THE ROLE OF RECORD VALUES IN STATISTICAL INFERENCE: A REVIEW ARTICLE

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Abstract

The record values data have received the attention of researchers in statistics for over seven decades. Through these decades the records have played a significant and widely utilized role for statistical inference in parameter estimation, predicting future values, hypothesis tests, as well as stress-strength tests, and characterizing distributions. In this paper, the types of record values, some distributional properties, and statistical inferences of record values and their applications are reviewed. The purpose of this paper is to shed light on the role of record values in statistical inference. Therefore, we will examine this issue from two perspectives, the first perspective being estimation and the second perspective being prediction. These are through some of the most important lifetime distributions are Exponential, Weibull, Gumbel, Geometric, Pareto, Generalized exponential, Rayleigh, Lomax, and Nadarajah-Haghighi distributions. I hope that the findings of this paper will be useful for researchers in various fields and lead to further enhancement of research in record values theory and its applications.

Keywords: maximum likelihood estimation; maximum likelihood predictor; record values; point prediction; probability distribution; Bayesian estimation; Bayesian prediction

1. Introduction

In statistics, a record value or record statistic is the largest or smallest value obtained from a sequence of random variables. Record values arise naturally in both theoretical and practical areas of probability and statistics. On the practical side, Record values are of interest and importance in several branches of studies such as, hydrology, seismology, psychology, medicine, engineering. All of us constantly hear of new records being created in events such as stock market prices, rainfall, temperature, flood level, athletic events, oil, and mining surveys etc. In any field, whenever a new high or a new low value is observed, in connection with the phenomena under study, it becomes a part of history and will be called as a record. On the theoretical side, various statistical inference procedures such as point or interval estimation and prediction as well as hypothesis testing can be developed based on observed record sequences.

Records become extremely important and necessary in some cases, including when we only want to study the value of the events that exceed the previous ones, or when observations are destroyed by experimental tests, or it is impossible to obtain a complete sample. Overall, records can be useful in any situation where there is a need to track and analyze data over time.

The aim of this study to shed light on the role of record values in statistical inference. The rest of this paper reviews the records from the following aspects: Section 2 introduces the definition of record values and their distributional properties. Section 3 introduces a literature review on uses of record values in statistical inference. Section 4 reviews the record values data in practical applications. Section 5 reviews the computer software for records. Section 6 discusses the future works. Section 7 introduces some conclusions.

2. Definition of Record Values and their Distributional Properties

The formal study of record value theory probably started with the pioneering paper by Chandler [1]. In this section, the definition of upper and lower and k-th record values and their distributional properties are introduced.

2.1 Upper Record Values

Let $X_1, X_2, \dots X_n, \dots$ be a sequence of independent and identically distributed (iid) random variables that have cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Let $Y_n = max\{X_1, X_2, \dots, X_n\}$ for $n \geq 1$. We say X_j is an upper record value of this sequence if $Y_j > Y_{j-1}$, $j > 1$. Thus X_j will be called an upper record value if its value exceeds that of all previous observations. The first record $Y_1 = X_1$ is called the trivial record.

The times at which records appear are of interest and are called record times. The random variables $U(0) = 1$, and $U(m) = min\{j: j > U(m-1),\ X_j > X_{U(m-1)}\}$ are called the upper record times, and the sequence $\{U_{(m)}, m \geq 0\}$ is called the sequence of upper record times.

The sequence of inter-record times, denoted by $\{T_n\ ,\ n \geq 1\}$, is defined as $T_n = U(n+1) - U(n), n = 1,2, \dots$

Many distributional properties of upper record values in the sequence of iid continuous random variables $X_1, X_2, \dots,\ X_{U(m)}$ with cdf F(x) and pdf f(x) have been expressed in terms of the function $\mathrm{R(x)} = -\ln[1 - \mathrm{F(x)}]$. The pdf of the upper record value $X_{U(m)}$,(see Arnold, et al. [2]) is

$$f_m(x) = \frac{(\mathrm{R(x)})^{m-1}}{(m-1)!} f(x), \qquad -\infty < x < \infty \tag{2.1}$$

and the joint pdf of the first (m) upper record values $X_{U(1)} = x_1, X_{U(2)} = x_2, \dots,\ X_{U(m)} = x_m$ is given by

$$f_{1,2,\dots,m}(x_1, x_2, \dots, x_m) = f(x_m) \prod_{i=1}^{m-1} \frac{f(x_i)}{1 - F(x_i)}, \tag{2.2}$$

and the joint pdf of the upper record values $X_{U(n)}$ and $X_{U(m)}$ $(n < m)$ is

$$f_{n,m}(x, y) = \frac{[\mathrm{R(x)}]^{n-1}}{(n-1)!\,(m-n-1)} \cdot \frac{f(x)}{1 - F(x)} \cdot [\mathrm{R(y)} - \mathrm{R(x)}]^{m-n-1} f(y),$$
$$-\infty < x < y < \infty, \qquad n = 0,1, \cdots, n < m, \tag{2.3}$$

the conditional probability density function of the upper record values $X_{U(j)}$ given $X_{U(i)}$ can be expressed as follows

$$f(x_j | x_i) = \frac{\left(R(x_j) - R(x_i)\right)^{j-i-1}}{(j-i-1)} \frac{f(x_j; \theta)}{1 - F(x_i; \theta)},$$
$$-\infty < x_i < x_j < \infty \tag{2.4}$$

where $R(.) = -\ln(1 - F(.))$

2.2 Lower Record Values

Let $X_1, X_2, \dots X_n, \dots$ be a sequence of iid random variables from a continuous distribution with cdf $F(x)$ and pdf $f(x)$. Let $Y_n = min\{X_1, X_2, \dots, X_n\}$ for $n \geq 1$. We say X_j is a lower record value of this sequence if $Y_j < Y_{j-1}$, $j > 1$. Thus X_j will be called a lower record value if its value is lower than of all previous observations. By definition X_1 is a lower record value. The times at which record appear are of interest which called a record times.

