EXPONENTIATED WEIBULL DISTRIBUTION: BAYESIAN ESTIMATION USING PROGRESSIVE TYPE I INTERVAL CENSORING

M. KUMAR, K P ASWATHI

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Department of Mathematics, National Institute of Technology Calicut, 673601, Kerala, India
mahesh@nitc.ac.in, aswathichithra01@gmail.com

Abstract

A three-parameter distribution known as the Generalized Weibull (GW) or Exponentiated Weibull distribution is studied in this work. We construct Baye's estimators for the unknown parameters and present reliability function using progressive type I interval censoring data. Two different loss functions, namely, squared error loss and general entropy loss functions are applied to derive Baye's estimators. It is observed that there is no closed-form solution for Baye's estimators as well as for MLE. Hence, Lindley's approximation procedure is applied to obtain Bayesian estimator of unknown parameters, and Newton Rapson method is employed to obtain MLE's numerically. The corresponding reliability function is derived. Monte Carlo simulation is used to obtain MLE. Further, the performance of MLE and Bayes estimators are compared in terms of their respective MSE and Relative errors. It is noted by numerical computation that MLE's performs better than Bayes estimators. In addition to this, Bayes estimators obtained using Squared error loss function and general entropy loss function are compared. It is observed through numerical computation that general entropy loss function is better in terms of MSE.

Keywords: Bayesian inference, Exponentiated Weibull distribution, Lindley's approximation, Maximum likelihood function, Monte Carlo simulation, Relative error.

1. Introduction

When it comes to analyzing data and adapting it to practical situations, statistical distributions are crucial. Weibull or Gamma distributions are typically employed to fi the data in real-world scenarios. In survival analysis, the Gamma distribution has more major applications than all other distributions. But the main drawback of Gamma distribution is that the survival function cannot be obtained in closed form unless the shape parameter is an integer. This makes Weibull distribution more popular than Gamma distribution. Its survival function and failure rate are simple and easy to analyze. And this distribution is easy to handle the censoring data because of that, in recent years Weibull distribution is more popular in analyzing lifetime data. The Exponentiated Weibull distribution (EW) or Generalized Weibull distribution, was firs described by [24] as a way to extend the Weibull family of two parameters by one more shape parameter. This distribution yields better fit than classic models such as exponential, gamma, Weibull, and log-normal distribution. Owing to its flexibilit in modeling a wide range of industrial data, the EW distribution may be widely and efficientl applied in reliability applications. The fundamental feature of this family is that it supports bathtub-shaped as well as unimodal hazard rates, in addition to numerous monotone hazard rates. The applications of this distribution were firs developed by [24]. Using fi e different classical failure data sets obtained for the Bus-motor system, [25] demonstrated the potential unfulness and flexibilit of EW distribution. It is a sub-model of a generic class of exponentiated distributions suggested by [11]. Generalized Weibull distribution was used by [26] to model survival data. The reliability and survival functions of this distribution were studied by [23]. Further statistical features and the importance of this distribution are addressed by [29] and [28]. The moments of the EW distribution were determined by [8]. EW distribution was compared on two-parameter Weibull and Gamma distributions in [32] study with regard to the failure rate. Exponentiated Weibull family distributed lifetime data observed under Type I progressive interval censoring with random removals were analyzed by [6]. Bayesian estimate and prediction for the EW distribution using both informative and non-informative priors was examined by [21]. After fittin a Weibull distribution and an EW distribution to the wind speed data and determining the mean and variance, [9] estimated the parameter using the MLE method. The non-Bayesian estimators methods for parameters of EW distribution studied by [4].The discrete case of EW distribution studied by [30]. The entropy and stress-strength model of EW distribution studied by [3]. Numerical estimation of parameters of EW distribution based on generalized progressive hybrid censoring scheme studied by [10]. In recent years, estimation of EW distribution under progressive type II censored data studied by [22].

The fundamental feature of this family is that it supports bathtub-shaped as well as unimodal hazard rates, in addition to numerous monotone hazard rates. The EW distribution is define in the following way.

It has distribution function given by

$$F(x;\alpha,\beta,\lambda) = (1 - e^{-(\lambda x)^{\beta}})^{\alpha},\ x > 0 \ and\ \alpha,\beta,\lambda > 0 \tag{1}$$

and therefore its probability density function is of the form

$$f(x;\alpha,\beta,\lambda) = \alpha\beta\lambda^{\beta}x^{(\beta-1)}e^{-(\lambda x)^{\beta}}((1 - e^{-(\lambda x)^{\beta}})^{\alpha-1}) \tag{2}$$

The corresponding reliability function is given by

$$R(x;\alpha,\lambda) = 1 - (1 - e^{-(\lambda x)^{\beta}})^{\alpha} \tag{3}$$

and the hazard rate is

$$h(x) = \frac{f(x)}{1 - F(x)},\ x > 0 \tag{4}$$

Note here that, the shape parameters are α and β, and the scale parameter is λ.

Several well known distributions are particular cases of the EW distribution. For example, the Exponential distribution is the case when $\alpha = 1$and $\beta = 1$, the Weibull Distribution is define with $\alpha = 1$, Rayleigh Distribution with $\alpha = 1$ and $\beta = 2$, $\beta = 1$ Generalized Exponential (GE) Distribution studied by [12], [13], [15], [17] [18], [37] and [39]. $\beta = 2$ Two parameter Burr Type X or Exponentiated Rayleigh(ER) or Generalized Rayleigh(GR) Distribution studied by [2], [36], [16], [14], [43], [38], [5] and [27] among others. Fig.(1) and Fig.(2) represents the many forms of these distributions graphically .

It was discovered that the EW family is a very versatile family that may be utilized to describe many sorts of skewed lifetime data. In reliability analysis, censoring is quite prevalent. It occurs when specifi failure times for a subset of test units in an experiment are detected.

In industrial life testing and medical survival analysis, very often the object of interest is lost or withdrawn before failure or the object's lifetime is only known within an interval. Hence, the obtained sample is called a censored sample (or an incomplete sample). The most common censoring schemes are type-I censoring, type-II censoring and progressive censoring. For type-I censoring, life testing ends at a pre-scheduled time and for type-II censoring, life testing ends whenever the number of lifetimes is reached. In type-I and type-II censoring schemes, the tested items are allowed to be withdrawn only at the end-of-life testing. In the progressive censoring

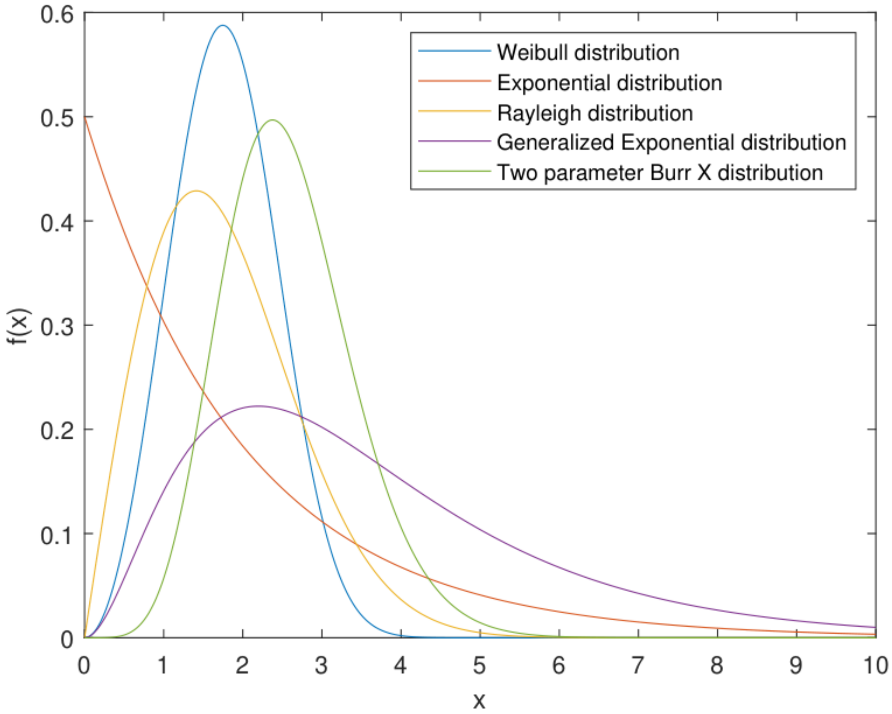

Figure 1: *Graph of EW distribution for different values of α, β and for fixed $\lambda = 0.5$*

