MARSHAL-OLKIN ALPHA POWER INVERSE RAYLEIGH DISTRIBUTION: PROPERTIES, ESTIMATION AND APPLICATIONS

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Abstract

In this study, a new three-parameter distribution is introduced by extending the two-parameter Alpha Power Inverse Rayleigh distribution using Marshall-Olkin G approach. The proposed Marshall-Olkin Generalized Alpha Power Inverse Rayleigh (MOAPIR) distribution generalizes the Marshall-Olkin Inverse Rayleigh, Alpha Power Inverse Rayleigh, and Inverse Rayleigh distribution. The characterization and statistical properties of the proposed distribution such as hazard rate function, reversed hazard rate function, quantiles, moments, and order statistics were derived. The estimation of the MOAPIR distribution parameters is derived using the maximum likelihood estimation method. The performance of the proposed distribution was compared with other competing distribution using two real-life data. The goodness of fit criteria and the distribution function curve showed that the proposed distribution provides a better fit than other competing distributions of the same family of heavily positive skewed distribution.

Keywords: Marshall-Olkin G family, Alpha Power Inverse Rayleigh distribution, Skewed distribution, distribution function, Statistical properties.

I. Introduction

Marshal-Olkin G method of generalization (MO-G) proposed by Marshall and Olkin [1] is often used to generate a new family of distributions. Using the cumulative distribution function (CDF) of any distribution of a random variable X, the cumulative function of the new family of distributions is obtained by

$$G(x;\theta) = \frac{G(x)}{\theta+(1-\theta)G(x)} \qquad \theta > 0, x \in \Re \tag{1}$$

where θ is the location parameter.

Its corresponding probability density function (PDF) is

$$g(x;\theta) = \frac{\theta g(x)}{[\theta+(1-\theta)G(x)]^2} \qquad \theta > 0, x \in \Re \tag{2}$$

Many authors such as; Ghitany [2], Ghitany et al.[3], Alice and Jose [4], Okasha and Kayid [5], Okasha et al. [6], Salah et al. [7], Gui [8], Krishna et al. [9], Al-Saiari et al. [10], Mahdavi and Kundu [11], Javed et al. [12], Maxwell et al. [13], Okasha et al. [14], Okasha et al. [15], Haj Ahmad and Almetwally [16], Abdul-Hadi et al. [17], Klakattawi et al. [18], and Aako et al. [19] have used MO-G to extend some base distributions by adding parameters to a well-established family of distribution to generate a new distribution.

This article proposes the generalization of APIR distribution proposed by Malik and Ahmad [20] based on the MO-G which we hereafter called the Marshall-Olkin Generalized Alpha Power Inverse Rayleigh (MOAPIR) distribution. The special cases and the statistical properties of MOAPIR were also presented. Furthermore, the method of maximum likelihood estimation was used to estimate the parameters of the proposed distribution and two data sets were used to demonstrate the performance of the proposed distribution in comparison with other competing distribution of the same family of distributions.

2. The Proposed Distribution

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables from the APIR distribution.

The cdf and pdf of the APIR distribution are presented in (3) and (4), respectively.

$$G_{APIR}(x;\alpha,\lambda) = \frac{\alpha^{e^{-\frac{\lambda}{x^2}}}-1}{\alpha-1}\ , \qquad x > 0, \alpha \neq 1, \lambda > 0\ , \tag{3}$$

$$g_{APIR}(x;\alpha,\lambda) = \frac{\log\alpha}{\alpha-1}\frac{2\lambda}{x^3}e^{-\frac{\lambda}{x^2}}\alpha^{e^{-\frac{\lambda}{x^2}}}, \qquad x > 0, \alpha \neq 1, \lambda > 0 \tag{4}$$

where α *and* λ are shape and scale parameters, respectively.

We applied the MO-G to the APIR distribution by inserting (3) into (1) and inserting (4) into (2) to have the CDF and PDF respectively, of a new generated distribution called the MOAPIR distribution.

