BAYES ESTIMATION OF CAPABILITY INDEX USING THREE-PARAMETER WEIBULL DISTRIBUTION

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Abstract

The process capability index is an important tool used in quality control and process improvement. Generally, the index is estimated under the assumption of a normal distribution, although some other distributions are also recommended in the literature. This paper instead considers a three-parameter Weibull distribution and obtains an estimate of the process capability index under the Bayesian framework. Bayesian development is based on the use of non-informative priors and the posterior sample-based inferences are drawn using an important Markov Chain Monte Carlo technique, namely, the Gibbs sampler algorithm. Finally, a numerical illustration based on two real datasets is provided.

Keywords: Process capability index, Gibbs sampler, Three-parameter Weibull distribution

1. Introduction

With the advancement of technology, there is an ever-increasing demand for high-quality products and services. Smart manufacturing process employing various advanced technologies facilitate automation, enhance productivity, improve maintenance and monitoring and reduce scope of human error. However, associated software products need to be examined for quality assurance.

The quality and reliability of the product can be assessed through various statistical tools, among which, process capability index (PCI) has been found propitious by the manufacturers as it is useful in assisting decision-making and boosting efforts in process performance. PCI is a measuring tool for accurately analysing the potential of a process and its performance. For quality control engineers, it is extremely important since it quantifies the relationship between the process's actual performance and the product's predetermined parameters. The index ascertains whether the process meets the defined manufacturing prerequisites. In this regard many capability indices have been developed so far (see, for example, [31], [11], [14] and [5]). The first index put forward in the literature was C_p, which simply calculates the span of the specifications relative to the six-sigma spread in the process (see [31]). As per this index, the process mean is centred between the lower and the upper specification limits. One of the major issues with this index is that it does not take into account the location of the process mean relative to the specifications. Moreover, if the process is not centred on the specification region, it would be possible to have a substantial percentage of the products with characteristics outside the specification limit although C_p may be high. In order to overcome this problem, [11] introduced another capability index, C_{pk}, which takes process centring into account in addition to the spread of the specifications relative to the six-sigma spread in the process. In other words, it measures the distance between the specification limits closest to the average from the quality characteristic of interest. Mathematically, C_p and C_{pk} can be defined as

$$C_p = \frac{USL - LSL}{6\sigma_p}, \tag{1}$$

$$C_{pk} = min(C_{pu}, C_{pl}), \tag{2}$$

where

$$C_{pu} = \frac{USL - \mu_p}{3\sigma_p}, \tag{3}$$

$$C_{pl} = \frac{\mu_p - LSL}{3\sigma_p}, \tag{4}$$

USL and LSL are the upper and lower specification limits, respectively, μ_p denotes the process mean and σ_p represents the process standard deviation.

Both of these PCIs are defined under two important assumptions, that is, the process is under statistical control and the quality characteristic of the process of interest is normally distributed (see [31]). Perhaps, because of these assumptions, a bulk of literature is available on the estimation of PCIs under the assumption of normality (see, for example, [1], [2], [13] and [23]). However, industrial processes are often not normally distributed and, for such scenarios, the values of conventional PCIs may be absurd and possibly misrepresent the quality of the product. For example, one may refer to [10], [27] and [24] for a systematic and detailed coverage. In order to remove this discrepancy, [3] proposed the quantile-based measure to estimate the capability index for non-normal distributions, which is given as under.

$$C_{pk} = min\left(\frac{USL - M}{U_p - M}, \frac{M - LSL}{M - L_p}\right), \tag{5}$$

where U_p, L_p and M are the 99.865th, 0.135th, and 50th percentiles of the target distribution, respectively, USL and LSL indicate upper and lower specification limits. A value of $C_{pk} < 1$ is unfavourable and indicates that the process is incapable, whereas, a value of $1 \leq C_{pk} \leq 1.33$ indicates that the process is barely capable and $C_{pk} \geq 1.33$ shows that the process is capable to meet the consumers' requirements.

Besides normality assumption, several developments can be seen in literature on non-normal assumptions as well. [3], [14], [17], [16], [22], [12], [9], [26] and [20] are some of the important among other references where capability indices are estimated under the assumption of non-normal distributions. A thorough literature review on the estimation of PCIs for non-normal datasets reveals that most of the developments are done using classical framework and only a few of them considered Bayesian approach for estimating capability index. Further, in statistical process control, most of the datasets lie at a particular location, generally far from zero, and, therefore, it becomes imperative to assess capability index by considering a model which has a location parameter even if one is dealing with non-normal data. To the best of our knowledge, there is no reference in the literature that entertains a non-normal model with location parameter for estimating the capability index. To bridge this gap, this paper considers a three-parameter Weibull distribution for estimating the capability index and performs a Bayes analysis of the distribution.

