RT&A, No 1 (77)
Volume 19, March 2024

A STUDY ON PARTIALLY ACCELERATED LIFE TEST MODEL FOR GENERALIZED INVERSE RAYLEIGH DISTRIBUTION UNDER ADAPTIVE TYPE-II PROGRESSIVE HYBRID CENSORING

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Abstract

Modeling and examination of lifetime phenomena are the main aspects of statistical work in a wide variety of scientific and industrial areas. The area of lifetime information analysis has developed and extended quickly with respect to methodology, theory, and fields of applications. The point and interval maximum-likelihood estimations of generalized inverse Rayleigh distribution (GIRD) parameters and the acceleration factor are considered in this work. The estimation procedure is carried out for a partially accelerated step-stress model under adaptive Type-II progressive hybrid censored data. The biases and the mean square errors of the maximum-likelihood estimators are computed to assess their performances in the occurrence of censoring developed in this study through a Monte Carlo simulation study.

Keywords: Partially accelerated life test, Generalized inverse Rayleigh distribution, Newton Raphson method, Adaptive Type-II Progressive Hybrid Censoring, Simulation study.

I. Introduction

The Partially accelerated life tests (PALTs) are applied by reliability practitioners profitably to calculate approximately the acceleration factor and thus gathering the accelerated information to ordinary surroundings. In a PALT, objects are experience in both regular and accelerated circumstances. Progressive-stress, step-stress, and constant-stress are the three types of PALTs. The assessment performed under these kinds of stress is called accelerated life test (ALT) or partially accelerated life test (PALT). In ALT, the components are placed under stress to obtain additional failures in a tiny time. The key postulation in ALT is that the mathematical model connecting the life span of the component and the stress is acknowledged or can be assumed. In various situations, such a model is neither identified nor assumed. That is, ALT information can't be gathered to ordinary use circumstances. So, in such situations, PALT is a more appropriate choice to be applied to calculate the statistical model's parameters. There are three types of PALT i.e. Constant stress PALT (CSPALT), step stress PALT (SSPALT) and progressive stress PALT (PSPALT).

In SSPALT, the test component initiates at ordinary use circumstances for a particular period. If it works successfully at that period, it is placed in stress. Stress continually increases until the examination components are unsuccessful or the examination is ended based on a confident censoring scheme. Rao [1] indicates that the step-stress technique can reduce the investigating period and save many human resources, substances, sources, and cash. In particular, SSPALT should be applied for a trustworthiness study to save time and wealth mainly, when the trial components are of superior reliability and have significant models.

In the present work, we combine an adaptive Type-II progressive hybrid censoring scheme with the step-stress PALT to obtain a step-stress PALT under adaptive Type-II progressive hybrid censored scheme with the GIRD as a lifetime model.

As pointed out by Lin et al. [2], many conditions in existence analysis and reliability research are available, in which components are lost or removed in the investigation prior to failure. The practitioner may not gainful idea about the failure times for all the elements under study. The information detected from this research is called censored information, and the scheme is called censoring scheme. The frequently applied censoring schemes are the Type-I and Type-II censoring scheme, for more details one may refer to Balakrishnan and Ng [3]. Many studies have discussed the hybrid censoring plan, which is a combination of Type-I and Type-II censoring schemes, with the associated statistical inference, see for example, Epstein [4], Balakrishnan and Kundu [5] Childs et al. [6], Gupta and Kundu [7], Kundu [8], Deyand Pradhan [9], and Salah el al. [10] among others. Due to the less flexibility of removing the components from the testing at any position other than the starting point, another censoring scheme was applied, which is called progressively Type-II hybrid censoring schemes. Table 1 summarizes a recent literature review of the different censoring schemes.

Table 1: *Related work to the proposed problem*

Author(s) Name	Method	Scheme	Failure Model	Strategy
Abdel-Ghaly et al. [11]	SSALT	Type-II	Pareto distribution	-
Alam et al. [12], Alam and Aquil [13]	CSPALT, SSPALT	Progressive censoring, Adaptive Type-II progressive hybrid censoring	Generalized inverted exponential distribution, Exponentiated Pareto distribution	Maintenance service policy
Abd El-Raheem [14, 15]	CSALT, CSALT	Complete sampling, Type-I censoring	Extension of the exponential distribution	-
Balakrishnan et al. [16]	SSALT	Type-II censoring	Exponential distribution	-
Xiaolin et al. [17]	SSPALT	Progressive Type-II hybrid censoring	Modified Weibull distribution	-
Alam and Aquil [18]	SSPALT	Progressive censoring	Generalized inverted exponential distribution	Maintenance service policy
Alam et al. [19]	SSPALT	Progressive censoring	Power function distribution	Maintenance service policy

Author(s) Name	Method	Scheme	Failure Model	Strategy
Ismail [20, 21]	SSPALT, SSPALT	Adaptive Type-II progressively hybrid censoring, Adaptive Type-I progressively hybrid censoring	Weibull distribution, Weibull distribution	-
Ismail [22]	SSPALT	Type-I progressive hybrid censoring	Weibull distribution	-
El-Sagheer and Ahsanullah [23]	SSPALT	Type-II-Progressive censoring	Lomax distribution	-
Zhou et al. [24]	SSALT	-	Copula function	Competing risk
Srivastava and Mittal [25]	SSPALT	Type-I and Type-II censorings	truncated logistic distribution	-
Proposed Work	SSPALT	Adaptive Type-II progressive hybrid censoring	Generalized Inverse Rayleigh distribution	-

The proposed study is motivated by two factors. The first aims to establish explicit formulas for the likelihood and log-likelihood functions under an adaptive Type-II progressive hybrid censoring scheme. The second is to apply a Monte Carlo simulation study to estimate the performance of the model parameter estimators with an adaptive Type-II progressive hybrid censoring scheme in terms of biases and mean squared errors. The authors presented a study on SSPALT utilizing adaptive Type-II progressive hybrid censoring where the lifespan of test items follows the two parameters GIRD in this work.

