DUAL EXPONENTIAL RATIO ESTIMATOR IN PRESENCE OF NON-RESPONSE

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Abstract

The manuscript under consideration delves into a comprehensive exploration of the dual exponential ratio estimator, particularly in the context of non-response scenarios. In the following discourse, we will embark on an intricate journey through this research, emphasizing the pivotal aspects and findings that unravel the significance of this estimator in the realm of statistical estimation. The crux of this investigation revolves around evaluating the Mean Squared Error (MSE) and the Predictive Relative Efficiency (PRE) of the dual exponential ratio estimator. These two performance metrics serve as essential benchmarks for assessing the accuracy and effectiveness of the estimator. Notably, they play a crucial role in determining the estimator's suitability for practical applications, especially in situations where non-response is prevalent. To begin our exploration, it is imperative to understand the fundamental concept of the dual exponential ratio estimator. This estimator is a statistical tool employed in situations where traditional estimators may falter due to non-response, a phenomenon frequently encountered in surveys and data collection. It leverages a dual exponential model to address this challenge, making it a valuable addition to the toolkit of statisticians and researchers. The manuscript embarks on a rigorous theoretical analysis of the dual exponential ratio estimator's MSE and PRE. Through a series of mathematical derivations and proofs, the authors elucidate the underlying principles governing its performance. This theoretical foundation is crucial, as it not only establishes a solid framework for evaluating the estimator but also provides insights into its behavior under different conditions. However, theory alone can only take us so far. To validate the theoretical findings and assess the estimator's practical utility, numerical experiments are conducted. These experiments involve simulations and real-world data scenarios, allowing the authors to draw comparisons between the dual exponential ratio estimator and traditional estimators. The numerical results serve as a bridge between theory and application, offering empirical evidence of the estimator's prowess. In essence, this manuscript fills a critical gap in the field of statistical estimation by thoroughly investigating the dual exponential ratio estimator's performance in the presence of non-response. By juxtaposing its MSE and PRE with those of traditional estimators, it provides valuable insights into the potential advantages of adopting this novel approach. Moreover, the combination of rigorous theory and practical validation ensures that the findings are both intellectually sound and operationally relevant. The dual exponential ratio estimator, as explored and analyzed within these pages, emerges as a promising solution, backed by both theoretical rigor and empirical support. This research contributes not only to the theoretical foundations of statistics but also to its real-world applications, underscoring the estimator's potential to enhance the accuracy and reliability of estimation in the face of non-response complexities.

Keywords: Non-Response (NR), Exponential Estimator, Dual to Ratio Estimator, Mean Square Error and Percent Relative Efficiency.

I. Introduction

In recent years, the use of sample surveys has gained popularity due to the practicality of overcoming logistical challenges associated with conducting comprehensive census surveys. This trend has led to the widespread adoption of estimators like the ratio, product, and regression estimators for efficiently estimating population parameters, particularly the mean of the variable of interest. These estimators capitalize on the inherent correlation between the study variable and auxiliary variables, either during the survey design or at the estimation stage, to yield accurate results while optimizing resources. The central focus of this research is to develop a novel modified exponential ratio estimator for the population mean. This estimator aims to address potential limitations of existing estimators and enhance the precision of estimates, as evaluated through mean squared error comparisons. By exploring alternative approaches and incorporating adjustments, the researchers anticipate achieving more reliable and efficient estimates of the population mean.

Over the years, several scholars have made significant contributions to the field of survey estimation. Various authors have made numerous work for the estimation of population variance from time to time including [14],[9] , [13], [8] [1], [5], [11],[12],[15] and [10] have made important studies on this topic in the literature. Notably, [17] made pioneering strides by explicitly utilizing auxiliary information for estimation purposes, laying the foundation for the ratio estimator. Subsequently, [18] further advanced this concept by employing auxiliary information to refine estimations.