scheme, the tested items are allowed to be withdrawn at some time before the end-of-life testing. See [7] for more information about progressive censoring combined with type-I or type-II and their applications. Using the concepts of progressive censoring, type I censoring, and interval censoring, [1] developed progressive type I interval censoring. Combining progressive censoring and type-II censoring, [18] and [34] investigated Bayesian inference for Weibull distribution and generalized exponential (GE) distribution, respectively. It should be emphasized that in many practical situations, unit lifetime is set on an interval, therefore type I interval censoring is beneficia in these instances (see,[1]). It may be noted that in real-life situations, the lifetime of units may not be recorded precisely due to some reasons, such as technical problems, non-availability of experimental resources or due to some unknown human errors, or some cost-saving measures employed by the industry. Thus such censored data generated can be used effectively in analyzing the reliability characteristics of well-known distribution, such as the more general class of distribution, namely, EW distribution, which gained lots of importance in recent times. The importance of progressive type-I interval censoring in handling practical problem has been studied by authors, namely, [6] and [19]. The concept of progressive type-I interval censoring to the Weibull distribution and compared many different estimation methods for two parameters in the Weibull distribution via simulation introduced by [31]. The recent study about progressive type I interval censoring is On inference and design under progressive type-I interval censoring scheme for inverse Gaussian lifetime model by [40]. A Study on the experimental design for the lifetime performance index of Rayleigh lifetime distribution under progressive type I interval censoring by [44]. Optimal design of accelerated life tests under progressive type I interval censoring with random removals by [46], and experimental design for progressive type I interval censoring on the lifetime performance index of Chen lifetime distribution by [45].

All the works available in the literature aims at obtaining estimators of parameters of EW distribution based upon, either data obtain from complete censoring or from type I censoring, type II censoring, hybrid censoring, etc. No work in the literature addresses the estimation of parameters of EW distribution based upon progressive type I interval-censored data. Therefore we

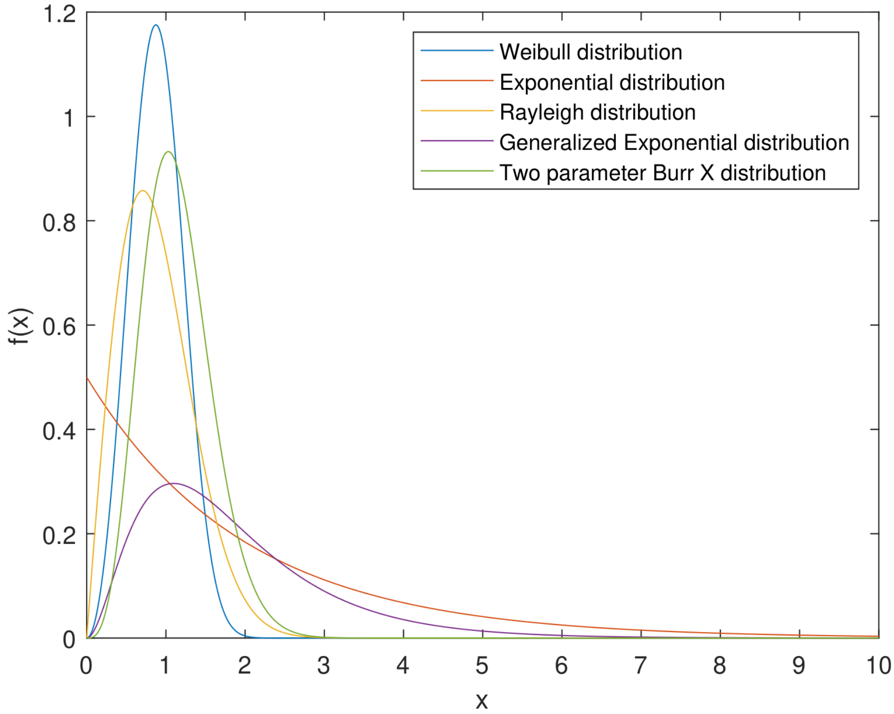

Figure 2: *Graph of EW distribution for different values of α, β and for fixed $\lambda = 1$*

consider in the next sections the derivation of MLE and Bayes estimators from data obtained via progressive type I interval censoring for EW distribution. Section 2 provides a brief fundamental required for obtaining estimators based on censored data. Some simulation results and discussion based upon the results obtained are presented in Section 3. The conclusion and future scope of research are given in Section 4.

2. Bayesian estimation using progressive type I interval censored data

In this section, we discuss the brief overview of the terms used in this paper and the procedure of obtaining Baye's estimators for Parameters and reliability function of EW distribution.

2.1. Progressive type I interval censored data and the likelihood function

Statistical inference for exponential distributions using progressive type I interval censored data and pioneered type I interval censoring in a progressive censoring scheme developed by [1]. Under progressive type I interval censoring, observations are only known within two successively pre-scheduled timeframes, and items may be allowed to be deleted at pre-scheduled time points. The progressively type I interval censored sample may be generated in the following manner:
Let n units be put on a life testing platform simultaneously at time $t_0 = 0$ and under examination at m pre-specifie time periods $t_1 < t_2 < ... < t_m$ where t_m is the predetermined time to end the experiment. The number of failures X_i within $(t_{i-1}, t_i]$ is recorded and R_i surviving items are randomly removed from the life testing at the i^{th} inspection time, t_i, for $i = 1, 2, ..., m$. Because the number of surviving items, Y_i, is an random variable and the precise number of items removed at time schedule t_i should not be larger than Y_i, R_i might be calculated by a pre-specifie percentage of the remaining surviving units at t_i for given $i = 1, 2, ..., m$.
For example, given certain pre-specifie percentage values say, $p_1, p_2, ..., p_{m-1}$ and $p_m = 1$, R_i can be determined by using $R_i = floor[p_i Y_i]$ at each inspection time t_i, where $floor[x]$ yields

$x's$ biggest integer. Therefore, a progressive type-I interval censored sample with size n, can be denoted as $D = (X_i, R_i, t_i)_m, i = 1, 2, ..., m$. If $R_i = 0, i = 1, 2, ..., m-1$ and $R_m = n - \sum_{i=1}^{m} X_i$, then the type-I interval-censored sample gradually shrinks to the typical interval-censored sample.
Given the progressively type-I censored data, $D = (X_i, R_i, t_i)_m$ of size n, from a continuous lifetime distribution with CDF $F(t; \kappa)$, then the likelihood function is given as follows

$$L(D \mid \kappa) \propto \prod_{i=1}^{m} [F(t_i; \kappa) - F(t_{i-1}; \kappa)]^{X_i} [1 - F(t_i; \kappa)]^{R_i}, \quad (5)$$

where $t_0 = 0$ and θ is the parameter vector. The more details of progressive type I interval censoring can be seen in [33].
For the $EW(\alpha, \lambda, \beta)$, the likelihood function (5) can be define in the following manner:

$$L(D \mid \alpha, \lambda, \beta) \propto \prod_{i=1}^{m} [(1 - e^{-(\lambda t_i)^\beta})^\alpha - (1 - e^{-(\lambda t_{i-1})^\beta})^\alpha]^{X_i} [1 - (1 - e^{-(\lambda t_i)^\beta})^\alpha]^{R_i}. \quad (6)$$

The log-likelihood function is thus given by

$$l(\alpha, \lambda, \beta) \propto \sum_{i=1}^{m} X_i ln[(1 - e^{-(\lambda t_i)^\beta})^\alpha - (1 - e^{-(\lambda t_{i-1})^\beta})^\alpha] + R_i ln[1 - (1 - e^{-(\lambda t_i)^\beta})^\alpha]. \quad (7)$$

