INFERENCE ON THE TIME-DEPENDENT STRESS-STRENGTH RELIABILITY MODELS BASED ON FINITE MIXTURE MODELS

Krishnendu K. [1], Annie Sabitha Paul [2], Drisya M. [3] & Joby K. Jose [4]

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[1,4] Dept. of Statistical Sciences, Kannur University, Kerala 670567, India.
[2] Dept. of Applied Sciences, Govt. College of Engineering Kannur, Kerala 670563, India.
[3] Dept. of Statistics, Govt. Victoria College Palakkad, Kerala 678001, India.
[1] krishnendu61195@gmail.com , [2] anniesabithapaul@gmail.com,
[3] drisyam.m@gmail.com, [4] jobyk@kannuruniv.ac.in.

Abstract

Time-dependent stress-strength reliability engages with the chance of survival for systems with dynamic strength and/or dynamic stress. When a system is allowed to run continuously, each run will cause a change in the strength of the system. The repeated occurrence of stress on the system over each run will affect the survival capacity of the system. In this paper, we consider the distribution of time taken for the completion of a run by the system follows gamma and the stress or strength of the system follows a finite mixture of lifetime probability models. Here we consider two cases in which the first case deals with stress and strength following a finite mixture of Weibull distribution and in the second case the stress and strength is assumed to follow a finite mixture of the power-transformed half-logistic distribution. Moreover, the strength of the system is assumed to decrease by a constant and the stress acting on the system is assumed to increase by a constant over each run. We obtained the expression of the stress-strength reliability function and explained the ML and Bayesian methods for the estimation of the reliability at various time points.

Keywords: Time-dependent Stress-strength reliability, Gamma Renewal process, Finite mixture distribution, Expectation Maximization algorithm, Markov Chain Monte Carlo method.

1. Introduction

In reliability theory, stress-strength reliability measures the chance of the strength of a system to overcome the stress acting on it. Every object or individual has its own strength for survival. When they are subject to any kind of stress, they will survive only if their strength surpasses the stress. Stress-strength reliability model can be used to compare the effectiveness of two treatments, to compare the life length of two equipment, etc. Let Y denotes the random strength of the system under consideration and X is the stress acting on that system. Then the stress-strength reliability of the system is denoted by R and is defined as $R = P[X < Y]$.

The concept of stress-strength reliability theory was originated by Birnbaum [2]. Kotz et.al. [11] discussed point and interval estimation of stress-strength models using different approaches. Baklizi and Eidous [1] proposed an estimator of R based on kernel estimators of the densities of X and Y. Zhou [20] illustrated the estimation of R using the bootstrap method. Recently many authors discussed classical and Bayesian methods of estimating R for different probability models, see Pakdaman et al. [12] Xavier and Jose [15,16], Xavier *et al.* [17, 18] and Jose et.al. [7,10].

Nowadays, research on stress-strength reliability estimation focuses on the case where the stress, strength or both changes with respect to time, and hence the term time-dependent stress-strength reliability. Let $Y(t)$ represent the strength of a system at time t and $X(t)$ be the stress on the system at t. Under the time-dependent stress-strength reliability model, we are interested in the estimation of the stress-strength reliability function

$$R(t) = P[X(t) < Y(t)], \tag{1}$$

which gives the chance of survival of the system at time t. For example, quite often we have to download files to mobile phones. The downloaded files consume the memory space of the phone corresponding to the size of that file. It will cause a reduction in the speed of functioning of the phone. So each time we download a new file, the number of files piled up in the phone memory which will reduce the functioning speed of the phone.Time dependent stress-strength reliability models were studied in Yadav [19], Gopalan and Venkateswarlu [5, 6], Eryilmaz [4] and Siju and Kumar [13, 14], Jose and Drisya [8, 9] and Drisya et al. [3].

Time-dependent stress-strength reliability engages with the chance of survival for systems with dynamic strength and/or dynamic stress. When a system is allowed to run continuously, each run will cause a change in the strength of the system. The repeated occurrence of stress on the system over each run will affect the survival capacity of the system. In this paper, we consider the distribution of time taken for the completion of a run by the system follows gamma and the stress or strength of the system follows a finite mixture of lifetime probability models. Here we consider two cases in which the first case deals with stress and strength following a finite mixture of Weibull distribution and in the second case the stress and strength are assumed to follow a finite mixture of the power-transformed half-logistic distribution. Moreover, the strength of the system is assumed to decrease by a constant and the stress acting on the system is assumed to increase by a constant over each run.

This paper is organized as follows. Estimation of stress-strength reliability function with gamma cycle times under random fixed stress and strength is discussed in Section 2. The expressions for stress-strength reliability function under a finite mixture of Weibull and a finite mixture of power-transformed half-logistic distributions are also derived. A brief description of the EM algorithm for estimating R(t) is given in Section 3 with numerical illustrations based on simulated data. Computation of the Bayes estimate of R(t)using the Markov Chain Monte Carlo method is illustrated in Section 4 with a numerical illustration based on simulated data.

2. Estimation of $R(t)$ based on finite mixture distribution

Consider a system that is allowed to work continuously. The system executes several runs during the time period of observation say $(0, t)$. The time taken for completion of a run by the system is a random variable and we call it cycle time. In this paper, we assume that the cycle times are gamma-distributed. Hence the total number of runs within the entire time period will have a renewal process. Let the cycle time Z follows gamma distribution with p.d.f.,

$$f(z) = \frac{a^k z^{k-1} e^{-az}}{(k-1)!}; z \geq 0. \tag{2}$$

Then the number of runs during the time interval (0,t), say N(t)has the following distribution.

$$\begin{aligned} P_n(t) &= p[N(t) = n] \\ &= e^{-at} \sum_{r=nk}^{(n+1)k-1} \frac{(at)^r}{r!}; n = 0, 1, 2, ... \end{aligned} \tag{3}$$

Let X_j be the stress imposed on the system during j^{th} cycle time and the corresponding strength of the system be Y_j. Also let the initial strength of the system, say Y_0 be a continuous random variable with density function $h(y_0)$ and the initial stress on the system X_0 also be a continuous random variable with p.d.f $g(x_0)$. The system is allowed to run continuously and when it runs, its strength decreases by a_0 and the stress increases by b_0 on completion of each run. Hence, the probability that the system works after n runs is given by

$$\begin{aligned} R_n &= P((X_1 < Y_1) \cap (X_2 < Y_2) \cap \cdots \cap (X_n < Y_n)) \\ &= P((x_0 + b_0 < y_0 - a_0) \cap (x_0 + 2b_0 < y_0 - 2a_0) \cap \cdots \cap (x_0 + nb_0 < y_0 - na_0)) \\ &= P(x_0 + n(a_0 + b_0) < y_0) \\ &= \int_0^\infty \int_{x_0+n(a_0+b_0)}^\infty h(y_0) g(x_0) dy_0 dx_0 \end{aligned} \tag{4}$$

Therefore the reliability of the system at time t is

$$\begin{aligned} R(t) &= \sum_{n=0}^\infty P_n(t) R_n \\ &= \sum_{n=0}^\infty P_n(t) \int_0^\infty \int_{x_0+n(a_0+b_0)}^\infty h(y_0) g(x_0) dy_0 dx_0 \quad (5) \\ &= \sum_{n=0}^\infty e^{-at} \sum_{r=nk}^{(n+1)k-1} \frac{(at)^r}{r!} \int_0^\infty \int_{x_0+n(a_0+b_0)}^\infty h(y_0) g(x_0) dy_0 dx_0 \quad (6) \end{aligned}$$

In particular, consider the case that stress acting on the system do not vary throughout the observation period as well as the strength of the system decreases by a constant say, a_0. Then the probability of functioning of the system after n runs is given by

$$\begin{aligned} R_n &= P[(X_1 < Y_1) \cap (X_2 < Y_2) \cap \cdots \cap (X_n < Y_n)] \\ &= P[(x_0 < Y_0 - a_0) \cap (x_0 < Y_0 - 2a_0) \cap \cdots \cap (x_0 < Y_0 - na_0)] \\ &= P[(x_0 + na_0 < Y_0)] \\ &= \int_{x_0+na_0}^\infty h(y_0) dy_0 \end{aligned} \tag{7}$$

Therefore, the value of $R(t)$ can be obtained as

$$\begin{aligned} R(t) &= \sum_{n=0}^\infty P_n(t) R_n \\ &= \sum_{n=0}^\infty e^{-at} \sum_{r=nk}^{(n+1)k-1} \frac{(at)^r}{r!} \int_{x_0+na_0}^\infty h(y_0) dy_0. \end{aligned}$$

2.1. R(t) based on finite mixture Weibull distribution

Let the initial strength of the system follow a mixture of Weibull distributions with p.d.f.