If X is a random variable from MOAPIR distribution, we shall denote as $X \sim MOAPIR(\alpha,\lambda,\theta)$. The CDF of MOAPIR is

$$G_{MOAPIR}(x) = \begin{cases} \frac{\alpha^{e^{-\lambda x^{-2}}}-1}{\theta(\alpha-1)+(1-\theta)\left(\alpha^{e^{-\lambda x^{-2}}}-1\right)}, & x > 0, \alpha \neq 1, \lambda > 0, \theta > 0 \\ 0, & = 1 \end{cases} \tag{5}$$

and the corresponding PDF of MOAPIR distribution is

$$g_{MOAPIR}(x) = \begin{cases} \frac{(\alpha-1)2\lambda\theta\log(\alpha)x^{-3}e^{-\lambda x^{-2}}\alpha^{e^{-\lambda x^{-2}}}}{\left[(\alpha-1)\theta+(1-\theta)\left(\alpha^{e^{-\lambda x^{-2}}}-1\right)\right]^2} & x>0, \alpha \neq 1, \lambda>0, \theta>0 \\ 0, & =1 \end{cases} \quad (6)$$

To have a useful linear representation of the pdf of the proposed distribution, we used the generalized binomial expansion (GBE) in (7) and the power series in (8)

$$(1-z)^2 = \sum_{k=0}^{\infty}(k+1)z^2, |z|<1, \quad (7)$$

$$\alpha^z = \sum_{m=0}^{\infty}(\log(\alpha))^m z^m \quad (8)$$

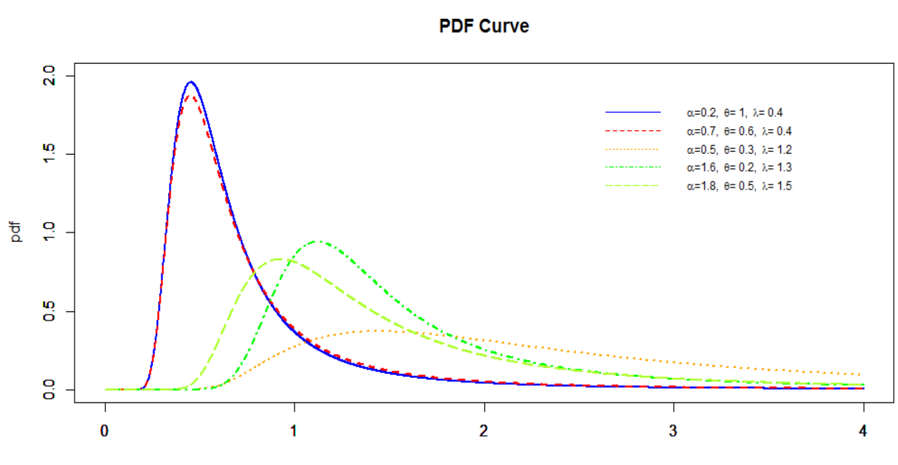
Applying the concept of GBE and power series in (7) and (8) into (6) if (α >0 and $\alpha\neq$ 1), then we have

$$g_{MOAPIR}(x;\alpha,\lambda,\theta) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{\infty} W_{k,j,m} 2\lambda(m+1)x^{-3}e^{-(m+1)\lambda x^{-2}}, \quad (9)$$

where

$$W_{k,j,m} = \begin{cases} (-1)^j \binom{k}{j}(k+1)\frac{(\theta-1)^k(k-j+1)^m(\log(\alpha))^{m+1}}{\theta^{k+1}(\alpha-1)^{k+1}(m+1)!}, & \theta>1 \\ (-1)^j \binom{k}{j}(k+1)\frac{(1-\theta)^k(j+1)^m(\log(\alpha))^{m+1}}{\theta^{k+1}(\alpha-1)^{k+1}(m+1)!}, & 0<\theta<1 \end{cases} \quad (10)$$

For some selected values of the parameters of MOAPIR, the cumulative distribution function and probability distribution function curves are presented in Figure 1. This is to show patterns of the behaviour of the parameters of the proposed distribution.



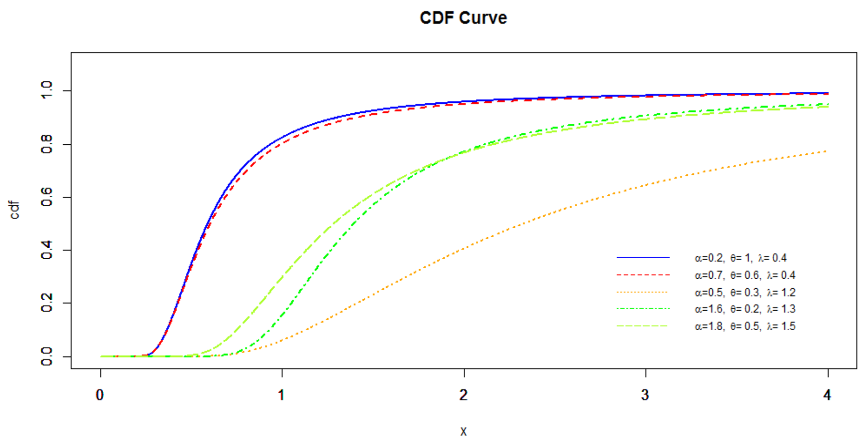


***Figure 1**: Plots of the PDF and CDF of MOPAIR distribution for selected values of the parameters*

Figure 1 shows that MOAPIR is a skewed and unimodal distribution, in addition, the CDF values lies between 0 and 1 is an indication that MOAPIR has a true PDF.