The Weibull distribution is an important distribution that has received enough attention in the field of reliability and quality control. Its versatility stems from the fact that it incorporates increasing, decreasing and stable hazard rates for different values of its shape parameter (see [18] and [15], etc). The literature on the analysis of Weibull distribution has considered both two-parameter and three-parameter form of model where the former model is defined without a threshold parameter. The two-parameter Weibull distribution is comparatively easier to deal with as compared to three-parameter model form and, therefore, the literature on both classical and Bayes analysis of two-parameter Weibull distribution is available in bulk (see, for example, [19], [28], [15], [25], among others). On the other hand, the three-parameter Weibull distribution is much richer because of the involvement of a threshold parameter although its analysis is slightly more challenging due to sometime unusual behaviour of the likelihood function, especially when the shape parameter is less than unity (see also [30] and [32]). As a result, this model is comparatively less entertained in the literature. [30], [32] and [28] are some of the important references among others where this form of the model is explored.

As mentioned, this paper is an attempt to provide Bayes analysis of the three-parameter Weibull distribution with ultimate objective of finding the estimate of PCI. The entire development is done using non-informative priors for the model parameters. It is seen that the resulting posterior is analytically intractable to draw exact posterior based inferences and, therefore, the paper utilizes an important Markov Chain Monte Carlo (MCMC) procedure, namely the Gibbs sampler algorithm, to simulate posterior samples and draw the sample based inferences including those of PCI. Finally, the proposed methodology is numerically illustrated on the basis of two real datasets from a juice manufacturing company.

The plan of the paper is as follows. The next section briefly describes the three-parameter Weibull model and its Bayesian formulation. Section 3 provides numerical illustration based on two real datasets. Finally, a brief conclusion is provided in the last section.

2. Model Formulation

2.1. Likelihood function

The probability density function (pdf) of the three-parameter Weibull distribution is

$$f(x|\theta,\beta,\mu) = \frac{\beta}{\theta}\left(\frac{x-\mu}{\theta}\right)^{\beta-1} exp\left[-\left(\frac{x-\mu}{\theta}\right)^{\beta}\right], \qquad x>\mu; \quad \theta,\beta,\mu>0 \tag{6}$$

where θ, β and μ are the scale, shape and location parameters, respectively. The distribution exhibits increasing hazard rate for $\beta > 1$, decreasing hazard rate for $\beta < 1$ and, for $\beta = 1$, the distribution reduces to two-parameter exponential model possessing constant hazard rate. Let us use the notation $W(\theta,\beta,\mu)$ to denote the three-parameter Weibull distribution given in (6). The reliability function and the hazard function of $W(\theta,\beta,\mu)$ at time t are, respectively, given by

$$R(t) = exp\left[-\left(\frac{t-\mu}{\theta}\right)^{\beta}\right], \tag{7}$$

and

$$h(t) = \frac{\beta}{\theta}\left(\frac{t-\mu}{\theta}\right)^{\beta-1}. \tag{8}$$

Similarly, the expressions for U_p, L_p and M for the model $W(\theta,\beta,\mu)$ can be written as

$$U_p = \theta[2.86967]^{\frac{1}{\beta}} + \mu, \tag{9}$$

$$L_p = \theta[0.00058]^{\frac{1}{\beta}} + \mu, \tag{10}$$

and

$$M = \theta[ln\ 2]^{\frac{1}{\beta}} + \mu, \tag{11}$$

respectively.

Let us now assume that an experiment consisting of n units is being conducted and let $\underline{x} = (x_i; i = 1,2,...,n)$ be the resulting observations. Then, the likelihood function for the dataset $\underline{x}$ can be expressed as

$$L(\underline{x}|\theta,\beta,\mu) = \left(\frac{\beta}{\theta}\right)^{n} \prod_{i=1}^{n}\left(\frac{x_i-\mu}{\theta}\right)^{\beta-1} exp\left[-\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\theta}\right)^{n}\right]. \tag{12}$$

2.2. Bayesian formulation

To conduct Bayesian analysis, it is essential to specify prior distribution for the parameters of the entertained model. Several types of priors are proposed in the literature for the Weibull parameters. The paper, however, considers joint non-informative prior as suggested by [32] and the same is given as

$$g(\theta,\beta,\mu) \propto \frac{1}{\theta\beta}. \tag{13}$$

Obviously, the parameter μ is assigned a constant prior over the positive real space.