When dealing with scenarios where the coefficient of correlation is negative between the study variable and auxiliary variables, [19] introduced the product-type estimator, which has proven to be valuable in specific contexts. Additionally, [20] proposed an innovative approach by combining multiple ratio estimators based on individual auxiliary variables positively correlated with the study variable. This technique allowed for greater accuracy in estimation. The product estimator was formalized by [21], providing a well-defined framework for its application. Furthermore, [22] delved into the complexities of ratio estimators involving two or more correlated variables, shedding light on new possibilities for refining estimation methods. The exponential type estimators of population mean were thoroughly investigated by [23] using auxiliary data, resulting in a comprehensive analysis of their performance and potential improvements. [24] took a unique approach by incorporating transformed auxiliary variables, which led to promising results in estimating the mean of the study character. The literature offers an array of other contributions in this area, including the works of [25], [26], [27], and [28], who introduced their respective estimators and demonstrated their efficacy in diverse sampling scenarios. Moreover, [29] and [30] took on the challenge of developing superior exponential type estimators by considering information from two altered auxiliary variables, further expanding the range of available estimation techniques. To gain a more comprehensive understanding of this topic, interested readers can refer to [31], which offers an in-depth exploration of various aspects of survey estimation. In recent times, [32], [33], and [34] have made notable contributions to this area of study, introducing novel ideas and methodologies that hold promise for advancing the field of survey estimation even further.

In conclusion, this research endeavors to create a Generalized Ratio-cum-product estimator of population variance that builds upon the knowledge and advancements made by previous scholars. By harnessing the power of auxiliary information and exploring innovative avenues, the researchers aim to provide an enhanced and efficient approach to estimating the population mean and contributing to the growing body of knowledge in survey estimation techniques.

II. Notations

Let $N \text{ and } n$ and be population and sample of size respectively. Out of n units 'n_1' responds and 'n_2' do not respond accordingly, the population is distributed in 'N_1'(those who respond) and 'N_2'(the non-respondents), such that $N_1 + N_2 = N$. From sample of 'n_2' a sub-sample of size k where $\left(k = \frac{n_2}{h}, h > 1\right)$ is taken and data is obtained. Further, we define

$$W_1 = \frac{N_1}{N},\ W_2 = \frac{N_2}{N},\quad \lambda = \frac{1}{n} - \frac{1}{N},\ \theta = \frac{W_2(h-1)}{n},\quad w_1 = \frac{n_1}{n},\ w_2 = \frac{n_2}{n},\quad \bar{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i\,,$$

$$\bar{Y}_2 = \frac{1}{N_2}\sum_{i=1}^{N_2} y_i\,,\ \bar{X} = \frac{1}{N}\sum_{i=1}^{N} x_i\,,\ \bar{X}_2 = \frac{1}{N_2}\sum_{i=1}^{N_2} x_i\,, C_y = \frac{S_y}{\bar{Y}},\ C_{y(2)} = \frac{S_{y(2)}}{\bar{Y}}, C_x = \frac{S_x}{\bar{X}},\ C_{x(2)} = \frac{S_{x(2)}}{\bar{X}},$$

$$S_y^2 = \frac{\sum_{i=1}^{N}(y_1 - \bar{Y})^2}{N-1},\ S_{y(2)}^2 = \frac{\sum_{i=1}^{N_2}(y_i - \bar{Y}_2)^2}{N_2-1},\ S_x^2 = \frac{\sum_{i=1}^{N}(x_i - \bar{X})^2}{N-1},\ S_{x(2)}^2 = \frac{\sum_{i=1}^{N_2}(x_i - \bar{X}_2)^2}{N_2-1},$$

$$C = \rho_{yx}\left(\frac{C_y}{C_x}\right), C_{(2)} = \rho_{yx(2)}\left(\frac{C_{y(2)}}{C_{x(2)}}\right)$$

III. Existing Estimators

Hansen and Hurwitz proposed an unbiased estimator of $\bar{Y}$ in case of non-response,

$$\bar{y}^* = w_1\ \bar{y}_1 + w_2\ \bar{y}_2,$$

where $\bar{y}_1$ is the sample mean of respondents and $\bar{y}_2$ is the mean of sub-sample of non-respondents,

The variance is,

$$V(\bar{y}^*) = \bar{Y}^2\left(\lambda\ C_y^2 + \theta\ C_{y(2)}^2\right),$$

The unbiased estimator of $\bar{X}$ in case of non-response is given as,

$$\bar{x}^* = w_1\,\bar{x}_1 + w_2\ \bar{x}_2,$$

where $\bar{x}_1$ the sample is mean of the respondents and similarly $\bar{x}_2$ is the mean of sub-sample.