2.1 Sub-models of MOAPIR Distribution

To show that the proposed MOAPIR distribution is a generalisation distribution of family of distributions, we varied the value of the parameters of the distribution.

If we substitute α =1 in (6), then the expression will become

$$g_{MOIR}(x;\theta,\lambda) = \frac{2\lambda\theta x^{-3}e^{-\lambda x^{-2}}}{\left[\theta + (1-\theta)e^{-\lambda x^{-2}}\right]^2} \quad x > 0, \lambda > 0, \theta > 0$$

which is the pdf of the Marshall-Olkin Inverse Rayleigh (MOIR).

Similarly, if θ =1, then the expression in (6) will become

$$g_{APIRD}(x;\alpha,\lambda) = \frac{log\alpha}{\alpha-1}\frac{2\lambda}{x^3}e^{-\frac{\lambda}{x^2}}\alpha^{e^{-\frac{\lambda}{x^2}}} \qquad x > 0, \alpha \neq 1, \lambda > 0$$

which is the pdf of the Alpha Power Inverse Rayleigh (APIR) distribution proposed by Malik and Ahmad [20]. Also, when $\alpha = \theta = 1$, (6) will be reduced to the pdf of Inverse Rayleigh (IR) distribution proposed by Srinivasa, et al. [21] which is given by

$$g_{IR}(x;\lambda) = \frac{2\lambda}{x^3}e^{-\frac{\lambda^2}{x^2}} \qquad x, \lambda > 0$$

Thus, the proposed MOAPIR has been proven to be a generalization distribution of the APIR family of distributions.

2.2 Reliability Analysis

2.2.1 Survival Function

The survival function of MOAPIR distribution denoted by $R_{MOAPIR}(x)$ is derived using the expression presented in (11)

$$R_{\mathrm{MOAPIR}}(x) = \bar{G}(x) = 1 - G(x) \tag{11}$$

Substituting G(x) in (5) into (11), we have the Survival function of MOAPIR to be

$$R_{MOAPIR}(x) = \begin{cases} \dfrac{\alpha\theta\left(1-\alpha^{e^{-\lambda x^{-2}}-1}\right)}{\theta(\alpha-1)+(1-\theta)\left(\alpha^{e^{-\lambda x^{-2}}}-1\right)}, & x > 0, \alpha \neq 1, \lambda > 0, \theta > 0 \\ 0, & \alpha = 1 \end{cases} \tag{12}$$

2.2.2 Hazard Rate Function (HRF)

Let X be a random variable with pdf, $g(x)$ and cdf, $G(x)$, then the hazard rate function (HRF) is derived by solving $\frac{g(x)}{\bar{G}(x)}$.

Thus, if X is a MOAPIR random variable, then the HRF of the random variable X denoted by $h_{MOAPIR}(x)$ is

$$h_{MOAPIR}(x) = \begin{cases} \dfrac{(\alpha-1)2\lambda\log(\alpha)x^{-3}e^{-\lambda x^{-2}}\alpha^{e^{-\lambda x^{-2}}-1}}{\left[\theta(\alpha-1)+(1-\theta)\left(\alpha^{e^{-\lambda x^{-2}}}-1\right)\right]\left(1-\alpha^{e^{-\lambda x^{-2}}-1}\right)} & x > 0, \alpha \neq 1 \\ 0, & = 1 \end{cases} \tag{13}$$

where $\lambda > 0, \theta > 0$.

The pattern of the survival function and hazard rate function of the proposed distribution for various selected values of the distribution parameters are presented in Figure 2