The uniqueness of this work stems from the fact that no earlier research has been conducted in this area using the proposed censoring scheme for two parameters GIRD.

The present paper is arranged as; the model illustration and test procedure are presented in section 2. The point and interval estimation is presented in section 3. A simulation study is carried out in section 4 to check the performance of model parameters. The result based on the proposed problem and conclusion is provided in section 5. The real-life implementation of the proposed work is shown in section 6.

II. Model Illustration and Test Process

The GIRD is one of the most beneficial and important distribution within the inverted scale distributions. It has been considered as an appropriate failure model in life testing and reliability analysis, for more details about GIRD one may refer to Fatima et al. [26]. The GIRD has lots of uses in the area of reliability theories. The Probability density function (pdf) of GIRD presents by the following equation (1);

$$f(y,\lambda,\theta)=\frac{2\lambda}{\theta^2 y^3}e^{-(\theta y)^{-2}}\left[1-e^{-(\theta y)^{-2}}\right]^{\lambda-1};y,\lambda,\theta>0 \qquad (1)$$

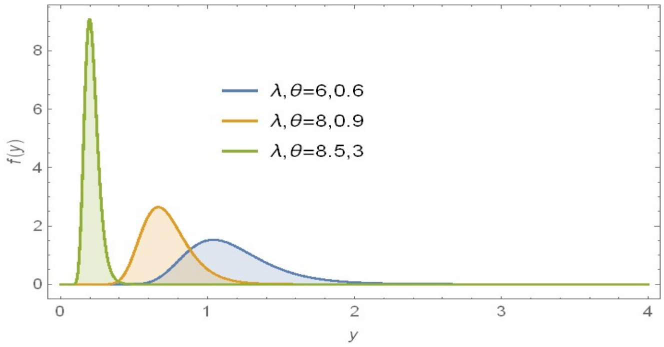

Figure 1: *Pdf pattern of GIRD*

The cumulative density function (cdf) of GIRD presents by the following equation (2);

$$F(y,\lambda,\theta) = 1 - \left[1 - e^{-(\theta y)^{-2}}\right]^{\lambda} ; y,\lambda,\theta > 0 \tag{2}$$

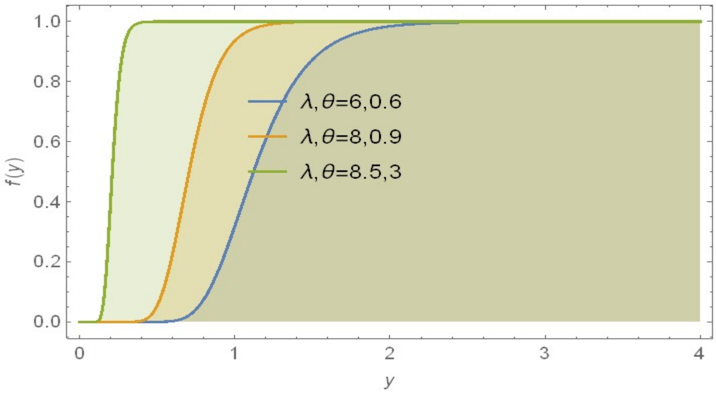

Figure 2: *Cdf pattern of GIRD*

The reliability function of GIRD is given by

$$S(y,\lambda,\theta) = \left[1 - e^{-(\theta y)^{-2}}\right]^{\lambda} \tag{3}$$

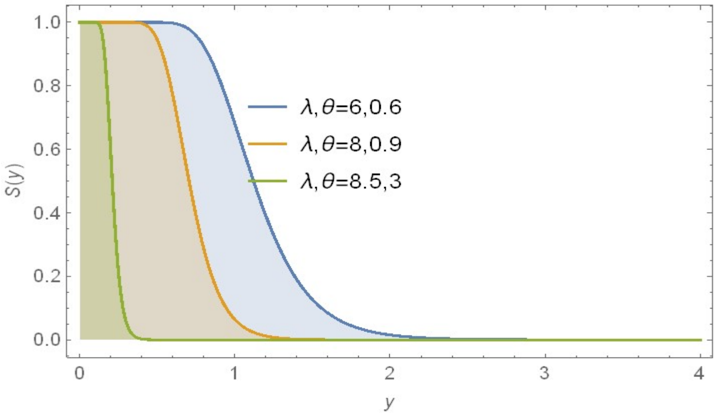

Figure 3: *Reliability pattern of GIRD*

The hazard rate function of GIRD is presented by the following equation:

$$h(y,\lambda,\theta)=\frac{2\lambda e^{-(\theta y)^{-2}}}{\theta^2 y^3\left[1-e^{-(\theta y)^{-2}}\right]} \tag{4}$$

