ON DIFFERENT CLASSICAL ESTIMATION APPROACHES FOR TYPE I HALF LOGISTIC-TOPP-LEONE- EXPONENTIAL DISTRIBUTION

Akeem Ajibola Adepoju[1], Sauta S. Abdulkadir[2], Danjuma Jibasen[3]

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[1]Department of Statistics, Faculty of Computing and Mathematical Sciences, Aliko Dangote University of Science and Technology, Wudil, Kano. Nigeria

[2,3]Department of Statistics, Faculty of Physical Sciences Modibbo Adama University, Yola, Adamawa. Nigeria

akeebola@gmail.com[1], ssabdulkadir@mautech.edu.ng[2], djibasen2001@gmail.com[3]

Abstract

This paper aims to propose six methods of parameter estimation in order to examine the behavior of the new Type I Half Logistic Topp-leone Exponential distribution. The methods taking into consideration are Maximum Likelihood, Anderson_Darling, Least Squares, Cramer_von_Mises, Maximum Product of Spacing, and Weighted Least Squares Methods. The results show that all the methods are consistent, since the estimates approach the true value of the parameters for all the methods. The bias, mean square error and mean relative estimates decay as the sample size is raised. The estimates of the six methods obtained for the model, indicated that MPS estimates is the closest to the true value of the parameters across the low, moderate and high sample sizes, invariable, the MPS produces the least biasness. Buttress more, the MPS produces the least MSE all through and remain the best estimator for low, moderate and high sample size of the model. Conclusively, MPS is the most consistent among the estimators for the model.

Keywords: Type I Half Logistic Topp-leone Exponential distribution, maximum likelihood, Anderson_Darling, least squares and weighted least squares Methods, Cramer_von_Mises, Maximum Product of Spacing

I. Introduction

Exponential (Exp) distribution is an important and commonly explored probability distribution both in univariate, bivariate and multivariate cases. The Type I Half Logistic Topp-leone Exponential (TIHLTLExp) distribution was proposed by [1] as a generalized distribution. The distribution is characterized with two shape parameters and a scale parameter. The hazard rate shapes of the distribution are monotonically increasing, monotonically decreasing and bathub in feature. It was revealed from the TIHLTLExp distribution analysis that the distribution potentiality is awesome in modeling a good number of life time data sets.

On the other hand, Exp distribution has witness different generalization where one or more shape parameter are introduced to extend it flexibility such can be found in the work of [2] where sine family was adopted to generalized the Exp distribution, in [3], the Type I half logistic exponentiated family was used to improve the Exp distribution. Other generalization of Exp distribution includes Half logistic-truncated exponential distribution [4], A new extension named Lehmann type–II G class of distributions: Exp distribution [5] Lomax Exp distribution [6], Lehmann Type II-Lomax Distribution [7], Exponentiated Gamma Exp Distribution [8], Exponentiated Weibull Exp distribution [9], Topp-Leone generalized Exp power series distribution [10], new extension of Exp distribution [11], Type II Half Logistic Exp Distribution [12], Gamma-exponentiated Exp distribution [13], and the Type II half logistic exponentiated family [14] to mention but few.

Various methods have been developed and applied to estimate the some newly developed distribution. No particular estimation method is the best all round. However, some methods perform better than the other depending on the behavior of the distribution parameters. Six different classical methods are considered in this article. The classical approaches such as Maximum Likelihood Estimator (MLE), Anderson–Darling Estimator (ADE), Cramér–von Mises (CVM), Maximum Product Spacing (MPS), Least Square Estimator (LSE), and Weighted Least Square Estimator (WLSE) are explored. Articles that adopted some estimation methods includes, type II exponentiated half-logistic-PLo (TIIEtHL-PLo) distribution by [15], Parameter estimation methods adopted are MLE, LSE, WLSE, MPS, CVM, and ADE in the study. Inference on Kavya–Manoharan Kumaraswamy distribution by [16], estimation of polynomial Exp family of distributions by [17], estimation comparison for extreme value distribution by [18], Classical and Bayesian Approach Estimation of Weibull-Exp Distribution by [19], estimation preference inverse rayleigh frechet model by [20], estimation methods in Tasks of processing measurement results by [21], comparison of estimation methods for the (Three-Parameter) Lindley distribution by [22]. MLE, OLS, WLS, MPS, and CVM methods, different estimation approaches for Type I half-logistic topp–leone distribution by [23], comparative study of estimation for Pareto distribution by [24], some estimation methods for lindley distribution, estimation methods include MME, MLE, resulting identification of MLE to be the best estimator by [25], also, the weibull distribution parameters, three methods such as the MLE, MME and LSE regression method were considered and compared, from the result, the MME method was superior [26], LSE of distribution functions [27], MPS estimation with preference to the lognormal distribution [28] and parameters estimation for the (three-parameter) Reflected Weibull model. The MME, MLE, Location and Scale Parameters free ML estimator (LSPEE). The data transformation is the basis for LSPEE, Mont Carlo simulations show that the LSPEE outperform MME and MLE. The TIHLTLExp distribution was a newly distribution developed, however, only two methods MLE and MPS were used for parameter estimation.