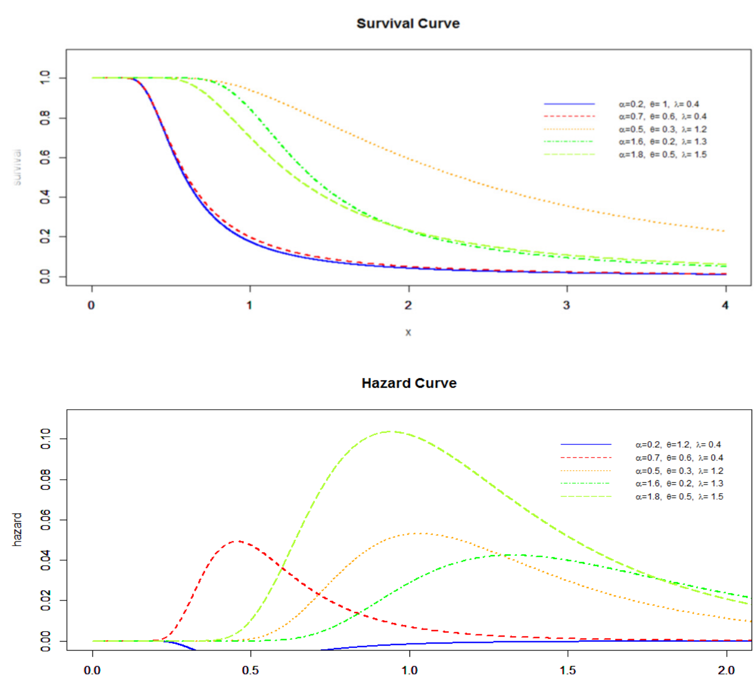


Figure 2: *The Survival Function and Hazard Rate Function Curves of MOAPIR distribution*

2.2.3 Reversed Hazard Rate Function

The reversed hazard rate (RHR) function of a random variable X from MOAPIR (α,θ,λ) distribution denoted as $r_{MOAPIR}(x)$ is derived to be:

$$r_{MOAPIR}(x) = \begin{cases} \dfrac{(\alpha-1)2\lambda\log(\alpha)x^{-3}e^{-\lambda x^{-2}}\alpha^{e^{-\lambda x^{-2}}}}{\left[\theta(\alpha-1)+(1-\theta)\left(\alpha^{e^{-\lambda x^{-2}}}-1\right)\right]\left(\alpha^{e^{-\lambda x^{-2}}}-1\right)} & x>0, \alpha\neq 1 \\ 0, & =1 \end{cases} \tag{14}$$

where $\lambda > 0, \theta > 0$

Figure 3 represents the RHRF curves for the MOAPIR (α,θ,λ) distribution for selected values of the distribution parameters

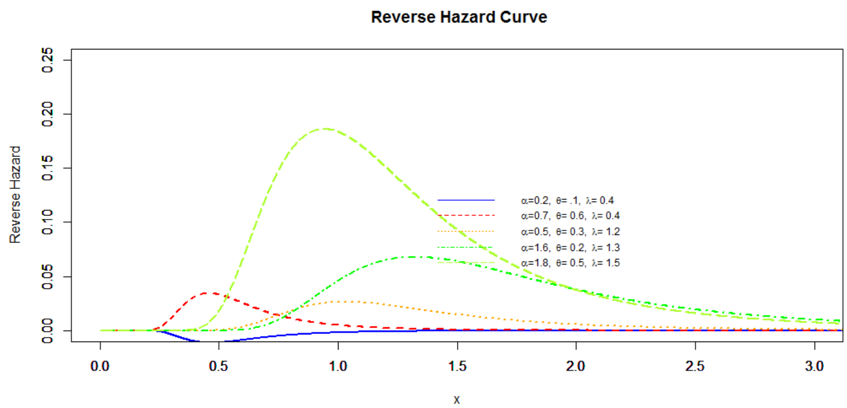


***Figure 3**: RHRF Curve of the MOAPIR distribution for some selected parameters values.*

2.3 Statistical Properties

In this section, we derived the statistical properties of the MOAPIR distribution. The properties derived are quantiles, median, mean, variance, order statistics, and range.

2.3.1 Quantiles

Quantiles explain how many values in a distribution are above or below a certain limit and define special part of a data set. The quantile of any distribution of a random variable X is given by solving the expression in (15)

$$G(x_q) = q,\ \ 0 < q < 1 \tag{15}$$

The qth quantile function is obtained by solving (16)

$$q = \frac{\alpha^{e^{-\lambda x^{-2}}}-1}{\theta(\alpha-1)+(1-\theta)\left(\alpha^{e^{-\lambda x^{-2}}}-1\right)} \tag{16}$$

Hence,

$$x_q = G^{-1}(q) = \left[\frac{1}{\lambda}\log\left(\frac{\log(\alpha)}{\log\left(\frac{1+(\alpha\theta-1)q}{1+(\theta-1)q}\right)}\right)\right]^{-\frac{1}{2}} \tag{17}$$

Using (17), we obtained the median, skewness, and kurtosis by determining the quantile of the MOAPIR distribution.