$$h(y_0) = \sum_{i=1}^{m_1} \pi_i \frac{\alpha}{\beta_i} y_0^{\alpha-1} e^{-y_0^\alpha/\beta_i}, y_0 \geq 0, \alpha > 0, 0 < \pi_i < 1, \beta_i > 0; i = 1, 2, ..., m_1. \tag{8}$$

and initial stress on the system follows a mixture of Weibull distribution with p.d.f.

$$g(x_0) = \sum_{j=1}^{m_2} p_j \frac{\alpha}{\theta_j} x_0^{\alpha-1} e^{-x_0^\alpha/\theta_j}, x_0 \geq 0, \alpha > 0, 0 < p_j < 1, \theta_j > 0; j = 1, 2, ..., m_2. \tag{9}$$

When the system runs, its strength decreases by a_0 and the stress increases by b_0 on completion of each run. The time taken for completion of a run is assumed to be a gamma variate. Then the chance for survival of the system after n runs is

$$R_n = \sum_{i=1}^{m_1} \pi_i \sum_{j=1}^{m_2} p_j e^{-(n(a_0+b_0))^\alpha/\beta_i}; n = 1, 2, ... \tag{10}$$

with

$$R_0 = \sum_{i=1}^{m_1} \pi_i \sum_{j=1}^{m_2} p_j \frac{\beta_i}{\beta_i + \theta_j} \tag{11}$$

Then the corresponding stress-strength reliability function is obtained as

$$\begin{aligned} R(t) &= e^{-at} \sum_{r=0}^{k-1} \frac{(at)^r}{r!} \sum_{i=1}^{m_1} \pi_i \sum_{j=1}^{m_2} p_j \frac{\beta_i}{\beta_i + \theta_j} \\ &+ \sum_{n=1}^{\infty} e^{-at} \sum_{r=nk}^{(n+1)k-1} \frac{(at)^r}{r!} \sum_{i=1}^{m_1} \pi_i \sum_{j=1}^{m_2} p_j e^{-(n(a_0+b_0))^{\alpha}/\beta_i} \end{aligned} \tag{12}$$

Change in $R(t)$ corresponding to change in different parameters stress and strength distributions are given in Figure 1. From the figure, it is clear that the value of $R(t)$ increases with an

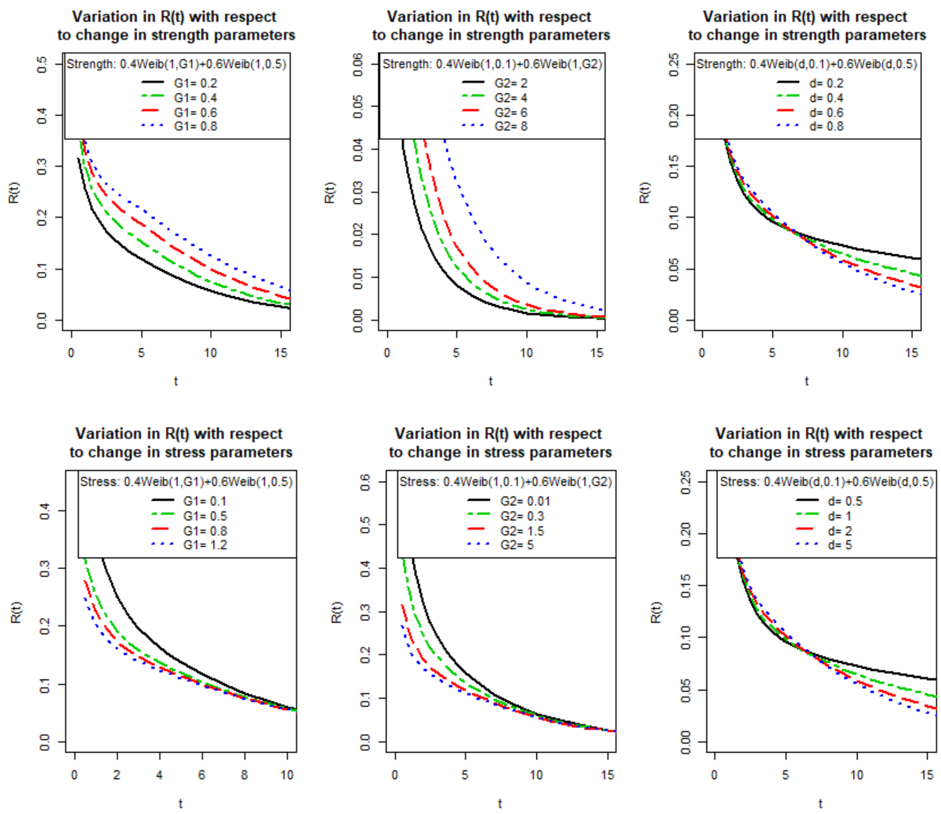

Figure 1: *Variation in $R(t)$ corresponding to change in parameters*

increase in shape parameter values and decreases with an increase in scale parameter values of strength when the initial strength of the the system is Weibull-distributed. Also $R(t)$ increases with an increase in shape parameter values of stress distribution.

As a particular case assume that the strength of the system has a mixture Weibull distribution with parameters $(\alpha, \beta_i); i = 1, 2, ...m_1$, and the stress is fixed. Then the chance of the system working after the completion of n runs is,

$$R_n = e^{-(x_0+na_0)^{\alpha}/\beta} \tag{13}$$

and the corresponding stress strength reliability is obtained as

$$R(t) = \sum_{n=0}^{\infty} e^{-at} \sum_{r=nk}^{(n+1)k-1} \frac{(at)^r}{r!} \sum_{i=1}^{m_1} \pi_i e^{-(x_0+na_0)^{\alpha}/\beta_i} \quad (14)$$

$$= \sum_{i=1}^{m_1} \pi_i \sum_{n=0}^{\infty} e^{-at} \sum_{r=nk}^{(n+1)k-1} \frac{(at)^r}{r!} e^{-(x_0+na_0)^{\alpha}/\beta_i} \quad (15)$$

2.2. R(t) based on finite a mixture of power transformed half logistic distribution

The p.d.f of the power transformed half-logistic distribution (Xavier and Jose (2020)) is given by

$$f(y) = \begin{cases} 2\delta\gamma y^{\gamma-1} e^{-\delta y^{\gamma}} \left(1+e^{-\delta y^{\gamma}}\right)^{-2}, & 0 \le y < \infty; \delta > 0; \gamma > 0. \\ 0 & \text{otherwise.} \end{cases} \quad (16)$$

Now, let us assume that initial strength (Y_0) of the system follows a mixture of power transformed half logistic distribution with p.d.f

$$h(y_0) = \sum_{i=1}^{m_1} \pi_i 2\delta_{1i}\gamma_{1i} y_0^{\gamma_{1i}-1} e^{-\delta_1 y_0^{\gamma_{1i}}} (1+e^{-\delta_{1i} y_0^{\gamma_{1i}}})^{-2}, \quad (17)$$

$0 \le y_0 < \infty, \delta_{1i} > 0, 0 < \pi_i < 1, \gamma_{1i} > 0, ; i = 1, 2, ..., m_1$. It is also assumed that initial stress on the system (X_0) follows the mixture of power transformed half logistic distribution with p.d.f

$$g(x_0) = \sum_{j=1}^{m2} p_j 2\delta_{2j}\gamma_{2j} x_0^{\gamma_{2j}-1} e^{-\delta_{2j} x_0^{\gamma_{2j}}} (1+e^{-\delta_{2j} x_0^{\gamma_{2j}}})^{-2}, \quad (18)$$

$0 \le x_0 < \infty, \delta_{2j} > 0, 0 < p_j < 1, \gamma_{2j} > 0, ; j = 1, 2, ..., m_2$.