This paper aims to investigate the behaviour of the TIHLTLExp model parameters using six estimation methods. The motivation for this study is the determination of the best model parameter's estimator for low, moderate and high sample size of the TIHLTLExp distribution.

II. Methods

2.0 Method of parameter estimation of TIHLTLExp distribution

In this section, we introduced the cumulative distribution function (cdf) and probability density function (pdf) of the Type Half Logistic Topp-leone Exponential Distribution.

$$F_{TIHLTLExp}\left(x;\beta,\theta,\lambda\right)=\frac{1-\left[1-\left[1-\left(e^{-\lambda x}\right)^{2}\right]^{\theta}\right]^{\beta}}{1+\left[1-\left[1-\left(e^{-\lambda x}\right)^{2}\right]^{\theta}\right]^{\beta}} \tag{1}$$

$$f_{TIHLTLExp}\left(x;\beta,\theta,\lambda\right)=\frac{4\beta\theta\lambda\left(e^{-\lambda x}\right)^{2}\left[1-\left(e^{-\lambda x}\right)^{2}\right]^{\theta-1}\left[1-\left[1-\left(e^{-\lambda x}\right)^{2}\right]^{\theta}\right]^{\beta-1}}{\left[1+\left[1-\left[1-\left(e^{-\lambda x}\right)^{2}\right]^{\theta}\right]^{\beta}\right]^{2}} \tag{2}$$

The method employed to be used to estimate the parameter include: MLE, ADE, CVM, MPS, LSE, and WLSE

2.1 Maximum Likelihood Estimation (MLE)

MLE is one of the widely explored estimation approaches. It is adopted in estimating the parameters of the TIHLTLExp model. if we randomly sampled X_i where $i=1,\ldots,n$, obtained from the TIHLTLExp distribution with parameter $\Omega=\beta,\theta,\lambda$. The log-likelihood function $L(\Omega)$ of (1) is obtained as

$$\begin{aligned}L(\Omega)&=n\log 4+n\log\beta+n\log\theta+n\log\lambda+2\sum_{i=0}^{n}\log\left(e^{-\lambda x}\right)+(\theta-1)\sum_{i=0}^{n}\log\left(1-\left(e^{-\lambda x}\right)^{2}\right)\\&+(\beta-1)\sum_{i=0}^{n}\log\left(1-\left(1-\left(e^{-\lambda x}\right)^{2}\right)^{\theta}\right)-2\sum_{i=0}^{n}\log\left(1+\left(1-\left(1-\left(e^{-\lambda x}\right)^{2}\right)^{\theta}\right)^{\beta}\right)\end{aligned}, \tag{3}$$

By differentiating $L(\Omega)$ in (3) with respect to β,θ and λ, and the results set to zero will provide the estimators. Thus,

$$\frac{\delta L(\Omega)}{\delta\beta}=\frac{n}{\beta}+\sum_{i=0}^{n}\log\left(1-\left(1-\left(e^{-\lambda x}\right)^{2}\right)^{\theta}\right)-\frac{\beta\log\sum_{i=0}^{n}\left(1-\left(1-\left(e^{-\lambda x}\right)^{2}\right)^{\theta}\right)}{\sum_{i=0}^{n}\left(1+\left(1-\left(1-\left(e^{-\lambda x}\right)^{2}\right)^{\theta}\right)^{\beta}\right)}=0 \tag{4}$$

$$\begin{aligned}\frac{\delta L(\Omega)}{\delta\theta}&=\frac{n}{\theta}+\sum_{i=0}^{n}\log\left(1-\left(e^{-\lambda x}\right)^{2}\right)+\frac{\theta(\beta-1)\sum_{i=0}^{n}\log\left(1-\left(1-\left(e^{-\lambda x}\right)^{2}\right)\right)}{\left(1-\left(1-\left(e^{-\lambda x}\right)^{2}\right)^{\theta}\right)}\\&-2\beta\sum_{i=0}^{n}\left(1+\left(1-\left(1-\left(e^{-\lambda x}\right)^{2}\right)^{\theta}\right)^{\beta-1}\right)\frac{\theta\sum_{i=0}^{n}\log\left(1-\left(1-\left(e^{-\lambda x}\right)^{2}\right)\right)}{\left(1-\left(1-\left(e^{-\lambda x}\right)^{2}\right)^{\theta}\right)^{\beta}}=0\end{aligned} \tag{5}$$

$$\frac{\delta L(\Omega)}{\delta\lambda}=\frac{n}{\lambda}-\frac{2\lambda e^{-\lambda x}x}{\left(e^{-\lambda x}\right)}+\frac{2\lambda(\theta-1)e^{-\lambda x}x}{\left(1-\left(e^{-\lambda x}\right)^{2}\right)}+\frac{2\theta\lambda(\beta-1)\left(1-\left(e^{-\lambda x}\right)^{2}\right)^{\theta-1}e^{-\lambda x}x}{\left(1-\left(1-\left(e^{-\lambda x}\right)^{2}\right)^{\theta}\right)}$$
$$-\frac{2\beta\left(1-\left(1-\left(e^{-\lambda x}\right)^{2}\right)^{\theta}\right)^{\beta-1}\left(1-\left(e^{-\lambda x}\right)^{2}\right)^{\theta-1}e^{-\lambda x}x}{\left(1+\left(1-\left(1-\left(e^{-\lambda x}\right)^{2}\right)^{\theta}\right)^{\beta}\right)} \tag{6}$$