The random variables $L(0) = 1$, and $L(m) = min\{j: j > L(m-1),\ X_j < X_{L(m-1)}\}$ are called the lower record times, and the sequence $\{L_{(m)}, m \geq 0\}$ is called the sequence of lower record times.

Many distributional properties of lower record values in the sequence of iid continuous random variables $X_1, X_2, \dots,\ X_{L(m)}$ with cdf F(x) and pdf f(x) have been expressed in terms of the function $G(x) = -\ln F(x)$. The pdf of the lower record value $X_{L(m)}$, is

$$f_m(x) = \frac{\left(G(x)\right)^{m-1}}{(m-1)!} f(x), \qquad -\infty < x < \infty \tag{2.5}$$

and the joint pdf of the first (m) lower record values $X_{L(1)} = x_1, X_{L(2)} = x_2, \dots, X_{L(m)} = x_m$ is given by

$$f_{1,2,\dots,m}(x_1, x_2, \dots, x_m) = f\left(x_{(m)}\right) \prod_{i=1}^{m-1} \frac{f\left(x_{(i)}\right)}{F\left(x_{(i)}\right)}, \tag{2.6}$$

$$-\infty < x_m < x_{m-1} < \dots < x_1 < \infty$$

and the joint pdf of the lower record values $X_{L(s)}$ and $X_{L(r)}$ $(r < s)$ is

$$f_{r,s}(x, y) = \frac{[G(x)]^{r-1}}{(r-1)!\,(s-r-1)!} \cdot \frac{f(x)}{F(x)} [G(y) - G(x)]^{s-r-1} f(y) \tag{2.7}$$

$$-\infty < y < x < \infty$$

where $x = X_{L(r)}$ and $y = X_{L(s)}$.

and the conditional pdf of the lower record values $X_{L(j)}$ given $X_{L(i)}$ can be expressed as follow

$$f(x_j | x_i) = \frac{\left(G(x_j) - G(x_i)\right)^{j-i-1}}{(j-i-1)!} \frac{f(x_j;\theta)}{F(x_i;\theta)}, \qquad -\infty < x_j < x_i < \infty \tag{2.8}$$

where $G(.) = -\ln F(.)$ for more details, see for example, Ahsanullah and Nevzorov [3].

2.3 The K-th Upper Record Values

Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables with a cdf $F(x)$ and pdf $f(x)$. The j-th order statistic of the sample $X_1, X_2, \dots, X_n$ is denoted by $X_{j:n}$. For a fixed positive integer k, Dziubdziela and Kopociński [4] defined the sequence $\left\{U_n^{(k)}, n \geq 1\right\}$ of k-th upper record times for the sequence $\{X_n, n \geq 1\}$ as follows:

$$U_1^{(k)} = 1$$

$$U_{n+1}^{(k)} = min\left\{j > U_n^{(k)}: X_{j:j+k-1} > X_{U_n^{(k)}:U_n^{(k)}+k-1}\right\},$$

Then the sequence $\left\{Y_n^{(k)}, n \geq 1\right\}$, where $Y_n^{(k)} = X_{U_n^{(k)}:U_n^{(k)}+k-1}$ is called a sequence of k-th upper record values of $\{X_n, n \geq 1\}$. For convenience, we also take $Y_0^{(k)} = 0$. Note that for k = 1, we get the usual upper record values as defined in Chandler [1].

The pdf of $Y_n^{(k)}$ (n ≥ 1) as given by Grudzien [5] is

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x), \quad -\infty < x < \infty \tag{2.9}$$

and the joint pdf of $Y_m^{(k)}$ and $Y_n^{(k)}$, $1 \leq m < n,\ n \geq 2$, is

$$f_{Y_m^{(k)},Y_n^{(k)}}(x,y) = \frac{k^n}{(m-1)!\,(n-m-1)!}[-\ln \bar{F}(y) + \ln \bar{F}(x)]^{n-m-1} \times [-\ln \bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}[\bar{F}(y)]^{k-1}f(y), \quad x < y \tag{2.10}$$

2.4 The K-th Lower Record Values

Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables with a cdf $F(x)$ and pdf $f(x)$. The j-th order statistic of a sample $X_1, X_2, \dots, X_n$ is denoted by $X_{j:n}$. For a fixed positive integer k, we defined the sequence $\{L_k(n), n \geq 1\}$ as k-th lower record times of $\{X_n, n \geq 1\}$ as follows:

$$L_k(n) = 1$$
$$L_k(n+1) = min\{j > L_k(n): X_{k:L_k(n)+k-1} > X_{k:j+k-1}\},$$

The sequence $\{Y_n^{(k)}, n \geq 1\}$, where $Y_n^{(k)} = X_{k:L_k(n)+k-1}$ is called a sequence of k-th lower record values of $\{X_n, n \geq 1\}$. For convenience, we also take $Y_0^{(k)} = 0$. Note that for k = 1, we get the usual lower record values as defined in Chandler [1].