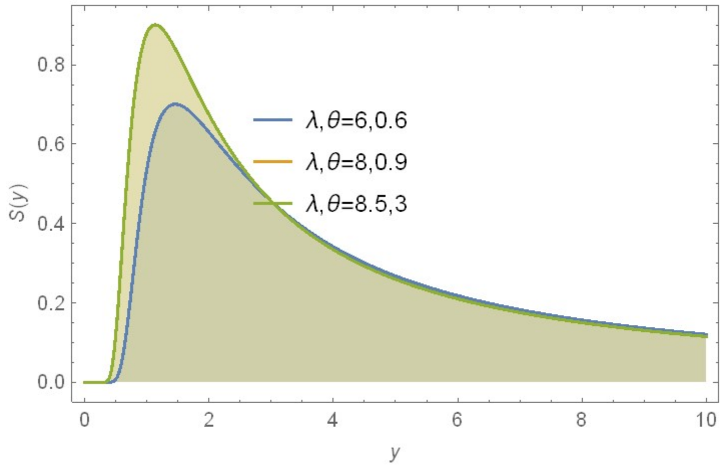

Figure 4: *Hazard pattern of GIRD*

Figure 1 shows that the Pdf of GIRD is positively skewed, while the shape of Cdf is increasing as shown in Figure 2. Figure 3 and 4 show the Reliability and Hazard shapes of GIRD for different values of λ *and* θ. The figure 3 shows that the reliability function of GIRD is downward skewed for different values of λ,θ, it becomes flatter and flatter as the shape parameter is increased. The behavior of instantaneous failure rate of the GIRD has an upside-down bathtub shape curve.

The unimodel hazard rate function shows the possibility of decreasing failures as soon as the product has passed a particular moment, during some kind of stress on that product. Thus, the GIRD shows excellent statistical performance and can be a better model to fit real data in many scientific fields.

Kumar and Garg [27] handled an estimation of parameters of GIRD based on randomly censored trials. Bakoban and Abubaker [28] presented the assumption of GIRD with real information applications. Bakoban and Abubaker [29] also proposed a study on the estimation of parameters of GIRD using progressive Type-II censoring.

Under SSPALT the pdf of Y can be written as:

$$f(y)=\begin{cases} 0, & y\le 0 \\ f_1(y)=f(y,\lambda,\theta), & 0<y\le\tau \\ f_2(y), & y>\tau \end{cases} \tag{5}$$

where, $f_2(y)=\dfrac{2\lambda\beta}{\theta^2 y^3}e^{-(\theta(\tau+\beta(y-\tau)))^{-2}}\left[1-e^{-(\theta(\tau+\beta(y-\tau)))^{-2}}\right]^{\lambda-1};y,\beta,\lambda,\theta>0$

$f_2(y)$ is attained by applying variable transformation that is projected by DeGroot and Goel [30] and the procedure is given in the following equation:

$$Y = \begin{cases} T \quad if\ T \le \tau \\ \tau + \beta^{-1}(T-\tau) \quad if\ T > \tau \end{cases} \tag{6}$$

In equation (6), T is the life span of the article in normal operating circumstances, while τ is the time at which stress is changed (i.e., stress change time) and β is the acceleration factor.

In life testing analysis, the Type-I and Type-II censoring ideas are the mainly well-liked and widespread plans. These schemes explain as follows, suppose there are n apparatus situate on the investigation, then under Type-I censoring, the investigation carries on until reached a pre-specified point κ . But in Type-II censoring, the investigation carries on until a reached a pre-specified quantity of components $m(\le n)$. We cannot take out components from the live experimentation at any moment and any position except the opening point in these two schemes. This is the key weakness of these two schemes. To remove this weakness, progressive Type-II censoring or progressively Type-II hybrid censoring another censoring comes in light. Hybrid censoring is the mixture of two censorings, i.e., Type-I and Type-II censorings. So, the progressive Type-II censoring scheme is described as follows;

In the progressive Type-II censoring scheme environment, the reliability practitioner presets the number of components to be unsuccessful (say m) out of the total number of components n, placed under analysis. At the moment, when initial failure happens, R_1 components among $n-1$ leftover (surviving) components are randomly taken off from the life analysis. Similarly, R_2 of the leftover $n-2-R_1$ examination components are eliminated from the analysis at the moment of the second failure. This practice continues until the mth failure is reached. All the leftover $R_m = n-m-R_1-R_2-...-R_m-1$ surviving examination components are eliminated from the examination at this point. The R_i units are situating before the work. The assumption related to progressive censoring and progressively is proposed by many authors such as Balakrishnan [31], Balakrishnan and Agrawala [32], etc.

If a life examination experiment stops randomly at a moment $\min(Y_{m:m:n}, \varphi)$, where $1 \le m \le n$, $0 < \varphi < \infty$ are determined prior to the experiment, and $Y_{1:m:n} \le Y_{2:m:n} \le ... \le Y_{m:m:n}$ are the ordered lifetimes consequential from the study, then $(R_1, R_2, ..., R_m)$ are called progressively hybrid censoring (PHC) scheme. If the mth progressive censored unit occurs before the point φ $(\varphi > Y_{m:m:n})$, then the investigation ends at the moment $Y_{m:m:n}$. Else, the examination will end at the moment φ , where $Y_{j:m:n} < \varphi < Y_{j+1:m:n}$, hence all the leftover $(n - \sum_{i=1}^{j} R_i - j)$ existing units are censored at φ . Here j is a random variable and denotes the number of unsuccessful units up to φ . The reliability engineer comes with the problem of different censoring schemes, and the practitioner may observe a tiny test size (even it is equal to zero). So, this is not possible to happen with standard suggestion procedures to obtain good results. To remove such type of drawback, another censoring comes in light called adaptive censored samples. This was commenced by Ng et al. [33].