2.2 Anderson–Darling Estimates (ADE)

The ADE was introduced by [30]. Applying ADE method for the TIHLTLExp distribution parameter $\Omega=\beta,\theta,\lambda$

$$ADE_{\Omega}=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log\left[F_{TIHLTLExp}\left(x_{(i)};\beta,\theta,\lambda\right)\right]+\log\left[1-F_{TIHLTLExp}\left(x_{(n+1-i)};\beta,\theta,\lambda\right)\right]\right\} \tag{7}$$

$$ADE_{\Omega}=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left[\frac{1-\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}{1+\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}\right]+\log\left[\frac{1-\left[1-\left[1-\left(e^{-\lambda x_{(n+1-i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}{1+\left[1-\left[1-\left(e^{-\lambda x_{(n+1-i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}\right]\right] \tag{8}$$

Thus, the estimates can be easily obtained by differentiating (8) with respect to. β,θ and λ set the results to zero.

$$\frac{\delta ADE_{\Omega}}{\delta\beta}=-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\varpi_{i}^{(p1)}(\beta,\theta,\lambda)}{\left[F_{TIHLTLExp}\left(x_{(i)};\beta,\theta,\lambda\right)\right]}-\frac{\varpi_{n+1-i}^{(p1)}(\beta,\theta,\lambda)}{\left[1-F_{TIHLTLExp}\left(x_{(n+1-i)};\beta,\theta,\lambda\right)\right]}\right]=0 \tag{9}$$

$$\frac{\delta ADE_{\Omega}}{\delta\theta}=-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\varpi_{i}^{(p2)}(\beta,\theta,\lambda)}{\left[F_{TIHLTLExp}\left(x_{(i)};\beta,\theta,\lambda\right)\right]}-\frac{\varpi_{n+1-i}^{(p2)}(\beta,\theta,\lambda)}{\left[1-F_{TIHLTLExp}\left(x_{(n+1-i)};\beta,\theta,\lambda\right)\right]}\right]=0 \tag{10}$$

$$\frac{\delta ADE_{\Omega}}{\delta\lambda}=-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\varpi_{i}^{(p3)}(\beta,\theta,\lambda)}{\left[F_{TIHLTLExp}\left(x_{(i)};\beta,\theta,\lambda\right)\right]}-\frac{\varpi_{n+1-i}^{(p3)}(\beta,\theta,\lambda)}{\left[1-F_{TIHLTLExp}\left(x_{(n+1-i)};\beta,\theta,\lambda\right)\right]}\right]=0 \tag{11}$$

where

$$\varpi_{i}^{(p1)}(\beta,\theta,\lambda)=\frac{2\beta\log\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]\left[\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]\right]}{\left[1-\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}\right]\left[1+\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}\right]^{3}}, \tag{12}$$

$$\varpi_i^{(p2)}\left(\beta,\theta,\lambda\right)=\frac{2\beta\theta\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^2\right]^\theta\right]^\beta\log\left[1-\left(e^{-\lambda x_{(i)}}\right)^2\right]}{\left[1+\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^2\right]^\theta\right]^\beta\right]^2} \tag{13}$$

$$\varpi_i^{(p3)}\left(\beta,\theta,\lambda\right)=\frac{4\beta\theta\left[1-\left[1-\left(e^{-\lambda x}\right)^2\right]^\theta\right]^{2\beta-1}\left[1-\left(e^{-\lambda x}\right)^2\right]^{\theta-1}xe^{-\lambda x}}{\left[1+\left[1-\left[1-\left(e^{-\lambda x}\right)^2\right]^\theta\right]^\beta\right]^2} \tag{14}$$

$$\varpi_{n+1-i}^{(p1)}\left(\beta,\theta,\lambda\right)=\frac{2\beta\log\left[1-\left[1-\left(e^{-\lambda x_{(n+1-i)}}\right)^2\right]^\theta\right]\left[\left[1-\left[1-\left(e^{-\lambda x_{(n+1-i)}}\right)^2\right]^\theta\right]\right]}{\left[1-\left[1-\left[1-\left(e^{-\lambda x_{(n+1-i)}}\right)^2\right]^\theta\right]^\beta\right]\left[1+\left[1-\left[1-\left(e^{-\lambda x_{(n+1-i)}}\right)^2\right]^\theta\right]^\beta\right]^3} \tag{15}$$