The pdf of Y(k)n (n ≥ 1) as given by Grudzien [5] is

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!}[-\ln F(x)]^{n-1}[F(x)]^{k-1}f(x), \quad n \geq 1 \tag{2.11}$$

and the joint pdf of $Y_m^{(k)}$ and $Y_n^{(k)}$, $1 \leq m < n,\ n \geq 2$, is

$$f_{Y_m^{(k)},Y_n^{(k)}}(x,y) = \frac{k^n}{(m-1)!\,(n-m-1)!}[-\ln F(y) + \ln F(x)]^{n-m-1} \times [-\ln F(x)]^{m-1}\frac{f(x)}{F(x)}[F(y)]^{k-1}f(y), \quad y < x \tag{2.12}$$

3. Literature Review of Statistical Inference of Record Values

This section presents a review of the statistical literature to emphasize the role of record values in statistical inference. Therefore, we will examine this issue from two perspectives, the first perspective being estimation and the second perspective being prediction. These are through some of the certain distributions are Exponential, Weibull, Gumbel, Geometric, Pareto, Generalized exponential, Rayleigh, Lomax, and Nadarajah-Haghighi distributions.

3.1 Review of Previous Studies on Estimation based on Record Values

In this subsection, previous studies on estimation problems based on record values for some of the certain distributions are reviewed.

3.1.1 Exponential distribution

Jaheen [6] obtained the ML and empirical Bayes estimate for the parameter of the exponential model based on record statistics. The estimate is obtained using the squared error loss and Varian's linear-exponential (LINEX) loss functions. Ahmadi and Doostparast [7] presented Bayes estimation when the data consist of k record values from a two-parameter exponential distribution under linear exponential loss function. Balakrishnan and Stepanov [8] discussed the Fisher information contained in records. In the case when the initial distribution belongs to the exponential family. Doostparast [9] derived the Bayesian and non-Bayesian estimates for the two parameters of the exponential distribution based on lower record values, with respect to the squared error (SE) and LINEX loss functions, and then compared with together. Arnold, et al. [2] obtained the ML estimates and best linear unbiased estimator (BLUE) for the exponential distribution. Wu [10] presented the interval estimation for the scale parameter of two-parameter exponential distribution using upper record values. In addition, two methods for the joint confidence region of two parameters are proposed. Asgharzadeh, et al. [11] proposed two families of optimal confidence regions for the location and scale parameters of the two-parameter exponential distribution based on upper records. Constrained optimization problems are used to find the smallest-area confidence regions for the exponential parameters with a specified confidence level. Baklizi [12] considered the stress-strength reliability when the available data is in the form of record values from the one parameter and two parameters exponential distribution. The ML estimators and the associated confidence intervals are derived. Ahsanullah and Aliev [13] considered several distributional properties of the upper records from the exponential distribution and presented some characterizations of the exponential distribution.

3.1.2 Weibull distribution

Abd-El-Hakim and Sultan [14] obtained the maximum likelihood estimators for the location and scale parameters of Weibull distribution based on upper record values. Soliman, et al. [15] discussed a Bayesian analysis in the context of record statistics values from the two-parameter Weibull distribution. The ML and the Bayes estimates based on record values are derived for the two unknown parameters and some survival time parameters e.g., reliability and hazard functions. The Bayes estimates are obtained based on a conjugate prior for the scale parameter and a discrete prior for the shape parameter of this model. This is done with respect to both symmetric loss function (squared error loss), and asymmetric loss function (linear-exponential (LINEX)) loss function. Jafari and Zakerzadeh [16] proposed a simple and exact test and a confidence interval for the shape parameter. In addition to a generalized confidence interval, a generalized test variable is derived for the scale parameter when the shape parameter is unknown. The paper presents a simple and exact joint confidence region as well. Wang and Ye [17] investigated point estimation and confidence intervals estimation for the Weibull distribution based on record data. The uniformly minimum variance unbiased estimator for the Weibull shape is derived. Based on this estimator, a bias-corrected estimator for the Weibull scale is obtained and it is shown to have much smaller bias and mean squared error compared with the maximum likelihood estimator. Confidence intervals for parameters and reliability characteristics of interest are constructed using pivotal or generalized pivotal quantities. Then the results are extended to the stress–strength model involving two Weibull populations with different parameter values. Construction of confidence intervals for the stress–strength reliability is discussed. Raqab, et al. [18] considered the problem of the estimation for the 3-parameter Weibull distribution based on record data. The maximum likelihood method is used for the estimation of all parameters involved in the model. Hassan, et al. [19] investigated the estimation of multicomponent stress-strength reliability following Weibull distribution based on upper record values. Al-Duais [20] developed a LINEX loss function to estimate the parameters and reliability function of the Weibull distribution based on upper record values when both shape and scale parameters are unknown. They performed this by merging a weight into LINEX to produce a new loss function called the weighted linear exponential (WLINEX) loss function. Then, they utilized WLINEX to derive the parameters and reliability function of the Weibull distribution. The results revealed that the proposed method is the best for estimating parameters and has good performance for estimating reliability.