In this scheme, the observed quantity of failed units m is prefixed and the investigation moment is unlocked to run over the moment κ . The investigation will continue along with pre-determined progressive censoring schemes $(R_1, R_2, R_3, ..., R_m)$ if $Y_{m:m:} < \varphi$, otherwise, the ongoing units (on work units), which following the $(j+1)th$ to $(m-1)th$ experimental failures, are not uninvolved from the analysis. All the surviving units $R_m = n - m - \sum_{i=1}^{j} R_i$ are taken back from the test at the stage $Y_{m:m:n}$ if m observed failures are obtained, i.e. $R_{j+1} = \ldots = R_{m-1} = 0$. The progressive Type-II censoring takes place if $n \to \infty$ and the conventional Type-II censoring takes place, if $n \to 0$.

If the practitioner is free to vary the value κ, then this kind of censoring proposal is known as an adaptive progressively Type-II hybrid censoring (APHCT-II) scheme. This variation in κ is completed to regulate the most advantageous of pointed investigation time and a better opportunity of supervising various failures.

III. Estimation Process

Let $Y_1, Y_2, \ldots, Y_n$ be a life span of n independently and identically distributed units following the GIRD. $y_{1:m:n} < y_{2:m:n} < \ldots y_{n_u:m:n} \le \tau < y_{n_u+1:m:n} \le \varphi < y_{J+1:m:n} < \ldots < y_{m:n:n}$ are completely observed (ordered) lifetimes. Both point and confidence interval estimation is presented in the following subsections:

I. Point Estimation

In this section, we used the maximum likelihood estimation method. Under the SSPALT the likelihood function with APHCT-II for GIRD based on m observed lifetime data takes the following form;

$$L(\lambda,\theta,\beta) \propto \prod_{i=1}^{m} f_1(y_{i:m:n}) f_2(y_{i:m:n}) \prod_{i=1}^{J} \left(S_1(\tau)\right)^{R_i} \left(S_2(y_{m:m:n})\right)^{(n-m-\sum_{i=1}^{J} R_i)} \tag{7}$$

where, $S(\tau) = \left[1 - e^{-(\theta\tau)^{-2}}\right]^{\lambda}$, $S(y_{m:m:n}) = \left[1 - e^{-(\theta(\tau+\beta(y_{m:m:n}-\tau)))^{-2}}\right]^{\lambda}$

$\ln L = \ln L(\lambda,\theta,\beta)$

n_u is the amount of components that are unsuccessful in the normal circumstance and n_a is the number of components that are unsuccessful in accelerated circumstance.

The log-likelihood function takes the following form;

$$\begin{aligned}\ln L = & -\sum_{i=1}^{m} (\theta y_i)^{-2} + (\lambda - 1)\sum_{i=1}^{m} \ln\left[1 - e^{-(\theta y_i)^{-2}}\right] - 3\sum_{i=1}^{m} \ln\left[\tau + \beta(y_i - \tau)\right] m \left[\ln\left(\frac{4\lambda^2\beta}{\theta^4}\right)\right] \\ & + \sum_{i=1}^{J} \lambda R_i \ln\left[1 - e^{-(\theta\tau)^{-2}}\right] - 3\sum_{i=1}^{m} \ln(y_i) - \sum_{i=1}^{m} \left(\theta(\tau + \beta(y_i - \tau))\right)^{-2} \\ & + \sum_{i=1}^{J} \lambda\left(n - m - \sum_{i=1}^{J} R_i\right) \ln\left[1 - e^{-(\theta(\tau+\beta(y_{m:m:n}-\tau)))^{-2}}\right] + (\lambda - 1)\sum_{i=1}^{m} \ln\left[1 - e^{-(\theta(\tau+\beta(y_i-\tau)))^{-2}}\right]\end{aligned} \tag{8}$$

To obtain maximum likelihood estimates (MLEs) of model parameters and acceleration factor, we differentiate the above equation for parameters ϕ, λ and β equating to zero.

where, $\sigma_i = \tau + \beta(y_i - \tau)$, $\sigma_{m:m:n} = \tau + \beta(y_{m:m:n} - \tau)$, $J = n_u + n_a$

$$\frac{\partial \ln L}{\partial \lambda} = \left(\frac{2m}{\lambda}\right) + \sum_{i=1}^{m} \ln\left[1 - e^{-(\theta y_i)^{-2}}\right] + \sum_{i=1}^{m} \ln\left[1 - e^{-(\theta \sigma_i)^{-2}}\right] + \sum_{i=1}^{J} R_i \ln\left[1 - e^{-(\theta \tau)^{-2}}\right] + \sum_{i=1}^{J}\left(n - m - \sum_{i=1}^{J} R_i\right)\ln\left[1 - e^{-(\theta \sigma_m)^{-2}}\right] = 0 \tag{9}$$

$$\frac{\partial \ln L}{\partial \theta} = -\frac{4m}{\theta} + 2\sum_{i=1}^{m} \theta^{-3}(y_i)^{-2} - 2(\lambda-1)\theta^{-3}\sum_{i=1}^{m} \frac{e^{-(\theta y_i)^{-2}}\, y_i^{-2}}{\left(1 - e^{-(\theta y_i)^{-2}}\right)} - \sum_{i=1}^{m} (\sigma_i)^{-2}\theta^{-3} - 2(\lambda-1)\theta^{-3}\sum_{i=1}^{m} \frac{(\sigma_i)^{-2} e^{-(\theta\sigma_i)^{-2}}}{1 - e^{-(\theta\sigma_i)^{-2}}} - 2\sum_{i=1}^{J} \lambda R_i \frac{e^{-(\theta\tau)^{-2}}\, \tau^{-2}\theta^{-3}}{1 - e^{-(\theta\tau)^{-2}}} - 2\sum_{i=1}^{J} \lambda\theta^{-3}\left(n - m - \sum_{i=1}^{J} R_i\right)\left[\frac{e^{-(\theta\sigma_m)^{-2}}\,(\sigma_m)^{-2}}{1 - e^{-(\theta\sigma_m)^{-2}}}\right] = 0 \tag{10}$$