The updated belief in the form of posterior distribution can be obtained by combining the prior distribution as specified in (13) with the likelihood function given in (12) via Bayes theorem. The joint posterior up to proportionality can, therefore, be written as

$$p(\theta,\beta,\mu|\underline{x}) \propto \frac{\beta^{n-1}}{\theta^{n\beta+1}} \prod_{i=1}^{n}(x_i-\mu)^{\beta-1} exp\left[-\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\theta}\right)^{\beta}\right]; \quad \theta>0, \beta>0, \mu<min(\underline{x}). \tag{14}$$

Obviously, the posterior given in (14) is analytically intractable and, therefore, one has to proceed with some approximation or simulation based alternative approaches for drawing the desired inferences from the posterior. As mentioned, this paper considers Gibbs sampler algorithm, an important MCMC procedure, because of its straightforwardness and ease of implementation. The algorithm requires specification of low-dimensional full conditionals for simulating the high dimensional posterior where both full conditionals and the posterior need to be specified up to proportionality only. The algorithm starts with the appropriately chosen initial values for the variates and then simulates the full conditionals one by one in a cyclic fashion with most recent available values for all the given variates at every stage. Obviously, the appropriately chosen initial values are updated after the first cycle of iteration from all the full conditionals. The process is continued for a large number of cycles until some systematic pattern of convergence is achieved among the generating variates. Moreover, it can be easily seen that the posterior (14) results in three one-dimensional full conditionals corresponding to θ, β and μ and these full conditionals can be easily simulated resulting in an easy implementation of the Gibbs sampler algorithm. For further details on the algorithm, one can refer to [7], [6] and [32], among others.

Coming on to the full conditionals derived from (14), it can be seen that the full conditional for θ happens to be the kernel of gamma distribution after appropriate transformation and, hence, θ can be easily generated from a gamma generating routine (see [4]). The full conditional of β can be seen to be log concave and, therefore, β can be simulated using adaptive rejection sampling procedure (see [8]). The generation of μ from its full conditional is based on the rejection algorithm using the envelope density $g_1(\mu|\beta,x_1) = \left(\frac{\beta}{x_1^{\beta}}\right)(x_1-\mu)^{(\beta-1)}$; $x_1 > \mu$, where x_1 is minimum of $(x_i; i=1,2,...,n)$ (see [32] for further details).

3. Numerical Illustration

For numerical illustration of the proposed formulation, the paper considers two real datasets on the weights (in grams) of thirty juice packs of grape and strawberry flavours. In the discussion that follows, the dataset on weights of juice packs of grape flavour is referred to as the *Data*1 whereas that of strawberry flavour is referred to as the *Data*2. The two datasets are presented in Table 1 and these are actually collected to assess the process of filling powdered juice bags. The two datasets were first reported by [21] where the authors analysed the datasets under the assumption of normal distribution and evaluated C_{pk} by considering the specification limits as: LSL= 18.0 and USL= 22.0. These specification limits were specified in accordance with the guidelines provided by the National Institute of Metrology, Quality and Technology (INMETRO), the Brazilian organisation responsible for the quality control.

Before proceeding with the analysis of datasets, let us plot the control charts with the specification limits of 18.0 and 22.0. The control charts are presented in Figure 1 where the red line corresponds to *Data*1 and the blue line corresponds to *Data*2. Moreover, the specification limits 18.0 and 22.0 suggest that the process must hover around the mean of these specification limits although the Figure 1 clearly suggests that the process is not centred around its mean. In fact, there are certain values that lie outside the provided range, which ultimately suggest that the process is out of control.

Table 1: *Data on weights (in grams) of juice packs*

Data1				
21.011	20.635	21.732	21.333	20.587
20.587	21.784	21.088	20.997	21.100
22.155	21.116	20.707	20.413	20.822
20.883	20.930	20.908	20.897	20.486
20.935	21.867	20.814	20.795	21.520
20.537	21.438	20.621	20.975	20.919
Data2				
22.572	21.376	20.768	21.833	19.970
21.583	21.813	22.025	20.892	20.241
21.816	21.232	21.730	20.529	21.435
21.106	20.519	21.263	20.684	21.233
19.624	21.150	20.962	21.024	20.316
21.942	21.495	20.819	20.973	21.115