3.1.3 Gumbel distribution

Ahsanullah [21] obtained ML, best linear invariant and minimum variance unbiased (MVU) estimators of the Gumbel location and scale parameters. Mousa, et al. [22] obtained the Bayesian estimators for the two parameters of the Gumbel distribution based on lower record values. Malinowska and Szynal [23] obtained Bayesian estimation for the two parameters of a Gumbel distribution based on k-th lower record values. Seo and Kim [24] addressed inference problems for

Gumbel distribution when the available data are lower record values. they first derive unbiased estimators of unknown parameters, and then, they construct an exact confidence interval for the scale parameter by deriving certain properties and pivotal quantities. For Bayesian inference, they derive noninformative priors such as the Jeffreys and reference priors for unknown parameters and examine whether they satisfy the probability matching criteria; then, they apply them to develop objective Bayesian analysis. Asgharzadeh, et al. [25] presented exact confidence intervals and joint confidence regions for the parameters of Gumbel distribution based on record data. Exact confidence intervals and joint confidence regions for the parameters of inverse Weibull distribution are also discussed. Three numerical examples with climate data are presented to illustrate the proposed methods.

3.1.4 Geometric distribution

Ahsanullah and Holland [26] discussed some distributional properties of the record values of non-identically distributed random variables having geometric distributions. Three theorems dealing with the characterization of the geometric distribution based on these distributional properties are presented. The unique minimum variance unbiased estimators of some functions of the parameters of the distribution are studied. Ahmadi and Doostparast [7] obtained Bayesian and non-Bayesian estimators of the parameter of geometric distribution based on upper record values. Okasha and Wang [27] E-Bayesian and Bayesian methods have been used for estimating the parameter, reliability, and hazard functions of the geometric distribution based on upper record value samples. Francis, et al. [28] obtained the shrinkage estimate of $R = P(X \leq Y)$ when X the stress and Y the strength are independent geometric variable and the sample on Y the strength is upper records.

3.1.5 Pareto distribution

Arnold and Press [29] discussed the Bayesian estimation for Pareto data based on record values. El-Qasem [30] used the upper record values to obtain the ML estimator for the uniform, the exponential and the Pareto distribution with one parameter. Sultan and Moshref [31] obtained the best linear unbiased estimates for the location and scale parameters of record values from the generalized Pareto distribution. Raqab, et al. [32], Raqab [33] obtained the ML and Bayes estimators from the two-parameter Pareto distribution for the two unknown parameters based on record values. Doostparast, et al. [34] on the basis of record values from the two-parameter Pareto distribution, ML and Bayes estimators as well as credible regions are developed for the two parameters of the Pareto distribution. Ahsanullah and Shakil [35] established some new results on the characterizations of the Pareto distribution by upper record values. Azhad, et al. [36] discussed inferences about the multicomponent stress strength reliability are drawn under the assumption that strength and stress follow independent Pareto distribution under the setup of upper record values. The ML estimator, Bayes estimator under squared error and LINEX loss functions, of multicomponent stress-strength reliability are constructed.

3.1.6 Generalized exponential distribution

Jaheen [6] derived Bayes and empirical Bayes estimators for the one-parameter of the generalized exponential distribution based on lower record values. These estimates are obtained based on squared error and LINEX loss functions. Madi and Raqab [37] used the importance sampling to estimate the model parameters. Baklizi [38] considered the ML and Bayesian estimation of the stress-strength reliability based on lower record values from the generalized exponential distribution. Confidence intervals, exact and approximate, as well as the Bayesian credible sets for the stress-strength reliability are obtained. Dey, et al. [39] derived the ML estimates and the Bayes estimates based on lower records for the unknown parameters of the generalized exponential distribution. The Bayesian estimation of the parameters of the generalized exponential distribution has been studied with respect to both symmetric and asymmetric loss functions. They have also derived the Bayes interval. Sana and Faizan [40] obtained ML estimators for the two unknown parameters of the generalized exponential distribution based on lower record values. They also obtained the Bayes estimators of the unknown parameters using Lindley's approximation under symmetric and asymmetric loss functions.

3.1.7 Rayleigh distribution

Balakrishnan and Chan [41] derived explicit expressions for the means, variances and covariances from a Rayleigh distribution. They also established some recurrence relationships for the single and product moments. These results are then used to derive explicitly the best linear unbiased estimators for the scale-parameter as well as the location-scale parameter cases. Hendi, et al. [42] obtained the Bayes estimators for the parameter, reliability function, and failure rate function based on upper record values of Rayleigh distribution. These estimators are obtained on the basis of square error and LINEX loss functions. Soliman and Al-Aboud [43] obtained the estimators of the parameter of Rayleigh distribution based on upper record values, Bayesian and non-Bayesian approaches have been used to obtain the estimators of the parameter, and some lifetime parameters such as the reliability and hazard functions. Ahsanullah and Shakil [44] established some results on characterizations of Rayleigh distribution based on order statistics and record values. Seo, et al. [45] provided the exact confidence intervals for unknown by providing some pivotal quantities in the two-parameter Rayleigh distribution based on the upper record values. Finally, the validity of the proposed inference methods was examined from Monte Carlo simulations and real data. Seo and Kim [46] provided an objective Bayesian analysis method based on the objective priors (the Jeffreys and reference priors, and the second-order PMP) for unknown parameters of the two-parameter Rayleigh distribution when the upper record values are observed. Abdi and Asgharzadeh [47] presented exact joint confidence regions for the parameters of the Rayleigh distribution based on record data. By providing some appropriate pivotal quantities, they construct several joint confidence regions for the Rayleigh parameters. These joint confidence regions are useful for constructing confidence regions for functions of the unknown parameters.

