M/M/∞ QUEUE WITH IMPATIENT CUSTOMERS

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Abstract

In this paper we proposed an M/M/∞ queue with impatient customers. Generally, customers are impatient due to long waits in queue but in this work, we consider the case when customers are not impatient due to long waits but they are impatient due to the poor quality of service. We model and analyze this queueing system by using continued fraction technique and obtained the probability mass function of the customers present in the system in time dependent form. Also, we calculate the average queue size. Finally, some graphical representations are given to illustrate the model.

Keywords: M/M/∞ Queue, Transient Solution, Impatient Customers, Laplace Transform.

1. Introduction

We present an M/M/∞ queue with impatient customers. The impatient behavior of customer is common in many real life queueing situations such as in hospital during emergencies, inventory systems, telecommunications system etc.When the waiting time is sufficiently large or intolerable the customers may become impatient and decide to leave (i.e.balk or renege) the system before being served. The study of queueing models with impatient customers play an important role in many revenue generating queueing system. There is an extensive literature available on queues with impatient customers(see e.g.,[4], [5], [14]). First attempt in this field was made by Haight [14]. After that, Al-Seedy et.al.[2] obtained the transient behavior of an M/M/1 queue with balking. The single server Markovian queue with reneging was proposed by Haight [15]. Ancker et.al.([4], [5]) considered an M/M/1/N queue with both balking and reneging simultaneously. Multi-server queueing model with impatient customers was investigated by Varshney et.al.[23]. Time dependent solution of the M/M/c queueing model was proposed by Al-Seedy et.al.[3]. The concept of balking with heterogeneous servers have been proposed by Abou El-Ata [6] and Singh [19]. Queues with catastrophes and impatient customers have been investigated by various authors. Yechiali [24] consider the case of impatient customers when server is down due to catastrophes. Sudhesh [22] extends the work of Yechiali [24] and obtained the time dependent solution. Altman and Yechiali [1] considered an infinite server queueing system with impatient customers under the situation where servers are free and doing some additional task. A GI/G/1 queue with disaster and customer impatient was studied by Chakravarthy [12]. Customers impatient due to priority has been analyzed by Choi et. al.[11]. Sudhesh et. al.[20] obtained the transient solution of two heterogeneous servers queue with impatient customers when server is down due to the occurrence of breakdown.Vacation queueing model are also analyzed with impatient behavior of customers by various researchers. Ammar [8] obtained the transient solution of a waiting server,vacation queueing model with impatient customers. Sampath et. al. [21] extends the work of Ammar [8] by considering multiple vacation in place of single vacation. Perel and Yechiali [17] give the steady sate solution of an M/M/c (c=1, 1 <c <∞, c= ∞) queue with slow server and impatient behavior of customers in two random phases. Generally it is assumed that customers are impatient due to long waits in queue but in this work authors considered that customers may be impatient due to slow service rate. In our work we also consider that customers may be impatient due to poor quality of service provided by the servers and obtained the time dependent solution of an M/M/∞ queue with impatient customers. Generally, in self service models or an infinite server models there is no question of impatient behavior of customers because the entering customers immediately get service and there is no waiting line in the system. But in our case the customer is impatient not due to long waits but due to the quality of the service provided by the server. It may possible that a customer is impatient due to poor quality of service. The motivation for studying this model comes from the field of telecommunications. Let us consider a university campus which is providing a free Wi-Fi service for their students at their campus. Every student entering in the campus may use the free Wi-Fi service. The service starts as soon as he joins the campus and carries until the end of the campus. A small university campus contains thousands of such self served customers who are using this service. Each customer receives identical quality of signals of Wi-Fi connection. The quality of signals may vary and fluctuate randomly.The poor quality of signals causes the impatient behavior of customers.Whenever a customer enters into the campus and finds poor quality of signals of the Wi-Fi connection, he may decide to leave the system without getting served i.e. customer balk from the system. On other hand, he joins the service but leaves the system due to poor quality of Wi-Fi connection, this also becomes a case of customer renege from the system. Hence, our operating model is a suitable preposition.