Hence, R_n is given by

$$R_n = 4\sum_{i=1}^{m_1} \pi_i \sum_{j=1}^{m2} p_j \delta_{2j}\gamma_{2j} \times \int_0^{\infty} [1-(1+e^{-\delta_{1i}(x_0+n(a_0+b_0))^{\gamma_{1i}}})^{-1}] x_0^{\gamma_{2j}-1} e^{-\delta_{2j} x_0^{\gamma_{2j}}} (1+e^{-\delta_{2j} x_0^{\gamma_{2j}}})^{-2} dx_0. \quad (19)$$

Then, the stress-strength reliability is given by

$$R(t) = 4\sum_{n=0}^{\infty} e^{-at} \sum_{r=nk}^{(n+1)k-1} \frac{(at)^r}{r!} \sum_{i=1}^{m_1} \pi_i \sum_{j=1}^{m2} p_j \delta_{2j}\gamma_{2j} \times \int_0^{\infty} [1-(1+e^{-\delta_{1i}(x_0+n(a_0+b_0))^{\gamma_{1i}}})^{-1}] x_0^{\gamma_{2j}-1} e^{-\delta_{2j} x_0^{\gamma_{2j}}} (1+e^{-\delta_{2j} x_0^{\gamma_{2j}}})^{-2} dx_0 \quad (20)$$

Change in R(t) corresponding to change in different parameters stress and strength distributions are given in Figure 2. From the graph, when the stress and strength parameters follow a mixture of power transformed half logistic distribution, the increase in the parameters results in a decrease in the $R(t)$ and after a point, they converge. Particularly when stress is fixed and strength of the system has a mixture of power transformed half logistic distribution with parameters $(\delta_i, \gamma_i) : i = 1, 2, ..., m_1$, the chance of the system working after the completion of n runs is,

$$R_n = \sum_{i=1}^{m_1} 2\pi_i [1-(1+e^{-\delta_i (x_0+na_0)^{\gamma_i}})^{-1}]. \quad (21)$$

Then, corresponding $R(t)$ is given by

$$R(t) = \sum_{n=0}^{\infty} e^{-at} \sum_{r=nk}^{(n+1)k-1} \frac{(at)^r}{r!} \sum_{i=1}^{m_1} 2\pi_i [1-(1+e^{-\delta_i (x_0+na_0)\gamma_i})^{-1}].$$

$$= \sum_{i=1}^{m_1} \pi_i \sum_{n=0}^{\infty} e^{-at} \sum_{r=nk}^{(n+1)k-1} \frac{(at)^r}{r!} 2[1-(1+e^{-\delta_i (x_0+na_0)^{\gamma_i}})^{-1}]. \quad (22)$$

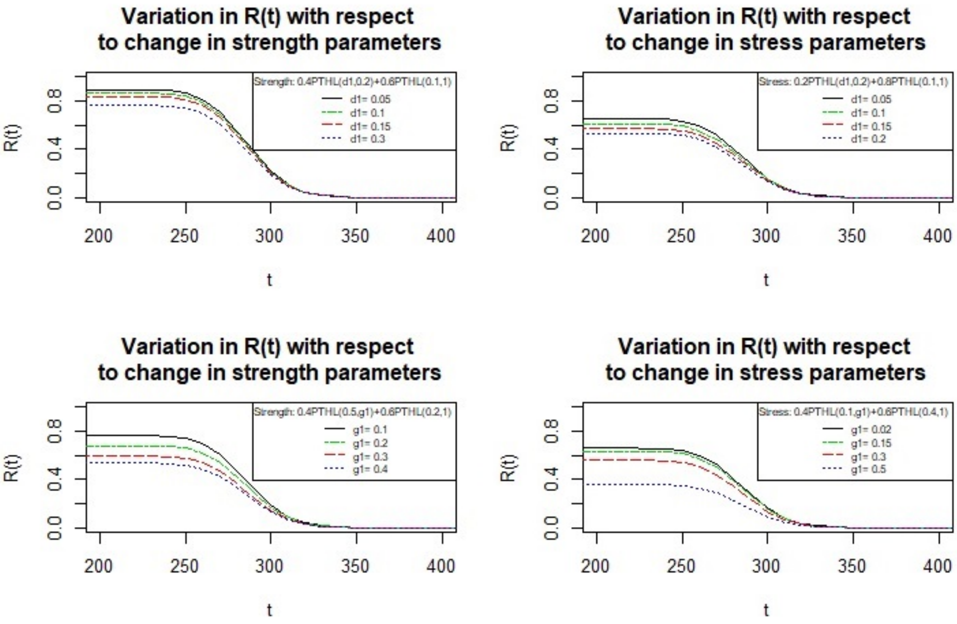

Figure 2: *Variation in $R(t)$ corresponding to change in stress and strength parameters*

3. ML Estimation of R(t) using EM Algorithm

In this section, we describe the ML estimation of the reliability function. Assume that the strength and the stress follow a finite mixture distribution with densities h(y) and g(x) respectively. where

$$h(y) = \sum_{i=1}^{m_1} \pi_i h_i(y), \quad 0 < \pi_i < 1, \ \sum_{i=1}^{m_1} \pi_i = 1. \tag{23}$$

and

$$g(x) = \sum_{j=1}^{m_2} p_j g_j(x), \quad 0 < p_j < 1, \ \sum_{i=1}^{m_2} p_j = 1. \tag{24}$$

The cycle time follows a gamma distribution with p.d.f.

$$f(z) = \frac{a^k z^{k-1} e^{-az}}{(k-1)!}, \quad z \geq 0. \tag{25}$$

Let $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_m)$ and $(z_1, z_2, ..., z_r)$ be random samples on stress, strength and cycle time respectively. Then the joint likelihood function is

$$L = \prod_{i=1}^{n} g(x_i) \prod_{j=1}^{m} h(y_j) \prod_{t=1}^{r} f(z_t) \tag{26}$$

and the corresponding log-likelihood function

$$\begin{aligned} l &= \sum_{i=1}^{n} log\ g(x_i) + \sum_{j=1}^{m} log\ h(y_j) + \sum_{t=1}^{r} log\ f(z_t) \\ &= l_1 + l_2 + l_3 \end{aligned} \tag{27}$$

As the log-likelihood function is the sum of log-likelihoods corresponding to the random samples of stress, strength as well as cycle time respectively and since the parameters are independent the stress, strength, and cycle time parameters can be obtained by maximizing corresponding log-likelihood function. The ML estimates of stress and strength parameters can be computed by by using Expectation - Maximization algorithm.

3.1. ML Estimation of R(t) based on a finite mixture of Weibull distribution

Assuming that the cycle time distribution follows gamma distribution and the initial stress follows a finite mixture of Weibull distribution with parameters $(\alpha, \beta_i), i = 1, 2, ..., m_1$ and strength follows finite mixture of Weibull distribution with parameters $(\alpha, \theta_j), j = 1, 2, ..., m_2$ expression for the stress strength reliability function is derived in the previous section. We can estimate the stress, strength and cycle time parameters separately. The ML estimates of stress and strength parameters can be computed by by using EM algorithm. Here we summarize the EM algorithm for computing the parameters of a finite mixture of Weibull distribution. Consider the strength data consists of n independent and identically distributed observations $(y_1, y_2, ..., y_n)$ from a finite a mixture of Weibull distribution with p.d.f.

$$h(y, \alpha, \beta) = \sum_{i=1}^{m_1} \pi_i h_i(y, \alpha, \beta_i), \quad \beta = (\beta_i; i = 1, 2, ..., m_1)$$

Where

$$h_i(y, \alpha, \beta_i) = \frac{\alpha}{\beta_i} y^{\alpha-1} e^{-\frac{y^\alpha}{\beta_i}}; y > 0, \alpha > 0, \beta_i > 0; i = 1, 2, ..., m_1$$

The associated log-likelihood function is

$$L(y, \alpha, \beta)) = \sum_{j=1}^{n} log\ h(y, \alpha, \beta). \tag{28}$$

The MLE of $\widehat{\alpha}$, $\widehat{\beta}$ is determined such that

$$L(y, \widehat{\alpha}, \widehat{\beta}) = sup_{\alpha,\beta} L(y, \alpha, \beta). \tag{29}$$

Define a variable z_{ij} such that $z_{ij} = 1$ if j^{th} unit of the sample comes from the i^{th} component and $z_{ij} = 0$ otherwise. Since each component comes from exactly one component, we have $\sum_{i=1}^{k} z_{ij} = 1, \quad \pi_i = P[z_{ij} = 1]$.

$$Y_i | z_{ij=1} \sim \text{Weibull}(\alpha, \beta_i), i = 1, 2, ..., m_1.$$

In missing data setup y can be considered as incomplete data and $x = (x_1, x_2, ..., x_n)$ where $x_j = (y_j, z_j)$ and $z_j = (z_{ij}, i = 1, 2, ..., m_1)$ as a complete data set. The density function corresponding to the observations in the complete data set is

$$h_c(x_j, \alpha, \beta) = h_c(y_j, z_j, \alpha, \beta) = \sum_{i=1}^{m_1} \pi_i I_{z_{ij}} h_i(y_j, \alpha, \beta_i). \tag{30}$$

and the likelihood function is

$$L_c(x, \alpha, \beta_i) = \sum_{j=1}^{n} log\ h_c(x_j, \alpha, \beta). \tag{31}$$