To obtain the First Quantile, substituting q = 0.25 in (17), then we obtained

$$x_{0.25} = G^{-1}(0.25) = \left[\frac{1}{\lambda}\log\left(\frac{\log(\alpha)}{\log\left(\frac{0.75+0.25\alpha\theta}{0.75+0.25\theta}\right)}\right)\right]^{-\frac{1}{2}} \tag{18}$$

For the Median = Q_2 = P_{50}, we use q = 0.50 in (17) and obtained

$$\text{Median} = x_{0.5} = G^{-1}(0.5) = \left[\frac{1}{\lambda}\log\left(\frac{\log(\alpha)}{\log\left(\frac{1+\alpha\theta}{1+\theta}\right)}\right)\right]^{-\frac{1}{2}} \tag{19}$$

For the Third Quantile = Q_3 = P_{75}, we use q = 0.75 in (17) and obtained

$$Q_3 = x_{0.75} = G^{-1}(0.75) = \left[\frac{1}{\lambda}\log\left(\frac{\log(\alpha)}{\log\left(\frac{0.25+0.75\alpha\theta}{0.25+0.75\theta}\right)}\right)\right]^{-\frac{1}{2}} \tag{20}$$

Using (18), (19) and (20), the Skewness (S_k) and Kurtosis (K_{MOAPIR}) of the MOAPIR distribution were obtained respectively as

$$S_{kMOAPIR} = \frac{G^{-1}(0.75)-2G^{-1}(0.5)+G^{-1}(0.25)}{G^{-1}(0.75)-G^{-1}(0.25)} \tag{21}$$

and

$$K_{MOAPIR} = \frac{G^{-1}(0.875)-G^{-1}(0.625)-G^{-1}(0.375)+G^{-1}(0.125)}{G^{-1}(0.75)-G^{-1}(0.25)} \tag{22}$$

2.3.2 Moments

Let X be a random variable that has MOAPIR $(\alpha, \lambda, \theta)$ distribution, the r[th] moments of X is defined as

$$E[X^r] = \int_0^\infty x^r g(x)dx \tag{23}$$

$$= \int_0^\infty x^r \frac{(\alpha-1)2\lambda\theta\log(\alpha)x^{-3}e^{-\lambda x^{-2}}\alpha^{e^{-\lambda x^{-2}}}}{\left[(\alpha-1)\theta+(1-\theta)\left(\alpha^{e^{-\lambda x^{-2}}}-1\right)\right]^2}\,dx \tag{24}$$

Using linear expressions of $g_{MOAPIR}(x)$ in (9) and (10), we have

$$E[X^r] = \int_0^\infty x^r \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{\infty} W_{k,j,m} 2\lambda(m+1)x^{-3}\,dx \tag{25}$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{\infty} W_{k,j,m}\big((m+1)\lambda\big)^{\frac{r}{2}}\Gamma\left(1-\frac{r}{2}\right) \tag{26}$$

From (25) and (26), the mean and variance of a random variable X from MOAPIR distribution are:

$$E[X] = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{\infty} W_{k,j,m}\big((m+1)\lambda\big)^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right) \tag{27}$$

and

$$V(X) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{\infty} W_{k,j,m}\big((m+1)\lambda\big) - \left(\Gamma\left(\frac{1}{2}\right)\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{\infty} W_{k,j,m}\big((m+1)\lambda\big)^{\frac{1}{2}}\right)^2 \tag{28}$$

2.3.3 Order statistics

The pdf of the i[th] order statistics $X_{i:n}$ of a random sample $X_1, X_2, \ldots, X_n$ is

$$g_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} g(x)[G(x)]^{i-1}[1-G(x)]^{n-i}, \quad x > 0, \alpha > 0, \lambda > 0 \qquad (29)$$

From (29), the pdf of the i^th^ order statistics $X_{i:n}$ of MOAPIR distribution is obtained to be

$$g_{i:n}(x) = \frac{2\lambda}{(i-1)!(n-i)!} \sum_{k_1=0}^{\infty} \sum_{j_1=0}^{n-i} \sum_{j_2=0}^{i+k_1-} W_{k_1,j_1,j_2} x^{-3} e^{\lambda x^{-2}} \alpha^{(n \ \ i-j_1)+(i+k_1 \ \ j_2)} e^{\lambda x^{-2}} \qquad (30)$$

where

$$W_{k_1,\ j_1,\ j_2} = (-1)^{j_1+j_2} \binom{n-i}{j_1} \binom{i+k_1-1}{j_2} \frac{\Gamma(\mathrm{n}+\mathrm{k}_1+1)}{\mathrm{k}_1!} \frac{(\theta-1)^{k_1}}{(\theta)^{i+k_1}}$$

2.3.4 Range of MOAPIR

Let $X_{(1)}, X_{(2)}, X_{(3)}, \ldots, X_{(n)}$ be the order statistics from the sample $X_1, X_2, X_3, \ldots, X_n$ of size n from a random variable that is of MOAPIR distribution, then the distribution of the range of the random variable X, R(x) can be obtained by solving $R(x) = g_{n:n}(x) - g_{1:n}(x)$