2. Mathematical Model

M/M/∞ queue with impatient customers is in operation. Arrivals occur one by one in a Poisson stream with mean rate α. There are infinite servers and service time are exponentially distributed with parameter β. Capacity of the system is infinite. After entering the system, the customers either decide to join the service with probability θ or balk with probability $1-\theta$, where $0 \leqslant \theta < 1$. After joining the service, if he finds a poor quality of signals of Wi-Fi connection then, the customer will wait for a certain length of time T, exponentially distributed with parameter γ, for improving the quality of service. If it has not improved by then, the customer abandons and leave the system without getting complete service. Let $P_n(t)$ be the probability that the random variable $N(t)$ assumes the value n i.e.
$P_n(t) = P(N(t) = n)$

3. Transient Solution

In this section, we provide the transient solution of the presented queueing model. For this, the differential- difference equations are given as:

$$P_0'(t) = -(\alpha\theta)P_0(t) + (\beta+\gamma)P_1(t) \tag{1}$$

$$P_n'(t) = -\left(\alpha\theta + n(\beta+\gamma)\right)P_n(t) + (n+1)\left(\beta+\gamma\right)P_{n+1}(t) + \theta\alpha P_{n-1}(t), n \geq 1. \tag{2}$$

Initially, at t=0,

$$P_n(0) = \begin{cases} 1 & \text{if } n = 0; \\ 0 & \text{otherwise}. \end{cases} \tag{3}$$

Laplace transformation of Eq.(2) with initial condition Eq.(3) results the following equation

$$\left(s + \theta\alpha + n(\beta+\gamma)\right)P_n^*(s) = (n+1)\left(\beta+\gamma\right)P_{n+1}^*(s) + \theta\alpha P_{n-1}^*(s) \tag{4}$$

After simplification, Eq.(4), gives

$$\frac{P_n^*(s)}{P_{n-1}^*(s)} = \frac{\theta\alpha}{(s + \theta\alpha + n(\beta+\gamma) - (n+1)(\beta+\gamma)\frac{P_{n+1}^*(s)}{P_n^*(s)}} \tag{5}$$

$$= \cfrac{\frac{\theta\alpha}{\beta+\gamma}}{\left(\frac{s}{\beta+\gamma}+\frac{\theta\alpha}{\beta+\gamma}+n\right)-\cfrac{(n+1)\frac{\theta\alpha}{\beta+\gamma}}{\left(\frac{s}{\beta+\gamma}+\frac{\theta\alpha}{\beta+\gamma}+(n+1)\right)-\cfrac{(n+2)\frac{\theta\alpha}{\beta+\gamma}}{\left(\frac{s}{\beta+\gamma}+\frac{\theta\alpha}{\beta+\gamma}+(n+2)\right)-\cdots}}} \tag{6}$$

Now using the identity given by Lorentzen and Waadeland [16]

$$\frac{{}_1F_1(c+1;q+1;z)}{{}_1F_1(c;q;z)} = \frac{q}{q-z+}\frac{(c+1)z}{q-z+1+}\frac{(c+2)z}{q-z+2+}\cdots \tag{7}$$

Use of Eq.(7) in Eq.(6), gives

$$\frac{P_n^*(s)}{P_{n-1}^*(s)} = \frac{\theta\alpha}{(\beta+\gamma)}\frac{{}_1F_1(n+1;\frac{s}{\beta+\gamma}+n+1;\frac{-\theta\alpha}{\beta+\gamma})}{\left(\frac{s}{\beta+\gamma}+n\right)\ {}_1F_1(n;\frac{s}{\beta+\gamma}+n;\frac{-\theta\alpha}{\beta+\gamma})}, \tag{8}$$

therefore for $n \geq 1$, we have

$$P_n^*(s) = \left(\frac{\theta\alpha}{(\beta+\gamma)}\right)^n \frac{{}_1F_1(n+1;\frac{s}{\beta+\gamma}+n+1;\frac{-\theta\alpha}{\beta+\gamma})}{\prod_{i=1}^n\left(\frac{s}{\beta+\gamma}+i\right)\ {}_1F_1(1;\frac{s}{\beta+\gamma}+1;\frac{-\theta\alpha}{\beta+\gamma})}P_0^*(s), \tag{9}$$

$$P_n^*(s) = \zeta_n^*(s)P_0^*(s), \tag{10}$$

where

$$\zeta_n^*(s) = \left(\frac{\theta\alpha}{(\beta+\gamma)}\right)^n \frac{{}_1F_1(n+1;\frac{s}{\beta+\gamma}+n+1;\frac{-\theta\alpha}{\beta+\gamma})}{\prod_{i=1}^n\left(\frac{s}{\beta+\gamma}+i\right)\ {}_1F_1(1;\frac{s}{\beta+\gamma}+1;\frac{-\theta\alpha}{\beta+\gamma})}. \tag{11}$$