$$\varpi_{n+1-i}^{(p2)}\left(\beta,\theta,\lambda\right)=\varpi_i^{(p2)}\left(\beta,\theta,\lambda\right)=\frac{2\beta\theta\left[1-\left[1-\left(e^{-\lambda x_{(n+1-i)}}\right)^2\right]^\theta\right]^\beta\log\left[1-\left(e^{-\lambda x_{(n+1-i)}}\right)^2\right]}{\left[1+\left[1-\left[1-\left(e^{-\lambda x_{(n+1-i)}}\right)^2\right]^\theta\right]^\beta\right]^2} \tag{16}$$

$$\varpi_{n+1-i}^{(p3)}\left(\beta,\theta,\lambda\right)=\frac{4\beta\theta\left[1-\left[1-\left(e^{-\lambda x_{(n+1-i)}}\right)^2\right]^\theta\right]^{2\beta-1}\left[1-\left(e^{-\lambda x_{(n+1-i)}}\right)^2\right]^{\theta-1}x_{(n+1-i)}e^{-\lambda x_{(n+1-i)}}}{\left[1+\left[1-\left[1-\left(e^{-\lambda x_{(n+1-i)}}\right)^2\right]^\theta\right]^\beta\right]^2}, \tag{17}$$

2.3 Cramér-von Mises Estimators. (CVM)

CVM was proposed [31]. The concept of this approach is to minimize the following function with respect to parameter $\Omega=\beta,\theta,\lambda$. The CVM distance function for TIHLTLExp distribution is defined by

$$CVM_\Omega=\frac{1}{12n}+\sum_{i=1}^{n}\left[F_{TIHLTLExp}\left(x_{(i)};\beta,\theta,\lambda\right)-\frac{2i-1}{2n}\right]^2=\frac{1}{12n}+\sum_{i=1}^{n}\left[\left(\frac{1-\left[1-\left[1-\left(e^{-\lambda x}\right)^2\right]^\theta\right]^\beta}{1+\left[1-\left[1-\left(e^{-\lambda x}\right)^2\right]^\theta\right]^\beta}\right)-\frac{2i-1}{2n}\right]^2 \tag{18}$$

Thus, the estimates of the TIHLTLExp distribution parameter under CVM method is obtained by differentiating the (18) with respect to β,θ and λ and set it to zero.

$$\frac{\delta CVM_{\Omega}}{\delta\beta}=2\sum_{i=1}^{n}\varpi_{i}^{(p1)}(\beta,\theta,\lambda)\left[\left(\frac{1-\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}{1+\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}\right)-\frac{2i-1}{2n}\right]=0 \quad (19)$$

$$\frac{\delta CVM_{\Omega}}{\delta\theta}=2\sum_{i=1}^{n}\varpi_{i}^{(p2)}(\beta,\theta,\lambda)\left[\left(\frac{1-\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}{1+\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}\right)-\frac{2i-1}{2n}\right]=0 \quad (20)$$

$$\frac{\delta CVM_{\Omega}}{\delta\lambda}=2\sum_{i=1}^{n}\varpi_{i}^{(p3)}(\beta,\theta,\lambda)\left[\left(\frac{1-\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}{1+\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}\right)-\frac{2i-1}{2n}\right]=0 \quad (21)$$

$\varpi_{i}^{(p1)}(\beta,\theta,\lambda)$, $\varpi_{i}^{(p2)}(\beta,\theta,\lambda)$ and $\varpi_{i}^{(p3)}(\beta,\theta,\lambda)$ are defined in (12), (13) and (14) respectively. The (31) provides more details

2.3 Maximum Product of Spacing (MPS)

The MPS approach of estimating the TIHLTLExp distribution parameters $\Omega=\beta,\theta,\lambda$ are produced by maximizing the equations below with respect to the parameters:

$$MPS_{\Omega}=\frac{1}{n+1}\sum_{i=1}^{n+1}\log(G_i) \quad (22)$$

Where

$$G_i=F_{TIHLTLExp}\left(x_{(i)};\beta,\theta,\lambda\right)-F_{TIHLTLExp}\left(x_{(i-1)};\beta,\theta,\lambda\right)$$

$$F_{TIHLTLExp}\left(x_{(0)};\beta,\theta,\lambda\right)=0,\ F_{TIHLTLExp}\left(x_{(n+1)};\beta,\theta,\lambda\right)=1$$

and

$$\sum_{i=1}^{n+1}G_i=1$$

Thus the MPS_{Ω} estimates are obtained by differentiating the equation (22) with respect to the parameters

where $F_{TIHLTLExp}\left(x_{(i)};\beta,\theta,\lambda\right)$ is the cdf of the TIHLTLExp distribution defined in (1)