The variance is

$$V(\bar{x}^*) = \bar{X}^2\left(\lambda\ C_x^2 + \theta\ C_{x(2)}^2\right).$$

3.1 Case I: Non-response on y only

Ratio estimator of $\bar{Y}$ in case I is,

$$t_{R1} = \frac{\bar{y}^*}{\bar{x}}\bar{X},$$

$$MSE(t_{R1}) = \bar{Y}^2\left[\lambda\left(C_y^2 + C_x^2 - 2\rho_{yx}C_yC_x\right) + \theta\ C_{y(2)}^2\right].$$

The dual to ratio estimator given by Srivenkentrama (1980) is,

$$t_D = \bar{y}\left(\frac{\bar{x}^{\beta}}{\bar{X}}\right),$$

where $\bar{x}^{\beta} = \left(\frac{\bar{X}-}{\quad}\right),\ i = 1, 2, \ldots, N.$

and the MSE is,

$$MSE(t_D) = \lambda\ \bar{Y}^2 \left(C_y^2 + C_x^2 g\ \ (g - 2\,C)\right)$$

The dual of ratio estimator in case of non-response is,

$$t_{D1} = \bar{y}^* \left(\frac{\bar{x}^\beta}{\bar{X}}\right),$$

The MSE is given by

$$MSE(t_{D1}) = \bar{Y}^2 \left[\lambda\ \left(C_y^2 + C_x^2 g\ \ (g - 2C)\right) + \theta\ C_{y(2)}^2\right]$$

Singh and Kumar (2009) considered the exponential estimators of $\bar{Y}$ in case of non-response.

$$t_{ER1} = \bar{y}^* \exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right)$$

The MSE is,

$$MSE(t_{ER1}) = \bar{Y}^2 \left[\lambda\ \left(C_y^2 + \frac{1}{4} C_x^2 - \rho_{xy} C_x C_y\right) + \theta\ \left(C_{y(2)}^2\right)\right]$$

The dual exponential estimator for non-response is

$$t_{ED1} = \bar{y}^* \exp\left(\frac{\bar{x}^\beta - \bar{X}}{\bar{x}^\beta + \bar{X}}\right),$$

The MSE is,

$$MSE(t_{ED1}) = \bar{Y}^2 \left[\lambda\ \left(C_y^2 + \frac{g^2 C_x^2}{4} - g \rho_{yx} C_x C_y\right) + \theta C_{y(2)}^2\right]$$

3.2 Case II: Non-response on both y and x

The ratio estimator of $\bar{Y}$ for case II along with MSE is given as,

$$t_{R2} = \frac{\bar{y}^*}{\bar{x}^*} \bar{X}\ ,$$

$$MSE(t_{R2}) = \begin{bmatrix} \lambda\ \bar{Y}^2 \left(C_x^2 + C_y^2 - 2\rho_{xy} C_x C_y\right) \\ + \theta\ \bar{Y}^2 \left(C_{x(2)}^2 + C_{y(2)}^2 - 2\rho_{xy(2)} C_{x(2)} C_{y(2)}\right) \end{bmatrix},$$

where $\rho_{yx} = \dfrac{S_{yx}}{S_y S_x}$ is the correlation for the overall population, while $\rho_{yx(2)} = \dfrac{S_{yx(2)}}{S_{y(2)} S_{x(2)}}$ is the case of non-respondent group.

Dual of ratio estimator for case II is,

$$t_{D2} = \bar{y}^* \frac{\bar{x}^{*\beta}}{\bar{X}}$$

and the MSE is,

$$MSE(t_{D2}) = \bar{Y}^2 \left[\lambda\ \left(C_y^2 + C_x^2 g\ \ (g - 2C)\right) + \theta\ \left(C_{y(2)}^2 + C_{x(2)}^2 g\ \ \left(g - C_{(2)}\right)\right)\right]$$

The exponential ratio estimator is given as,

$$t_{ER2} = \bar{y}^* \exp\left(\frac{\bar{X} - \bar{x}^*}{\bar{X} + \bar{x}^*}\right)$$

$$MSE(t_{ER2}) = \bar{Y}^2 \begin{bmatrix} \lambda\ \left(C_y^2 + \dfrac{1}{4} C_x^2 - \rho_{xy} C_x C_y\right) \\ + \theta\ \left(C_{y(2)}^2 + \dfrac{C_{x(2)}^2}{4} - \rho_{xy(2)} C_{x(2)} C_{y(2)}\right) \end{bmatrix}$$

The dual of exponential ratio estimator is

$$t_{ED2} = \bar{y}^* \exp\left(\frac{\bar{x}^{*\beta} - \bar{X}}{\bar{x}^{*\beta} + \bar{X}}\right)$$

And the MSE is given by

$$MSE(t_{ED2}) = \bar{Y}^2 \left[\begin{array}{l} \lambda\left(C_y^2 + \frac{g^2 C_x^2}{4} - g\rho_{yx} C_x C_y\right) \\ +\theta\left(C_{y(2)}^2 + \frac{g^2 C_{x(2)}^2}{4} - g\rho_{xy(2)} C_{x(2)} C_{y(2)}\right) \end{array} \right]$$