It is well known that

$$\sum_{n=0}^{\infty} P_n^*(s) = \frac{1}{s}, \tag{12}$$

by the use of Eq.(10) in Eq.(12), we get

$$P_0^*(s) = \frac{1}{s}\left[1+\sum_{n=1}^{\infty}\zeta_n^*(s)\right]^{-1} \tag{13}$$

$$P_0^*(s) = \frac{1}{s}\left[\sum_{k=0}^{\infty}\left(\sum_{n=1}^{\infty}\zeta_n^*(s)\right)^k\right], \tag{14}$$

after taking inverse Laplace transform of Eq.(10), we get

$$P_n(t) = \zeta_n(t) * P_0(t), \tag{15}$$

where the symbol * denotes the convolution and

$$P_0(t) = \int_0^t \sum_{k=0}^{\infty}\left(\sum_{n=1}^{\infty}\zeta_n(y)\right)^k dy. \tag{16}$$

Next we derive the expression for $\zeta_n(t)$, where $\zeta_n(t)$ represents the inverse Laplace transform of $\zeta_n^*(s)$.
From Eq.(11)

$$\zeta_n^*(s) = \left(\frac{\theta\alpha}{(\beta+\gamma)}\right)^n \frac{{}_1F_1(n+1;\frac{s}{\beta+\gamma}+n+1;\frac{-\theta\alpha}{\beta+\gamma})}{\prod_{i=1}^n\left(\frac{s}{\beta+\gamma}+i\right)\ {}_1F_1(1;\frac{s}{\beta+\gamma}+1;\frac{-\theta\alpha}{\beta+\gamma})}.$$

It is well known that

$$ {}_1F_1(n+1; \frac{s}{\beta+\gamma}+n+1; \frac{-\theta\alpha}{\beta+\gamma}) = \sum_{k=0}^{\infty} \frac{(n+1)_k \left(-\frac{-\theta\alpha}{\beta+\gamma}\right)^k}{(\frac{s}{\beta+\gamma}+n+1)_k \, k!} $$

where $(\beta)_k$ represents the Pochhammer symbol,i.e.

$$ (\beta)_k = \begin{cases} 1 & \text{if } k=0; \\ \beta(\beta+1)(\beta+2)...(\beta+k-1) & \text{if } k=1,2,3,.... \end{cases} $$

Therefore

$$ \frac{{}_1F_1(n+1; \frac{s}{\beta+\gamma}+n+1; \frac{-\theta\alpha}{\beta+\gamma})}{\prod_{i=1}^{n}\left(\frac{s}{\beta+\gamma}+i\right)} = \sum_{k=0}^{\infty} \frac{\binom{n+k}{k}\left(-\frac{\theta\alpha}{\beta+\gamma}\right)^k}{\prod_{i=1}^{n+k}\left(\frac{s}{\beta+\gamma}+i\right)} $$

Applying partial fraction expansion, the above equation can be written as

$$ \frac{{}_1F_1(n+1; \frac{s}{\beta+\gamma}+n+1; \frac{-\theta\alpha}{\beta+\gamma})}{\prod_{i=1}^{n}\left(\frac{s}{\beta+\gamma}+i\right)} = (\beta+\gamma)\sum_{k=0}^{\infty}\binom{n+k}{k}\left(-\frac{\theta\alpha}{\beta+\gamma}\right)^k \sum_{i=1}^{n+k} \frac{(-1)^{i-1}}{(n+k-i)!\,(i-1)!\,(s+i(\beta+\gamma))}. \tag{17} $$

Also

$$ {}_1F_1(1; \frac{s}{\beta+\gamma}+1; \frac{-\theta\alpha}{\beta+\gamma}) = \sum_{k=0}^{\infty}(-\theta\alpha)^k\, d_k^*(s), $$

where

$$ d_k^*(s) = \frac{1}{\prod_{i=1}^{k}(s+i(\beta+\gamma))} \quad \textit{and} \quad d_0^*(s)=1. $$