2.2. Maximum likelihood function

In this section, we discuss the Maximum likelihood estimation to estimate unknown parameters α, λ, β, and the reliability function $R(t)$ for EW distribution define in (1) using the numerical method.
By setting the derivatives of the log likelihood function with respective to α, λ or β to zero, the MLEs of α, λ and β are the solutions to the following likelihood equations

$$\begin{aligned} \sum_{i=1}^{m} \left[X_i \left(\frac{\frac{\partial F_i}{\partial \alpha} - \frac{\partial F_{i-1}}{\partial \alpha}}{F_i - F_{i-1}} \right) \right] &= \sum_{i=1}^{m} \left[R_i \left(\frac{\frac{\partial F_i}{\partial \alpha}}{1 - F_i} \right) \right] \\ \sum_{i=1}^{m} \left[X_i \left(\frac{\frac{\partial F_i}{\partial \lambda} - \frac{\partial F_{i-1}}{\partial \lambda}}{F_i - F_{i-1}} \right) \right] &= \sum_{i=1}^{m} \left[R_i \left(\frac{\frac{\partial F_i}{\partial \lambda}}{1 - F_i} \right) \right] \end{aligned}$$

and

$$\sum_{i=1}^{m} \left[X_i \left(\frac{\frac{\partial F_i}{\partial \beta} - \frac{\partial F_{i-1}}{\partial \beta}}{F_i - F_{i-1}} \right) \right] = \sum_{i=1}^{m} \left[R_i \left(\frac{\frac{\partial F_i}{\partial \beta}}{1 - F_i} \right) \right]$$

There is no closed form of the solution to the above equations and numerical methods can be used to obtain the MLEs from the above likelihood equations. Since there is no closed form of the MLE, Newton-Raphson method is introduced as follows for findin the MLEs of α, λ and β.
One of the most used methods for optimization in statistics is the Newton-Raphson method(or Newton™s rule). Assume that l only involves a one-dimensional parameter and that $\overline{\vartheta}$ is our current best guess on the maximum of $l(\vartheta)$. $l(\vartheta)$ can be approximated by employing a Taylor series expansion around $\overline{\vartheta}$. Hence we have

$$\bar{l}_{\overline{\vartheta}}(\vartheta) = l(\overline{\vartheta}) + l'(\overline{\vartheta})(\vartheta - \overline{\vartheta}) + \frac{1}{2} l''(\overline{\vartheta})(\vartheta - \overline{\vartheta})^2.$$

When ϑ is close to $\overline{\vartheta}$, the difference $l(\vartheta) - \bar{l}_{(\overline{\vartheta})}(\vartheta)$ is small. The maximum value of $\bar{l}_{(\overline{\vartheta})}(\vartheta)$ is closer to the maximum value of $l(\vartheta)$ than $l(\overline{\vartheta})$.
The gradient of $\bar{l}_{(\overline{\vartheta})}(\vartheta)$ at ϑ is

$$\bar{l}'_{(\overline{\vartheta})}(\vartheta) = l'(\overline{\vartheta}) + l''(\overline{\vartheta})(\vartheta - \overline{\vartheta})$$

and the Hessian or second derivative is

$$\bar{l}''_{(\overline{\vartheta})}(\vartheta) = l''(\overline{\vartheta}).$$

At the point $\overline{\vartheta}$, $l(\vartheta)$ and $\bar{l}_{(\overline{\vartheta})}(\vartheta)$ have equal firs and second derivatives. In the case of log likelihood function Hessian is same as the minus of observed information evaluated at $\vartheta = \overline{\vartheta}$, $l''(\overline{\vartheta}) = -J(\overline{\vartheta})$. In the optimum point of the approximation, $\bar{l}_{(\overline{\vartheta})}(\vartheta)$ has a gradient equal to zero, giving the following equation:

$$l''(\overline{\vartheta})(\vartheta - \overline{\vartheta}) = -l'(\overline{\vartheta}).$$

Solving with respect to ϑ, we get

$$\vartheta = \overline{\vartheta} - \frac{l'(\overline{\vartheta})}{l''(\overline{\vartheta})}.$$

This gives a procedure for optimizing $\bar{l}_{(\overline{\vartheta})}(\vartheta)$. An iterative procedure for optimizing $l(\vartheta)$ is given by

$$\vartheta^{(s+1)} = \vartheta^{(s)} - \frac{l'(\vartheta^{(s)})}{l''(\vartheta^{(s)})}$$

which is the Newton-Raphson Method. The procedure is run until there is no significan difference between $\vartheta^{(s)}$ and $\vartheta^{(s+1)}$.
When $l(\vartheta)$ is a log likelihood function, this algorithm can be written as

$$\vartheta^{(s+1)} = \vartheta^{(s)} - \frac{s(\vartheta^{(s)})}{J(\vartheta^{(s)})}$$

where $s(\vartheta)$ is the score function while $J(\vartheta)$ is the observed information matrix.

2.3. Bayesian Estimation

In this section, we discuss the Bayesian technique to estimate unknown parameters α, λ, β, and the reliability function $R(t)$ using the Squared error loss and general entropy loss functions.
Assume that all parameters, namely, α, λ and β of EW distributions are unknown and independent. We address the problem of constructing Baye's estimators for these parameters. We assume non-informative priors for α and β, and conjugate prior for λ. The reason for choosing these prior forms is duo to their simplicity of in obtaining mathematically treatable posterior distributions. We observe that such priors are successfully applied by many authors, namely, [[33] and [35]]. The following equations give respective definition of prior densities.

$$\pi_1(\alpha) = \frac{1}{\alpha}, \quad \alpha > 0 \tag{8}$$

$$\pi_2(\lambda) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}, \quad \lambda > 0, a, b > 0 \tag{9}$$

and

$$\pi_3(\beta) = \frac{1}{\beta}, \quad \beta > 0 \tag{10}$$

respectively where $\Gamma(.)$ is the gamma function.
We consider two different form of loss functions in estimating the parameters of EW density. The firs one is a symmetric loss function, the squared error loss function(SEL), which is given by

$$L_1(\zeta, \hat{\zeta}) = (\hat{\zeta} - \zeta)^2, \tag{11}$$

where $\hat{\zeta}$ is the estimate of parameter ζ. Then the Bayesian estimate of any function $q = q(\alpha, \lambda, \beta)$ is obtained by considering following equation

$$\hat{q} = E(q \mid D) = \frac{\int_\alpha \int_\lambda \int_\beta q(\alpha, \lambda, \beta) l(\alpha, \lambda, \beta) \pi_1(\alpha) \pi_2(\lambda) \pi_3(\beta) d\alpha d\lambda d\beta}{\int_\alpha \int_\lambda \int_\beta l(\alpha, \lambda, \beta) \pi_1(\alpha) \pi_2(\lambda) \pi_3(\beta) d\alpha d\lambda d\beta} \tag{12}$$

The second loss function, is the generalization of the Entropy loss used by several authors ([41] and [42]). The General Entropy loss(GEL) is defin as:

$$L_2(\zeta, \hat{\zeta}) \propto \left(\frac{\hat{\zeta}}{\zeta}\right)^c - c\, log \frac{\hat{\zeta}}{\zeta} - 1, \tag{13}$$

where $\hat{\zeta}$ is an estimate of parameter ζ. It may be noted that when $c > 0$, a positive error causes more serious consequences than a negative error. On the other hand, when $c < 0$, a negative error causes more serious consequences than a positive error. Then the Bayesian estimator of $q(\alpha, \lambda, \beta)$ under this general entropy loss function is

$$\hat{q}_{GEL} = [E(q^{-c})]^{-\frac{1}{c}}, \tag{14}$$

provided that $E(q^{-c})$ exists and is finite It can be shown that, when $c = 1$, the Bayes estimate (12) coincides with the Bayes estimate under the weighted squared-error loss function. Similarly, when $c = -1$ the Bayes estimate (14) coincides with the Bayes estimate under squared error loss function. The equations (12) and (14) cannot be solved for obtaining closed form solutions. Hence, we resort to well known Lindley approximation [20] procedure to evaluate the ratio of integrals involved in (12) and (14). Note that the Lindley approximation procedure is successively employed by authors, such as [18] to obtain Bayesian estimators. Next, the Bayesian posterior expection function of a parameter vector η, say $h(\eta)$ is obtained by using the following equation

$$\hat{h}_B = E(h(\eta) \mid D) = \frac{\int_\eta h(\eta) l(\eta) \pi(\eta) d\eta}{\int_\eta l(\eta) \pi(\eta) d\eta}, \tag{15}$$