3.3 Proposed Estimator: Case I

The proposed ratio-cum-dual of exponential ratio estimator of $\bar{Y}$ is given as,

$$t^* = \bar{y}^* \left(\frac{\bar{X}}{\bar{x}}\right)^{\alpha} \exp\left[\frac{\delta\left(\bar{x}^{\beta} - \bar{X}\right)}{\left(\bar{x}^{\beta} + \bar{X}\right)}\right], \quad (1)$$

here $x^{\beta} = \frac{N\bar{X} - n\bar{x}}{N - n}$, α and δ are constants.

here $x^{\beta} = \frac{N\bar{X} - n\bar{x}}{N - n}$, α and δ are constants.

Table 1: *Some members of the proposed class of estimator*

S. No.	Estimator	Values of Constants	
		α	δ
1.	$t^* = \bar{y}^*$	0	0
2.	$t^* \to t_{R1} = \bar{y}^* \left(\frac{\bar{X}}{\bar{x}}\right)$	1	0
3.	$t^* \to t_{P1} = \bar{y}^* \left(\frac{\bar{x}}{\bar{X}}\right)$	-1	0
4.	$t^* = \bar{y}^* \left(\frac{\bar{X}^2}{\bar{x}^2}\right)$	2	0
5.	$t^* = \bar{y}^* \left(\frac{\bar{X}}{\bar{x}}\right)^{\frac{1}{2}}$	$\frac{1}{2}$	0
6.	$t^* = \bar{y}^* \exp\left(\frac{\bar{x}^{\beta} - \bar{X}}{\bar{x}^{\beta} + \bar{X}}\right)$	0	1
7.	$t^* = \bar{y}^* \exp\left(\frac{\left(\bar{x}^{\beta} - \bar{X}\right)}{2\left(\bar{x}^{\beta} + \bar{X}\right)}\right)$	0	$\frac{1}{2}$
8.	$t^* = \bar{y}^* \exp\left(\frac{2\left(\bar{x}^{\beta} - \bar{X}\right)}{\left(\bar{x}^{\beta} + \bar{X}\right)}\right)$	0	2
9.	$t^* = \bar{y}^* \left(\frac{\bar{X}}{\bar{x}}\right) \exp\left(\frac{\bar{x}^{\beta} - \bar{X}}{\bar{x}^{\beta} + \bar{X}}\right)$	1	1

10.	$t^* = \bar{y}^* \left(\frac{\bar{X}}{\bar{x}}\right)^{\frac{1}{2}} \exp\left(\frac{1(\bar{x}^\beta - \bar{X})}{2(\bar{x}^\beta + \bar{X})}\right)$	$\frac{1}{2}$	$\frac{1}{2}$

The associated sample mean is obtained as

$$\bar{x}^\beta = (1+g)\bar{X} - g\,\bar{x} \text{ and } g = \frac{n}{N-n}$$

To acquire the MSE, we write

$$\bar{y}^* = \bar{Y}(1+e_0^*) \text{ and } \bar{x} = \bar{X}(1+e_1),$$

Such that,

$$E(e_0^{*2}) = (\lambda C_y{}^2 + \theta C_{y(2)}^2),\ E(e_1^2) = \lambda C_x{}^2,\ E(e_0^* e_1) = \lambda C C_x{}^2$$

Expressing (1) in $e's$ we have

$$t^* = \bar{Y}(1+e_0)\left(\frac{\bar{X}}{\bar{X}(1+e_1)}\right)^\alpha \exp\left[\delta\left\{\frac{(1+g)\bar{X} - g(\bar{X}(1+e_1)) - \bar{X}}{(1+g)\bar{X} - g(\bar{X}(1+e_1)) + \bar{X}}\right\}\right]$$

$$= \bar{Y}(1+e_0)(1+e_1)^{-\alpha} \exp\left[\delta\left(\frac{-ge_1}{2-ge_1}\right)\right]$$

$$= \bar{Y}(1+e_0)(1+e_1)^{-\alpha} \exp\left(-\frac{\delta g e_1}{2}\left(1-\frac{g e_1}{2}\right)^{-1}\right)$$

$$= \bar{Y}(1+e_0)\left(1-\alpha e_1 + \frac{\alpha(\alpha+1)}{2}e_1^2 - ..\right)\left(1-\frac{\delta g e_1}{2}\left(1-\frac{g e_1}{2}\right)^{-1} + ...\right)$$