2.4 Least Square Estimates (LSE)

LSE was introduced by [32]. The LSE of the TIHLTLExp distribution parameters $\Omega=\beta,\theta,\lambda$ are obtained by minimizing, the equation below. The LSE function is defined by

$$LSE_{\Omega}=\sum_{i=1}^{n}\left[F_{TIHLTLExp}\left(x_{(i)};\beta,\theta,\lambda\right)-\frac{i}{n+1}\right]^{2}=\sum_{i=1}^{n}\left[\left(\frac{1-\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}{1+\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}\right)-\frac{i}{n+1}\right]^{2} \tag{23}$$

Thus, the LSE can be obtained by differentiating equation [23] with respect to the β,θ and λ, and set it to zero

$$\frac{\delta LSE_{\Omega}}{\delta\beta}=2\sum_{i=1}^{n}\varpi_{i}^{(p1)}\left(\beta,\theta,\lambda\right)\left[\left(\frac{1-\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}{1+\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}\right)-\frac{i}{n+1}\right]=0 \tag{24}$$

$$\frac{\delta LSE_{\Omega}}{\delta\theta}=2\sum_{i=1}^{n}\varpi_{i}^{(p2)}\left(\beta,\theta,\lambda\right)\left[\left(\frac{1-\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}{1+\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}\right)-\frac{i}{n+1}\right]=0 \tag{25}$$

$$\frac{\delta LSE_{\Omega}}{\delta\lambda}=2\sum_{i=1}^{n}\varpi_{i}^{(p3)}\left(\beta,\theta,\lambda\right)\left[\left(\frac{1-\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}{1+\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^{2}\right]^{\theta}\right]^{\beta}}\right)-\frac{i}{n+1}\right]=0 \tag{26}$$

$\varpi_{i}^{(p1)}\left(\beta,\theta,\lambda\right)$, $\varpi_{i}^{(p2)}\left(\beta,\theta,\lambda\right)$ and $\varpi_{i}^{(p3)}\left(\beta,\theta,\lambda\right)$ are defined in (12), (13) and (14) respectively.

2.5 Weighted Least Square Estimates (WLSE)

Similarly, the WLSE was introduced by [32]. The WLSE of the TIHLTLExp distribution parameters $\Omega=\beta,\theta,\lambda$ are produced by minimizing the equation below with respect to the β,θ and λ. The WLSE function is defined by

$$WLSE_{\Omega}=\sum_{i=1}^{n}\left(\frac{\left(n+1\right)^{2}\left(n+1\right)}{i\left(n+1-i\right)}\right)\left[F_{TIHLTLExp}\left(x_{(i)};\beta,\theta,\lambda\right)-\frac{i}{n+1}\right]^{2} \tag{27}$$

$$\frac{\delta WLSE_{\Omega}}{\delta\theta}=2\sum_{i=1}^{n}\varpi_i^{(p2)}(\beta,\theta,\lambda)\left[\left(\frac{1-\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^2\right]^{\theta}\right]^{\beta}}{1+\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^2\right]^{\theta}\right]^{\beta}}\right)-\frac{i}{n+1}\right]=0 \quad (28)$$

$$\frac{\delta WLSE_{\Omega}}{\delta\beta}=2\sum_{i=1}^{n}\varpi_i^{(p1)}(\beta,\theta,\lambda)\left[\left(\frac{1-\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^2\right]^{\theta}\right]^{\beta}}{1+\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^2\right]^{\theta}\right]^{\beta}}\right)-\frac{i}{n+1}\right]=0 \quad (29)$$

$$\frac{\delta WLSE_{\Omega}}{\delta\lambda}=2\sum_{i=1}^{n}\varpi_i^{(p3)}(\beta,\theta,\lambda)\left[\left(\frac{1-\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^2\right]^{\theta}\right]^{\beta}}{1+\left[1-\left[1-\left(e^{-\lambda x_{(i)}}\right)^2\right]^{\theta}\right]^{\beta}}\right)-\frac{i}{n+1}\right]=0 \quad (30)$$

$\varpi_i^{(p1)}(\beta,\theta,\lambda)$, $\varpi_i^{(p2)}(\beta,\theta,\lambda)$ and $\varpi_i^{(p3)}(\beta,\theta,\lambda)$ are defined in (12), (13) and (14) respectively.

III. Results

3.1 Simulation study

Now, the performance of the MLE, ADE, CVM, MPS, LSE and WLSE method is investigated for TIHLTLExp parameters through Monte Carlo simulation study while considering 10,000 replications. Data were generated with different sample sizes (10,30,50,100,200). The estimates, Bias, Mean square error (MSE) and Mean relative estimate were obtained by R software. Thus, obtained as follows

Table 1: *Estimates of different estimation methods for parameter lambda=1.5, theta=1 and beta=1*