$$\frac{\partial \ln L}{\partial \beta} = 2\theta\sum_{i=1}^{m} \frac{(\sigma_i - \tau)(\theta\sigma_i)^{-3}}{\beta} - 2(\lambda-1)\theta\sum_{i=1}^{m} \frac{(\sigma_i - \tau)(\theta\sigma_i)^{-3} e^{-(\theta\sigma_i)^{-2}}}{\beta\left(1 - e^{-(\theta\sigma_i)^{-2}}\right)} - \sum_{i=1}^{J} \lambda\theta\left(n - m - \sum_{i=1}^{J} R_i\right)\frac{(\sigma_m - \tau)(\theta\sigma_m)^{-3} e^{-(\theta\sigma_m)^{-2}}}{\beta\left(1 - e^{-(\theta\sigma_m)^{-2}}\right)} + \frac{m}{\beta} - 3\sum_{i=1}^{m} \frac{(\sigma_i - \tau)}{\beta\sigma_i} = 0 \tag{11}$$

It is an impossible task to solve the above equations manually. Hence, an iterative procedure called the Newton Raphson technique is used to get the MLE of the model parameters and acceleration factor.

II. Interval Estimation

The interval estimation for the model parameters and acceleration factor based on APHCT-II is obtained. The asymptotic distribution of MLE λ, θ and β takes the following form presented in the following equations.

$$\left((\hat{\lambda} - \lambda), (\hat{\theta} - \theta)(\hat{\beta} - \beta)\right) \rightarrow N\left(0, I^{-1}(\lambda, \theta, \beta)\right) \tag{12}$$

The above procedure is suggested by Miller and Nelson [34]. $I^{-1}(\lambda, \theta, \beta)$ denotes the variance-covariance matrix of λ, θ and β. The 3×3 matrix I^{-1} which is approximately equal to I and the elements $I_{ij}^{-1}, (\lambda, \theta, \beta)$, $i = 1,2,3$; $j = 1,2,3$, closed to $I_{ij}(\hat{\lambda}, \hat{\theta}, \hat{\beta})$, under the APHCT-II are given as.

where, $\frac{\partial^2 \ln L}{\partial \lambda^2} = -\left(\frac{2m}{\lambda^2}\right)$

$$\frac{\partial^2 \ln L}{\partial\lambda\partial\theta} = -2\theta^{-3}\sum_{i=1}^{m}\frac{y_i^{-2}e^{-(\theta y_i)^{-2}}}{\left(1-e^{-(\theta y_i)^{-2}}\right)} - 2\theta^{-3}\sum_{i=1}^{m}\frac{e^{-(\theta\sigma_i)^{-2}}(\sigma_i)^{-2}}{\left(1-e^{-(\theta\sigma_i)^{-2}}\right)} - 2\theta^{-3}\sum_{i=1}^{J}\left(n-m-\sum_{i=1}^{J}R_i\right)\frac{(\sigma_m)^{-2}e^{-(\theta\sigma_m)^{-2}}}{\left(1-e^{-(\theta\sigma_m)^{-2}}\right)}$$
$$-2\theta^{-3}\tau^{-2}\sum_{i=1}^{J}R_i\frac{e^{-(\theta\tau)^{-2}}}{\left(1-e^{-(\theta\tau)^{-2}}\right)}$$

$$\frac{\partial^2 \ln L}{\partial\theta^2} = \frac{4m}{\theta^2} - 6\sum_{i=1}^{m}\theta^{-4}(y_i)^{-2} - 2(\lambda-1)\theta^{-3}\sum_{i=1}^{m}\frac{e^{-(\theta y_i)^{-2}}y_i^{-2}}{\left(1-e^{-(\theta y_i)^{-2}}\right)}\left[-3\theta^{-1}+2\theta^{-3}y_i^{-2}-\frac{2y_i^{-2}\theta^{-3}e^{-(\theta y_i)^{-2}}}{1-e^{-(\theta y_i)^{-2}}}\right]$$
$$+3\sum_{i=1}^{m}\left((\sigma_i)^{-2}\theta^{-4} - 2(\lambda-1)\theta^{-3}\sum_{i=1}^{m}\frac{(\sigma_i)^{-2}e^{-(\theta\sigma_i)^{-2}}}{1-e^{-(\theta\sigma_i)^{-2}}} - 2\sum_{i=1}^{J}\lambda R_i\frac{e^{-(\theta\tau)^{-2}}\tau^{-2}\theta^{-3}}{1-e^{-(\theta\tau)^{-2}}}\times\right.$$
$$\left[-3\theta^{-1}+2\theta^{-3}\tau^{-2}-\frac{2\tau^{-2}\theta^{-3}e^{-(\theta\tau)^{-2}}}{1-e^{-(\theta\tau)^{-2}}}\right] - 2\sum_{i=1}^{J}\lambda\theta^{-3}\left(n-m-\sum_{i=1}^{J}R_i\right)\times$$
$$\left[\frac{e^{-(\theta\sigma_m)^{-2}}(\sigma_m)^{-2}}{1-e^{-(\theta\sigma_m)^{-2}}}\right]\left[-3\theta^{-1}+2\theta^{-3}(\sigma_m)^{-2}-\frac{2\theta^{-3}e^{-(\theta\sigma_m)^{-2}}(\sigma_m)^{-2}}{1-e^{-(\theta\sigma_m)^{-2}}}\right]$$