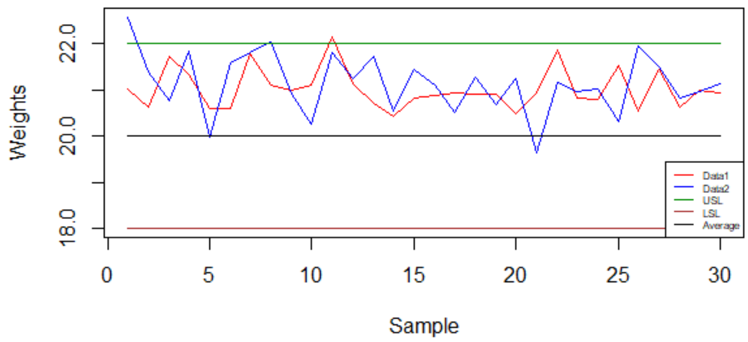

Figure 1: *Control chart for the two datasets.*

Further, before carrying out the Bayes analysis of the considered datasets, let us check the compatibility of two datasets with the assumed model (6). The compatibility was examined based on Kolmogorov-Smirnov (KS) test statistic which was evaluated using maximum likelihood (ML) estimates of the model parameters. It may be noted that the ML estimates for θ, β and μ were found to be 0.693, 1.475 and 20.391, respectively, for *Data*1 and 2.635, 4.244 and 18.737, respectively, for *Data*2. Finally, for *Data*1, the KS statistic was found to be 0.110 with the corresponding p-value as 0.860 while for *Data*2, the KS statistic was 0.066 with the corresponding p-value as 0.998. Obviously, the two datasets provide good compatibility with the model $W(\theta, \beta, \mu)$.

For performing the Bayes analysis, the Gibbs sampler algorithm was implemented on the posterior (14) as per details given in subsection 2.2. Convergence monitoring was done using ergodic averages, obtained separately for each of the three variates, using a single long run of the iterating chain. It was found that $50K$ iterations were good enough for getting stationarity behaviour of the ergodic averages. Once the convergence was assessed, equally spaced observations at a gap of 10 were chosen to make auto correlation negligibly small. In this way, a posterior sample of size $1K$ was taken from the marginal posterior of each of θ, β and μ (see also [29] and [32]). Once the samples of θ, β and μ are obtained, the same can be used in (9)-(11) by substitution to get the corresponding samples of size $1K$ from the posterior of each of U_p, L_p and M. Finally, the samples of U_p, L_p and M so obtained can be used to get the posterior samples of size $1K$ corresponding to C_{pk} given in (5).

Table 2: *Estimated posterior summaries for θ, β, μ and C_{pk}*

		Estimated Posterior Summaries				
Datasets	**Parameters**	**Mean**	**Median**	**Mode**	**0.95 HPDI**	
Data1	θ	0.701	0.700	0.698	0.587	0.816
	β	1.493	1.490	1.483	1.215	1.794
	μ	20.384	20.386	20.391	20.350	20.412
	C_{pk}	1.227	1.221	1.211	0.931	1.532
Data2	θ	2.663	2.589	2.439	1.770	3.725
	β	4.259	4.139	3.901	2.522	6.242
	μ	18.706	18.780	18.928	17.715	19.561
	C_{pk}	0.870	0.869	0.867	0.695	1.053

Table 2 provides a few important posterior based summaries of different posterior characteristics corresponding to various entertained model parameters, each estimated on the basis of corresponding $1K$ posterior samples. These summaries are shown in the form of estimated posterior mean, median, mode and the highest posterior density intervals with 0.95 coverage probability (0.95 HPDI) for each of the two datasets. It can be observed from Table 2 that the estimated posterior mean, median and mode corresponding to each parameter for both the datasets are quite close to each other, implying that the posterior distributions are approximately symmetric. Furthermore, the width of 0.95 HPDIs for all the parameters are quite small indicating less variability in the estimated values of the parameters and, hence, ensuring the consistency of the estimated values. An important finding presented in Table 2 is that $1 \leq C_{pk} \leq 1.33$ for $Data1$, indicating that the process is barely capable whereas for $Data2$ $C_{pk} < 1$ implying that the process is incapable and requires further improvement. A similar conclusion was drawn on the basis of control charts shown in Figure 1.

4. Conclusion

Technological advancements have typically led to an expansion of the industry, wherein the need for high-quality goods and services is reinforced by a competitive environment. From this vantage point, industries that deal with manufacturing are always susceptible to manufacturing process failures leading to the products that may not meet the desired specifications. The manufacturing sector has made extensive use of PCIs, providing a numerical gauge of a process's ability to produce goods that satisfy the factory-set quality standards. In estimating PCIs, more often the assumption is made that the data are generated randomly using a normal model. Nonetheless, asymmetric data are found in many circumstances. This paper has successfully demonstrated the utility of the three-parameter Weibull model in estimating the aforesaid index. Further, the Bayesian methodology developed in the paper is also found to offer the intended inferences in a routine manner. The inferential results show that the process pertaining to $Data1$ is barely capable while that of $Data2$ is incapable to offer the desired quality assurance.

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