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EXACT AND CONDITIONAL BOUNDS FOR GENERALIZED CUMULATIVE ENTROPY

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Abstract

The differential entropy is a natural analog of the Shannon entropy for discrete distributions in respect to absolutely continuous distributions (with density). In modern studies, many other kinds of entropy have been introduced and analyzed, including various cumulative entropies, which are based not on the density but on the (cumulative) distribution function of random variable. Such characteristics can be used, for example, in computer vision, reliability theory, risk analysis, etc. We consider some generalizations of cumulative entropy, for a wide class of entropy generators. We use the methods of probability theory, calculus of variations and Cauchy–Bunyakovsky–Schwarz inequality. In the class of centered and normalized random variables, exact and conditional bounds are found as well as the distributions on which they are attained. By conditional bounds we understand bounds for one generalized cumulative entropy given the value of another entropy (in the class of random variables with zero mean and unit variance). This problem is analogous to the previously posed and partly solved problem on conditional bounds for expectations of sample maxima when we know the expected maximum of a sample of another size or expected maxima of two smaller samples.

Keywords: cumulative entropy, exact bounds, conditional bounds, calculus of variations

1. Introduction

The *differential entropy* is a natural analog of the Shannon entropy for discrete distributions in respect to absolutely continuous distributions [14, 6]. For a random variable X with probability density function $p(x)$, it is given by

$$H(X) = -\int_{-\infty}^{+\infty} p(x) \ln p(x)\, dx.$$

For a given variance σ^2, the differential entropy attains its maximum on Gaussian distributions $\mathcal{N}(\mu, \sigma^2)$ [14, §20]; then

$$H(X) = \frac{1 + \ln(2\pi\sigma^2)}{2}.$$

In modern studies, many other kinds of entropy have been introduced and analyzed, including various *cumulative entropies*, which are based not on the density but on the (cumulative) distribution function. Such characteristics can be used, for example, in computer vision [13], reliability theory and risk analysis [4, 5], etc. Even medical applications have been noted [1].

In [13], for nonnegative random variables there was introduced the *cumulative residual entropy* (CRE)

$$\mathcal{E}(X) = -\int_0^{+\infty} \bar{F}(x) \ln \bar{F}(x)\, dx,$$

where $\bar{F}(x) = 1 - F(x)$, F being the (cumulative) distribution function (CDF) of a random variable X, and in [4] there was introduced the *cumulative entropy* (CE)

$$\mathcal{CE}(X) = -\int_0^{+\infty} F(x) \ln F(x)\, dx,$$

which was afterwards also called the *direct* cumulative entropy (in contrast to the residual one). In such expressions it is assumed that $0 \ln 0 = 0$.

It is clear that these functionals can be extended from nonnegative to arbitrary random variables by taking integrals over the entire axis:

$$\mathcal{E}(X) = -\int_{-\infty}^{+\infty} \bar{F}(x) \ln \bar{F}(x)\, dx, \qquad \mathcal{CE}(X) = -\int_{-\infty}^{+\infty} F(x) \ln F(x)\, dx. \tag{1}$$

In the general case, the integrals may both converge or diverge.

For these cumulative entropies, there is symmetry

$$\mathcal{E}(X) = \mathcal{CE}(-X). \tag{2}$$

Note that cumulative entropies (as well as the differential entropy) are traditionally written as numerical characteristics of a random variable X, though they actually depend on its distribution function F only.

In [3], representations for $\mathcal{E}(X)$ and $\mathcal{CE}(X)$ through moments of order statistics (using the power series expansion of the logarithm) have been obtained and upper bounds on these entropies were constructed assuming that X has mean μ and variance σ^2 (taking into account classical estimates for order statistics [7, 8]).

Namely, there were obtained the inequality [3, Theorem 1]

$$\mathcal{E}(X) \leq \sum_{n=1}^{+\infty} \frac{\sigma}{(n+1)\sqrt{2n+1}} \approx 1.21\sigma, \tag{3}$$

which is also valid for $\mathcal{CE}(X)$ due to symmetry (2), and the inequality [3, Theorem 3]

$$\mathcal{E}(X) + \mathcal{CE}(X) \leq \sum_{n=1}^{+\infty} \frac{\sigma\sqrt{2}}{n\sqrt{n+1}} \approx 3.09\sigma. \tag{4}$$

Also, various classes of generalized cumulative entropies have been considered [9, 10].