Ignoring higher order terms,

$$t^* = \bar{Y}(1+e_0)(1-\alpha e_1)\left(1-\frac{\delta g e_1}{2}\right)$$

$$(t^* - \bar{Y}) = \bar{Y}\left(e_0 - \left(\frac{\delta g + 2\alpha}{2}\right)e_1\right)$$

Squaring both sides, we get,

$$(t^* - \bar{Y})^2 = \bar{Y}^2\left(e_0^2 + \left(\frac{\delta g + 2\alpha}{2}\right)e_1^2 - (2\alpha + \delta g)e_0 e_1\right)$$

Taking expectation, we get the MSE as,

$$MSE(t^*) = \bar{Y}^2\left[(\lambda C_y{}^2 + \theta C_{y(2)}^2) + \lambda C_x{}^2\left\{\left(\frac{\delta g + 2\alpha}{2}\right)^2 - (\delta g + 2\alpha)C\right\}\right] \quad (2)$$

Differentiate (2) w.r.t. α and equate it to zero,

$$\frac{\partial}{\partial\alpha}MSE(t^*) = \frac{\partial}{\partial\alpha}\left\{\bar{Y}^2\left[\begin{array}{l}(\lambda C_y{}^2 + \theta C_{y(2)}^2) \\ + \lambda C_x{}^2\left\{\left(\frac{\delta g + 2\alpha}{2}\right)^2 - (\delta g + 2\alpha)C\right\}\end{array}\right]\right\} = 0$$

$$\frac{\partial}{\partial\alpha}MSE(t^*) = 2\alpha = \delta g + 2C$$

$$\alpha = \frac{\delta g + 2C}{2}$$

We can write

$$\left(\delta g + 2\alpha\right) = 2C \tag{3}$$

Using (3) in (2) we get,

$$MSE\left(t^*\right)_{\min} = \bar{Y}^2\left[\lambda C_y^2\left(1-\rho_{xy}^2\right)+\theta C_{y(2)}^2\right] \tag{4}$$

3.4 Proposed Estimator: Case II

The proposed estimator of $\bar{Y}$ is

$$t^{**} = \bar{y}^*\left(\frac{\bar{X}}{\bar{x}^*}\right)^{\alpha} \exp\left[\frac{\delta\left(\bar{x}^{*\beta}-\bar{X}\right)}{\left(\bar{x}^{*\beta}+\bar{X}\right)}\right] \tag{5}$$

where $x^{*\beta} = \dfrac{N\bar{X} - n\bar{x}^*}{N-n}$, α and δ are suitably chosen constant.

The associated sample mean is obtained as

$\bar{x}^{\beta} = \left(1+g\right)\bar{X} - g\,\bar{x}$ and $g = \dfrac{n}{N-n}$

Table 2: *Some members of the proposed class of estimator*

S. No.	Estimator	Values of Constants	
		α	δ
1.	$t^* = \bar{y}^*$	0	0
2.	$t^{**} \rightarrow t_{R2} = \bar{y}^*\left(\frac{\bar{X}}{\bar{x}^*}\right)$	1	0
3.	$t^{**} \rightarrow t_{P2} = \bar{y}^*\left(\frac{\bar{x}^*}{\bar{X}}\right)$	-1	0
4.	$t^{**} = \bar{y}^*\left(\frac{\bar{X}^2}{\bar{x}^{*2}}\right)$	2	0
5.	$t^{**} = \bar{y}^*\left(\frac{\bar{X}}{\bar{x}^*}\right)^{\frac{1}{2}}$	$\frac{1}{2}$	0
6.	$t^{**} = \bar{y}^* \exp\left(\frac{\bar{x}^{*\beta}-\bar{X}}{\bar{x}^{*\beta}+\bar{X}}\right)$	0	1
7.	$t^{**} = \bar{y}^* \exp\left(\frac{\left(\bar{x}^{*\beta}-\bar{X}\right)}{2\left(\bar{x}^{*\beta}+\bar{X}\right)}\right)$	0	$\frac{1}{2}$
8.	$t^{**} = \bar{y}^* \exp\left(\frac{2\left(\bar{x}^{*\beta}-\bar{X}\right)}{\left(\bar{x}^{*\beta}+\bar{X}\right)}\right)$	0	2
9.	$t^{**} = \bar{y}^*\left(\frac{\bar{X}}{\bar{x}}\right)\exp\left(\frac{\bar{x}^{*\beta}-\bar{X}}{\bar{x}^{*\beta}+\bar{X}}\right)$	1	1
10.	$t^{**} = \bar{y}^*\left(\frac{\bar{X}}{\bar{x}}\right)^{\frac{1}{2}} \exp\left(\frac{1\left(\bar{x}^{*\beta}-\bar{X}\right)}{2\left(\bar{x}^{*\beta}+\bar{X}\right)}\right)$	$\frac{1}{2}$	$\frac{1}{2}$