3.1.8 Lomax distribution

Lee and Lim [48] characterized the Lomax distribution by conditional expectations of record values. Nasiri and Hosseini [49] obtained ML estimation based on records and a proper prior distribution to attain a Bayes estimation (both informative and non-informative) based on records for quadratic loss and squared error loss functions. The study considers the shortest confidence interval and highest posterior distribution confidence interval based on records. Mahmoud, et al. [50] considered the Bayes estimators of the unknown parameters of the Lomax distribution under the assumptions of gamma priors on both the shape and scale parameters. The Bayes estimators cannot be obtained in explicit forms. So, they propose Markov Chain Monte Carlo (MCMC) techniques to generate samples from the posterior distributions and in turn computing the Bayes estimators. Point estimation and confidence intervals based on ML and bootstrap methods are also used. Mahmoud, et al. [51] addressed the problem of estimating $R = P[Y < X]$ for the Lomax distributions, and classical and MCMC Bayesian analysis for R were developed when both samples on X and Y are in the form of upper record values, observed from the Lomax distribution. Hassan and Zaky [52] considered estimation of entropy for Lomax distribution based on upper record values. Bayesian estimator of Shannon entropy is discussed under informative and non-informative priors. The entropy Bayesian estimator and the corresponding credible interval based on a LINEX, squared error loss functions are derived.

3.1.9 Nadarajah-Haghighi distribution

Selim [53] discussed maximum likelihood and Bayes estimation of the two unknown parameters of Nadarajah and Haghighi distribution based on record values. It assumed that in Bayes case, the unknown parameters of Nadarajah and Haghighi distribution have gamma prior densities. Lindley approximation is exploited to obtain point estimators for the unknown parameters. Sana and Faizan [54] discussed maximum likelihood and Bayes estimation of the two unknown parameters of Nadarajah and Haghighi distribution based on record values. Different Bayes estimates are derived under squared error, balanced squared error and general entropy loss functions by using Jeffreys' prior information and extension of Jeffreys' prior information. Tierney and Kadane approximation method used to compute these estimates. MirMostafaee, et al. [55] obtained exact explicit expressions as well as several recurrence relations for the single and product moments of record values and then these results are used to compute the means, variances and the covariances of the upper record values. Also, these calculated moments are used to find the best linear unbiased estimators of the location and scale parameters of NH distribution. Confidence intervals for the unknown parameters are also discussed.

3.1.10 General classes of distributions

Abu-Youssef [56] characterized general classes of continuous distribution by considering the conditional expectation of function of record values. The specific distribution considered as a particular case of the general class of distribution are Weibull, Pareto, power function, Burr, beta of the first kind, Cauchy, rectangular, Rayleigh, Lomax, and inverse Weibull distributions. Ahmadi and Doostparast [7] obtained Bayesian estimation for the two parameters of some life distributions, including Exponential, Weibull, Pareto and Burr type XII, based on upper record values. Ahmadi, et al. [57] showed how to develop Bayes estimation in the context of upper k-record data from a semi-parametric class of distributions that includes several well-known lifetime distributions such as exponential, Weibull (one parameter), Pareto and Burr type XII under some balanced type of loss functions. Malinowska and Szynal [58] characterized general classes of continuous distributions by the conditional expectation of the kth lower record values. Specific distributions inverse exponential, inverse Weibull, inverse Pareto, negative exponential, negative Weibull, negative Pareto, negative power, Gumbel, exponentiated-Weibull, loglogistic, Burr X, inverse Burr XII and inverse paralogistic distributions.

3.2 Reviewing Previous Studies on Prediction based on Recorded Values

In this section, to accentuate the role of recorded values in statistical prediction, we will review the literature on prediction problems based on record values for some of the certain distributions.

3.2.1 Exponential distribution

Ahsanullah [59] obtained best linear unbiased predictor and best linear invariance predictor of future records X_s based on $X_i, 1 \leq i \leq m$, for $m < s$, using the standard least squares theory. Dunsmore [60] studied the problem of predicting future records from the Bayesian viewpoint and derived classical results for the exponential and the gamma models. Awad and Raqab [61] considered the prediction problem of the future nth record value based on the first m (m<n) observed record values from one parameter exponential distribution. Jaheen [6] obtained empirical Bayes prediction bounds for future record values. Ahmadi and Doostparast [7] presented Bayes prediction procedures when the data consist of k record values from a two-parameter exponential distribution under linear exponential loss function. Ahmadi and MirMostafaee [62] studied the

problem of predicting future records based on observed order statistics from two-parameter exponential distribution. The prediction intervals for the future order statistics as well as for the total lifetime in a future sample of size m from two parameter exponential distribution are obtained on the basis of the first n records coming from the same distribution. Asgharzadeh, et al. [11] proposed two families of optimal confidence regions for the location and scale parameters of the two-parameter exponential distribution based on upper records.