In particular, in [9] there were introduced the *cumulative residual STM (Sharma–Taneja–Mittal) entropy*

$$SR_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \int_0^{\infty} (\bar{F}^{\alpha}(x) - \bar{F}^{\beta}(x))\, dx, \quad \alpha, \beta > 0, \quad \alpha \neq \beta,$$

and the *cumulative STM entropy*

$$SP_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \int_0^{\infty} (F^{\alpha}(x) - F^{\beta}(x))\, dx, \quad \alpha, \beta > 0, \quad \alpha \neq \beta.$$

Clearly, they can also be extended from nonnegative to arbitrary random variables:

$$SR_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \int_{-\infty}^{+\infty} (\bar{F}^{\alpha}(x) - \bar{F}^{\beta}(x))\, dx,$$
$$SP_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \int_{-\infty}^{+\infty} (F^{\alpha}(x) - F^{\beta}(x))\, dx,$$
$$\alpha, \beta > 0, \quad \alpha \neq \beta.$$

In [10], for a broad class of generalized cumulative entropies, optimal distributions (with given means and variances) that maximize these entropies (i.e., give their exact upper limits) have been obtained by methods of calculus of variations; however, the corresponding maximum values of the entropies have not been derived. If they are derived, for example, for $\mathcal{E}(X)$, $\mathcal{CE}(X)$, and $\mathcal{E}(X)+\mathcal{CE}(X)$, it turns out that these bounds are stronger than (3) and (4).

We will consider for simplicity the class of distributions with zero mean and unit variance. Following [10], one can easily deduce that the maximum value of $\mathcal{E}(X)$ is 1 and the maximum is attained at the shifted exponential distribution with the CDF

$$F(x) = 1 - e^{-(x+1)}, \quad x \geq -1; \tag{5}$$

the maximum value of $\mathcal{CE}(X)$ is the same, and it is attained at the distribution with the CDF

$$F(x) = e^{x-1}, \quad x \leq 1; \tag{6}$$

and the maximum value of $\mathcal{E}(X)+\mathcal{CE}(X)$ is $\pi/\sqrt{3} \approx 1.81$, the maximum being attained at the logistic distribution with the CDF

$$F(x) = \frac{e^{\pi x/\sqrt{3}}}{e^{\pi x/\sqrt{3}}+1}. \tag{7}$$

Next we formulate a simple statement that allows us to obtain an upper bound on the generalized cumulative entropy without deriving the corresponding optimal distribution; we will demonstrate it by an example of the cumulative residual STM-entropy.

Then we solve a new problem about the range in which one generalized cumulative entropy of a random variable can lie provided that another entropy of this random variable is known (for random variables with zero mean and unit variance). Besides the general theorem, we in detail analyze the case of the relationship of the entropies $\mathcal{E}(X)$ and $\mathcal{CE}(X)$.

This problem is analogous to the previously posed and partly solved problem on conditional bounds for expectation of sample maxima when we know the expected maximum of a sample of another size [11] or the expected maxima of two smaller samples [12]. In this case, the corresponding characteristics are also expressed as integral functionals of the distribution function.

From the point of view of calculus of variations, the arising problems belong to the class of isoperimetric problems and are solved by the method of Lagrange multipliers (Euler–Lagrange equations).

2. Main Results

Consider the class CN of centered and normalized random variables, i.e.,

$$CN = \{X : \mathbf{E}X = 0,\ \mathbf{Var}X = 1\}.$$

It is clear that for all the above-mentioned entropies, in order to establish bounds, it suffices to consider random variables in this class. Indeed, let a random variable X have mean μ and variance σ^2; then it admits a representation $X = \mu + \sigma X_0$ with $X_0 \in CN$, and it follows from definition (1) that $\mathcal{E}(X) = \sigma\mathcal{E}(X_0)$, and so on.