$$\frac{\partial^2 \ln L}{\partial\theta\partial\beta} = -2\sum_{i=1}^{m}(\sigma_i)^{-3}(\sigma_i-\tau)\beta^{-1}\theta^{-3} - 2(\lambda-1)\theta^{-3}\sum_{i=1}^{m}\frac{(\sigma_i)^{-2}e^{-(\theta\sigma_i)^{-2}}}{1-e^{-(\theta\sigma_i)^{-2}}}\times$$
$$\left[-2\frac{(\sigma_i-\tau)}{(\sigma_i)\beta}+2\theta^{-2}(\sigma_i-\tau)\beta^{-1}(\theta\sigma_i)^{-3}+\frac{2\theta^{-2}(\sigma_i-\tau)\beta^{-1}(\sigma_i)^{-3}e^{-(\theta\sigma_i)^{-2}}}{1-e^{-(\theta\sigma_i)^{-2}}}\right] - 2\sum_{i=1}^{J}\lambda\theta^{-3}\left(n-m-\sum_{i=1}^{J}R_i\right)$$
$$\left[-2\frac{(\sigma_m-\tau)}{(\sigma_m)\beta}+2\theta^{-2}(\sigma_m-\tau)\beta^{-1}(\theta\sigma_m)^{-3}+\frac{2\theta^{-2}\sigma_m\beta^{-1}(\sigma_m)^{-3}e^{-(\theta\sigma_m)^{-2}}}{1-e^{-(\theta\sigma_m)^{-2}}}\right]\left[\frac{e^{-(\theta\sigma_m)^{-2}}(\sigma_m)^{-2}}{1-e^{-(\theta\sigma_m)^{-2}}}\right]$$

$$\frac{\partial^2 \ln L}{\partial\beta^2} = -\frac{m}{\beta^2} + 3\sum_{i=1}^{m}\frac{(\sigma_i-\tau)^2}{\beta^2(\sigma_i)^2} - 6\theta^{-2}\sum_{i=1}^{m}(\sigma_i)^{-4}\beta^{-2}(\sigma_i-\tau)^2 - 2(\lambda-1)\theta\times$$
$$\sum_{i=1}^{m}\frac{(\sigma_i-\tau)\beta^{-1}(\theta\sigma_i)^{-3}e^{-(\theta\sigma_i)^{-2}}}{1-e^{-(\theta\sigma_i)^{-2}}}\left[2\theta^{-2}(\sigma_i-\tau)\beta^{-1}(\sigma_i)^{-3} - \frac{3(\sigma_i-\tau)}{\beta\sigma_i} - \frac{2\theta^{-2}(\sigma_i)^{-3}(\sigma_i-\tau)\beta^{-1}e^{-(\theta\sigma_i)^{-2}}}{1-e^{-(\theta\sigma_i)^{-2}}}\right]$$
$$-\sum_{i=1}^{J}\lambda\theta\left(n-m-\sum_{i=1}^{J}R_i\right)\frac{(\sigma_m-\tau)\beta^{-1}(\theta\sigma_m)^{-3}e^{-(\theta\sigma_m)^{-2}}}{1-e^{-(\theta\sigma_m)^{-2}}}\times$$
$$\left[-\frac{3(\sigma_m-\tau)\beta^{-1}}{\sigma_m}+2\theta^{-2}\beta^{-1}\sigma_m(\sigma_m)^{-3}-\frac{2\theta^{-2}(\sigma_m)^{-3}(\sigma_m-\tau)\beta^{-1}e^{-(\theta\sigma_m)^{-2}}}{1-e^{-(\theta\sigma_m)^{-2}}}\right]$$

$$\frac{\partial^2 \ln L}{\partial\beta\partial\lambda} = -2\theta\sum_{i=1}^{m}\frac{(\sigma_i-\tau)(\theta\sigma_i)^{-3}e^{-(\theta\sigma_i)^{-2}}}{\beta\left(1-e^{-(\theta\sigma_i)^{-2}}\right)} - \sum_{i=1}^{J}\theta\left(n-m-\sum_{i=1}^{J}R_i\right)\frac{(\sigma_m-\tau)(\theta\sigma_m)^{-3}e^{-(\theta\sigma_m)^{-2}}}{\beta\left(1-e^{-(\theta\sigma_m)^{-2}}\right)}$$

The $100(1-\pi)\%$ approximated two-sided limits of confidence for parameters λ, θ and β

are given as:

$$\hat{\lambda} \pm Z_{\pi/2} \sqrt[-1]{I_{11}^{-1}(\hat{\lambda},\hat{\theta},\hat{\beta})}, \ \hat{\theta} \pm Z_{\pi/2} \sqrt[-1]{I_{22}^{-1}(\hat{\lambda},\hat{\theta},\hat{\beta})} \text{ and } \hat{\beta} \pm Z_{\pi/2} \sqrt[-1]{I_{33}^{-1}(\hat{\lambda},\hat{\theta},\hat{\beta})}$$

IV. Simulation study

Since it is theoretically not achievable to evaluate the presentation of different censorings for different values of model parameters. For this job, many software and simulation techniques are used. In this segment, the Monte-Carlo simulation procedure is applied to evaluate the efficiency of the MLEs. This efficiency is recorded based on the mean squared error (MSE) and bias of the MLEs. The following three progressive censorings are chosen for this assignment;