3.2.2 Weibull distribution

Soliman, et al. [15] derived Bayesian predictive density function for Weibull distribution, which is necessary to obtain bounds for predictive interval of future record. Paul and Thomas [63] studied prediction of a future record for Weibull distribution using best linear unbiased predictor. Raqab, et al. [18] considered the problem of prediction for the 3-parameter Weibull distribution based on record data. The ML method is used for the joint prediction of future records along with the estimation of all parameters involved in the model. The existence and uniqueness of the MLPs of future records as well as the PMLEs of all unknown quantities were discussed in detail. Volovskiy and Kamps [64] considered point prediction of future record values from a sequence of independent and identically distributed two-parameter Weibull random variables using the maximum likelihood method. Two likelihood functions for prediction, the predictive and the observed predictive likelihood functions, are considered and the associated predictors are derived. Mean squared error and Pitman closeness criterion are used for comparing the prediction procedures.

3.2.3 Gumbel distribution

Ahsanullah [65] gave two types of predictors of the n-th record value based on the first m ($m<n$) record values. Mousa, et al. [22] obtained the Bayesian predictions, either point or interval, for future lower record values. Malinowska and Szynal [23] obtained Bayesian prediction, either point or interval, for future n-th lower record values. Seo and Kim [24] addressed inference problems for Gumbel distribution when the available data are lower record values. They first derive unbiased estimators of unknown parameters, and then, they construct a predictive interval for the next lower value by deriving certain properties and pivotal quantities.

3.2.4 Geometric distribution

Ahsanullah and Holland [26] discussed some distributional properties of the record values of non-identically distributed random variables having geometric distributions. Three theorems dealing with the characterization of the geometric distribution based on these distributional properties are presented. Also various predictors of the nth record valued utilizing the first $m(m<n)$ record values are studied. Ahmadi and Doostparast [7] considered Bayesian and non-Bayesian prediction, either point or interval, of geometric distribution based on the past record values observed.

3.2.5 Pareto distribution

Arnold and Press [29] discussed the Bayesian prediction for Pareto data based on record values. Madi and Raqab [66] used the Bayesian approach to establish future predictions for the Pareto records. Raqab, et al. [32], Raqab [33] used the Bayesian approach to predicting future record values, either point or interval, from the Pareto distribution based on the past record values observed. Also, the ML prediction of the future records and other classical methods are used for obtaining prediction intervals for the future records. Paul and Thomas [67] studied prediction of future records of Pareto distribution by using best linear unbiased predictors. Shafay, et al. [68] discussed the problem of prediction of the two-parameter Pareto distribution from a future sample. The Bayesian approach is applied to construct predictors based on observed k-record values for the cases when the future sample size is fixed and when it is random. Several Bayesian prediction intervals are derived.

3.2.6 Generalized exponential distribution

Jaheen [6] obtained the prediction bounds for future lower record values from the generalized exponential distribution by using Bayes and empirical Bayes techniques. Madi and Raqab [37] described and used a Bayesian parametric approach to predict the behavior of further Los Angeles rainfall records from generalized exponential distribution. Importance sampling is used to estimate the model parameters, and the Gibbs and Metropolis samplers are used to implement the prediction procedure. Dey, et al. [39] derived the Bayes interval and discussed the Bayesian prediction intervals of the future record values based on the observed record values. Vidović [69] investigated Bayesian point predictors of order statistics from a future sample based on the k-th lower record values from generalized exponential distribution. Sana and Faizan [40] derived the Bayesian prediction for the future record values from generalized exponential distribution.

3.2.7 Rayleigh distribution

Balakrishnan and Chan [41] developed the prediction of a future record value and the test for superiority of the current record values from a Rayleigh distribution. Soliman and Al-Aboud [43] obtained Bayesian prediction intervals of the future record values from Rayleigh distribution. Seo, et al. [45] provided the exact predictive intervals for the future upper record values by providing some pivotal quantities in the two-parameter Rayleigh distribution based on the upper record values. Finally, the validity of the proposed inference methods was examined from Monte Carlo simulations and real data. Seo and Kim [46] provided an objective Bayesian analysis method based on the objective priors (the Jeffreys and reference priors, and the second-order PMP) for unknown parameters of the two-parameter Rayleigh distribution when the upper record values are observed. Abdi and Asgharzadeh [47] presented exact joint confidence regions for the parameters of the Rayleigh distribution based on record data. By providing some appropriate pivotal quantities, they construct several joint confidence regions for the Rayleigh parameters. These joint confidence regions are useful for constructing confidence regions for functions of the unknown parameters.

3.2.8 Lomax distribution

Volovskiy and Kamps [70] Point prediction of future record values from a sequence of independent and identically distributed Pareto and Lomax random variables is addressed. The focus is on likelihood-based prediction techniques; in particular, the maximum likelihood as well as the maximum observed likelihood prediction principles are invoked to derive predictors. Moreover, one-sided prediction intervals are also addressed.