Introduce a notation for the generalized inverse distribution function (also called the quantile function)

$$x(u) = \inf\{x : F(x) \geq u\}, \quad u \in [0,1],$$

where F is the CDF of the random variable X. Then

$$X \stackrel{d}{=} x(U),$$

where U is uniformly distributed on $[0,1]$, and the condition $X \in CN$ is equivalent to the following constraints on $x(u)$:

$$\mathbf{E}X = \int_0^1 x(u)\,du = 0, \qquad \mathbf{Var}X = \int_0^1 x^2(u)\,du = 1,$$

where the function $x(u)$, $u \in [0,1]$, is nondecreasing and right continuous.

We will consider functions g (entropy generators) satisfying the following conditions:

$(*)$ $g(u)$ is a nonnegative continuous concave function on $[0,1]$ which is piecewise smooth on $(0,1)$, with $g(0) = g(1) = 0$, and such that

$$G = \int_0^1 \left(g'(u)\right)^2 du < \infty.$$

Introduce generalized cumulative entropies represented by the integral (if it converges)

$$\mathcal{E}_g(X) = \int_{-\infty}^{+\infty} g(\bar{F}(x))\, dx, \tag{8}$$

where F is the CDF of the random variable X.

Using integration by parts and the change of variables $u = F(x)$, we can obtain the following representations:

$$\mathcal{E}_g(X) = -\int_{-\infty}^{+\infty} x\, dg(\bar{F}(x)) = \int_0^1 x(u) g'(1-u)\, du = \int_0^1 g(1-u)\, dx(u),$$

which gives a particular case of the generalized cumulative Φ-entropy

$$CE_\Phi(F) = \int_0^1 \Phi(u)\, dx(u)$$

introduced in [10], with the only difference that in [10] it was not required that $\Phi(0) = \Phi(1) = 0$ (though it was actually the case in all examples considered there).

Definition (8) also implies $\mathcal{E}_g(X) = \sigma \mathcal{E}_g(X_0)$, $X_0 = (X-\mu)/\sigma$, $\sigma > 0$.

Proposition 1. *Let g satisfy condition $(*)$; then*

$$\max_{X \in CN} \mathcal{E}_g(X) = \sqrt{G}.$$

The proposition follows from the fact that according to [10, Theorem 1] this maximum is attained at the distribution with the inverse CDF

$$x(u) = \frac{g'(1-u)}{\sqrt{G}}, \quad u \in [0,1].$$

Corollary 1. Let $1/2 < \min\{\alpha, \beta\} \le 1$, $\alpha \ne \beta$; then

$$\max_{X \in CN} SR_{\alpha,\beta} = \sqrt{\frac{2\alpha\beta - \alpha - \beta + 1}{(2\alpha-1)(2\beta-1)(\alpha+\beta-1)}}, \tag{9}$$

and the maximum is attained at the distribution with the inverse CDF

$$x(u) = \frac{\alpha(1-u)^{\alpha-1} - \beta(1-u)^{\beta-1}}{(\beta-\alpha)\sqrt{G}}, \quad u \in [0,1]. \tag{10}$$

In this case an optimal distribution F is not found explicitly, but it can be obtained, for example, for $\alpha = 1$ or $\beta = 1$, when all expressions become simpler (this was made in [10]).

Note that for $\min\{\alpha,\beta\} > 1$ the concavity condition for g is violated, and for $0 < \min\{\alpha,\beta\} \le 1/2$ the entropy $SP_{\alpha,\beta}(X)$ may take infinitely large values on $X \in CN$ (when the corresponding integrals diverge).

Clearly, analogous statements hold as well for $SP_{\alpha,\beta}$, since $SP_{\alpha,\beta}(X) = SR_{\alpha,\beta}(-X)$.

Theorem 1. Assume that g_1 and g_2 satisfy conditions $(*)$, the integrals

$$G_{ij} = \int_0^1 g_i'(u) g_j'(u)\, du, \quad 1 \le i,j \le 2,$$

are introduced, and it is known that $\mathcal{E}_{g_2}(X) = t$. Then for all $X \in CN$ we have

$$\mathcal{E}_{g_1}(X) \leq \frac{1}{G_{22}} \left(G_{12}t + \sqrt{(G_{11}G_{22} - G_{12}^2)(G_{22} - t^2)} \right), \tag{11}$$

and this bound is tight if the function

$$\tilde{x}(u) = \lambda_1 g_1'(1-u) + \lambda_2 g_2'(1-u),$$

where

$$\lambda_1 = \sqrt{\frac{G_{22} - t^2}{G_{11}G_{22} - G_{12}^2}}, \quad \lambda_2 = \frac{t - \lambda_1 G_{12}}{G_{22}}, \tag{12}$$

is nondecreasing on $(0,1)$; then $\tilde{x}(u)$ defines the distribution on which the bound is attained.