	Estimation methods					
n	**MLE**	**AD**	**CVM**	**MPS**	**LS**	**WLS**
10	1.65 1.71 1.45	1.63 1.39 1.33	1.80 2.28 1.48	1.42 1.18 1.14	1.57 1.55 1.29	1.55 1.38 1.34
30	1.46 1.19 1.22	1.46 1.17 1.21	1.50 1.21 1.23	1.43 1.07 1.07	1.48 1.18 1.19	1.36 1.15 1.28
50	1.42 1.13 1.20	1.42 1.11 1.18	1.49 1.15 1.17	1.45 1.06 1.05	1.46 1.12 1.16	1.29 1.10 1.33
100	1.40 1.08 1.17	1.40 1.07 1.17	1.45 1.09 1.15	1.47 1.04 1.04	1.44 1.08 1.14	1.27 1.08 1.31
200	1.40 1.06 1.14	1.41 1.05 1.14	1.43 1.06 1.14	1.48 1.03 1.03	1.43 1.05 1.11	1.24 1.05 1.32

Table 2: *Bias of different estimation methods for parameter lambda=1.5, theta=1 and beta=1*

	Estimation methods					
n	**MLE**	**AD**	**CVM**	**MPS**	**LS**	**WLS**
10	0.61 0.71 0.45	0.47 0.39 0.33	0.61 1.28 0.48	0.29 0.19 0.14	0.38 0.55 0.29	0.44 0.38 0.34
30	0.29 0.19 0.22	0.26 0.18 0.21	0.27 0.21 0.23	0.15 0.07 0.07	0.23 0.18 0.19	0.31 0.15 0.28
50	0.24 0.13 0.20	0.19 0.10 0.18	0.21 0.15 0.17	0.11 0.06 0.05	0.19 0.12 0.16	0.33 0.11 0.33
100	0.19 0.08 0.17	0.18 0.07 0.17	0.16 0.09 0.15	0.07 0.04 0.04	0.15 0.08 0.14	0.31 0.08 0.31
200	0.16 0.06 0.14	0.15 0.05 0.14	0.14 0.06 0.14	0.05 0.03 0.03	0.12 0.05 0.11	0.32 0.05 0.32

Table 3: *Mean square error of different estimation methods for parameter lambda=1.5, theta=1 and beta=1*

	Estimation methods					
n	**MLE**	**AD**	**CVM**	**MPS**	**LS**	**WLS**
10	2.22 2.57 0.81	1.02 0.81 0.45	3.26 2.18 0.92	0.21 0.26 0.10	0.55 0.59 0.39	1.01 1.18 0.43
30	0.16 0.12 0.18	0.15 0.12 0.17	0.18 0.17 0.21	0.05 0.03 0.03	0.12 0.14 0.16	0.16 0.10 0.19
50	0.11 0.05 0.14	0.08 0.04 0.11	0.10 0.08 0.13	0.02 0.02 0.01	0.09 0.06 0.11	0.17 0.04 0.23
100	0.08 0.01 0.10	0.07 0.02 0.10	0.06 0.03 0.09	0.01 0.01 0.01	0.06 0.02 0.09	0.14 0.02 0.20
200	0.06 0.01 0.08	0.06 0.01 0.07	0.05 0.01 0.08	0.01 0.00 0.00	0.04 0.01 0.06	0.15 0.01 0.19

Table 4: *Mean relative estimates of different estimation methods for parameter lambda=1.5, theta=1 and beta=1*

	Estimation methods					
n	**MLE**	**AD**	**CVM**	**MPS**	**LS**	**WLS**
10	0.41 0.71 0.45	0.31 0.39 0.33	0.40 1.28 0.48	0.19 0.19 0.14	0.26 0.60 0.29	0.29 0.38 0.34
30	0.20 0.19 0.22	0.18 0.18 0.21	0.18 0.21 0.23	0.10 0.07 0.07	0.16 0.18 0.19	0.21 0.15 0.28
50	0.16 0.13 0.20	0.13 0.10 0.18	0.14 0.15 0.17	0.07 0.06 0.05	0.13 0.12 0.16	0.22 0.11 0.33
100	0.13 0.08 0.17	0.12 0.07 0.17	0.10 0.09 0.15	0.05 0.04 0.04	0.10 0.08 0.14	0.21 0.08 0.31
200	0.11 0.06 0.14	0.10 0.05 0.14	0.09 0.06 0.14	0.03 0.03 0.03	0.08 0.05 0.11	0.21 0.05 0.32

Table 5: *Mean square error ranking for different estimation methods for parameter lambda=1.5, theta=1 and beta=1*

	Estimation methods					
n	**MLE**	**AD**	**CVM**	**MPS**	**LS**	**WLS**
10	5+5+6=16^{5}	4+4+3=11^{4}	6+6+5=17^{6}	1+1+1=3^{1}	2+2+2=6^{2}	*3+3+4=10^{3}*
30	4.5+3.5+4=12^{5}	3+3.5+3=9.5^{3}	6+6+6=18^{6}	1+1+1=3^{1}	2+5+2=9^{2}	*4.5+2+5=11.5^{4}*
50	5+3+4=12^{4}	2+1.5+2.5=6^{2}	4+5+4=13^{5}	1+1+1=3^{1}	3 +4+2.5=9.5^{3}	*6+1.5+6=13.5^{6}*
100	5+1.5+4.5=11$^{3.5}$	4+4+4.5=12.5^{5}	2.5+6+2.5=11$^{3.5}$	1+1.5+1=3.5^{1}	2.5+4+2.5=9^{2}	*6+4+6=16^{6}*
200	4.5+4+4.5=13^{5}	4.5+4+3=11.5$^{3.5}$	3+4+4.5=11.5$^{3.5}$	1+1+1=3^{1}	2+4+2=8^{2}	*6+4+6=16^{6}*