- Scheme (I) $R_1 = R_2 = R_3 = ... = R_{m-1}, \ R_m = n - m$
- Scheme (II) $R_1 = n - m, \ R_2 = R_3 = R_4 ... = 0$
- Scheme (III) $R_1 = R_2 = R_3 = ... = R_{m-1}, \ R_m = n - 2m + 1$

For this task, 1000 simulation-based on MSEs and biases are estimated. The steps for this procedure are;

- The values of parameters $n, m, \tau, \varphi, \ \lambda, \theta$ and β are specified first.
- After selecting the parameter values, we generate a random sample from GIRD of size n by the inverse CDF method in both situations (regular and accelerated circumstances).
- Generate the PHC sample for the parameters $n, m, \tau, \varphi, \lambda, \theta$ and β by using the technique discussed in equation (6).
- The sample data set for the APHCT-II is;
  $y_{1:m:n} < x_{y:m:n} < ... y_{n_u:m:n} \le \tau < y_{n_u+1:m:n} \le \varphi < y_{J+1:m:n} < ... < y_{m:n:n}$
- Find the values of the MSEs and the biases associated with MLEs of the parameters, the computing values are presented in Table 2,3,4 and 5 at different values of parameters.

Table 2: *The average MSEs and biases for $\lambda, \theta, \beta, \tau$ and φ are set at 0.9, 1.4, 1.76, 2.4 and 6*

$(n.m)$	Schemes	Values of λ		Values of θ		Values of β	
		Bias	MSE	Bias	MSE	Bias	MSE
(50,12)	1	0.856	0.929	0.498	0.638	0.574	0.684
	2	0.911	0.998	0.685	0.694	0.633	0.693
	3	0.743	0.873	0.584	0.658	0.593	0.709
(70,12)	1	0.502	0.577	0.476	0.609	0.543	0.644
	2	0.587	0.676	0.632	0.698	0.600	0.676
	3	0.522	0.611	0.564	0.650	0.578	0.687
(90,12)	1	0.411	0.500	0.386	0.521	0.465	0.565
	2	0.599	0.680	0.658	0.705	0.533	0.590
	3	0.431	0.534	0.489	0.580	0.498	0.577
(50,20)	1	0.343	0.344	0.300	0.467	0.398	0.511
	2	0.445	0.587	0.499	0.612	0.466	0.554
	3	0.365	0.398	0.387	0.513	0.440	0.534
(70,20)	1	0.233	0.231	0.190	0.376	0.300	0.432
	2	0.342	0.498	0.409	0.546	0.376	0.467
	3	0.287	0.280	0.298	0.412	0.333	0.442

$(n.m)$	Schemes	Values of λ		Values of θ		Values of β	
		Bias	MSE	Bias	MSE	Bias	MSE
(90,20)	1	0.143	0.154	0.122	0.287	0.198	0.365
	2	0.234	0.409	0.265	0.387	0.287	0.398
	3	0.188	0.190	0.198	0.322	0.209	0.370

Table 3: *Average values of MSEs and biases when* $\lambda, \theta, \beta, \tau$ *and* φ *are set at 0.7, 1.4, 1.76, 2.4 and 9*

$(n.m)$	Schemes	Values of λ		Values of θ		Values of β	
		Bias	MSE	Bias	MSE	Bias	MSE
(50,12)	1	0.387	0.508	0.609	0.673	0.548	0.698
	2	0.435	0.580	0.655	0.705	0.715	0.775
	3	0.477	0.546	0.640	0.642	0.658	0.739
(70,12)	1	0.323	0.410	0.456	0.521	0.512	0.574
	2	0.408	0.498	0.509	0.578	0.598	0.687
	3	0.397	0.433	0.480	0.547	0.545	0.632
(90,12)	1	0.276	0.324	0.387	0.454	0.431	0.511
	2	0.322	0.413	0.433	0.517	0.508	0.596
	3	0.299	0.356	0.410	0.489	0.474	0.541
(50,20)	1	0.197	0.250	0.311	0.386	0.324	0.434
	2	0.354	0.431	0.465	0.530	0.534	0.608
	3	0.218	0.288	0.327	0.416	0.419	0.487
(70,20)	1	0.113	0.176	0.232	0.318	0.243	0.353
	2	0.265	0.334	0.379	0.464	0.465	0.533
	3	0.175	0.212	0.248	0.354	0.325	0.397
(90,20)	1	0.007	0.119	0.146	0.243	0.154	0.265
	2	0.175	0.254	0.299	0.385	0.386	0.421
	3	0.108	0.175	0.186	0.278	0.256	0.290

Table 4: *Average values of MSEs and biases when* $\lambda, \theta, \beta, \tau$ *and* φ *are set at 0.7, 1.4, 1.76, 2.8 and 9*

$(n.m)$	Schemes	Values of λ		Values of θ		Values of β	
		Bias	MSE	Bias	MSE	Bias	MSE
(50,12)	1	0.334	0.398	0.387	0.465	0.480	0.602
	2	0.387	0.446	0.445	0.576	0.587	0.715
	3	0.354	0.431	0.412	0.543	0.535	0.675
(70,12)	1	0.296	0.344	0.297	0.387	0.429	0.519
	2	0.320	0.400	0.365	0.487	0.519	0.630
	3	0.312	0.386	0.345	0.438	0.482	0.567
(90,12)	1	0.230	0.278	0.204	0.316	0.349	0.430
	2	0.266	0.342	0.295	0.416	0.451	0.579
	3	0.240	0.294	0.256	0.398	0.380	0.483
(50,20)	1	0.187	0.238	0.138	0.253	0.227	0.341
	2	0.287	0.360	0.305	0.436	0.465	0.583
	3	0.209	0.267	0.178	0.303	0.283	0.425
(70,20)	1	0.129	0.186	0.008	0.180	0.145	0.265
	2	0.220	0.287	0.221	0.254	0.373	0.454
	3	0.148	0.202	0.120	0.228	0.220	0.374