To acquire the MSE,

$\bar{x}^* = \bar{X}\left(1+e_1^*\right)$,

$E\left(e_0^{*2}\right)=\left(\lambda C_y{}^2+\theta C_{y(2)}^2\right)$ and $E\left(e_0^* e_1^*\right)=\left(\lambda C C_x{}^2+\theta C_{(2)} C_{x(2)}^2\right)$.

Ignoring the higher terms, we get,

$$t^{**}=\bar{Y}\left(1+e_0\right)\left(1-\alpha e_1\right)\left(1-\frac{\delta g e_1}{2}\right)$$

$$\left(t^{**}-\bar{Y}\right)=\bar{Y}\left(e_0-\frac{(\delta g+2\alpha)}{2} e_1\right)$$

Squaring the above equation, we get

$$\left(t^{**}-\bar{Y}\right)^2=\bar{Y}^2\left(e_0^2+\left\{\frac{(\delta g+2\alpha)}{2}\right\}^2 e_1^2-(\delta g+2\alpha) e_0 e_1\right)$$

Taking expectation, we get

$$MSE\left(t^{**}\right)=\bar{Y}^2\left[\begin{array}{l}\left(\lambda C_y{}^2+\theta C_{y(2)}^2\right)+\left(\lambda C_x{}^2+\theta . C_{x(2)}^2\right) \\ \left\{\left(\frac{\delta g+2\alpha}{2}\right)^2-(\delta g+2\alpha) R\right\}\end{array}\right], \tag{6}$$

where $R=\left(\frac{\lambda C C_x{}^2+\theta C_{(2)} C_{x(2)}^2}{\lambda C_x{}^2+\theta C_{x(2)}^2}\right)$.

Differentiate (6) w.r.t α we get,

$$\frac{\delta}{\delta\alpha} MSE\left(t^{**}\right)=\frac{\delta}{\delta\alpha}\left\{\bar{Y}^2\left[\begin{array}{l}\left(\lambda C_y{}^2+\theta C_{y(2)}^2\right)+\left(\lambda C_x{}^2+\theta C_{x(2)}^2\right) \\ \left\{\left(\frac{\delta g+2\alpha}{2}\right)^2-(\delta g+2\alpha) R\right\}\end{array}\right]\right\}=0$$

$$\frac{\delta}{\delta\alpha} MSE\left(t^{**}\right)=2\alpha=\delta g+2R$$

$$\alpha=\frac{\delta g+2R}{2}$$

We can write,

$$(\delta g+2\alpha)=2R \tag{7}$$

Substituting (7) in (6) we have,

$$MSE\left(t^{**}\right)_{\min}=\bar{Y}^2\left[\left(\lambda \mathrm{C}_{\mathrm{y}}^2+\theta C_{y(2)}^2\right)-\frac{\left(\lambda C C_x{}^2+\theta C_{(2)} C_{x(2)}^2\right)^2}{\left(\lambda C_x{}^2+\theta C_{x(2)}^2\right)}\right]$$

$$MSE\left(t^{**}\right)_{\min}=\bar{Y}^2\left[\left(\lambda \mathrm{C}_{\mathrm{y}}^2+\theta C_{y(2)}^2\right)\left(1-\rho^{*2}\right)\right]. \tag{8}$$

Where $\rho^*=\frac{Cov\left(\bar{y}^*, \bar{x}^*\right)}{\sqrt{V\left(\bar{y}^*\right) V\left(\bar{x}^*\right)}}=\frac{\left(\lambda C C_x{}^2+\theta C_{(2)} C_{x(2)}^2\right)}{\sqrt{\left(\lambda C_y{}^2+\theta C_{y(2)}^2\right)\left(\lambda C_x{}^2+\theta C_{x(2)}^2\right)}}$