The EM algorithm iteratively maximizes $Q(\alpha, \beta | \alpha, \beta^{(t)}) = E(L_c(x, \alpha, \beta | y, \alpha, \beta^{(t)}))$ instead of maximizing $L(y, \alpha, \beta)$, where $\alpha, \beta^{(t)}$ is the current value at t and then compute the expectation

$$E_{\alpha,\beta^{(t)}}(L_c(x, \alpha, \beta)|y) = \sum_{j=1}^{n} \sum_{i=1}^{m_1} E_{\alpha,\beta_i^{(t)}}(z_{ij}|y)(log\ \pi_i + log\ h_i(y_j, \alpha, \beta_i)) \tag{32}$$

$$\begin{aligned} E_{\alpha,\beta_i^{(t)}}(z_{ij}|y) &= P_{\alpha,\beta_i^{(t)}}(z_{ij} = 1|y) \\ &= \frac{\pi_i^{(t)} h_i(y_j, \alpha, \beta_i)}{\sum_{i=1}^{m_1} \pi_i^{(t)} h_i(y_j, \alpha, \beta_i)}, j = 1, 2, ..., n; i = 1, 2, ..., m_1 \qquad (33) \\ &= \tau_{ij}(y_j, \alpha, \beta_i) \qquad (34) \end{aligned}$$

It is the posterior probability that j^{th} observation belongs to the i^{th} component in the t^{th} iteration. Thus we have

$$Q(\alpha, \beta_i|\alpha, \beta^{(t)}) = \sum_{j=1}^{n}\sum_{i=1}^{m_1} \tau_{ij}(\alpha, \beta_i^{(t)})(log\ \pi_i + log\ h_i(y_i, \alpha, \beta_i)). \tag{35}$$

Hence the EM algorithm consists of the following two steps.

Step1.E-step: *Compute* $Q(\alpha, \beta|\alpha, \beta^{(t)})$
Step2.M-step:*Compute the value of* $\alpha, \beta^{(t+1)}$ *that maximizes* $Q(\alpha, \beta|\alpha, \beta^{(t)})$.

If τ_{ij} where observable posterior probabilities, then MLE of π is simply given by

$$\widehat{\pi}_i = \sum_{j=1}^{n} \frac{\tau_{ij}}{n}, \quad i = 1, 2, ..., m_1,$$

which is the proportion of the sample having arisen from the i^{th} component of the mixture.

For the $(t+1)^{th}$ update other parameters α and $(\beta_1, \beta_2, ..., \beta_{m_1})$, we have to obtain the solution of

$$\sum_{j=1}^{n}\sum_{i=1}^{m_1} \tau_{ij}(\pi^{(t)}) \frac{\partial}{\partial \alpha, \beta_i} log\ h_i(y_j, \alpha, \beta_i)) = 0 \tag{36}$$

We repeat the procedure until the desired accuracy is obtained. Hence we get the estimates of the strength parameters as:

$$\hat{\beta}_i(t+1) = \left[\frac{\sum_{j=1}^{n} \tau_{ij} y_j^{\alpha(t+1)}}{\sum_{j=1}^{n} \tau_{ij}} \right] \tag{37}$$

$$\hat{\alpha}(t+1) = n \left[\sum_{i=1}^{m_1} \frac{1}{\beta_i(t)} \sum_{j=1}^{n} \tau_{ij} y_j^{\hat{\alpha}(t)} log(y_j) - \sum_{i=1}^{m_1}\sum_{j=1}^{n} \tau_{ij} log(y_j) \right]^{-1} \tag{38}$$

Similarly, we can estimate the stress parameters. The ML estimates of gamma cycle time parameters can be obtained by standard procedures. Using the ML estimates of the stress, strength, and cycle time parameters and applying the invariance property of the ML estimators we can find the value of $R(t)$.

We use the Monte Carlo simulation technique to estimate R(t) for systems with initial strength and initial stress following Weibull mixture and cycle times following gamma distribution. We have done the entire numerical analysis using R. The numerical illustration of ML of R(t) with gamma cycle time with Weibull mixture initial stress and strength for different time values is given in Table 1. In which y_0 represent initial strength and x_0 represent initial stress of the system. For a fixed time interval, we draw samples for cycle time and the number of cycles based on the distributional assumption of cycle times. The maximum number of cycles up to which the total cycle time does not exceed the length of the time interval under consideration is taken as the number of runs during the time interval. The cycle time observed during each run constitutes the simulated sample of cycle times. The command rweibull helps in simulating samples from the Weibull distribution. Samples to represent initial stress and initial strength distributions, when both are mixtures of Weibull distributions are generated using this command. We repeat the entire simulation experiment 1,000 times.

From the table, it is clear that $R(t)$ decreases as the time increases, when the initial stress and strength of the system is distributed as a mixture of Weibull distribution with gamma cycle time.

Table 1: *ML Estimation of R(t) with Weibull mixture initial stress and strength*

Cycle time		Stress and Strength		a_0	x_0	t	$R(t)$
Parameters	Estimated	Parameters	Estimated				
G(0.5,2)	$a = 0.4881$	Y_0 :0.8W(0.3,0.6)+	$\alpha = 0.2890$	1	0.02	10	0.2230
	$k = 2.0840$	0.2W(0.3,2)	$\theta = (0.5892, 1.9454)$			25	0.1539
		X_0 :0.3W(0.3,1)+	$\alpha = 0.2954$			50	0.1175
		0.7W(0.3,0.3)	$\beta = (1.1603, 0.2957)$			75	0.0989
						100	0.0869
G(0.5,4)	$a = 0.5231$	Y_0 :0.6W(5,0.3)+	$\alpha = 5.0982$	0.001	0.08	10	0.9155
	$k = 3.1780$	0.4W(5,2)	$\theta = (0.2992, 2.1114)$			25	0.9154
		X_0 :0.3W(5,0.1)+	$\alpha = 5.1089$			50	0.6168
		0.7W(5,0.2)	$\beta = (0.1008, 0.2292)$			75	0.2228
						100	0.0498
G(1,2)	$a = 0.5231$	Y_0 :0.4W(2,1.2)+	$\alpha = 2.0008$	0.02	0.05	10	0.8699
	$k = 3.1780$	0.6W(2,4)	$\theta = (1.3100, 4.1216)$			25	0.8938
		X_0 :0.7W(2,4)+	$\alpha = 1.9279$			50	0.7896
		0.3W(2,2.5)	$\beta = (4.3704, 2.6518)$			75	0.6559
						100	0.5188
G(1,4)	$a = 0.9910$	Y_0 :0.2W(1.2,0.8)+	$\alpha = 1.2056$	0.1	0.05	10	0.5913
	$k = 3.9622$	0.8W(1.2,2.4)	$\theta = (0.7759, 2.3253)$			25	0.4231
		X_0 :0.3W(1.2,4)+	$\alpha = 1.1128$			50	0.2357
		0.7W(1.2,3.2)	$\beta = (3.9000, 3.9901)$			75	0.1292
						100	0.0693

3.2. ML Estimation of R(t) based on a finite mixture of power-transformed half-logistic distribution

By assuming that the cycle time follows gamma distribution and the initial stress and strength follow the mixture of power transformed half logistic distribution with parameters $(\delta_{1i}, \gamma_{1i}), i = 1, 2, ..., m_1$ and $(\delta_{2j}, \gamma_{2j}), j = 1, 2, ..., m_2$ respectively, the corresponding stress-strength reliability is given in the previous section. Now, consider independent and identically distributed strength observations $y = (y_1, y_2, ..., y_n)$ from a finite a mixture of power transformed half logistic mixture with p.d.f.

$$h(y, \delta_1, \gamma_1) = \sum_{i=1}^{m_1} \pi_i h_i(y, \delta_{1i}, \gamma_{1i}).$$

Where

$$h_i(y) = \begin{cases} 2\delta_{1i}\gamma_{1i} y^{\gamma_{1i}-1} e^{-\delta_{1i} y^{\gamma_{1i}}} \left(1 + e^{-\delta_{1i} y^{\gamma_{1i}}}\right)^{-2}, & 0 \le y < \infty; \delta_{1i} > 0; \gamma_{1i} > 0 \\ 0, & \text{otherwise} \end{cases} \tag{39}$$

$i = 1, 2, ..., m_1$. Using the EM algorithm explained earlier, we get the ML estimates of the strength parameters as

$$\pi_i = \frac{\sum_{j=1}^{n} \tau_{ij}}{n}, i = 1, 2, ..., m_1. \tag{40}$$

$$\hat{\delta_{1i}} = \frac{\sum_{j=1}^{n} \tau_i(y_j; \delta_{1i}, \gamma_{1i})}{\sum_{j=1}^{n} (\tau_i(y_j; \delta_{1i}, \gamma_{1i}) y_j^{\gamma_{1i}} [1 - \frac{2e^{-\delta_{1i} y_j^{\gamma_{1i}}}}{1+e^{-\delta_{1i} y_j^{\gamma_{1i}}}}])}; i = 1, 2, ..., m_1. \tag{41}$$

$$\hat{\gamma_{1i}} = \frac{\sum_{j=1}^{n} \tau_i(y_j; \delta_{1i}, \gamma_{1i})}{\sum_{j=1}^{n} (\tau_i(y_j; \delta_{1i}, \gamma_{1i}) log(y_j) [\delta_{1i} y_j^{\gamma_{1i}} (1 - \frac{2e^{-\delta_{1i} y_j^{\gamma_{1i}}}}{1+e^{-\delta_{1i} y_j^{\gamma_{1i}}}}) - 1])}; i = 1, 2, ..., m_1. \tag{42}$$

Similarly, we can find the stress estimates and hence we can find $R(t)$ by the estimated parameters.