Recall that in the above expression $l(\eta)$ denotes log likelyhood function, $\pi(\eta)$ denotes prior density and D denotes the data obtained using progressive type I interval censoring.
By [20], if n, the sample size is sufficientl large, every ratio of the integral of the form,

$$\begin{aligned} \hat{h} &= E[v(\eta_1, \eta_2, \eta_3)] \\ &= \frac{\int_{\eta_1, \eta_2, \eta_3} v(\eta_1, \eta_2, \eta_3) e^{l(\eta_1, \eta_2, \eta_3) + G(\eta_1, \eta_2, \eta_3)} d(\eta_1, \eta_2, \eta_3)}{\int_{\eta_1, \eta_2, \eta_3} e^{l(\eta_1, \eta_2, \eta_3) + G(\eta_1, \eta_2, \eta_3)} d(\eta_1, \eta_2, \eta_3)} \end{aligned}$$

where
$v(\eta) = v(\eta_1, \eta_2, \eta_3)$ is a function of η_1, η_2 or η_3 only,
$l(\eta_1, \eta_2, \eta_3)$ is log of likelihood function,
and $G(\eta_1, \eta_2, \eta_3)$ is log joint prior of η_1, η_2 and η_3,
can be evaluated as

$$\begin{aligned} \hat{h} = v(\hat{\eta_1}, \hat{\eta_2}, \hat{\eta_3}) + (v_1 a_1 + v_2 a_2 + v_3 a_3 + a_4 + a_5) + \frac{1}{2}[A(v_1\sigma_{11} + v_2\sigma_{12} + v_3\sigma_{13}) + \\ B(v_1\sigma_{21} + v_2\sigma_{22} + v_3\sigma_{23}) + C(v_1\sigma_{31} + v_2\sigma_{32} + v_3\sigma_{33})] \end{aligned}$$

where
$\hat{\eta_1}, \hat{\eta_2}$ and $\hat{\eta_3}$ are the MLE of η_1, η_2 and η_3 respectively.

$$\begin{aligned}
a_i &= \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3}, \quad i = 1, 2, 3, \\
a_4 &= v_{12}\sigma_{12} + v_{13}\sigma_{13} + v_{23}\sigma_{23}, \\
a_5 &= \frac{1}{2}(v_{11}\sigma_{11} + v_{22}\sigma_{22} + v_{33}\sigma_{33}), \\
A &= \sigma_{11}l_{111} + 2\sigma_{12}l_{121} + 2\sigma_{13}l_{131} + 2\sigma_{23}l_{231} + \sigma_{22}l_{221} + \sigma_{33}l_{331}, \\
B &= \sigma_{11}l_{112} + 2\sigma_{12}l_{122} + 2\sigma_{13}l_{132} + 2\sigma_{23}l_{232} + \sigma_{22}l_{222} + \sigma_{33}l_{332}, \\
C &= \sigma_{11}l_{113} + 2\sigma_{12}l_{123} + 2\sigma_{13}l_{133} + 2\sigma_{23}l_{233} + \sigma_{22}l_{223} + \sigma_{33}l_{333}
\end{aligned}$$

and subscripts 1,2,3 on the right-hand sides refer to η_1, η_2, η_3 respectively and,

$$\begin{aligned}
\rho_i &= \frac{\partial\rho}{\partial\eta_i}, \quad v_i = \frac{\partial v(\eta_1,\eta_2,\eta_3)}{\partial\eta_i}, \; i = 1, 2, 3, \\
v_{ij} &= \frac{\partial^2 v(\eta_1,\eta_2,\eta_3)}{\partial\eta_i\partial\eta_j}, \; i, j = 1, 2, 3,
\end{aligned}$$

$$l_{ij} = \frac{\partial^2 l(\eta_1,\eta_2,\eta_3)}{\partial\eta_i\partial\eta_j}, \; i, j = 1, 2, 3, \tag{16}$$

$$l_{ijk} = \frac{\partial^3 l(\eta_1,\eta_2,\eta_3)}{\partial\eta_i\partial\eta_j\partial\eta_k}, \; i, j, k = 1, 2, 3, \tag{17}$$

and σ_{ij} is the $(i, j)^{th}$ element of the inverse of the matrix $\{l_{ij}\}$, which is given by

$$I(\alpha,\lambda,\beta) = \begin{bmatrix} -\frac{\partial^2 l}{\partial\alpha^2} & -\frac{\partial^2 l}{\partial\alpha\partial\lambda} & -\frac{\partial^2 l}{\partial\alpha\partial\beta} \\ -\frac{\partial^2 l}{\partial\lambda\partial\alpha} & -\frac{\partial^2 l}{\partial\lambda^2} & -\frac{\partial^2 l}{\partial\lambda\partial\beta} \\ -\frac{\partial^2 l}{\partial\alpha\partial\beta} & -\frac{\partial^2 l}{\partial\beta\partial\lambda} & -\frac{\partial^2 l}{\partial\beta^2} \end{bmatrix}$$

Now by equations, (8), (9) and (10), by using independence of α, λ, β, the joint prior distribution of there three parameters is given by

$$\pi(\alpha,\lambda,\beta) = \frac{b^a\lambda^{a-1}e^{-b\lambda}}{\beta\alpha\Gamma(a)}, \; \alpha, \lambda, \beta > 0, a, b > 0. \tag{18}$$

Let

$$\begin{aligned}
\rho &= \ln\pi(\alpha,\lambda,\beta) \\
&= a\ln b + (a-1)\ln\lambda - b\lambda - \ln\beta - \ln\alpha - \ln\Gamma(a).
\end{aligned} \tag{19}$$

Differentiating (19) with respect to α, λ, β respectively, we have

$$\rho_1 = -\frac{1}{\alpha}, \;\; \rho_2 = \frac{a-1}{\lambda} - b, \;\; \rho_3 = -\frac{1}{\beta}.$$

Observe that while performing progressive type I interval censoring, there are 'm' pre-specifie time periods, say, $t_1 < t_2 < ... < t_m$, where t_m is pre-specifie stopping time of experiment. Now let us defin the pdf for EW distribution for $1 \le i \le m$ as $F_i = (1 - e^{-(\lambda x)^\beta})^\alpha \;\; i = 1, 2, 3, ..., m$. Now from the expression (5) we have

$$l \propto \sum_{i=1}^{m} \{X_i \ln[F_i - F_{i-1}] + R_i \ln[1 - F_i]\}$$

Then,

$$
\begin{aligned}
l_1 &= \sum_{i=1}^{m}\left[X_i\left(\frac{\frac{\partial F_i}{\partial\alpha}-\frac{\partial F_{i-1}}{\partial\alpha}}{F_i-F_{i-1}}\right)-R_i\left(\frac{\frac{\partial F_i}{\partial\alpha}}{1-F_i}\right)\right]\\
l_2 &= \sum_{i=1}^{m}\left[X_i\left(\frac{\frac{\partial F_i}{\partial\lambda}-\frac{\partial F_{i-1}}{\partial\lambda}}{F_i-F_{i-1}}\right)-R_i\left(\frac{\frac{\partial F_i}{\partial\lambda}}{1-F_i}\right)\right]\\
l_3 &= \sum_{i=1}^{m}\left[X_i\left(\frac{\frac{\partial F_i}{\partial\beta}-\frac{\partial F_{i-1}}{\partial\beta}}{F_i-F_{i-1}}\right)-R_i\left(\frac{\frac{\partial F_i}{\partial\beta}}{1-F_i}\right)\right]
\end{aligned}
$$