By the use of the identity given in Srivastava and Kashyap [18]

$$ \frac{1}{{}_1F_1(1; \frac{s}{\beta+\gamma}+1; \frac{-\theta\alpha}{\beta+\gamma})} = \sum_{k=0}^{\infty}(\theta\alpha)^k e_k^*(s), \tag{18} $$

where $e_0^*(s)=1$, and for k=1,2,3,...

$$ e_k^*(s) = \begin{vmatrix} d_1^*(s) & 1 & & . & . & . & & \\ d_2^*(s) & d_1^*(s) & 1 & . & . & . & & \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ d_{k-1}^*(s) & d_{k-2}^*(s) & d_{k-3}^*(s) & . & . & . & d_1^*(s) & 1 \\ d_k^*(s) & d_{k-1}^*(s) & d_{k-2}^*(s) & . & . & . & d_2^*(s) & d_1^*(s) \end{vmatrix} $$

$$ = \sum_{j=1}^{k}(-1)^{j-1} d_j^*(s) e_{k-j}^*(s). $$

By substituting Eq.(17) and Eq.(18) in Eq.(11), we get

$$ \zeta_n^*(s) = (\theta\alpha)^n \sum_{j=0}^{\infty}(-\theta\alpha)^j \binom{n+j}{j} d_{n+j}^*(s) \sum_{k=0}^{\infty}(\theta\alpha)^k e_k^*(s). $$

On inversion, we obtain

$$ \zeta_n(t) = (\theta\alpha)^n \sum_{j=0}^{\infty}(-\theta\alpha)^j \binom{n+j}{j} d_{n+j}(t) \sum_{k=0}^{\infty}(\theta\alpha)^k e_k(t), \tag{19} $$

where

$$d_k(t) = \frac{1}{(\beta+\gamma)^{k-1}} \sum_{i=1}^{k} \frac{(-1)^{i-1}}{(k-i)!\,(i-1)!} e^{-i(\beta+\gamma)t}, k = 1, 2, 3, ...,$$

and

$$e_k(t) = \sum_{i=1}^{k} (-1)^{i-1} d_i(t) * e_{k-i}(t), \quad k = 2, 3, 4, ...; \quad e_1(t) = d_1(t)$$

4. Time Dependent Moments

4.1. Mean

Let A(t) represents the average value of the random variable N(t) , therefore

$$A(t) = E(N(t)) = \sum_{n=1}^{\infty} nP_n(t) \tag{20}$$

Initially, at t=0, Eq(20) gives

$$A(0) = 0,$$

which implies

$$A'(t) = \sum_{n=1}^{\infty} nP_n'(t), \tag{21}$$

where$A'(t)$ denote the differentiation of $A(t)$.
Application of Eq.(2) in Eq.(21),after some calculation gives

$$A'(t) + (\beta+\gamma)A(t) - \theta\alpha = 0. \tag{22}$$

which is a linear differential equation in $A(t)$, whose solution gives

$$A(t) = \frac{\theta\alpha}{\beta+\gamma}[1 - e^{-(\beta+\gamma)t}] \tag{23}$$

4.2. Variance

Let Var(N(t)) represents the variance of the random variable N(t), therefore

$$Var(N(t)) = E[N(t) - E(N(t)]^2$$

which may be written as

$$Var(N(t)) = b(t) - [A(t)]^2, \tag{24}$$

where

$$b(t) = E(N^2(t)) = \sum_{n=1}^{\infty} n^2 P_n(t),$$

with

$$b(0) = 0,$$

also

$$b'(t) = \sum_{n=1}^{\infty} n^2 P_n'(t) \tag{25}$$

substitution of $P_n'(t)$ in Eq.(25), after some calculation results in the form of a linear differential equation in $b(t)$ i.e.

$$b'(t) = -(2\beta+\gamma)b(t) + (2\theta\alpha+\beta+\gamma)M(t) + \theta\alpha \tag{26}$$

which after integration gives

$$
\begin{aligned}
b(t) = & \frac{(2\theta\alpha + \beta + \gamma)\theta\alpha(\beta - (\gamma + 3\beta)e^{-(\gamma+2\beta)t} + (\gamma + 2\beta))e^{-(\gamma+\beta)t}}{\beta(\gamma + 2\beta)(\gamma + \beta)} \\
& + \frac{\theta\alpha}{(\gamma + 2\beta)}[1 - e^{-(\gamma+2\beta)t}]. \qquad (27)
\end{aligned}
$$

Substitution of Eq.(27) in Eq.(24), gives the expression of Var(N(t)).