$(n.m)$	Schemes	Values of λ		Values of θ		Values of β	
		Bias	MSE	Bias	MSE	Bias	MSE
(90,20)	1	0.007	0.129	0.004	0.109	0.100	0.187
	2	0.139	0.198	0.188	0.169	0.270	0.378
	3	0.102	0.149	0.009	0.134	0.139	0.190

Table 5: *Average values of MSEs and biases when* $\lambda, \theta, \beta, \tau$ *and* φ *are set at 1.5, 1.4, 1.76, 2.4 and 9*

$(n.m)$	Schemes	Values of λ		Values of θ		Values of β	
		Bias	MSE	Bias	MSE	Bias	MSE
(50,12)	1	0.560	0.593	0.593	0.656	0.644	0.734
	2	0.677	0.765	0.709	0.788	0.723	0.797
	3	0.600	0.712	0.650	0.693	0.687	0.755
(70,12)	1	0.476	0.499	0.486	0.575	0.563	0.665
	2	0.588	0.691	0.628	0.690	0.659	0.687
	3	0.523	0.633	0.576	0.620	0.581	0.671
(90,12)	1	0.410	0.453	0.399	0.484	0.487	0.556
	2	0.593	0.698	0.645	0.705	0.667	0.710
	3	0.447	0.560	0.480	0.563	0.530	0.600
(50,20)	1	0.334	0.407	0.311	0.422	0.435	0.523
	2	0.450	0.599	0.513	0.567	0.574	0.616
	3	0.389	0.523	0.419	0.497	0.467	0.586
(70,20)	1	0.254	0.306	0.223	0.375	0.370	0.455
	2	0.334	0.492	0.460	0.500	0.479	0.544
	3	0.319	0.345	0.280	0.386	0.417	0.478
(90,20)	1	0.130	0.233	0.155	0.284	0.245	0.374
	2	0.252	0.364	0.359	0.433	0.332	0.407
	3	0.209	0.288	0.197	0.357	0.300	0.431

V. Application in Real Life Situation

SSPALT is now the most significant procedure of reviewing item trustworthiness rapidly, and the blueprint of capable investigation plans is a serious step to guarantee that SSPALTs can evaluate the item reliability correctly, quickly, and cheaply. With the encouragement of the national approach of civil-military integration, SSPALT will be mostly applied in the research and development of a variety of manufactured goods, and the SSPALT plan design hypothesis will face more challenges. To assist engineers in selecting suitable hypotheses and to stimulate researchers to build up the theories necessary in manufacturing, with the focal point on the demands for theory investigation that happen from the execution of SSPALT, this study reviews and summarizes the expansion of the SSPALT plan. The expansion of the theory and technique for setting up the most favorable SSPALT for shape-scale distribution, which is the most functional and grown-up theory of designing the optimal SSPALT, are explained in detail. Based on the theory of convenience for engineers to choose suitable techniques according to the troubles that originate in practice, this discussed will help to review the progress of optimal ALT plan design theory by taking the engineering problems occurring from the ALT execution as the key thread, provides strategy on choosing suitable theories for engineers, and suggests views about the vital solved theory problems for researchers.

A real life data set is commenced to demonstrate how the ML estimation method works in practice based on real life data set from Nelson [35]. Table 6 is presented the data set and the data set is correspond to the oil breakdown period of insulating fluid under two stress stages (34 kilovolt (kv) and 36 kv), considering the data set under 34 kv as data under ordinary stress condition. Before further proceeding, we test the strength of GRID to fit the data listed in Table 6 using Kolmogorov-Smirnov (K-S) test statistic and its corresponding p-value for each stress stage. The outcome is presented in Table 7. We can observe that the GRID fits better to the given data in the two stress stages because the p-values are greater than 0.05. The MLEs, p-values and K-S statistic are presented in Table 7.

Table 6: *Stress values and complete failure data*

Stress (in kv)	Complete failure data
34	0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89
36	0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.9, 3.67, 3.99, 5.35, 13.77, 25.5

Table 7: *MLEs of the parameters, p-value and K-S statistic*

Parameters	Stress (in kv)	K-S	p-value
$\lambda = 1.2065, \theta = 3.0873, \beta = 1.2189$	34	0.1562	0.5422
	36	0.1752	0.1290

VI. Results and Conclusion

From Tables 2 to 5, it is concluded that the MLE is consistent and asymptotically normally distributed and one can realize that the biases and MSEs decrease as sample size increase for different values of parameters, which proves the efficiency of MLE.

The study deals with SSPALT by using an adaptive Type-II progressively hybrid censoring scheme for GIRD with a maximum likelihood estimation procedure. The numerical values of MLEs of distribution parameters are attained using the Newton-Raphson technique, and the performances of parameters are recorded in terms of MSEs and biases. Superb efficiency in estimating distribution parameters is examined under APHCT-II due to the huge sample size attained. So, APHCT-II is an excellent option for reliability practitioners to attain a greater efficiency of the distribution parameters. In the future, this work can be extended for different failure distributions under the Bayesian atmosphere.

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