Note that by Proposition 1 we have $\mathcal{E}_{g_2}^2(X) \leq G_{22}$, so the radicand is always nonnegative. The functions $g_1'(1-u)$ and $g_2'(1-u)$ are nondecreasing, and $\lambda_1 \geq 0$; however, the nondecreasing condition for $\tilde{x}(u)$ can be violated when $\lambda_2 < 0$.

For the sequel, it would be convenient to introduce the notation for the constant

$$p = \frac{\pi^2}{6} - 1 \approx 0.645.$$

Corollary 2. For all $X \in CN$ we have

$$\mathcal{E}(X) \leq p\,\mathcal{CE}(X) + \sqrt{(1-p^2)(1 - \mathcal{CE}^2(X))}, \tag{13}$$

and this bound is tight if $\mathcal{CE}(X) \geq p$.

By symmetry (2) of the entropies, we also have

$$\mathcal{CE}(X) \leq p\,\mathcal{E}(X) + \sqrt{(1-p^2)(1 - \mathcal{E}^2(X))},$$

and this bound is tight if $\mathcal{E}(X) \geq p$. By inverting the inequality, we can also obtain a lower bound

$$\mathcal{E}(X) \geq p\,\mathcal{CE}(X) - \sqrt{(1-p^2)(1 - \mathcal{CE}^2(X))}$$

in the range $\mathcal{CE}(X) \geq \sqrt{1-p^2} \approx 0.764$ where this bound is nonnegative (but we cannot claim that it is tight). Similarly, a lower bound for $\mathcal{CE}(X)$ can be found.

The question of what is the upper bound when $\tilde{x}(u)$ is not nondecreasing remains open. In this case we deal with a problem of not the calculus of variations but optimal control (with an additional condition $x'(u) \geq 0$), which is much more complicated. One can also apply an approach to establishing (not tight) bounds using special families of distributions, as was done in [11]. This approach is exploited in the proof of the following theorem

Theorem 2. For any $0 < t < p$ we have

$$\max_{X \in CN,\, \mathcal{CE}(X) = t} \mathcal{E}(X) \geq \sqrt{\frac{1-a}{1+a}}\,(1 - \ln(1-a)),$$

where a is a unique solution on $(0,1)$ of the equation [1]

$$-\frac{a(\ln a - 1) - \mathrm{Li}_2(1-a) + 1}{\sqrt{1-a^2}} = t.$$

By symmetry (2), an analogous estimate holds for $\mathcal{CE}(X)$ given $\mathcal{E}(X)$, whence one can obtain a lower estimate for the maximum of $\mathcal{E}(X)$ given $\mathcal{CE}(X)$.

Figure 1 represents plots of the obtained bounds for the entropies $\mathcal{E}(X)$ and $\mathcal{CE}(X)$. In bold, we highlight the interval where the bound (13) is tight; the dotted line shows the bound of Theorem 2. Points of the bound marked by the triangle, star, and circle correspond to the distributions (5), (6), and (7). In the ranges $\mathcal{CE}(X) < p$ and $\mathcal{E}(X) < p$, true bounds lie somewhere in between the solid and dotted lines. Establishing them deserves further investigation.

[1] Here, $\mathrm{Li}_m(z) = \sum_{n=1}^{\infty} z^n / n^m$ is the polylogarithm of order m.

