RT&A, No 1 (77)
Volume 19, March 2024

M/M/C QUEUE WITH MULTIPLE WORKING VACATIONS AND SINGLE WORKING VACATION UNDER ENCOURAGED ARRIVAL WITH IMPATIENT CUSTOMERS

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Abstract

This paper demonstrates an M/M/C queuing model with Multiple working vacations and also single working vacation under encouraged arrival with impatient customers. The queuing model with the servers adopting multiple working vacation policy and single working vacation are determined separately and it is observed that the servers during working vacation(s) will be serving the customers at a slower service rate when compared during regular busy period. In addition to the above conditions, if there is a rapid increase in the customers' arrival i.e, if encouraged arrival occurs and due to this sudden growth of the queue, there may be a impatience in the behaviour of the customer. With these considerations, an M/M/C Queuing model is analysed with two vacation policies separately by applying PGF method and thus the performance measures for an M/M/C Queue with Multiple Working Vacations and Single Working Vacation under Encouraged arrival with impatient customers are evaluated.

Keywords: Multiple Working Vacations(MWV), Single Working Vacation (SWV),Encouraged Arrival, Impatient behaviour ,Performance Measures

1. Introduction

In our daily life, we meet up with the scenario of waiting in queues to get our work done, for example - to make bank deposit, mail a package, obtain food in cafeteria etc. Waiting in queue is a matter of personal annoyance and it also costs the amount of time that we waste by waiting in queues. It may affect the efficienc of the service provided and is a major factor in both the quality of life and also affecting the efficienc of a nation's economy. Great inefficiencies also occur because of waiting.

For example, making machines wait to be repaired may result in less production, delay in telecommunication transmission due to saturated lines may cause data glitches etc. In fact, we have become accustomed to considerable amounts of waiting. Origin of Queuing theory in research was contributed by Agner Krarup Erlang, who created models to describe the system of incoming calls at the Copenhagen Telephone Exchange Company .

An M/M/s queuing system in which the servers undergoing vacation was analysed in [7]. In Queuing vacation policy, an overview of some general decomposition results were attained and the methodology used to obtain those results for two vacation models were analysed in [3]. Moreover, the literature on statistical analysis of queuing systems were briefl discussed in [2].

It can be observed that in numerous industrial sector, the concept of Queuing with servers' vacation is implemented. An M/G/1 Queue with vacation policy used in the scenarios like maintenance of production systems, where machines or equipment mainly degrade while being operated were evaluated and for such queuing model,an explicit expression for the distribution of the time it takes until the specifie amount of work has been served were derived in [1].

In General, Systems with vacations are usually modeled and analyzed by queuing theory. An approach for modeling and analyzing finite-sou ce multi-server systems with single and multiple vacations of servers or all stations were presented using the Generalized Stochastic Petri nets model in [11]. During any service,the servers may undergo breakdown simultaneously both in regular busy period and working vacation period due to the failure of a main control unit. This scenario was discussed by modeling and analysing a Markovian multiserver finite buffer queue under synchronous working vacation policy in [5].

A multiserver queuing system with customers′ impatience until the end of service under single and multiple vacation policies were examined in [6]. Situations like arrival of the customers following Poisson distribution but the general distribution followed by the administration rendering service with various vacations were detailedly discussed in [10].

The concept of impatient behaviours like balking and reneging with the availability of heterogeneous servers in an M/M/c queue was analysed in [16]. Moreover, the time-dependent system size probabilities were derived explicitly using generating function and also the time-dependent mean, variance, busy period distribution and steady-state probabilities were also obtained. In addition to this, performance of an M/M/c/K Queuing Models applied in Healthcare Things for Medical Monitoring were evaluated in [14].

The impatient nature of the customer during any service may be expressed if there is a delay in the service and the delay may be due to lack of servers or slow service provided. Queues with slow servers and impatient customers were considered and the the mean queue size were derived. Also, Several extreme cases were investigated and numerical results are presented in [12].

An M/M/1 queue with single and multiple working vacations with impatient customers were studied and Closed-form solutions and various performance measures like, the mean queue lengths and the mean waiting times were derived and the stochastic decomposition properties were verifie for both multiple and single working vacation cases in [13]. Likely, the impatient behaviour of the customers with with single and multiple synchronous working vacations in an M/M/C queue was analysedin [9]. Performance nature of a Markovian Queue with Impatient Customers and Working Vacation were derived in [8].