Using (30), the range of MOAPIR random variable is derived to be

$$R(x) = \frac{2\lambda}{(n-1)!} \Big(\sum_{k_1=0}^{\infty} \sum_{j_2=0}^{n+k_1-1} W_{k_1,j_2} x^{-3} e^{\lambda x^{-2}} \alpha^{(n+k_1-j_2)} e^{\lambda x^{-2}} - \sum_{k_1=0}^{\infty} \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{k_1} W_{k_1,j_1,j_2} x^{-3} e^{\lambda x^{-2}} \alpha^{(n-1-j_1)+(1+k_1-j_2)} e^{\lambda x^{-2}} \Big) \qquad (31)$$

2.4. Estimation of Parameters of MOAPIR Distribution

The parameters of the proposed distribution were derived using the maximum likelihood estimation approach. Let $X_1, \ldots, X_n$ be a random sample of size n from MOAPIR distribution, then the likelihood function of the MOAPIR distribution, $L(x/\alpha, \lambda, \theta)$ is

$$L(x/\alpha, \lambda, \theta) = \prod_{i=1}^{n} g(x_i) = \frac{(\alpha-1)^n (\log(\alpha))^n \lambda^n 2^n \theta^n e^{-\lambda \sum_{i=1}^{n} x_i} \alpha^{\sum_{i=1}^{n} e^{-\lambda x_i^{-2}}} \prod_{i=1}^{n} x_i^{-3}}{\prod_{i=1}^{n} \left[(\alpha-1)\theta + (1-\theta)\left(\alpha^{e^{-\lambda x_i^{-2}}} - 1 \right) \right]^2} \qquad (32)$$

By taking logarithm of the likelihood function, we have

$$\ell(x/\alpha, \lambda, \theta) = n\log\big((\alpha-1)\log(\alpha)\, 2\lambda\theta\big) - \lambda \sum_{i=1}^{n} x_i^{-2} + \log(\alpha) \sum_{i=1}^{n} e^{-\lambda x_i^{-2}} - 3\sum_{i=1}^{n} \log(x_i) - 2\sum_{i=1}^{n} \log\left[(\alpha-1)\theta + (1-\theta)\left(\alpha^{e^{-\lambda x_i^{-2}}} - 1 \right) \right] \qquad (33)$$

To obtain the MLEs of α, λ and θ, we differentiate the expression in (33) with respect to α, λ and θ. Thus, we have

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha-1} + \frac{n}{\alpha \log(\alpha)} + \frac{1}{\alpha} \sum_{i=1}^{n} e^{-\lambda x_i^{-2}} - 2\sum_{i=1}^{n} \frac{\theta + (1-\theta) e^{-\lambda x_i^{-2}} \alpha^{e^{-\lambda x_i^{-2}}} - 1}{(\alpha-1)\theta + (1-\theta)\left(\alpha^{e^{-\lambda x_i^{-2}}} - 1 \right)} \qquad (34)$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i-1}^{n} x_i^{-2} - \log(\alpha) \sum_{i=1}^{n} x_i^{-2} e^{-\lambda x_i^{-2}} - 2\sum_{i=1}^{n} \frac{(1-\theta)\log(\alpha) x_i^{-2} e^{-\lambda x_i^{-2}} \alpha^{e^{-\lambda x_i^{-2}}}}{(\alpha-1)\theta + (1-\theta)\left(\alpha^{e^{-\lambda x_i^{-2}}} - 1 \right)} \qquad (35)$$

and

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - 2\sum_{i-1}^{n} \frac{\alpha - \alpha^{e^{-\lambda x_i^{-2}}}}{(\alpha-1)\theta + (1-\theta)\left(\alpha^{e^{-\lambda x_i^{-2}}} - 1 \right)} \qquad (36)$$

Solving (34), (35) and (36) by equating them to zero, we have

$$\frac{n}{\alpha-1}+\frac{n}{\alpha\log(\alpha)}+\frac{1}{\alpha}\sum_{i=1}^{n}e^{-\lambda x_i^{-2}}-2\sum_{i=1}^{n}\frac{\theta+(1-\theta)e^{-\lambda x_i^{-2}}\alpha^{e^{-\lambda x_i^{-2}}}-1}{(\alpha-1)\theta+(1-\theta)\left(\alpha^{e^{-\lambda x_i^{-2}}}-1\right)}=0 \tag{37}$$