From equation (16), the values of $l_{ij}, (i,j=1,2,3)$ can be obtained as follows

$$
l_{11}=\sum_{i=1}^{m}\left\{X_i\left[\frac{(F_i-F_{i-1})\left(\frac{\partial^2F_i}{\partial\alpha^2}-\frac{\partial^2F_{i-1}}{\partial\alpha^2}\right)-\left(\frac{\partial F_i}{\partial\alpha}-\frac{\partial F_{i-1}}{\partial\alpha}\right)^2}{(F_i-F_{i-1})^2}\right]-R_i\left[\frac{(1-F_i)\frac{\partial^2F_i}{\partial\alpha^2}+\left(\frac{\partial F_i}{\partial\alpha}\right)^2}{(1-F_i)^2}\right]\right\},
$$

$$
l_{12}=\sum_{i=1}^{m}\left\{X_i\left[\frac{(F_i-F_{i-1})\left(\frac{\partial^2F_i}{\partial\alpha\partial\lambda}-\frac{\partial^2F_{i-1}}{\partial\alpha\partial\lambda}\right)-\left(\frac{\partial F_i}{\partial\alpha}-\frac{\partial F_{i-1}}{\partial\alpha}\right)\left(\frac{\partial F_i}{\partial\lambda}-\frac{\partial F_{i-1}}{\partial\lambda}\right)}{(F_i-F_{i-1})^2}\right]-R_i\left[\frac{(1-F_i)\frac{\partial^2F_i}{\partial\alpha\lambda}+\left(\frac{\partial F_i}{\partial\alpha}\right)\left(\frac{\partial F_i}{\partial\lambda}\right)}{(1-F_i)^2}\right]\right\}=l_{21},
$$

$$
l_{13}=\sum_{i=1}^{m}\left\{X_i\left[\frac{(F_i-F_{i-1})\left(\frac{\partial^2F_i}{\partial\alpha\partial\beta}-\frac{\partial^2F_{i-1}}{\partial\alpha\partial\beta}\right)-\left(\frac{\partial F_i}{\partial\alpha}-\frac{\partial F_{i-1}}{\partial\alpha}\right)\left(\frac{\partial F_i}{\partial\beta}-\frac{\partial F_{i-1}}{\partial\beta}\right)}{(F_i-F_{i-1})^2}\right]-R_i\left[\frac{(1-F_i)\frac{\partial^2F_i}{\partial\alpha\beta}+\left(\frac{\partial F_i}{\partial\alpha}\right)\left(\frac{\partial F_i}{\partial\beta}\right)}{(1-F_i)^2}\right]\right\}=l_{31},
$$

$$
l_{22}=\sum_{i=1}^{m}\left\{X_i\left[\frac{(F_i-F_{i-1})\left(\frac{\partial^2F_i}{\partial\lambda^2}-\frac{\partial^2F_{i-1}}{\partial\lambda^2}\right)-\left(\frac{\partial F_i}{\partial\lambda}-\frac{\partial F_{i-1}}{\partial\lambda}\right)^2}{(F_i-F_{i-1})^2}\right]-R_i\left[\frac{(1-F_i)\frac{\partial^2F_i}{\partial\lambda^2}+\left(\frac{\partial F_i}{\partial\lambda}\right)^2}{(1-F_i)^2}\right]\right\},
$$

$$
l_{23}=\sum_{i=1}^{m}\left\{X_i\left[\frac{(F_i-F_{i-1})\left(\frac{\partial^2F_i}{\partial\lambda\partial\beta}-\frac{\partial^2F_{i-1}}{\partial\lambda\partial\beta}\right)-\left(\frac{\partial F_i}{\partial\lambda}-\frac{\partial F_{i-1}}{\partial\lambda}\right)\left(\frac{\partial F_i}{\partial\beta}-\frac{\partial F_{i-1}}{\partial\beta}\right)}{(F_i-F_{i-1})^2}\right]-R_i\left[\frac{(1-F_i)\frac{\partial^2F_i}{\partial\lambda\beta}+\left(\frac{\partial F_i}{\partial\lambda}\right)\left(\frac{\partial F_i}{\partial\beta}\right)}{(1-F_i)^2}\right]\right\}=l_{32},
$$

$$l_{33} = \sum_{i=1}^{m} \left\{ X_i \left[\frac{(F_i - F_{i-1}) \left(\frac{\partial^2 F_i}{\partial \beta^2} - \frac{\partial^2 F_{i-1}}{\partial \beta^2} \right) - \left(\frac{\partial F_i}{\partial \beta} - \frac{\partial F_{i-1}}{\partial \beta} \right)^2}{(F_i - F_{i-1})^2} \right] - R_i \left[\frac{(1 - F_i) \frac{\partial^2 F_i}{\partial \beta^2} + \left(\frac{\partial F_i}{\partial \beta} \right)^2}{(1 - F_i)^2} \right] \right\}.$$

Similarly, from equation (17), the values for $l_{ijk}(i, j, k = 1, 2, 3)$ can be obtained.
Now we proceed to obtain Bayes estimators of the parameters α, λ, β of EW distribution function, and the reliability function $R(t)$ under squared error loss function. Recall that $v(\hat{\alpha_s}, \hat{\lambda_s}, \hat{\beta_s})$ denotes a function MLE's for α, λ, β. Hence we present here the Bayes estimators of α, λ, β and $R(t)$ via following equations:

- $v(\hat{\alpha}, \hat{\lambda}, \hat{\beta}) = \hat{\alpha}$ then

$$\hat{\alpha_s} = \hat{\alpha} - \frac{1}{\hat{\alpha}}\sigma_{11} + \frac{a - 1 - b\hat{\lambda}}{\hat{\lambda}}\sigma_{12} - \frac{1}{\hat{\beta}}\sigma_{13} + \frac{1}{2}\left[A\sigma_{11} + B\sigma_{21} + C\sigma_{31}\right], \tag{20}$$

- $v(\hat{\alpha}, \hat{\lambda}, \hat{\beta}) = \hat{\lambda}$ then

$$\hat{\lambda_s} = \hat{\lambda} - \frac{1}{\hat{\alpha}}\sigma_{21} + \frac{a - 1 - b\hat{\lambda}}{\hat{\lambda}}\sigma_{22} - \frac{1}{\hat{\beta}}\sigma_{23} + \frac{1}{2}\left[A\sigma_{12} + B\sigma_{22} + C\sigma_{32}\right], \tag{21}$$

- $v(\hat{\alpha}, \hat{\lambda}, \hat{\beta}) = \hat{\beta}$ then

$$\hat{\beta_s} = \hat{\beta} - \frac{1}{\hat{\alpha}}\sigma_{31} + \frac{a - 1 - b\hat{\lambda}}{\hat{\lambda}}\sigma_{32} - \frac{1}{\hat{\beta}}\sigma_{33} + \frac{1}{2}\left[A\sigma_{13} + B\sigma_{23} + C\sigma_{33}\right], \tag{22}$$

- $v(\hat{\alpha}, \hat{\lambda}, \hat{\beta}) = \hat{R(x)}$ then

$$\hat{R_s} = \hat{R} + (\hat{R_1}a_1 + \hat{R_2}a_2 + \hat{R_3}a_3 + a_4 + a_5) + \frac{1}{2}\Big[A(\hat{R_1}\sigma_{11} + \hat{R_2}\sigma_{12} + \hat{R_3}\sigma_{13}) + B(\hat{R_1}\sigma_{21} + \hat{R_2}\sigma_{22} + \hat{R_3}\sigma_{23}) + C(\hat{R_1}\sigma_{31} + \hat{R_2}\sigma_{32} + \hat{R_3}\sigma_{33})\Big], \tag{23}$$

where,

$$\begin{aligned} \hat{R_1} &= \frac{\partial \hat{R}}{\partial \hat{\alpha}} \\ &= -\left(1 - e^{-(\hat{\lambda}x)^{\hat{\beta}}}\right)^{\hat{\alpha}} \log\left(1 - e^{-(\hat{\lambda}x)^{\hat{\beta}}}\right), \\ \hat{R_2} &= \frac{\partial \hat{R}}{\partial \hat{\lambda}} \\ &= \hat{\alpha}\hat{\beta}x\left(-e^{-(\hat{\lambda}x)^{\hat{\beta}}}\right)(\hat{\lambda}x)^{\hat{\beta}-1}\left(1 - e^{-(\hat{\lambda}x)^{\hat{\beta}}}\right)^{\hat{\alpha}-1}, \\ \hat{R_3} &= \frac{\partial \hat{R}}{\partial \hat{\beta}} \\ &= \hat{\alpha}\left(-e^{-(\hat{\lambda}x)^{\hat{\beta}}}\right)(\hat{\lambda}x)^{\hat{\beta}}\log(\hat{\lambda}x)\left(1 - e^{-(\hat{\lambda}x)^{\hat{\beta}}}\right)^{\hat{\alpha}-1}. \end{aligned}$$