We use the Monte Carlo simulation technique to estimate R(t) for systems with initial strength and initial stress following a finite mixture of power-transformed half logistic distribution and cycle times following Gamma distribution. Table 2 gives the estimated value of R(t) with gamma cycle time with power transformed half logistic mixture initial stress and strength for different time values. The package bayesmeta available in R software allows sampling from half-logistic distribution. Then sample from power transformed half logistic distribution is simulated using simple conversion techniques.

Table 2: *ML Estimation of R(t) with PTHL mixture initial stress and strength*

Cycle time		Stress and Strength		a_0	x_0	t	$R(t)$
Parameters	Estimated	Parameters	Estimated				
G(0.5,1)	$a = 0.4885$	Y_0 :0.6PTHL(5,0.3)	$\delta_2 = (5.0927, 4.9764)$	0.001	0.008	10	0.00756
	$k = 0.9687$	+0.4PTHL(0.4,2)	$\gamma_2 = (0.1067, 0.2025)$			50	0.00755
		X_0 : 0.3PTHL(5,0.1)	$\delta_1 = (9.2026, 2.0041)$			100	0.00755
		+0.7PTHL(5,0.2)	$\gamma_1 = (1.1615, 6.0581)$			150	0.00753
						200	0.00367
G(0.5,2)	$a = 0.9522$	Y_0 :0.7PTHL(8,0.5)	$\delta_2 = (7.3349, 1.1391)$	0.001	0.08	10	0.0629
	$k = 1.9138$	+0.3PTHL(0.5,2.5)	$\gamma_2 = (0.5245, 1.8925)$			50	0.0629
		X_0 :0.2PTHL(4,0.5)	$\delta_1 = (3.8315, 5.0771)$			125	0.0626
		+0.8PTHL(5,0.2)	$\gamma_1 = (0.1008, 0.2292)$			140	0.0525
						150	0.0313
G(1,1)	$a = 1.0097$	Y_0 :0.2PTHL(2,0.2)	$\delta_2 = (1.9903, 3.9043)$	0.002	0.005	10	0.1067
	$k = 1.0029$	+0.8PTHL(4,2.4)	$\gamma_2 = (0.1961, 2.4533)$			50	0.1067
		X_0 :0.6PTHL(4,2)	$\delta_1 = (3.9133, 5.1319)$			75	0.1063
		+0.4PTHL(4.6,2)	$\gamma_1 = (2.1198, 1.9368)$			100	0.0490
						125	0.0008
G(1,2)	$a = 1.0099$	Y_0 :0.1PTHL(3,2.4)	$\delta_2 = (2.6875, 3.0715)$	0.002	0.005	10	0.1038
	$k = 1.9968$	+0.9PTHL(3,1.2)	$\gamma_2 = (2.2066, 1.1926)$			50	0.1038
		X_0 :0.8PTHL(2.5,1.1)	$\delta_1 = (2.4427, 5.3425)$			75	0.1034
		+0.2PTHL(5,2)	$\gamma_1 = (1.1208, 2.0481)$			100	0.0477
						125	0.0008

From this table, we can see that, $R(t)$ decreases as time increases, when the initial stress and strength of the system is distributed as a mixture of power transformed half logistic distribution with gamma cycle time.

4. Bayesian Estimation of R(t) using MCMC method

In this section, we describe the Bayesian estimation of the reliability function. The stress and strength follow a finite mixture distribution with densities g(x) and h(y) respectively and the cycle time follows a gamma distribution. Let $(x_1, x_2, ..., x_n)$, $(y_1, y_2, ..., y_m)$ and $(z_1, z_2, ..., z_r)$ be random samples on stress, strength, and cycle time respectively. Then, the joint likelihood function is

$$L = \prod_{i=1}^{n} g(x_i) \prod_{j=1}^{m} h(y_j) \prod_{t=1}^{r} f(z_t) \tag{43}$$

where

$$g(x) = \sum_{j=1}^{m_2} p_j g_j(x), \quad 0 < p_j < 1, \ \sum_{j=1}^{m_2} p_j = 1. \tag{44}$$

and

$$h(y) = \sum_{i=1}^{m_1} \pi_i h_i(y), \quad 0 < \pi_i < 1, \ \sum_{i=1}^{m_1} \pi_i = 1. \tag{45}$$

The cycle time follows a gamma distribution with p.d.f.

$$f(z) = \frac{a^k z^{k-1} e^{-az}}{(k-1)!}, \quad z \geq 0. \tag{46}$$

We assume prior probabilities corresponding to each parameter to get a Bayesian estimate of the reliability function.

4.1. Bayesian Estimation of R(t) based on a finite mixture of Weibull distribution

Let the cycle time follow a gamma distribution with parameters (a, k) and stress and strength of the system follow a mixture of Weibull distribution with parameters $(\alpha, \beta_j), j = 1, 2, .., m_2$ and $(\delta, \gamma_i), i = 1, 2, .., m_1$ respectively. The expression for stress-strength reliability is given in section 2. Here we discuss the estimation of the parameters by the Bayesian estimation method. Treating Z_i as the auxiliary variable, such that

$$X_j|Z_j = i \sim g_i(x, \alpha, \beta_i) \; and \; p(Z_j = i) = p_i, j = 1, 2, .., n, i = 1, 2, .., m_2.$$

$$Y_j|Z_j = i \sim h_i(y, \delta, \gamma_i) \; and \; p(Z_j = i) = \pi_i, j = 1, 2, .., m, i = 1, 2, .., m_1.$$

Where

$$g_i(x) = \frac{\alpha}{\beta_i} x^{\alpha-1} e^{-x^{\alpha}/\beta_i}, x \geq 0, \alpha > 0, \beta_i > 0; i = 1, 2, ..., m_1. \tag{47}$$

and

$$h_i(y) = \frac{\alpha}{\theta_j} y^{\alpha-1} e^{y^{\alpha}/\theta_j}, y \geq 0, \alpha > 0, \theta_j > 0; j = 1, 2, ..., m_2. \tag{48}$$

We can simplify the likelihood function into the form,

$$\begin{aligned} L &= \prod_{t=1}^{r} \frac{a^k z_t^{k-1} e^{-az_t}}{(k-1)!} \prod_{i=1}^{m_2} \left\{ \pi_i^{n1_i} (\frac{\alpha}{\beta_i})^{n1_i} \left(\prod_{j=1}^{n} x_j^{z_{ij}} \right)^{\alpha-1} e^{\frac{1}{\beta_i} \sum_{j=1}^{n} (z_{ij} x_j^{\alpha})} \right\} \\ & \prod_{k=1}^{m_1} \left\{ p_k^{n2_k} (\frac{\delta}{\gamma_k})^{n2_k} \left(\prod_{l=1}^{n} y_l^{z_{kl}} \right)^{\delta-1} e^{\frac{1}{\gamma_k} \sum_{j=1}^{n} (z_{kl} y_l^{\delta})} \right\} \end{aligned} \tag{49}$$

We fix the Dirichlet prior distribution for $\pi = (\pi_1, \pi_2, ..., \pi_{m_1})$ and $p = (p_1, p_2, ..., p_{m_2})$, gamma prior for $\beta_i, \gamma_k; i = 1, 2, .., m_2, k = 1, 2, .., m_1$ non-informative prior for α, δ, a and k.The variable z_{ij} is such that $z_{ij} = 1$ if j^{th} unit of the sample comes from the i^{th} component and $z_{ij} = 0$ otherwise. Also $n_{1i} = \sum_{j=1}^{m_2} z_{ij}$ and $n_{2k} = \sum_{j=1}^{m_1} z_{kj}$.
Hence

$$\begin{aligned} \pi &\sim \text{Dirichlet}(\mu_{11}, \mu_{12}, ..., \mu_{1m_2}) \\ p &\sim \text{Dirichlet}(\mu_{21}, \mu_{22}, ..., \mu_{2m_1}) \\ \pi_{3i}(\beta_i) &\propto \beta_i^{a1_i-1} e^{-b1_i\beta_i}; i = 1, 2, .., m_2 \\ \pi_{4k}(\gamma_k) &\propto \gamma_k^{a_2k-1} e^{-b_2k\gamma_k}; k = 1, 2, .., m_1 \\ \pi_5(\alpha), \pi_6(\delta), \pi_7(a), \pi_8(k) &\propto 1 \end{aligned}$$

where $\mu_1 = (\mu_{11}, \mu_{12}, ..., \mu_{1m_1})$, $\mu_2 = (\mu_{21}, \mu_{22}, ..., \mu_{2m_2})$, $(a_{1i}, a_{2i}); i = 1, 2, .., m_2$ and $(b_{1j}, b_{2j}); j = 1, 2, .., m_1$ are the hyper-parameters. Since the cycle time parameters have a non-informative prior, their estimates coincide with the ML estimates. The joint prior distribution of $\pi, p, \beta, \gamma, \alpha$, and δ can be written as,

$$g(\pi, p, \beta, \gamma, \alpha, \delta) \propto \prod_{i=1}^{m_2} p_i^{\mu1_i-1} \beta_i^{(a1_i-1)} e^{b1_i\beta_i} \prod_{j=1}^{m_1} \pi_j^{\mu2_i-1} \gamma_j^{(a2_j-1)} e^{b2_j\gamma_j} \tag{50}$$