$$\frac{n}{\lambda}-\sum_{i-1}^{n}x_i^{-2}-\log(\alpha)\sum_{i=1}^{n}x_i^{-2}e^{-\lambda x_i^{-2}}-2\sum_{i=1}^{n}\frac{(1-\theta)\log(\alpha)x_i^{-2}e^{-\lambda x_i^{-2}}\alpha^{e^{-\lambda x_i^{-2}}}}{(\alpha-1)\theta+(1-\theta)\left(\alpha^{e^{-\lambda x_i^{-2}}}-1\right)}=0 \tag{38}$$

and

$$\frac{n}{\theta}-2\sum_{i-1}^{n}\frac{\alpha-\alpha^{e^{-\lambda x_i^{-2}}}}{(\alpha-1)\theta+(1-\theta)\left(\alpha^{e^{-\lambda x_i^{-2}}}-1\right)}=0 \tag{39}$$

The MLE of α, λ and θ can not be obtained by solving (37), (38), and (39) analytically. Hence the Newton-Raphson iterative method would be used to accomplish the task of estimating the parameters.

3.Determination of Flexibility of the Proposed Distribution

To access the flexibility of the proposed distribution, the MOAPIR distribution is compared with three competing distributions by using two real life data sets. The distributions considered in this study are the Marshall Olkin Alpha Power Inverse Exponential (MOAPIE), Alpha Power Inverse Rayleigh (APIR), and Inverse Rayleigh (IR) distributions.

Data Set I is on life of fatigue fracture of Kevlar 373/epoxy that are subjected to constant pressure at the 90% stress level until all had failed (Ogunde et al. [22]) and **Data Set II** is on the relief times of twenty patients receiving an analgesic as reported by Gross and Clark [23].

The summary statistics of the two datasets are presented in Table 1 and the density plot of the datasets along with the empirical density plots of the considered distributions are presented in Figures 4.

Table 1: *Summary Statistics of Datasets*

Data set	Min	Q1	Median	Q3	Mean	Variance	Max	Skewness	Kurtosis
I (n=76)	0.0251	0.0905	1.7361	2.2960	1.9590	2.4774	9.0960	1.9406	4.9474
II (n=20)	1.10	1.475	1.7	2.05	1.90	0.4958	4.10	1.5924	2.3465