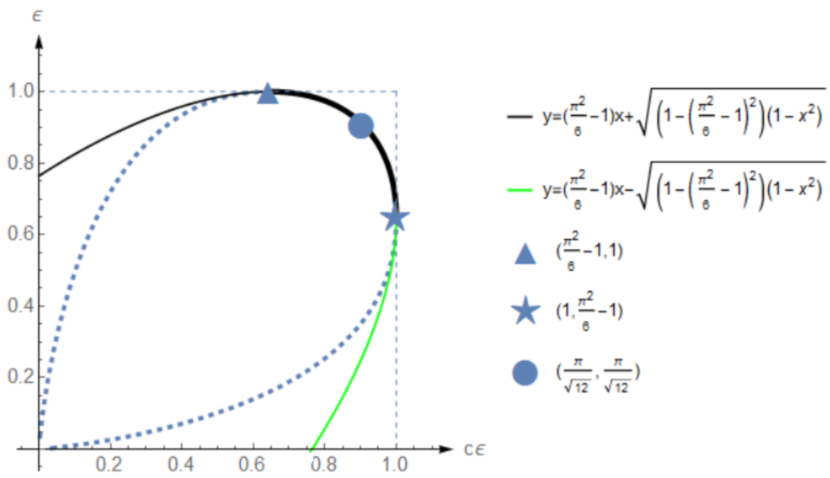

Figure 1: *Plots of the bounds for the entropies $\mathcal{E}(X)$ and $\mathcal{CE}(X)$.*

3. Proofs

Proof of Corollary 1. Let, for definiteness, $\alpha < \beta$; then $1/2 < \alpha \le 1$. Put $g(u) = (u^\alpha - u^\beta)/(\beta - \alpha)$; then

$$g'(u) = \frac{\alpha u^{\alpha-1} - \beta u^{\beta-1}}{\beta - \alpha},$$
$$g''(u) = \frac{\alpha(\alpha-1)u^{\alpha-2} - \beta(\beta-1)u^{\beta-2}}{\beta-\alpha} < 0, \quad u \in (0,1).$$

We obtain

$$\begin{aligned} G &= \int_0^1 \left(\frac{\alpha u^{\alpha-1} - \beta u^{\beta-1}}{\beta-\alpha}\right)^2 du \\ &= \frac{1}{(\beta-\alpha)^2}\int_0^1 (\alpha^2 u^{2\alpha-2} - 2\alpha\beta u^{\alpha+\beta-2} + \beta^2 u^{2\beta-2})\, du \\ &= \frac{1}{(\beta-\alpha)^2}\left\{\frac{\alpha^2}{2\alpha-1} - \frac{2\alpha\beta}{\alpha+\beta-1} + \frac{\beta^2}{2\beta-1}\right\} \\ &= \frac{2\alpha\beta - \alpha - \beta + 1}{(2\alpha-1)(2\beta-1)(\alpha+\beta-1)} \end{aligned} \tag{14}$$

and equations (9) and (10).

Proof of Theorem 1. By considering the Lagrangian

$$\mathcal{L} = \int_0^1 \Big(\lambda_1 x(u) g_1'(1-u) + \lambda_2 x(u) g_2'(1-u) + \lambda_3 x(u) + \lambda_4 x^2(u)\Big)\, du,$$

we obtain the Euler–Lagrange equation

$$\lambda_1 g_1'(1-u) + \lambda_2 g_2'(1-u) + \lambda_3 + 2\lambda_4 x(u) = 0,$$

wher e we may without loss of generality takee $\lambda_4 = -1/2$.

Thus, we will seek for a function

$$\tilde{x}(u) = \lambda_1 g_1'(1-u) + \lambda_2 g_2'(1-u) + \lambda_3$$

satisfying the conditions

$$\int_0^1 \tilde{x}(u)\, du = 0, \qquad \int_0^1 \tilde{x}^2(u)\, du = 1, \qquad \int_0^1 \tilde{x}(u) g_2'(1-u)\, du = t.$$

The first condition, taking into account that $g_i(0) = g_i(1) = 0$, $i = 1, 2$, gives $\lambda_3 = 0$; the second and third yield a system of equations

$$\begin{cases} G_{11}\lambda_1^2 + 2G_{12}\lambda_1\lambda_2 + G_{22}\lambda_2^2 = 1, \\ G_{12}\lambda_1 + G_{22}\lambda_2 = t; \end{cases}$$

by solving this system with respect to λ_1 and λ_2, we obtain (12).