Where $\beta = (\beta_1, \beta_2, ..., \beta_{m_2})$ and $\gamma = (\gamma_1, \gamma_2, ..., \gamma_{m_1})$. The posterior probability is given by,

$$\begin{aligned} h(\pi, p, \beta, \gamma, \alpha, \delta | x, y, z_{ij}) \propto & \prod_{i=1}^{m_2} p_i^{\mu 1_i - 1} \beta_i^{(a1_i - 1)} e^{b1_i \beta_i} \prod_{j=1}^{m_1} \pi_j^{\mu 2_i - 1} \gamma_j^{(a2_j - 1)} e^{b2_j \gamma_j} \\ & \prod_{t=1}^{r} \frac{a^k z_t^{k-1} e^{-a z_t}}{(k-1)!} \prod_{i=1}^{m_2} \left\{ \pi_i^{n1_i} (\frac{\alpha}{\beta_i})^{n1_i} \left(\prod_{j=1}^{n} x_j^{z_{ij}} \right)^{\alpha - 1} e^{\frac{1}{\beta_i} \sum_{j=1}^{n} (z_{ij} x_j^{\alpha})} \right\} \\ & \prod_{k=1}^{m_1} \left\{ p_k^{n2_k} (\frac{\delta}{\gamma_k})^{n2_k} \left(\prod_{l=1}^{n} y_l^{z_{kl}} \right)^{\delta - 1} e^{\frac{1}{\gamma_k} \sum_{j=1}^{n} (z_{kl} y_l^{\delta})} \right\} \end{aligned} \tag{51}$$

Then, the conditional posterior distributions of $\pi, p, \beta, \gamma, \alpha$, and δ are:

$$\pi \sim \text{Dirichlet}(\mu_{11} + n_{11}, \mu_{12} + n_{12}, ..., \mu_{1m_2} + n_{1m_2}) \tag{52}$$

$$p \sim \text{Dirichlet}(\mu_{21} + n_{21}, \mu_{22} + n_{22}, ..., \mu_{2m_1} + n_{2m_1}) \tag{53}$$

$$\pi_1(\alpha | \beta, x, z) \propto \prod_{i=1}^{m_2} \left\{ \alpha^{n1_i} \left(\prod_{j=1}^{n} x_j^{z_{ij}} \right)^{\alpha - 1} e^{-\frac{1}{\beta_i} \sum_{j=1}^{n} x_j^{\alpha}} \right\} \tag{54}$$

$$\pi_2(\delta | \gamma, y, z) \propto \prod_{i=1}^{m_1} \left\{ \delta^{n2_i} \left(\prod_{j=1}^{n} y_j^{z_{ij}} \right)^{\delta - 1} e^{-\frac{1}{\gamma_i} \sum_{j=1}^{n} y_j^{\delta}} \right\} \tag{55}$$

$$\pi_{3i}(\beta_i | \alpha, \beta_i^*, x, z) \propto \beta_i^{-n1_i + a1_i - 1} e^{-\frac{1}{\beta_i} \sum_{j=1}^{n} x_j^{\alpha}} e^{-b1 \beta_i}; i = 1, 2, .., m_2 \tag{56}$$

$$\pi_{4i}(\gamma_i | \delta, \gamma_i^*, y, z) \propto \gamma_i^{-n2_i + a2_i - 1} e^{-\frac{1}{\gamma_i} \sum_{j=1}^{n} y_j^{\delta}} e^{-b2 \gamma_i}; i = 1, 2, .., m_1. \tag{57}$$

Where $\beta_i^* = \{\beta_i, i = 1, 2, i-1, i+1.., m_2\}$ and $\gamma_i^* = \{\gamma_i, i = 1, 2, i-1, i+1.., m_1\}$.

The posterior distributions of α, β_i, δ, and γ_i cannot be reduced analytically to a well-known distribution. So we use the Markov chain Monte Carlo method with Gibbs sampling under Metropolis-Hastings algorithm for computing Bayes estimate using the statistical software, R. The Metropolis-Hastings algorithm with chi-square proposal density is used for generating samples from $(\pi, p, \alpha, \beta, \delta, \gamma)$, where $\pi = (\pi_1, \pi_2, ..., \pi_{m_1})$, $p = (p_1, p_2, ..., p_{m_2})$ $\beta = \{\beta_i, i = 1, 2, .., m_2\}$, and $\gamma = \{\gamma_i, i = 1, 2, .., m_1\}$is given as follows.

ALGORITHM − 1 :

Step1. Set the initial values $(\pi^0, p^0, \alpha^o, \beta^0, \delta^0, \gamma^0)$
Step2.Generate z_{ij} values using sample x
Step3.Generate π^t
Step4. Using the proposal density $g(\alpha) \sim \chi^2_{(x)}$ where x is the d.f and choose $x = \alpha^{t-1}$ Generate another random variable y from the chi-square density g. Generate u from Uniform(0,1). If $u < \frac{\pi_1(y) g(x)}{\pi_1(x) g(y)}$ accept y and set $\alpha^t = y$; otherwise set $\alpha^t = x$
Step5. Using the proposal density $g(\beta_i) \sim \chi^2_{(x)}$ where x is the d.f and choose $x = \beta_i^{t-1}$ Generate another random variable y from the chi-square density g. Generate u from Uniform(0,1). If $u < \frac{\pi_{3i}(y) g(x)}{\pi_{3i}(x) g(y)}$ accept y and set $\beta_i^t = y$; otherwise set $\beta_i^t = x$. Repeat the procedure and generate $\beta_i^t, i = 1, 2, .., m_2$
Step6.Generate z_{ij} values using sample y
Step7.Generate p^t
Step8. Using the proposal density $g(\delta) \sim \chi^2_{(x)}$ where x is the d.f and choose $x = \delta^{t-1}$ Generate random variable y from the chi-square density g. Generate u from Uniform(0,1). If $u < \frac{\pi_2(y) g(x)}{\pi_2(x) g(y)}$ accept y and set $\delta^t = y$; otherwise set $\delta^t = x$

Step9. Using the proposal density $g(\gamma_i) \sim \chi^2_{(x)}$ where x is the d.f and choose $x = \gamma_i^{t-1}$ Generate random variable y from the chi-square density g. Generate u from Uniform(0,1). If $u < \frac{\pi_{4i}(y)g(x)}{\pi_{4i}(x)g(y)}$ accept y and set $\beta_i^t = y$; otherwise set $\beta_i^t = x$. Repeate the procedure and generate $\gamma_i^t, i = 1, 2, .., m_1$
Step10 Compute $R(t)$.
Step11 Increment t.

Table 3 provides the estimated values of R(t) by the Bayesian estimation method when the stress and strength of the system follow a mixture of two Weibull distributions with gamma cycle time. We assume that the mixture proportions π and p are known and the component parameters $(\alpha, \beta, \delta, \gamma)$ are unknown and are following gamma prior distributions. Also, we assume that cycle time parameters a and k follow non-informative prior. Since the cycle time parameters have a non-informative prior, their estimates coincide with the ML estimates. The table shows Bayes estimates of the parameters $(\alpha, \beta, \delta, \gamma)$ and Bayes estimate of the reliability function $R(t)$ for different time values corresponding to various sets of hyperparameter values. The table shows that R(t) decreases as time increases.