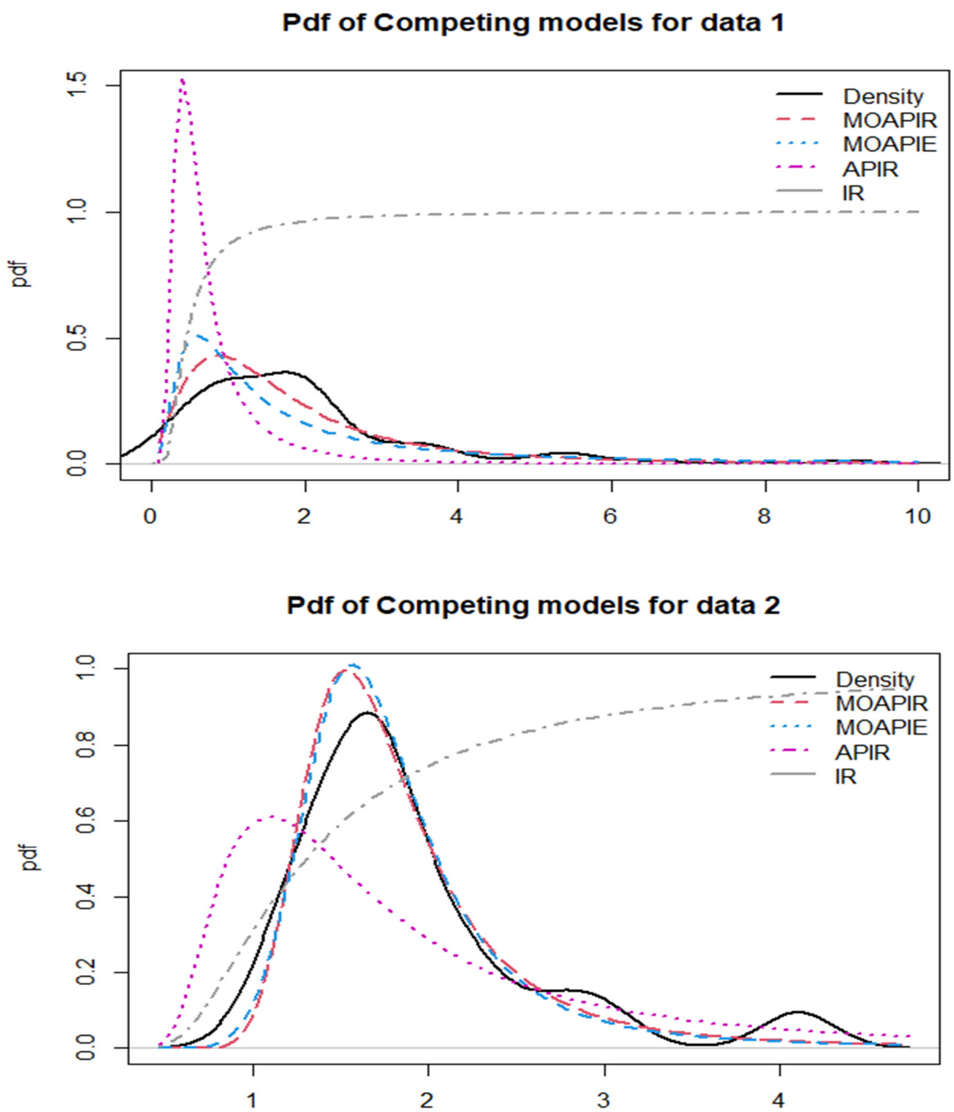


Figure 4: *Density Plot of Datasets I and II With MOAPIRD and other competing Distributions.*

3.1 Parameter Estimation and Goodness of Fit Test

Four criteria, namely, the log-likelihood values (-LL), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Hannan-Quinn information criterion (HQIC) are used to select the best fitted model to the two data sets under consideration. The model with minimum value of each of the four criteria is adjudged as the best fit for the datasets under study. The estimated values of the parameters of the four distributions and the goodness of fit criteria are presented in Table 2.

Table 2: Estimated parameters and Criteria for goodness of fit

		Parameters			- LL	AIC	BIC	HQIC
Data set	Distribution	α	λ	θ				
I	MOAPIR	7.652	0.001136	853.4	**124.84**	**255.69**	**262.67**	**258.5**
	MOAPIE	1253	0.2375	0.5040	139.90	285.80	292.79	288.6
	APIR	6883	0.0293	-	6007.67	12019.33	12024.0	12021.2
	IR	-	0.1406	-	211.48	424.96	427.29	425.89
II	MOAPIR	51.84	7.5567	0.0071	**15.51**	**37.02**	**40.01**	**37.6**
	MOAPIE	1.114	10.80	0.0017	15.65	37.29	40.27	37.86
	APIR	1	1.801	-	694.52	1393.05	1395.03	1393.43
	IR	-	1.1749	-	28.27	58.54	59.54	58.73

* the bold number represents the smallest value for each criterion.

4.Discussion

The cumulative distribution function and probability density function of the proposed MOAPIR distribution were given in (5) and (6) respectively. Figure 1 illustrates the shape of the distribution when its parameters were varied and it was clear that the distribution is a positively heavily skewed, unimodal and a true distribution function. The survival function curve reflected that higher value of λ, (λ>0.4) will destruct the expected shape of the hazard function. Similarly, it is crystal clear that the value of θ has no significant effect on the shape of the hazard function (see Figure 2). However, it was observed that variations in the values of the parameters significantly affect the pattern of the hazard rate function and the reversed hazard rate function (see Figure 2 and 3). Further analysis shows that θ and λ have no significant effects on the skewness and kurtosis of the distribution but have influence on the mean, median and variance of the distribution. The summary statistics in Table 1 shows that the two data sets are heavily positively skewed data. Furthermore, the density plot in Figure 4 for both data sets indicated that the two data sets are heavily positively skewed. The fitted distributions as shown in Figure 4, reflected that the proposed MOAPIR is a more suitable distribution than all other competing distributions considered in this study. The results from the performance indices namely, -LL, AIC, BIC and HQIC confirmed that the proposed MOAPIR best fit the two data sets considered in this paper than the MOAPIE, APIR and IR distributions.

5.Conclusion

In this paper, a new distribution called Marshall Olkin Alpha Power Inverse Rayleigh (MOAPIR) distribution was introduced. The pdf and cdf of the distribution were derived and some of its properties, such as hazard rate function, reversed hazard rate function, quantiles, moments, and order statistics were studied. The parameters of the proposed distribution were estimated using the Maximum likelihood estimation method. To access the flexibility of the proposed MOAPIR distribution with three competing distributions of the same family, namely the MOAPIE, APIR and IR distributions, two data sets were used. The results showed that the proposed MOAPIR distribution has minimum value of -LL, AIC, BIC and HQIC, and then, adjudged to be the best fit for the two data sets considered in this study. Therefore, the proposed distribution provides a better fit than other competing distributions of the same family of heavily positively skewed distribution.

Hence, for a heavily positive skewed data, the MOAPIR is a good distribution model to be used for further analysis.

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