Table 6: *Best estimation methods based on the Monte Carlo simulation study*

	Estimation methods				
Rank/n	**10**	**30**	**50**	**100**	**200**
1st	*MPS*	*MPS*	*MPS*	*MPS*	*MPS*
2nd	*LS*	*LS*	*AD*	*LS*	*LS*
3rd	*WLS*	*AD*	*LS*	*MLE/CVM*	*AD/CVM*
4th	*AD*	*WLS*	*MLE*	*MLE/CVM*	*AD/CVM*
5th	*MLE*	*MLE*	*CVM*	*AD*	*MLE*
6th	*CVM*	*CVM*	*WLS*	*WLS*	*WLS*

IV. Discussion

Table 1-6 is the illustration of simulation study conducted. The six methods (MLE, ADE, CVM, MPS, LS, WLS) explored in this article. The Table 1 reveals various estimates for the TIHLTLExp parameters across the six methods explored. The estimates of the estimation methods approach the true value of the parameters as the sample sizes increases. Table 2 illustrate the biases of the different methods explored, one can deduced that the biases reduces as the sample sizes increases. Table 3 illustrates the mean square error MSE, the MSE values decay as the sample sizes increases. It is evidenced that the Mean relative estimates of different estimation methods decay as the sample sizes increasing, this is illustrated in Table 4. It is evidence from the results that the six estimators possess consistency property.

The ranking of the performance of methods explored in this article is achieved in the Table 4. In Table 5, summation of the rank is done across the three parameters of the distribution. The preference of estimation methods is summarized in table 6 and the sample size are categorized as low (10,30), moderate (50) and finally high (100, 200). For the low, moderate and high sample sizes, the MPS is the best. The second best estimator for low and high sample sizes is LS and the second best estimator for moderate sample size is AD. However, the worst estimator for low sample size is CVM, while the worst estimator for moderate and high sample sizes is WLS. Conclusively, since MPS outperform other estimation methods at low, moderate and high sample sizes, it is suggested that MPS should be adopted for analyzing the TIHLTLExp model. Alternatively, LS could be consider for estimating low and high sample size while AD for moderate sample size.

References

[1] Adepoju, A. A., Abdulkadir, S. S., & Jibasen, D. (2023). The Type I Half Logistics-Topp-Leone-G Distribution Family: Model, its Properties and Applications. UMYU Scientifica, 2(4), 09-22.

[2] Isa A. M., Sule O. B., Ali B. A., Akeem A. A., and Ibrahim I. I. (2022). Sine-Exponential Distribution: Its Mathematical Properties and Application to Real Dataset. UMYU Scientifica, 1(1), 127 – 131.

[3] Bello O., A.,Doguwa S., I., Yahaya A., Jibril H., M. (2020). A Type I Half Logistic Exponentiated-G Family of Distributions: Properties and Application, Communication in Physical Sciences, 7(3):147-163.

[4] Gul A, Sandhu A.J., Farooq M., Adil M., Hassan Y., Khan F. (2023) Half logistic-truncated exponential distribution: Characteristics and applications. PLoS ONE 18(11): e0285992.

[5] Balogun, O. S., Arshad, M. Z., Iqbal, M. Z., & Ghamkhar, M. (2021). A new modified Lehmann type–II G class of distributions: exponential distribution with theory, simulation, and applications to the engineering sector. F1000Research, 10, 483

[6] Ijaz, M., Asim, S. M. & Alamgir (2019). Lomax exponential distribution with an application to real life data. Plos One 14, 12, pp. e0225827,

[7] Isa A. M., Kaigama A., Adepoju A. A., Bashiru S. O., Lehmann Type II-Lomax Distribution: Properties and Application to Real Data Set. (2023). Communication in Physical Sciences, 9(1):63 – 72

[8] Jabeen, S. & Para, B. (2018). Exponentiated Gamma Exponential Distribution. Sohag Journal of Mathematics: An International Journal, 5, 3, pp. 79-84.

[9] Elgarhy, M. Shakil, M & Kibria, B. M. G. (2017). Exponentiated WeibullExponential distribution with application. Application and Applied Mathematics: An International Journal, 12, 2, pp. 710-725.

[10] Kunjiratanachot, N.; Bodhisuwan, W.; Volodin, A. The Topp-Leone generalized exponential power series distribution with applications. (2018). J. Probab. Stat. Sci. 16, 197–208.