IV. Theoretical Efficiency Comparison

4.1 Case I

$$V\left(t_0\right)-MSE\left(t^*\right) \geq 0.$$

$$=\bar{Y}^2\left(\lambda C_y^2+\theta.C_{y(2)}^2\right)-\bar{Y}^2\left[\lambda.C_y^2\left(1-\rho_{xy}^2\right)+\theta.C_{y(2)}^2\right]\geq 0$$

$$=\bar{Y}^2\lambda C^2C_x^2\geq 0$$

$MSE\ (t_{R1})-MSE\ \left(t^*\right)\geq 0$.

$$=\bar{Y}^2\left[\lambda\left(C_y^2+C_x^2-2\rho_{yx}C_yC_x\right)+\theta C_{y(2)}^2\right]-\bar{Y}^2\left[\lambda.C_y^2\left(1-\rho_{xy}^2\right)+\theta.C_{y(2)}^2\right]\geq 0$$

$$=\bar{Y}^2\lambda\left(C_x-CC_x\right)^2\geq 0$$

$MSE\ (t_{ER1})-MSE\ \left(t^*\right)\geq 0$

$$=\bar{Y}^2\left[\lambda\left(C_y^2+\frac{1}{4}C_x^2-\rho_{xy}C_xC_y\right)+\theta\left(C_{y(2)}^2\right)\right]-\bar{Y}^2\left[\lambda.C_y^2\left(1-\rho_{xy}^2\right)+\theta.C_{y(2)}^2\right]\geq 0$$

$$=\bar{Y}^2\lambda\frac{C_x^2}{4}\left(1-2C\right)^2\geq 0$$

4.2 Case II

$V\left(t_0\right)-MSE\ \left(t^{**}\right)\geq 0$.

$$=\bar{Y}^2\left(\lambda C_y^2+\theta.C_{y(2)}^2\right)-\bar{Y}^2\left[\left(\lambda C_y^2+\theta.C_{y(2)}^2\right)-\frac{\left(\lambda CC_x{}^2+\theta.C_{(2)}C_{x(2)}^2\right)^2}{\left(\lambda C_x{}^2+\theta.C_{x(2)}^2\right)}\right]\geq 0$$

$$=\frac{\left(\lambda CC_x^2+\theta C_{(2)}C_{x(2)}^2\right)}{\left(\lambda C_x^2+\theta C_{x(2)}^2\right)}\geq 0$$

$MSE\left(t_{R2}\right)-MSE\left(t^{**}\right)\geq 0$.

$$=\left\{\begin{array}{l}\lambda\bar{Y}^2\left(C_x^2+C_y^2-2\rho_{xy}C_xC_y\right)+\theta\bar{Y}^2\left(C_{x(2)}^2+C_{y(2)}^2-2\rho_{xy(2)}C_{x(2)}C_{y(2)}\right)\\-\bar{Y}^2\left[\left(\lambda C_y^2+\theta.C_{y(2)}^2\right)-\frac{\left(\lambda CC_x{}^2+\theta.C_{(2)}C_{x(2)}^2\right)^2}{\left(\lambda C_x{}^2+\theta.C_{x(2)}^2\right)}\right]\end{array}\right\}\geq 0$$

$$=\bar{Y}^2\left[\left(\lambda C_x^2+\theta C_{x(2)}^2\right)-\left(\lambda CC_x^2+\theta C_2C_{x(2)}^2\right)\right]^2\geq 0$$

$MSE\left(t_{ER2}\right)-MSE\left(t^{**}\right)\geq 0$.

$$=\left\{\begin{array}{l}\bar{Y}^2\left[\lambda\left(C_y^2+\frac{1}{4}C_x^2-\rho_{xy}C_xC_y\right)+\theta\left(C_{y(2)}^2+\frac{C_{x(2)}^2}{4}-\rho_{xy(2)}C_{x(2)}C_{y(2)}\right)\right]\\-\bar{Y}^2\left[\left(\lambda C_y^2+\theta.C_{y(2)}^2\right)-\frac{\left(\lambda CC_x{}^2+\theta.C_{(2)}C_{x(2)}^2\right)^2}{\left(\lambda C_x{}^2+\theta.C_{x(2)}^2\right)}\right]\end{array}\right\}\geq 0$$

$$=\bar{Y}^2\left[\left(\lambda\frac{C_x^2}{2}+\theta\frac{C_{x(2)}^2}{2}\right)-\left(\lambda CC_x^2+\theta C_2C_{x(2)}^2\right)\right]^2\geq 0$$

V. Empirical Study

We have used two data sets.
Population I: Khare and Sinha (2004).
y: weight in kg of children,

x: chest circumference in cm of children
Population II: Satici and Kadilar (2011).
y: number of successful students,
x: number of teachers.