Next, we present Baye's estimators using GEL function. Let $\hat{\alpha_g}, \hat{\lambda_g}, \hat{\beta_g}$ and $\hat{R_g}$ denote Baye's estimators of α, λ, β and $R(t)$ respectively. The following steps, for various choice of $v(\hat{\alpha}, \hat{\lambda}, \hat{\beta})$ Bayes estimators for α, λ, β and $R(t)$ respectively,

- $v(\hat{\alpha}, \hat{\lambda}, \hat{\beta}) = \hat{\alpha}^{-c}$ then

$$\hat{\alpha_g} = \hat{\alpha}^{-c} - c\hat{\alpha}^{-(c+1)}\left(-\frac{1}{\hat{\alpha}}\sigma_{11} + \left(\frac{a-1}{\hat{\lambda}} - b\right)\sigma_{12} - \frac{1}{\hat{\beta}}\sigma_{13}\right) + \frac{1}{2}\left(c(c+1)\hat{\alpha}^{-(c+2)}\sigma_{11}\right) - \frac{c\hat{\alpha}^{-(c+1)}}{2}[A\sigma_{11} + B\sigma_{21} + C\sigma_{31}] \tag{24}$$

- $v(\hat{\alpha}, \hat{\lambda}, \hat{\beta}) = \hat{\lambda}^{-c}$ then

$$\hat{\lambda_g} = \hat{\lambda}^{-c} - c\hat{\lambda}^{-(c+1)}\left(-\frac{1}{\hat{\alpha}}\sigma_{21} + \left(\frac{a-1}{\hat{\lambda}} - b\right)\sigma_{22} - \frac{1}{\hat{\beta}}\sigma_{23}\right) + \frac{1}{2}\left(c(c+1)\hat{\lambda}^{-(c+2)}\sigma_{22}\right) - \frac{c\hat{\lambda}^{-(c+1)}}{2}[A\sigma_{12} + B\sigma_{22} + C\sigma_{32}] \tag{25}$$

- $v(\hat{\alpha}, \hat{\lambda}, \hat{\beta}) = \hat{\beta}^{-c}$ then

$$\hat{\beta_g} = \hat{\beta}^{-c} - c\hat{\beta}^{-(c+1)}\left(-\frac{1}{\hat{\alpha}}\sigma_{31} + \left(\frac{a-1}{\hat{\lambda}} - b\right)\sigma_{32} - \frac{1}{\hat{\beta}}\sigma_{33}\right) + \frac{1}{2}\left(c(c+1)\hat{\beta}^{-(c+2)}\sigma_{33}\right) - \frac{c\hat{\beta}^{-(c+1)}}{2}[A\sigma_{13} + B\sigma_{23} + C\sigma_{33}] \tag{26}$$

- $v(\hat{\alpha}, \hat{\lambda}, \hat{\beta}) = \hat{R}^{-c}$ then

$$\hat{R_g} = \hat{R}^{-c} + (\hat{R_1}a_1 + \hat{R_2}a_2 + \hat{R_3}a_3 + a_4 + a_5) + \frac{1}{2}\left[A(\hat{R_1}\sigma_{11} + \hat{R_2}\sigma_{12} + \hat{R_3}\sigma_{13}) + B(\hat{R_1}\sigma_{21} + \hat{R_2}\sigma_{22} + \hat{R_3}\sigma_{23}) + C(\hat{R_1}\sigma_{31} + \hat{R_2}\sigma_{32} + \hat{R_3}\sigma_{33})\right], \tag{27}$$

where

$$\hat{R_i} = \frac{\partial \hat{R}}{\partial \hat{\eta_i}}, i = 1, 2, 3 \ \ and \ \ (\hat{\eta_1}, \hat{\eta_2}, \hat{\eta_3}) = (\hat{\alpha}, \hat{\lambda}, \hat{\beta}).$$

Observe that all equations define above depends upon MLEs of α, λ and β. The detailed procedure for obtaining MLE is discussed in Section 2.2. Moreover, these MLEs don't have closed form studies. Note that we resorted to using Newton Raphson method for solving equations for obtaining MLEs numerically . Then next Section present the simulation study to obtain Bayes estimators for various parameters of EW distribution and the reliability function $R(t)$.

3. Simulation

In this Section, The results obtained in previous section, are illustrated by means of simulation. The data simulated by using R programming language are used to obtain Baye's estimators of parameters of EW distribution, namely, α, λ, β and $R(t)$. Further, the performance of these estimators are studied by computing their respective mean square error and standard deviation. The following subsection will describe the details of simulation procedure.

3.1. Simulation Algorithm

Let us assume that prior distribution for $\alpha \sim U(0,1), \lambda \sim Gamma(a, b)$ and $\beta \sim U(0,1)$ are chosen at random.

If the random variable U follows a uniform distribution in (0, 1), then $X = \left[-\frac{1}{\lambda}\log\left(1 - U^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{\beta}}$ follows the $GW(\alpha, \lambda, \beta)$. Next, progressive type-I interval censored sampling data, $D = (X_i, R_i, t_i)_m$, of the $GW(\alpha, \lambda, \beta)$, are generated as follows. First, the random variables, $U_1, U_2, ..., U_n, n \leq m$, are generated from $U(0,1)$, and then $GW(\alpha, \lambda, \beta)$ data $t'_1, t'_2, ..., t'_k, ..., t'_n$ are calculated by inverting

$t'_k = \left[-\frac{1}{\lambda}\log\left(1-U_k^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{\beta}}$. Now,the number, X_i, of failures within $(t_{(i-1)}, t_i]$ are generated and R_i surviving items are randomly removed from the testing based on the pre-specifie inspection times $t_1 < ... < t_m$ and the pre-specifie percentage $p = (p_1, p_2, ..., p_{m-1}, 1)$, respectively. The specifi steps are as given below.(see, Aggarwala [?])

- Set $X_0 = 0$ and $R_0 = 0$ and for $i = 1, 2, ..., m$
- $X_i \mid X_{i-1}, ..., X_0, R_{(i-1)}, ..., R_0 \sim rbinom\left(n - \sum_{i=1}^{j-1}(X_j + R_j), \frac{F_i - F_(i-1)}{1-F_(i-1)}\right)$
- $R_i \mid X_i, ..., X_0, R_{(i-1)}, ..., R_0 = floor\left[p_i * \left(n - \sum_{j=1}^{i} X_j - \sum_{j=1}^{i-1} R_j\right)\right]$

where rbinom(n,p) generates a random variable from the binomial distribution with parameters n and p.

3.2. Example

Let the priors $\alpha \sim U(0,1)$, $\lambda \sim Gamma(1,2)$ and $\beta \sim U(0,1)$ and a set of parameters α,λ and β are generated from these distributions. Let us assume that values for $\alpha = 0.4650936$, $\lambda = 0.09790184$, $\beta = 0.2090737$ and $R(t;\alpha,\lambda,\beta)_{t=1} = 0.1592157$ are selected from this set as true values. Let us assume that m=8.Then, the randomly generated data are chosen from the Uniform distribution U(0,1) as follows:

U=(0.8716594, 0.6916711, 0.3129649, 0.3065460, 0.7183383, 0.3928726, 0.4819814, 0.6090094)

To generate the inspection time set of the gradually type-I interval censored sample by appling $t'_k = \left[-\frac{1}{\lambda}\log\left(1-U_k^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{\beta}}$ is given by,

T=(0.4273016, 0.5336827, 6.341113, 10.02617, 63.84012, 108.4094, 223.2485, 595.9245)

To create distinct progressive type-I interval censored samples, four group sample sizes n=10,15,20,25,30,35,40,45 and fi e pre-specifie percentages p: $p_{(1)}$ and $p_{(2)}$ are considered, where

$$p_{(1)} = (0, 0, 0, 0, 0, 0, 0, 1),\ p_{(2)} = (0.1, 0, 0, 0, 0, 0, 0, 1)$$

In Tables 1 and 2, for specifie $p_{(1)}$ and $p_{(2)}$ in progressive type I interval censoring, relative error (Re) and mean square error (MSE) of Bayesian estimators under SEL function (B_S) and Linex Loss function (B_L) with $c = 0.5$, are permited. Note that Re is given by

$$Re = \frac{|\hat{g} - g|}{g}$$

and MSE is given by

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(\hat{g}_i - g_i)^2,$$

where $\hat{g}$ denote the MLEs or Bayesian estimates of g.

