071	0.5085	-	3.1002	- 115.1316	238.2632
KFr	0.0046	0.0248	0.0084	-	2.5480	- 130.8708	269.7416
GoFr	3.3750	2.0249	5.3750	-	3.3750	-111.8307	231.6613
ExF	6.5403	0.3166	-	9.4231	-	- 108.8879	223.7758
Fr	-	-	19.4876	0.2848	-	-139.9228	283.8457

In Table 3, you can find the outcomes of Maximum Likelihood Estimation for estimating the parameters of the TIIHLEtF distribution and five other comparator distributions. When considering the goodness of fit statistic AIC, it's worth noting that the new distribution displayed the lowest AIC value, indicating that it is the most appropriate fit for the hypertension patients' dataset. Furthermore, a visual examination of the fit, as depicted in Figure 4, reinforces the superiority of the new distribution over its comparator counterparts. Hence, the new distribution is confirmed as the optimal choice for modeling the data of instances of failure and the periods of service for a windshield.

8. CONCLUSION

In this article, we introduced and explored a novel distribution known as the Type II Half-Logistic Exponentiated Frechet Distribution, building upon the distribution family originally proposed by Bello *et al., [4]*. We conducted a thorough examination of various statistical components associated with this new distribution, including the explicit quantile function, probability-weighted moments, moments, generating function, reliability function, hazard function, and order statistics. The estimation of its parameters was carried out using the maximum likelihood technique. We presented simulation results to assess the performance of this new distribution, and we also compared it to well-established models. Furthermore, we applied it to analyze two real datasets to underscore the significance and versatility of the new distribution. The findings suggest that the new distribution outperforms the existing models considered, indicating its potential applicability in a wide range of practical applications for modeling data.

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