3.2.9 Nadarajah-Haghighi distribution

MirMostafaee, et al. [55] investigated based on the observed records, how to obtain best linear unbiased predictor for the future record values. prediction intervals for future records are also discussed. Selim [53] discussed the Bayesian and non-Bayesian predictions of both point and interval predictions of the future record values.

3.2.10 General classes of distributions

AL-Hussaini and Ahmad [71] obtained Bayesian prediction bounds for the nth future record value based on the one-sample scheme, all of the informative and future observations are assumed to be obtained from a general class of distributions which includes the Weibull, compound Weibull, Pareto, beta, Gompertz, compound Gompertz among other distributions. Ahmadi and Doostparast [7] obtained prediction, either point or interval, for future upper record values from a Bayesian view point of some life distributions, including Exponential, Weibull, Pareto and Burr type XII. Ahmadi, et al. [72] discussed the problem of predicting future k-records based on k-record data for a large class of distributions, which includes several well-known distributions such as: exponential, Weibull (one parameter), Pareto, Burr type XII, among others.

4. The Record Values Data in Applications

This section reviews the applications of records in various disciplines. The purpose of this review is to show the widespread use of real records data in statistical inference.

4.1 Applications on weather, rainfall, and floods

Raqab and Balakrishnan [73] considered the record values of daily temperatures (in degrees Fahrenheit) recorded at the National Center of Atmospheric Research (NCAR) during the year 2005. Nadar, et al. [74] considered the data set represents the monthly water capacity data from the Shasta reservoir in California, USA and were taken for the month of February from 1991 to 2010. Chacko and Mary [75] used the data which represent the records of the total annual rainfall (in inches) at Oxford, Eng-land, for the years 1858-1903. Seo and Song [76] analyzed two real data sets: one is the average annual temperatures (in degrees centigrade) recorded at Daejeon in Korea from 1969 to 2016). The other is carbon dioxide (CO_2) emissions in Trinidad and Tobago from 1971 to 2016. Volovskiy and Kamps [77] considered data collected by the German Federal Office of Hydrology in its role as a scientific advisor to the Federal Waterways and Shipping Administration. The data set contains hourly measurements (in cm) of water level for the time period from January 1918 to February 2019 collected at the measurement site Cuxhaven-Steubenhöft located at the river Elbe. Selim [53] considered the real data set which represent the total annual rainfall (in inches) during the month of January from 1880 to 1916 recorded at Los Angeles Civic Center. Asgharzadeh, et al. [78] analyzed the total annual rainfall (in inches) during March recorded at Los Angeles Civic Center from 1973 to 2006 (see the website of Los Angeles Almanac: www.laalman-ac.com/weather/we08aa.htm). Raqab, et al. [79] discussed the analysis of real life data representing the water level exceedances over the level 65m by the River Nidd at Hunsingore Weir which is located in North Yorkshire, England from 1934 to 1970. Tripathi, et al. [80], Awwad, et al. [81] considered a real data set regarding the March precipitation measured in inches, over a period of 30 years which was reported by Hinkley [82].

4.2 Applications in industry and life-testing

Salehi and Doostparast [83] considered a data set on life testing of an given electrical equipment, planned for quality control purposes. Singh, et al. [84] considered the data set represents the failure times (in h) of 59 conductors from an accelerated life test from Lawless [85]. Vidović [86] considered the case where failure and running times (1000 of cycles) of a sample of 30 units of a larger electrical system are under study. Wang, et al. [87] considered the real-life data set from Lawless [25, p. 3] which represents the times to breakdown of an electrical insulating fluid subjected to 30 kilovolts. Wu [10] considered the data for times between successive failures of air conditioning equipment in a Boeing 70 airplane.

4.3 Applications in health and medicine

Salehi and Doostparast [83] considered a data set representing the Hemoglobin of the Australian 102 men athletes data. Seo and Kim [88], Awwad, et al. [81] discussed the analysis of the data of the survival times in days of a group of lung cancer patients provided in Lawless. Kumar, et al. [89] reanalyzing Efron's data pertaining to a head-and-neck cancer clinical trial. EL-Sagheer, et al. [90], Fayyazishishavan and Kılıç Depren [91] used the data represents a COVID-19 data belonging to the Netherlands of 30 days, which recorded from 31 March to 30 April 2020.

4.4 Miscellaneous applications

Carlin and Gelfand [92] considered the record-breaking Olympic high jumps since 1896, as presented in the World Almanac and Book of Facts 1989 by Hoffman [93]. Tanış [94] considered the data includes of the monthly actual taxes revenue (in million Egyptian pounds) in Egypt from January 2006 to November 2010. Volovskiy and Kamps [70] applied the proposed prediction procedures to the well-known Danish reinsurance claims dataset to predict record fire losses. The data were collected at Copenhagen Reinsurance and consist of 2167 fire losses in millions of Danish Krone between 1980 and 1990.

5. Computer Software for Records

Software specialized in calculating record values is quite rare. In program R, for example, the built-in routines for computing the records are available in two packages:

5.1 Package "Records": This package includes Functions for producing lower k-record times; lower k-record values; upper k-record times; upper k-record values for given samples (See Appendix). https://CRAN.R-project.org/package=Records.