After an extensive study of the results thus obtained, conclusions are drawn regarding the behavior of the errors of estimators, which are summarized below graphically(see Figure 3- Figure 14) .

Table 1: *RE and MSE of the Example for fixed* $p = p_{(1)}$

		RE				MSE			
Item	n	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{R}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{R}$
MLE	10	0.9149	0.7544	0.2869	0.7365	0.0902	0.0098	0.0023	0.0154
	15	0.9703	0.0017	0.3649	0.9462	0.0231	0.0000	0.0027	0.0076
	20	0.9341	0.1479	0.3482	0.8148	0.0536	0.0002	0.0029	0.0097
	25	0.0909	0.8181	0909	0.1052	0.0000	0.1202	0.0005	0.0007
	30	0.5389	0.1793	0.5890	0.6659	0.0004	0.0022	0.0098	0.0161
	35	0.9808	0.1949	0.3735	0.9434	0.1066	0.0000	0.0102	0.0114
	40	0.4328	0.3296	0.6769	0.6602	0.0024	0.0061	0.0113	0.0176
	45	0.9366	0.0389	0.3552	0.9200	0.0833	0.0000	0.0085	0.0047
B_s	10	0.5954	0.5185	0.8065	0.5529	0.0382	0.2128	0.0179	0.0087
	15	1.0388	0.2083	1.2265	0.2629	0.0265	0.0336	0.3157	0.0589
	20	0.8271	1.2446	1.7592	0.5632	0.0420	0.0123	0.3435	0.0047
	25	0.4623	0.7895	0.4622	0.8032	0.0162	0.1044	1.2557	0.0381
	30	0.1860	0.2859	0.0094	0.0636	0.0000	0.5612	0.0000	0.0002
	35	0.6569	1.4812	0.2202	0.0294	0.0478	0.0007	0.0262	0.0000
	40	0.3758	0.6661	0.2903	0.0621	0.0000	0.0244	0.0021	0.0002
	45	0.0828	0.4764	1,5272	0.1365	0.0007	0.0000	0.1572	0.0001
B_g	10	0.1253	1.7028	1.1261	0.5510	0.0017	0.0498	0.0349	0.0086
	15	0.3227	0.3916	1.4387	0.2632	0.0026	0.1188	0.0434	0.0591
	20	0.0519	1.9679	1.2674	0.5606	0.0002	0.0699	0.0390	0.0046
	25	1.1834	0.2332	0.4580	0.2034	0.1062	0.0091	0.0123	0.2439
	30	0.0138	0.3522	0.7898	0.7545	0.0000	0.0084	0.0343	0.0207
	35	0.7894	1.8737	0.3078	0.2895	0.0690	0.0005	0.0069	0.0000
	40	1.1188	0.4942	1.2519	1.3603	0.0161	0.0137	0.0386	0.0745
	45	0.0184	0.4489	0.3615	0.1628	0.0000	0.0000	0.0088	0.0001

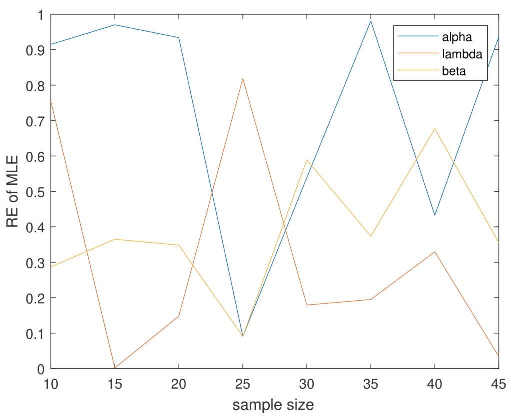

Figure 3: *Relative Error of MLE for* $p(1)$

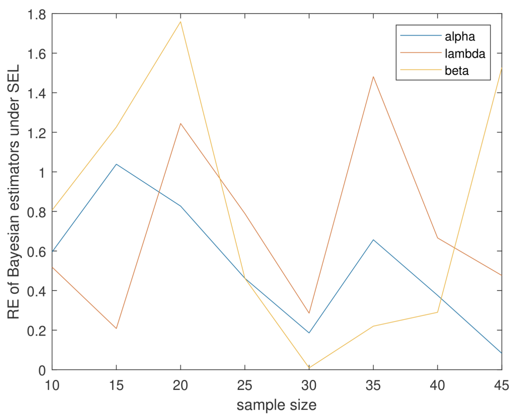

Figure 4: *Relative Error of* B_s *for* $p(1)$

Table 2: *RE and MSE of the Example for fixed $p = p_{(2)}$*

		RE				MSE			
Item	n	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{R}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{R}$
MLE	10	0.9009	0.3422	0.3621	0.7676	0.0837	0.0003	0.0066	0.0098
	15	0.9481	0.3285	0.3743	0.8974	0.0134	0.0004	0.0003	0.0070
	20	0.9152	0.4706	0.2872	0.7947	0.0629	0.0172	0.0014	0.0081
	25	0.9561	0.0317	0.3609	0.8741	0.0663	0.0000	0.0016	0.0085
	30	0.6986	0.4347	0.8745	0.7925	0.0252	0.0160	0.0822	0.0247
	35	0.9852	0.9342	0.6004	0.8572	0.0715	0.0013	0.0333	0.0127
	40	0.9431	0.3594	0.2985	0.8779	0.0284	0.0033	0.0007	0.0073
	45	0.9467	0.6701	0.2837	0.8926	0.0244	0.0007	0.0014	0.0059
B_s	10	0.0605	0.4409	0.3096	1.2529	0.0004	0.0005	0.0048	0.0262
	15	1.9421	0.1121	1.5398	1.9250	0.0563	0.0000	0.0249	0,0744
	20	0.1961	0.0578	0.4542	0.7478	0.0029	0.0003	0.0035	0.0072
	25	0.5377	0.2024	0.4913	0.2352	0.0210	0.0338	0.0030	0.0006
	30	0.3550	0.0855	0.6875	0.1301	0.0065	0.0999	0.0508	0.0006
	35	1.5559	1.5346	0.5596	1.3814	0.1783	0.3580	0.0289	0.0330
	40	0.512	0.7987	0.4375	1.0897	0.0084	0.0165	0.1403	0.0413
	45	0.4568	0.8569	0.1748	1.8877	0.0057	0.0314	0.0005	0.0265
B_g	10	0.0173	0.4638	0.5745	1.2778	0.0000	0.0005	0.0166	0.0272
	15	0.7190	1.6573	1.0796	1.1535	0.0077	0.0840	0.0086	0.0403
	20	0.1677	0.2676	1.7243	0.7769	0.0021	0.0056	0.0501	0.0078
	25	0.1207	0.3234	1.1491	0.2385	0.0011	0.0729	0.0582	0.0006
	30	0.5552	0.2137	0.0639	0.2737	0.0159	0.0038	0.0004	0.0000
	35	1.2341	1.1229	0.1248	1.3814	0.1122	0.1917	0.0014	0.0329
	40	0.4512	1.2013	0.3081	1.3114	0.0065	0.0372	0.0712	0.0466
	45	0.5546	2.6507	1.6666	1.9275	0.0083	0.0978	0.0488	0.0276