[11]Almarashi, A. M., Elgarhy, M., Elsehetry, M. M., Kibria, B. M., & Algarni, A. (2019). A new extension of exponential distribution with statistical properties and applications. Journal of Nonlinear Sciences & Applications (JNSA),12(3).

[12] Elgarhy M., Muhammad, A., & Ismat, P.(2018). Type II Half Logistic Exponential Distribution with Applications, Annals of Data Science.

[13] Ristic, M. M., & Balakrishnan, N. (2011). The gamma-exponentiated exponential distribution. Journal of Statistical Computation and Simulation, 82, 1191-1206.

[14] Bello O., A.,Doguwa S., I., Yahaya A., Jibril H., M. (2021). A Type II Half Logistic Exponentiated-G Family of Distributions with Applications in Survival Analysis, FUDMA Journal of Science, 5(3):177-190.

[15] Hassan E. A. A., Elgarhy M., Eldessouky E. A., Hassan O. H. M. Amin E. A. Almetwally, E. M. (2023). Different Estimation Methods for New Probability Distribution Approach Based on Environmental and Medical Data. Axioms. 12, 220.

[16] Alotaibi B. N., Elbatal I., Shrahili M., Al-Moisheer A., Elgarhy M., and Almetwally E. M.

(2023). Statistical inference for the Kavya–Manoharan Kumaraswamy model under ranked set sampling with applications. *Symmetry*, 15:3, 587.

[17] Mukherjee I.,Maiti S. S., Singh V. V. (2021). On estimation of the PDF and CDF of the one-parameter polynomial exponential familyof distributions. *Communications in statistics- Theory and Methods*. 52:1104-120.

[18] Yilmaz A., Kara M., Ozdemir O. (2021). Comparison of different estimation methods for extreme value distribution. Journal of applied statistics. 48, 13-15.

[19] Adepoju A. A., Usman M., Alkassim R. S., Sani S. S. Adamu K.. Parameter (shape) Estimation of Weibull-Exponential Distribution Using Classical and Bayansian Approach Under Different Loss Functions. (2021). Royal Statistical Society Nigeria Local Group Conference Proceedings. 182-190

[20] Adepoju A. A., Issa A. A.,, Magaji A. A., Nasir M. S., Aliyu A. M., Preference of bayesian methods over classical method in estimating the scale parameter of inverse rayleigh frechet distribution.(2021). Royal Statistical Society Nigeria Local Group Conference Proceedings. 158-167.

[21] Artyushenko V. M., Volovach V. I. ((2021)). Methods of Estimating the Form of the Probability Distribution Density in Tasks of Processing Measurement Results. International Conference on Automatics and Energy (ICAE 2021) Journal of Physics: Conference Series. 2096 012136

[22] Thamer M. K.., Zine R. (2021). Comparison of Five Methods to Estimate the Parameters for the Three-Parameter Lindley Distribution with Application to Life Data. Hindawi Computational and Mathematical Methods in Medicine Volume, Article ID 2689000, 14 pages

[23] ZeinEldin R. A., Chesneau C, Jamal F., Elgarhy M. (2019). Different estimation methods for Type I Half-Logistic Topp–Leone Distribution. *Mathematics*, 7, 985.

[24] Warsono, Gustavia E., Kurniasari D., Amanto, Antonio Y. (2019). On the Comparison of the Methods of Parameter Estimation for Pareto Distribution. IOP Conf. Series: Journal of Physics: Conf. Series 1338, 012042 e.

[25] Al-bayati R. S. S. (2018) Some Estimation Methods for Lindley Distribution M, [M.S. Thesis], College of Science, Mustansiriyah University, Baghdad, Iraq.

[26] Lei Y. (2008). Evaluation of three methods for estimating the Weibull distribution parameters of Chinese pine (Pinus tabulaeformis). Journal of forest science, 54, (12): 566–571

[27] Venkatraman J. J. S, S., and Wilson J. R., (1988) "Least-squares estimation of distribution functions in Johnson's translation system," Journal of Statistical Computation and Simulation, vol. 29, no. 4, pp. 271–297.

[28] Cheng R., Amin N. (1979). Maximum product of spacings estimation with application to the lognormal distribution, Communications in Statistics-Theory and Methods, Math. Report, p. 791.

[29] Macdonald P. (1971). An estimation procedure for mixtures of distributions. Journal of the Royal Statistical Society, vol. 20, pp. 102–107.

[30] Anderson, T.W.; Darling, D.A. (1952). Asymptotic theory of certain 'goodness-of- fit' criteria based on stochastic processes. Ann. Math. Stat. 23, 193–212.

[31]Macdonald, P.D.M. (1971). Comment on 'An estimation procedure for mixtures of distributions' by Choi and Bulgren. J. R. Stat. Soc. B, 33, 326–329.

[32] Swain, J.J.; Venkatraman, S.; Wilson, J.R. (1988). Least-squares estimation of distribution functions in johnson's translation system. J. Stat. Comput. Simul. 29, 271–297.