5.2 Package "RecordTest": This package includes statistical tools based on the probabilistic properties of the record occurrence in a sequence of independent and identically distributed continuous random variables. That is tools to prepare a time series as well as distribution-free trend and change-point tests and graphical tools to study the record occurrence. Details about the implemented tools can be found in Castillo-Mateo, et al. [95]. https://CRAN.R-project.org/package=RecordTest.

6. Future Work

Although much has been done with respect to record values theory, there is still scope for more work. Here, we discuss some open problems that the researchers may like to work on.

i. There is little work with respect to the theory of records for bivariate or multivariate random sequences. Therefore, we recommend more studies in this direction.
ii. Does the type of records affect the estimates of the parameters?. Tripathi, et al. [96] noted that the performance of the estimator depends on the type of records. However, the suitability of the type of record varies from one distribution to distribution. Therefore, this topic needs further study.
iii. Develop a methodology for conducting inference based on record values and record times. Where the record times and record values jointly contain considerably more information about distribution than do the record values alone, see Feuerverger and Hall [97].
iv. Instead of just using record values in inference, we suggest using the record values with their corresponding inter-record times, see Kızılaslan and Nadar [98], Arashi and Emadi [99].
v. Investigating the concept of records with respect to using the kernel density approach to characterize the behavior of records is an interesting extension of the theory of records. This approach will be very useful in cases where a classical distribution

cannot be identified to statistically fit the underlying data from which the record observations are obtained.

vi. Serious difficulties arise for statistical inference based on records due to the fact that the occurrences of record data are very rare in practical situations [since the mean of the number of records in a random sample of size n is equal to $1+2^{-1}+\cdots+n^{-1}$ (see Arnold, et al. [100])] and the expected waiting time is infinite for every record after the first. Although, these problems can be avoided if we consider the model of k-record statistics introduced by Dziubdziela and Kopociński [4]. However, research is still open to investigating this problem.

7. Conclusion

By reviewing the literature (we only mentioned some of them) on the record values in statistical inference, we conclude that the records have played a significant and widely utilized role for statistical inference in parameter estimation, predicting future values, hypothesis tests, as well as stress-strength tests and characterizing of distributions. For this purpose, various known statistical inference methods have been used, including Bayesian and non-Bayesian methods. We also conclude from the applications in previous studies that the records are not limited to a specific field, but rather comprehend all aspects of life including sports, health, medicine, insurance, economy, industry, climate, environment, floods, and rainfall. Overall, records play a critical role in statistical inference by providing the data needed to make informed decisions and draw accurate conclusions.

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Appendix: R program

```
rinfal<-c(11.30, 20.34, 13.13, 10.4, 12.11, 38.18, 9.21, 22.31, 14.05, 13.87, 19.28, 34.84, 13.36, 11.85, 26.28, 6.73, 16.11, 8.51, 16.86, 7.06, 5.59, 7.91, 16.29, 10.6, 19.32, 8.72, 19.52, 18.65, 19.3, 11.72, 19.18, 12.63, 16.18, 11.6, 13.42, 23.65, 17.05, 19.92, 15.26, 13.86, 8.58, 12.52, 13.71, 19.66, 9.59, 6.67, 7.38, 17.56, 17.44, 9.77, 12.66, 12.5, 12.53, 16.95, 11.84, 14.55, 21.66, 12.07, 22.41, 23.43, 13.06, 18.96, 32.76, 11.18, 19.17, 19.21, 11.58, 12.13, 12.61, 7.22, 7.99, 10.6, 8.21, 26.21, 9.46, 11.99, 11.94, 16, 9.54, 21.13, 5.58, 8.18, 4.85, 18.79, 8.38, 7.93, 13.69, 20.44, 22, 16.58, 27.47, 7.77, 12.32, 7.17, 21.26, 14.92, 14.35, 7.22, 12.31, 33.44, 19.67, 26.98, 8.98, 10.71, 31.25, 10.43, 12.82, 17.86, 7.66, 12.48, 8.08, 7.35, 11.47, 21, 27.36, 8.11, 24.35, 12.46, 12.4, 31.01, 9.09, 11.57, 17.94, 4.42, 16.49, 9.24, 37.25, 13.19, 3.21, 13.53, 9.08, 16.36, 20.2, 8.69, 5.85, 6.08, 8.52, 9.65, 19, 4.79, 18.82, 14.86, 5.82, 12.4, 27.85)
> library(Records)
> lower.record.times(rinfal, 1)
[1]   1   4   7  16  21  81  83 124 129
lower.record.values(rinfal, 1)
[1] 11.30 10.40  9.21  6.73  5.59  5.58  4.85  4.42  3.21
> upper.record.times(rinfal, 1)
[1]  1  2  6
> upper.record.values(rinfal, 1)
[1]   11.30  20.34   38.18
> lower.record.times(rinfal, 2)
[1]   2   3   4   7  16  18  20  21  46  81  83 124 129
> lower.record.values(rinfal, 2)
[1] 20.34 13.13 11.30 10.40  9.21  8.51  7.06  6.73  6.67  5.59  5.58  4.85  4.42
> upper.record.times(rinfal, 2)
[1]  2   3   6   8  12 127
> upper.record.values(rinfal, 2)
[1] 11.30 13.13 20.34 22.31 34.84 37.25
```