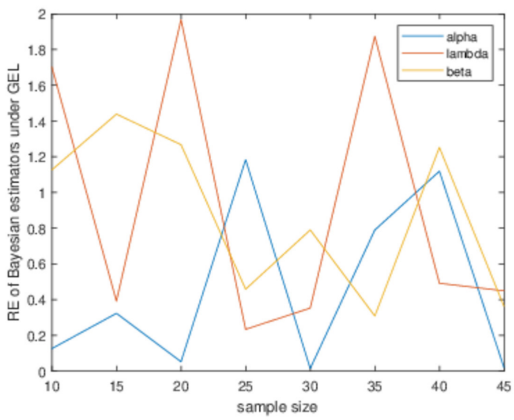

Figure 5: *Relative Error of B_g for $p(1)$*

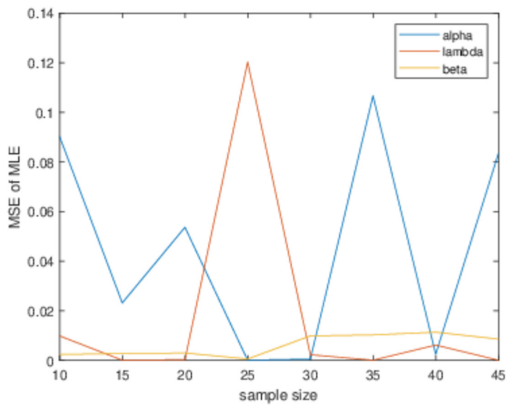

Figure 6: *Mean Squared Error of MLE for $p(1)$*

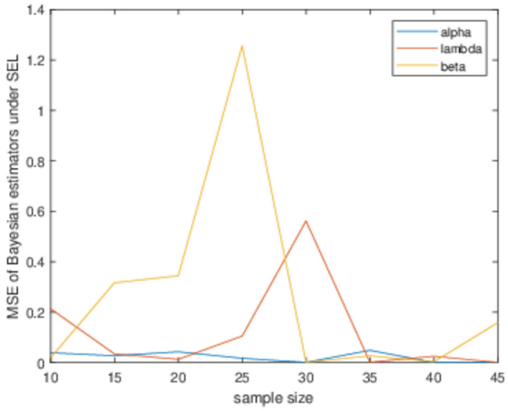

Figure 7: *Mean Squared Error of B_s for $p(1)$*

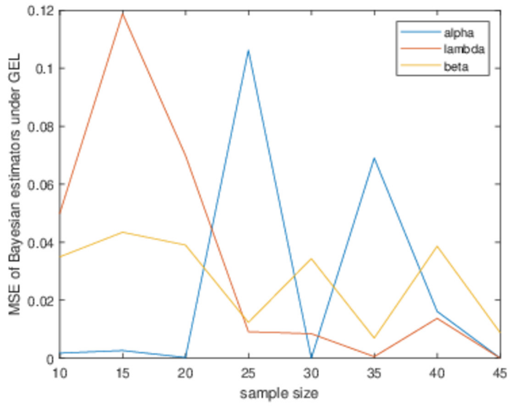

Figure 8: *Mean Squared Error of B_g for $p(1)$*

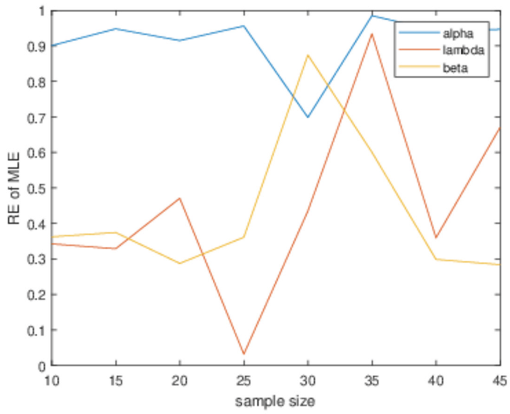

Figure 9: *Relative Error of MLE for $p(2)$*

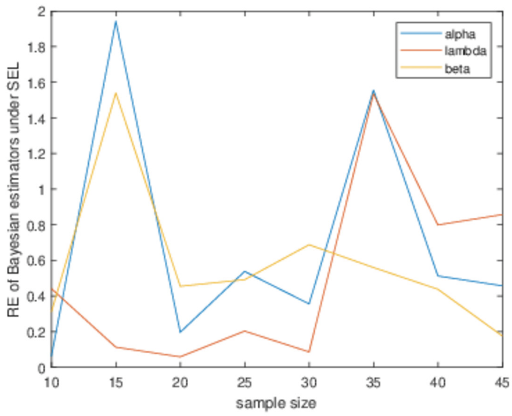

Figure 10: *Relative Error of B_s for $p(2)$*

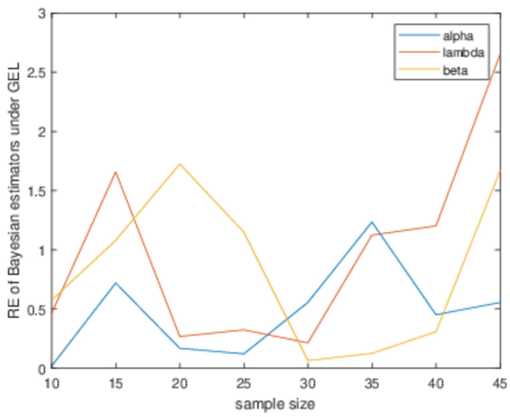

Figure 11: *Relative Error of B_g for $p(2)$*

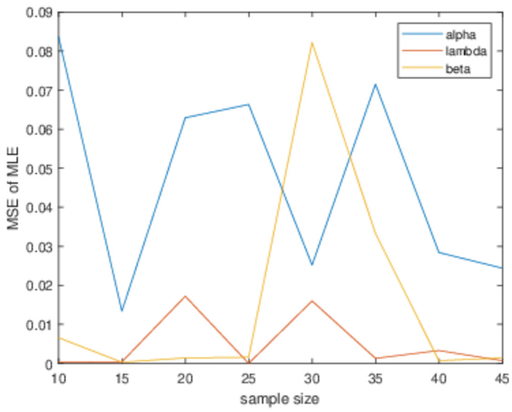

Figure 12: *Mean Squared Error of MLE for $p(2)$*

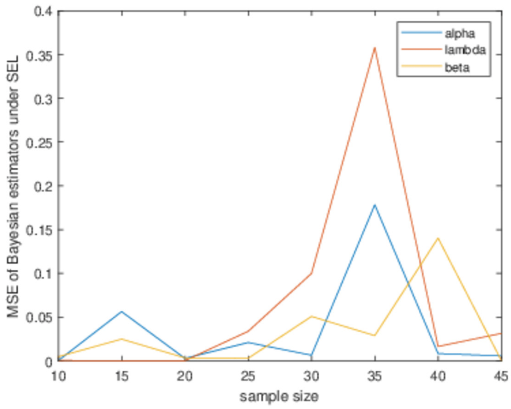

Figure 13: *Mean Squared Error of B_s for $p(2)$*

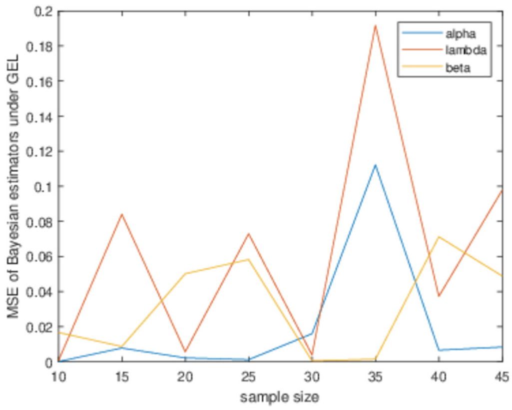

Figure 14: *Mean Squared Error of B_g for $p(2)$*

4. Conclusion

In this article, the performance of the proposed Bayes estimators has been compared to the maximum likelihood estimator of the $EWD(\alpha, \lambda, \beta)$ under the progressive type-I interval censoring based on the squared error loss function and general entropy loss function using Lindley's approximation. The simulation result indicates that this approach is better suited for small sample sizes. MLE is the best choice when compared to Bayesian estimators. From Table 1, it is observed that the general entropy loss function in Bayesian estimation is better as compared to the squared error loss function in terms of MSE. From Table 2, it is noted that the squared error loss function in Bayesian estimation is better as compared to the general entropy loss function in terms of MSE. It can be seen from Figures 4, 5, 10 and 11 that the RE of Bayes estimators show fluctuatio trend, and one can not see continuously decreasing or increasing trend for RE.

It is observed in practice, especially while modeling lifetime of electronic products, this three-parameter EW distribution describes the lifetime in the best possible way as compared to commonly used lifetime distributions such as Exponential distribution or Weibull distribution. Moreover, practically progressive type I interval censoring is the most convenient way of obtaining data of lifetimes as compared to traditional censoring schemes such as type I or type II or hybrid censoring. Further, the results obtained in this paper can be used for applications in the fiel of economics or analysis of clinical data in the medical field

The results obtained in this paper use the approximation process such as Lindley approximation to obtain Bayes estimators of parameters of EW distribution. As future scope of research an analytical solution for deriving Bayes estimators can be considered by using suitable choice of prior distributions.

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