Next, for any function $x(u)$ corresponding to $X \in CN$, by the Cauchy–Bunyakovsky–Schwarz inequality we obtain

$$\int_0^1 x(u)\tilde{x}(u)\,du = \lambda_1\,\mathcal{E}_{g_1}(X) + \lambda_2 t \le \left(\int_0^1 x^2(u)\,du\right)^{1/2}\left(\int_0^1 \tilde{x}^2(u)\,du\right)^{1/2} = 1, \qquad (15)$$

whence

$$\mathcal{E}_{g_1}(X) \le \frac{1-\lambda_2 t}{\lambda_1} = \frac{G_{22} - (t - \lambda_1 G_{12})t}{\lambda_1 G_{22}} = \frac{\lambda_1 G_{12} t + G_{22} - t^2}{\lambda_1 G_{22}}$$
$$= \frac{1}{G_{22}}\left(G_{12}t + \sqrt{(G_{11}G_{22} - G_{12}^2)(G_{22} - t^2)}\right).$$

If $\tilde{x}(u)$ is nondecreasing and thus corresponds to some distribution, then with $x(u) = \tilde{x}(u)$ inequality (15) turns into equality, and the bound is attained.

Proof of Corollary 2. We apply Theorem 1 in the case of $g_1(u) = -u\ln u$ and $g_2(u) = -(1-u)\ln(1-u)$; then, as we have already obtained, $G_{11} = G_{22} = 1$, and we find

$$G_{12} = -\int_0^1 (\ln u + 1)(\ln(1-u) + 1)\,du = p;$$

plugging this into (11), we obtain (13). In this case we have

$$\tilde{x}(u) = \lambda_1(-(\ln(1-u) + 1)) + \lambda_2(\ln u + 1),$$

where

$$\lambda_1 = \sqrt{\frac{1-t^2}{1-p^2}}, \quad \lambda_2 = t - p\lambda_1.$$

A necessary and sufficient condition for $\tilde{x}(u)$ to be nondecreasing on $(0, 1)$ is $\lambda_2 \ge 0$, which happens to be equivalent to the inequality $t \ge p$.

Proof of Theorem 2. Consider a family of random variables X_a^0, $a \in [0, 1)$, whose distribution is a mixture of zero (with probability a) and the standard exponential distribution (with probability $1 - a$). Then the inverse CDFs take the form

$$x_a^0(u) = \begin{cases} 0, & 0 \le u < a; \\ -\ln\dfrac{1-u}{1-a}, & a \le u < 1. \end{cases}$$

We have

$$\mathbf{E}X_a^0 = 1 - a, \qquad \mathbf{E}(X_a^0)^2 = 2(1-a), \qquad \mathbf{Var}X_a^0 = 2(1-a) - (1-a)^2 = 1 - a^2.$$

Put

$$X_a = \frac{X_a^0 - \mathbf{E}X_a^0}{\sqrt{\mathbf{Var}X_a^0}}.$$

Then $X_a \in CN$, $a \in [0, 1)$; X_0 has distribution (5); and $X_a \xrightarrow{d} 0$ as $a \to 1 - 0$.

Compute the corresponding entropies for $0 < a < 1$:

$$\begin{aligned}\mathcal{E}(X_a) &= \frac{\mathcal{E}(X_a^0)}{\sqrt{1-a^2}} = -\frac{1}{\sqrt{1-a^2}}\int_a^1 \ln\frac{1-u}{1-a}\left(\ln(1-u)+1\right)du \\ &= \frac{(1-a)(1-\ln(1-a))}{\sqrt{1-a^2}} = \sqrt{\frac{1-a}{1+a}}\,(1-\ln(1-a)), \\ \mathcal{CE}(X_a) &= \frac{\mathcal{CE}(X_a^0)}{\sqrt{1-a^2}} = \frac{1}{\sqrt{1-a^2}}\int_a^1 \ln\frac{1-u}{1-a}\left(\ln u+1\right)du \\ &= -\frac{a(\ln a-1)-\mathrm{Li}_2(1-a)+1}{\sqrt{1-a^2}},\end{aligned}$$

and $\mathcal{CE}(X_a)$ strictly decreases in the interval $0 < a < 1$.

Thus, from the values of the entropies on the family X_a, $a \in (0,1)$, we can obtain the estimate of Theorem 2.

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