Table 3: *Data Statistics*

Population I [Khare and Sinha (2004)]		Population II[Satici and Kadilar (2011)]	
$N = 95$	$N_1 = 71$	$N = 261$	$N_1 = 196$
$n = 35$	$N_2 = 24$	$n = 90$	$N_2 = 65$
$\bar{X} = 55.86$	$C_{xy(2)} = 0.00395$	$S_y = 415.1944$	$C_x = 1.7595$
$\bar{Y} = 19.5$	$\rho_{xy(2)} = 0.729$	$C_y = 1.8654$	$\rho_{xy} = 0.9705$
$C_x = 0.05860$	$C_{x(2)} = 0.05402$	$\bar{Y} = 222.57$	$S_{x(2)} = 376.48$
$\rho_{xy} = 0.85$	$C_{xy} = 0.00776$	$\bar{X} = 306.43$	$C_{x(2)} = 1.2285$
$C_y = 0.15613$	$C_{y(2)} = 0.12075$	$S_x = 539.1722$	$\rho_{xy(2)} = 0.9733$

Table 4: *The MSE and PRE's of the population I w.r.t unbiased estimator for case I*

Estimators	$h = 2$		$h = 3$		$h = 4$	
	MSE	PRE	MSE	PRE	MSE	PRE
t_0	0.2015	100	0.2395	100	0.2813	100
t_{R1}	0.1293	155.81	0.1593	150.31	0.1960	143.52
t_{ER1}	0.1594	126.41	0.1914	125.13	0.2264	124.21
$t^{*}_{(prop)}$	**0.0791**	**254.43**	**0.1172**	**204.27**	**0.1574**	**178.70**

Table 5: *The MSE and PRE's of population I w.r.t unbiased estimator for case II*

Estimators	$h = 2$		$h = 3$		$h = 4$	
	MSE	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**
t_0	0.2015	100	0.2395	100	0.2813	100
t_{R1}	0.1363	147.79	0.1743	137.35	0.2152	130.66
t_{ER1}	0.1654	121.81	0.2023	117.73	0.2447	114.93
$t^{*}_{(prop)}$	0.0832	242.16	0.1022	234.31	0.1228	228.92

Table 6: The MSE and PRE's population II w.r.t unbiased estimator for case I

Estimators	$h = 2$		$h = 3$		$h = 4$	
	MSE	PRE	MSE	PRE	MSE	PRE
t_0	1459.59	100	1664.28	100	1868.98	100
t_{R1}	278.57	523.95	483.27	344.37	687.96	271.67
t_{ER1}	589.96	247.40	796.66	209.43	999.36	187.02
$t^{*}_{(prop)}$	275.92	529.00	480.92	346.28	685.32	272.72

***Table 7:** The MSE and PRE's of population II w.r.t unbiased estimator for case II*

Estimators	$h = 2$		$h = 3$		$h = 4$	
	MSE	PRE	MSE	PRE	MSE	PRE
t_0	1459.59	100	1664.28	100	1868.98	100
t_{R2}	84.90	1719.15	95.92	1735.01	106.94	1747.60
t_{ER2}	441.20	330.82	497.13	334.38	553.06	337.93
t^{**}	82.77	1763.31	94.07	1769.19	105.21	1776.46

V. Conclusion

In the context of survey sampling and estimation, the ratio-cum-dual of exponential ratio estimator has been proposed as a valuable approach, particularly in cases involving non-response. This innovative method combines elements of the traditional ratio estimator and dual to improve estimation accuracy. To assess the performance of this novel estimator, an essential step is to compute the Mean Squared Error (MSE) expression. This metric provides insights into the estimator's precision and reliability in estimating population parameters. To further evaluate the efficacy of the suggested estimator, both theoretical and empirical analyses have been conducted. Theoretical assessments involve rigorous mathematical proofs and calculations, while empirical evaluations utilize real-world data to validate the estimator's practical utility. The synergy of these two evaluation approaches ensures a comprehensive understanding of the estimator's competence. Upon scrutinizing the results presented in the accompanying table, a compelling conclusion emerges. It is evident that the proposed estimator surpasses the existing estimators found in the literature in terms of efficiency. This conclusion is drawn from a careful consideration of the MSE values, which indicate that the proposed estimator consistently provides more accurate and precise estimates, even in the presence of non-response. Therefore, this study contributes to the field by introducing a superior estimator for survey sampling, offering improved accuracy and reliability in estimating population parameters.

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