Table 3: *Bayesian Estimation of R(t) with Weibull mixture initial stress strength*

Cycle time		Stress and Strength		a_0	x_0	t	$R(t)$
Parameters	Estimated	Parameters	Estimated				
G(0.5,2)	$a = 0.4881$	Y_0:0.8W(0.3,0.6)+	$\delta = 0.2890$	1	0.02	10	0.2230
	$k = 2.0840$	0.8W(0.3,2)	$\gamma = (0.5892, 1.9454)$			25	0.1539
		X_0:0.3W(0.3,1)+	$\alpha = 0.2954$			50	0.1175
		0.7W(0.3,0.3)	$\beta = (1.1603, 0.2957)$			75	0.0989
						100	0.0869
G(0.5,4)	$a = 0.5231$	Y_0:0.6W(5,0.3)+	$\delta = 5.0982$	0.001	0.08	10	0.9155
	$k = 3.1780$	0.4W(5,2)	$\gamma = (0.2992, 2.1114)$			25	0.9154
		X_0:0.3W(5,0.1)+	$\alpha = 5.1089$			50	0.6168
		0.7W(5,0.2)	$\beta = (0.1008, 0.2292)$			75	0.2228
						100	0.0498
G(1,2)	$a = 0.9975$	Y_0:0.3W(0.2,2)+	$\delta = 0.2005$	0.01	0.02	10	0.7598
	$k = 1.9738$	0.7W(0.2,5)	$\gamma = (0.8536, 8.6029)$			25	0.7208
		X_0:0.4W(0.2,0.9)+	$\alpha = 5.1089$			50	0.6868
		0.6W(0.2,8)	$\beta = (0.1008, 0.2292)$			75	0.6652
						100	0.6492
G(1,4)	$a = 0.9652$	Y_0:0.5W(2,0.2)+	$\delta = 2.0344$	0.001	0.05	10	0.7455
	$k = 3.7469$	0.5W(2,6)	$\gamma = (0.2093, 5.9099)$			25	0.6070
		X_0:0.5W(2,1)+	$\alpha = 1.9980$			50	0.4140
		0.5W(2,10)	$\beta = (1.0200, 9.8969)$			75	0.3370
						100	0.2908

4.2. Bayesian Estimation of R(t) based on a finite mixture of power-transformed half-logistic distribution

Let the cycle time follows a gamma distribution with parameters (a, k) and stress and strength of the system follow a mixture of power transformed half logistic distribution with parameters $(\delta_j, \gamma_j), j = 1, 2, .., m_2$ and $(\alpha_i, \theta_i), i = 1, 2, .., m_1$ respectively. The expression for stress-strength reliability is given in section 2. Here we discuss the estimation of the parameters by the Bayesian estimation method. Consider the auxiliary variable Z_j, such that

$$X_j|Z_j = i \sim g_i(x, \delta_i, \gamma_i) \text{ and } p(Z_j = i) = p_i, j = 1, 2, .., n, i = 1, 2, .., m_2$$

.

$$Y_j|Z_j = i \sim h_i(y, \alpha_i, \theta_i) \text{ and } p(Z_j = i) = \pi_i, j = 1, 2, .., m, i = 1, 2, .., m_1$$

.

Where

$$g_i(x) = \begin{cases} 2\delta_i\gamma_i x^{\gamma_i-1}e^{-\delta_i x^{\gamma_i}}\left(1+e^{-\delta_i x^{\gamma_i}}\right)^{-2}, & 0 \le x < \infty; \delta_i > 0; \gamma_i > 0. \\ 0 & \text{otherwise.} \end{cases} \quad (58)$$

$$f(y) = \begin{cases} 2\alpha_i\theta_i y^{\theta_i-1}e^{-\alpha_i y^{\theta_i}}\left(1+e^{-\alpha_i y^{\theta_i}}\right)^{-2}, & 0 \le y < \infty; \alpha_i > 0; \gamma_i > 0. \\ 0 & \text{otherwise.} \end{cases} \quad (59)$$

Then likelihood function L is

$$\begin{aligned} L &= \prod_{i=1}^{n}\frac{a^k y_i^{k-1}e^{-ay_i}}{(k-1)!} \\ &\prod_{i=1}^{m_2}\left\{\pi_i^{n1_i}2^{n1_i}\gamma_i^{n1_i}\delta_i^{n1_i}\left(\prod_{j=1}^{n}x_j^{z_{ij}}\right)^{\gamma_i-1}e^{-\delta_i\sum_{j=1}^{n}(z_{ij}x_j^{\gamma_i})}\prod_{j=1}^{n}\left[\left(1+e^{-\delta_i x_j^{\gamma_i}}\right)^{-2z_{ij}}\right]\right\} \\ &\prod_{k=1}^{m_1}\left\{p_k^{n2_k}2^{n2_k}\theta_k^{n2_k}\alpha_k^{n2_k}\left(\prod_{j=1}^{m}y_j^{z_{kj}}\right)^{\theta_k-1}e^{-\alpha_k\sum_{j=1}^{m}(z_{kj}y_j^{\theta_k})}\prod_{j=1}^{m}\left(1+e^{-\alpha_k y_j^{\theta_k}}\right)^{-2z_{kj}}\right\} \end{aligned} \quad (60)$$

We fix the Dirichlet prior distribution for $\pi = (\pi_1, \pi_2, ..., \pi_{m_1})$ and $p = (p_1, p_2, ..., p_{m_2})$, gamma prior for $\delta_i, \alpha_k; i = 1, 2, .., m_2, k = 1, 2, .., m_1$ non-informative prior for γ, θ, a and k. Since the cycle time parameters have a non-informative prior, their estimates coincide with the ML estimates. The variable z_{ij} is such that $z_{ij} = 1$ if j^{th} unit of the sample comes from the i^{th} component and $z_{ij} = 0$ otherwise. Also $n_{1i} = \sum_{j=1}^{m_2} z_{ij}$ and $n_{2k} = \sum_{j=1}^{m_1} z_{kj}$. Hence

$$\begin{aligned} \pi &\sim \text{Dirichlet}(\mu_{11}, \mu_{12}, ..., \mu_{1m_2}) \\ p &\sim \text{Dirichlet}(\mu_{21}, \mu_{22}, ..., \mu_{2m_1}) \\ \pi_{3i}(\delta_i) &\propto \delta_i^{a1_i-1}e^{-b1_i\delta_i}; \\ \pi_{4k}(\alpha_k) &\propto \alpha_k^{a2_k-1}e^{-b2_k\alpha_k}; k = 1, 2, .., m_1 \\ \pi_5(\gamma_i), \pi_6(\theta_k), \pi_7(a), \pi_8(k) &\propto 1; i = 1, 2, .., m_2, ; k = 1, 2, .., m_1 \end{aligned}$$

where $\mu_1 = (\mu_{11}, \mu_{12}, ..., \mu_{1m_1})$,$\mu_2 = (\mu_{21}, \mu_{22}, ..., \mu_{2m_2})$, $(a_{1i}, a_{2i}); i = 1, 2, .., m_2$ and $(b_{1j}, b_{2j}); j = 1, 2, .., m_1$ are the hyper-parameters. Since the cycle time parameters have a non-informative prior, their estimates coincide with the ML estimates.

Now proceeding as in the case of Bayesian estimation of R(t) based on the finite mixture of Weibull distribution discussed in the previous section we can easily obtain the conditional marginal distributions $\pi, p, \delta, \alpha, \gamma$, and θ. The conditional posterior distributions of $\pi, p, \delta, \alpha, \gamma$, and θ are:

$$\pi \sim \text{Dirichlet}(\mu_{11} + n_{11}, \mu_{12} + n_{12}, ..., \mu_{1m_2} + n_{1m_2}) \quad (61)$$

$$p \sim \text{Dirichlet}(\mu_{21} + n_{21}, \mu_{22} + n_{22}, ..., \mu_{2m_1} + n_{2m_1}) \quad (62)$$

$$\pi_{3i}(\delta_i|\gamma, \delta_i^*, x, z_{ij}) \propto \delta_i^{a1_i+n1_i-1}e^{-\left(\delta_i\sum_{j=1}^{n}\left(z_{ij}x_j^{\gamma_i}\right)+b1_i\delta_i\right)}\prod_{j=1}^{n}\left[1+e^{-\delta_i x_j^{\gamma_i}}\right]^{-2z_{ij}} \quad i = 1, 2, .., m_2. \quad (63)$$

$$\pi_{4k}(\alpha_k|\theta, \alpha_k^*, y, z_{kj}) \propto \alpha_k^{a2_k+n2_k-1}e^{-\left(\alpha_k\sum_{j=1}^{m}\left(z_{ij}y_j^{\theta_k}\right)+b2_k\alpha_k\right)}\prod_{j=1}^{m}\left[1+e^{-\alpha_k y_j^{\theta_k}}\right]^{-2z_{kj}} \quad k = 1, 2, .., m_1. \quad (64)$$

$$\pi_{5i}(\gamma_i|\delta,\gamma_i^*,x,z_{ij}) \propto \gamma_i^{n1_i}\left(\prod_{j=1}^{n} x_j^{z_{ij}}\right)^{\gamma_i-1} e^{-\left(\delta_i \sum_{j=1}^{n}\left(z_{ij}x_j^{\gamma_i}\right)\right)} \prod_{j=1}^{n}\left[1+e^{-\delta_i x_j^{\gamma_i}}\right]^{-2z_{ij}} : \quad i=1,2,..,m_2. \tag{65}$$

$$\pi_{6k}(\theta_k|\alpha,\theta_k^*,y,z_{kj}) \propto \theta_k^{n2_k}\left(\prod_{j=1}^{m} y_k^{z_{kj}}\right)^{\theta_k-1} e^{-\left(\alpha_k \sum_{j=1}^{m}\left(z_{kj}y_k^{\theta_k}\right)\right)} \prod_{j=1}^{m}\left[1+e^{-\alpha_k y_k^{\theta_k}}\right]^{-2z_{ij}} : \quad k=1,2,..,m_1. \tag{66}$$