5. Graphical Illustrations

In this section, we presents some graphical results to observe the time dependent behavior of various probabilities and average number of customers in the system.

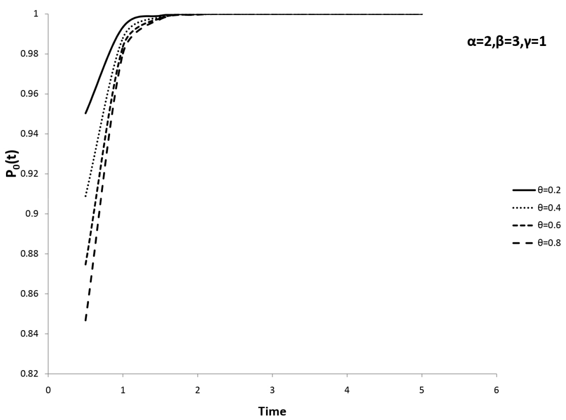

Figure 1: $P_0(t)$ *versus Time*

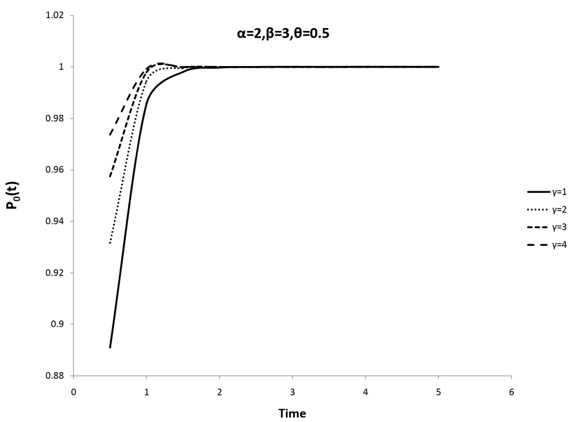

Figure 2: $P_0(t)$ *versus Time*

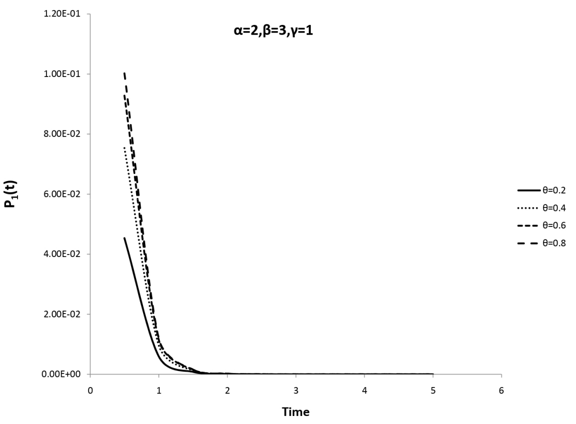

Figure 3: $P_1(t)$ *versus Time*

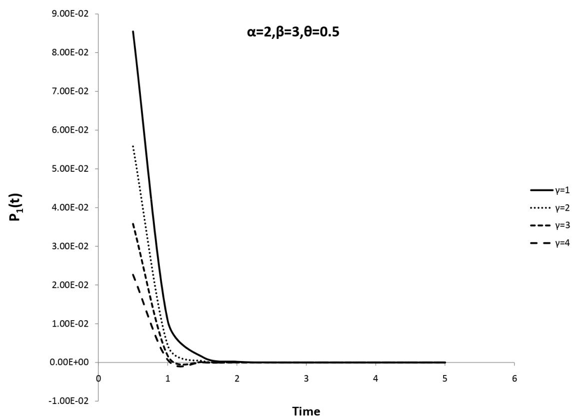

Figure 4: $P_1(t)$ *versus Time*

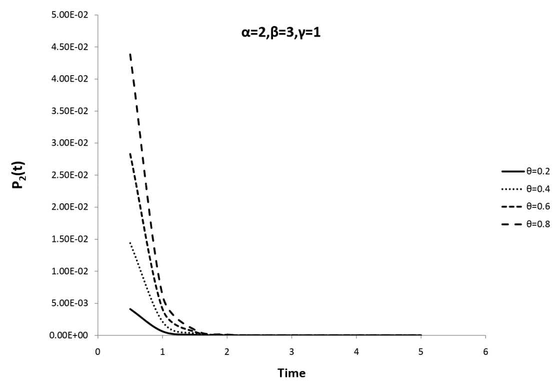

Figure 5: $P_2(t)$ *versus Time*

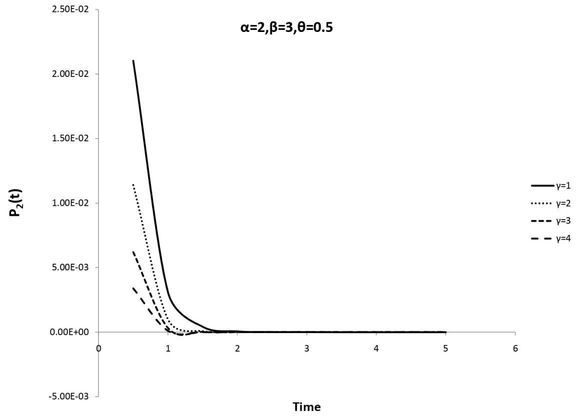

Figure 6: *$P_2(t)$ versus Time*

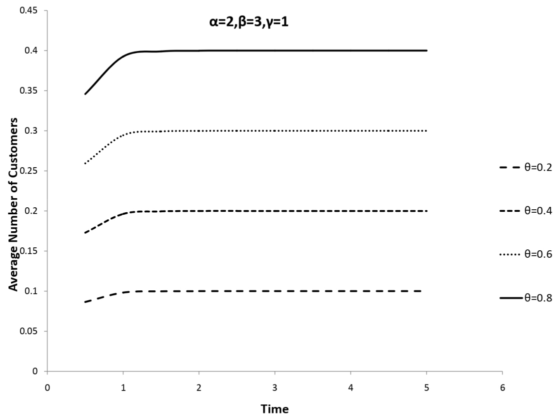

Figure 7: *Average Number of Customers versus Time*

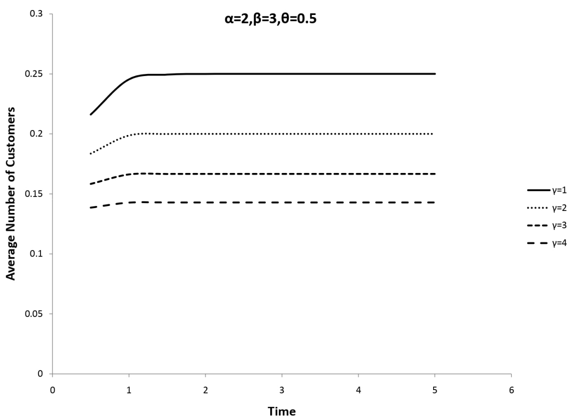

Figure 8: *Average Number of Customers versus Time*

Fig.(1 to 6) illustrates that as time increases, all the probability curves except $P_0(t)$ are decreases initially and then attain the steady state after $t = 2$. Also we notice that the probabilities $P_1(t)$ and $P_2(t)$ increases with increasing θ and the probability $P_0(t)$ decreases while we increase the value of θ. Further, we observe that if γ increases the probability of an empty system i.e. $P_0(t)$ increases while the other probabilities decrease. Fig.(7 and 8) explain the situation that the average number of customers in the system increases with time initially and then finally attains steady state. The average number of customers increase with the increasing values of θ and decrease with the increasing values of γ.

6. Conclusion

impatient behavior of customers is common in many real life queueing situations. Approximately, in all previous work available in the literature, it has been assumed that customers are impatient due to long waits in queue. But in the present study we analyze the case in which customers are impatient due to the quality of the service provided by the server. We have obtained the probability mass function of the number of customers in time dependent form. Also, we have determined the transient mean and variance of the number of customers. At the end, for observing the time dependent behavior of various probabilities, we provide some graphical illustrations.

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