Where $\delta_i^* = \{\delta_j, j=1,2,...,i-1,i+1..,m_2\}$, $\gamma_i^* = \{\gamma_j, j=1,2,...,i-1,i+1..,m_2\}$, $\alpha_k^* = \{\alpha_i, i=1,2,...,k-1,k+1..,m_1\}$ and $\theta_k^* = \{\theta_i, i=1,2,...,k-1,k+1..,m_1\}$.

Since the posterior distributions of $\alpha_k, \theta_k, \delta_i$, and γ_i cannot be reduced analytically to a well-known distribution, as done in the previous section, we use the Markov chain Monte Carlo method with Gibbs sampling under Metropolis-Hastings algorithm for computing Bayes estimates. We fix the proposal density as the chi-square distribution. The Metropolis-Hastings algorithm with chi-square proposal density is used for generating samples from $(\pi, p, \alpha, \theta, \delta, \gamma)$, where $\pi = (\pi_i, i=1,2,..,m_1)$, $p = (p_k, k=1,2,..,m_1)$ $\alpha = \{\alpha_k, k=1,2,..,m_1\}$, $\theta = \{\theta_k, k=1,2,..,m_1\}$, $\delta = \{\delta_i, i=1,2,..,m_2\}$ and $\gamma = \{\gamma_i, i=1,2,..,m_1\}$ is given as follows.

ALGORITHM − 2 :

Step1. Set the initial values $(\pi^0, p^0, \alpha^o, \theta^0, \delta^0, \gamma^0)$.
Step2.Generate z_{ij} values using sample x
Step3.Generate π^t.
Step4. Using the proposal density $g(\delta_i) \sim \chi^2_{(x)}$, where x is the d.f and choose $x = \delta_i^{t-1}$. Generate another random variable y from the chi-square density g. Generate u from Uniform(0,1). If $u < \frac{\pi_{3i}(y)g(x)}{\pi_{3i}(x)g(y)}$ accept y and set $\delta_i^t = y$; otherwise set $\delta_i^t = x$. Repeate the procedure and generate $\delta_i^t, i=1,2,..,m_2$.
Step5. Using the proposal density $g(\gamma_i) \sim \chi^2_{(x)}$, where x is the d.f and choose $x = \gamma_i^{t-1}$ Generate another random variable y from the chi-square density g. Generate u from Uniform(0,1). If $u < \frac{\pi_{5i}(y)g(x)}{\pi_{5i}(x)g(y)}$ accept y and set $\gamma_i^t = y$; otherwise set $\gamma_i^t = x$. Repeate the procedure and generate $\beta_i^t, i=1,2,..,m_2$.
Step6.Generate z_{ij} values using sample y.
Step7.Generate p^t.
Step8. Using the proposal density $g(\alpha_k) \sim \chi^2_{(x)}$, where x is the d.f and choose $x = \alpha_k^{t-1}$ Generate random variable y from the chi-square density g. Generate u from Uniform(0,1). If $u < \frac{\pi_{4k}(y)g(x)}{\pi_{4k}(x)g(y)}$ accept y and set $\alpha_k^t = y$; otherwise set $\alpha_k^t = x$. Repeate the procedure and generate $\alpha_k^t, k=1,2,..,m_1$.
Step9. Using the proposal density $g(\theta_k) \sim \chi^2_{(x)}$, where x is the d.f and choose $x = \theta_k^{t-1}$ Generate random variable y from the chi-square density g. Generate u from Uniform(0,1). If $u < \frac{\pi_{6k}(y)g(x)}{\pi_{6k}(x)g(y)}$ accept y and set $\theta_k^t = y$; otherwise set $\theta_k^t = x$. Repeate the procedure and generate $\theta_k^t, k=1,2,..,m_1$.
Step10 Compute $R(t)$.
Step11 Increment t.

Table 4 provides the estimated values of R(t) by the Bayesian estimation method when the stress and strength of the system follow a mixture of two power-transformed half-logistic distributions with gamma cycle time. We assume that the mixture proportions π and p are known and the component parameters $(\alpha, \theta, \delta, \gamma)$ are unknown and are following gamma prior distributions. Also, we assume that cycle time parameters a and k follow non-informative prior. Since the cycle time parameters have a non-informative prior, their estimates coincide with the ML estimates. The table shows Bayes estimates of the parameters $(\alpha, \theta, \delta, \gamma)$ and Bayes estimate of the reliability function $R(t)$ for different time values corresponding to various sets of hyperparameter values. The table shows that R(t) decreases as time increases, as we expected.

Table 4: *Bayesian Estimation of R(t) with PTHL mixture initial stress and strength*

Cycle time		Stress and Strength		a_0	x_0	t	$R(t)$
Parameters	Estimated	Parameters	Estimated				
G(0.5,1)	$a = 0.5110$	Y_0:0.3PTHL(0.2,5)	$\alpha = (0.2942, 1.7069)$	0.001	0.005	10	0.0184
	$k = 0.9955$	+0.7PTHL(2,6)	$\theta = (4.5235, 5.4221)$			75	0.0185
		X_0: 0.3PTHL(1,3)	$\delta = (1.8201, 2.2808)$			150	0.0184
		+0.7PTHL(2.2,4.5)	$\gamma = (6.1306, 4.6754)$			200	0.0166
						225	0.0090
G(0.5,1)	$a = 0.4759$	Y_0:0.6PTHL(2,6)	$\alpha = (1.8055, 6.7181)$	0.002	0.005	10	0.1851
	$k = 0.9559$	+0.4PTHL(4,2)	$\theta = (5.9235, 2.1754)$			75	0.1864
		X_0:0.4PTHL(1,3)	$\delta = (1.1301, 1.0738)$			150	0.1858
		+0.6PTHL(0.5,2)	$\gamma = (2.4231, 0.6529)$			200	0.0907
						225	0.0202
G(1,4)	$a = 0.9694$	Y_0:0.7PTHL(8,0.5)	$\alpha = (0.6985, 3.4123)$	0.001	0.005	10	0.2369
	$k = 3.9365$	+0.3PTHL(0.5,2.5)	$\theta = (1.8173, 4.9566)$			75	0.2394
		X_0:0.2PTHL(4,0.5)	$\delta = (0.5839, 0.2675)$			125	0.2335
		+0.8PTHL(5,0.2)	$\gamma = (2.2849, 1.6391)$			140	0.1487
						150	0.0589
G(1,4)	$a = 0.9674$	Strength:0.6PTHL(1.5,5)	$\alpha = (1.1917, 6.8528)$	0.002	0.08	10	0.0400
	$k = 3.8930$	+0.4PTHL(5,4)	$\theta = (6.2016, 3.7667)$			75	0.0404
		Stress:0.5PTHL(2.4,6)	$\delta = (1.92620.2490)$			125	0.0394
		+0.5PTHL(0.3,4)	$\gamma = (5.5248, 3.8446)$			140	0.0251
						150	0.0099

5. Conclusion

In this paper, we investigated the stress-strength reliability of a system. Here we considered a scenario where the stress and strength of the system follow a finite mixture distribution with gamma cycle time. Specifically, we examined the performance of the system under two types of finite mixture models: a finite mixture of Weibull distribution and a finite mixture of power-transformed half-logistic distribution. To estimate the reliability function R(t), we employed two methods: maximum likelihood (ML) estimation using the expectation-maximization (EM) algorithm and Bayesian estimation using the Markov Chain Monte Carlo (MCMC) method. We computed the estimates of R(t) for different time points corresponding to various sets of parameter values. Based on the graphs and tables presented in the paper, it can be observed that as time increases, the reliability function R(t) decreases when the stress and strength of the system follow a finite mixture of Weibull or power-transformed half-logistic distribution with gamma cycle time. This suggests that the system becomes less reliable or more prone to failure as time progresses.

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