It is obvious that in the case of any discounts or offers provided during any sale or if any sudden demand is created for a product or a service, then there will be a rapid growth in the arrival of the customers, which is termed as encouraged arrival. The concept of encouraged arrival in an M/M/c/N queuing systems with reneging, retention and Feedback customers were discussed in [15]. The stationary system size probabilities were obtained recursively for the above model, while the steady state behavior of the M/M/1/N queuing model with encouraged or discouraged arrivals and impatient customers are obtained in [4].

With the aid of the above discussed concepts, an M/M/C Queuing model during encouraged arrival undergoing single working vacation and multiple working vacations with impatient behaviour of the customers are analysed separately .

In this paper, between the two vacation policies analysed,multiple working vacation is considered firs in which if a server returns to an empty queue, then he goes for another vacation immediately ,thus working vacation occurs multiple times. Whereas, in the later vacation policy, the server takes only a single vacation each time. Thus for an M/M/C Queue during encouraged arrival with impatient behaviour undergoing multiple working vacation is derived with explicit formulations followed by the same queuing model with single working vacation.

2. Methods

An $M/M/c$ queuing model with encouraged arrival following multiple working vacations with impatient customers is considered. Customers arriving to be served follow Poisson process and the arrival rate is denoted by the parameter λ_w. If there is a sudden increase in the arrival of the customers,i.e., encouraged arrival occuring in the system follows poisson process with the encouraged arrival rate $\lambda_w(1+\delta)$.

Since the considered model denotes 'c' servers, there may be maximum of 'c' servers available, to serve the customers according to FCFS rule. When a customer arrives and find all the servers in the system are busy, then he needs to wait until he gets served and thus the waiting line or the queue begins.

The time taken for each server to complete the work during regular busy period follows exponential distribution and denoted with the service rate μ_w. Thus the traffi intensity or the stability of the system during regular busy period is considered as $\rho = \frac{\lambda_w(1+\delta)}{c\mu_w} < 1$

After completion of a service, if there is no customer in the system,then all the 'c' servers will take vacation promptly and the duration of working vacation for each servers is exponentially distributed with parameter η'. As all the servers in the system undergo vacation, even if a single customer arrives,then anyone of the server will return from his vacation and start serving the arrived customer .Thus the concept of working even during vacation for the arrival of customers is termed as working vacation period, and the service rate following exponential process during working vacation period is μ_{wv} and it is observed that the service rate during working vacation is slower than the regular busy period i.e., $\mu_{wv} < \mu_w$

It is obvious that if the servers return from their vacation and when the system is non empty, the service rate of the servers changes from $\mu_{wv} to \mu_v$ indicating that the regular busy period begins. Suppose, if the servers fin no customer waiting in the queue after returning from their vacation, they immediately leave for another vacation. In such cases,if a customer waits in the queue for a longer time, as all the 'c' servers are in working vacation period,he may become impatient in waiting and the impatient behaviour of the customer at the time T is is exponentially distributed with parameter γ_w which is considered to be independent of the customers in that moment.

The customer waiting in the queue may exit the queue and never returns if its service has not been completed before the time T expires. The inter arrival times, service times, vacation duration times and impatient time are all taken to be mutually independent. To construct this system, we defin a two dimensional continuous time discrete state Markov chain as $\{(M(t), N(t)), t \geq 0\}$ with state space s = $\{(0,0) \cup \{(n,j)\}, n \geq 1, j = 0,1\}$

Where M(t) denotes the total number of customers in the system at time t and N(t) denotes the state of the system at time t with

N(t) = {1 when the servers are in non-vacation period at time t } and

N(t) = {0 when the servers are in working vacation period at time t }.

2.1. Steady State Equations and its Solutions for Multiple Working Vacations Model:

The steady state transition probabilities are define by
$P_{nj} = P\{M(t) = n, N(t) = j\}, n \geq 0, j = 0,1$

Now, the set of balance equations as

$$\lambda_w(1+\delta)P_{00} = (\mu_{wv} + \gamma_w)P_{1,0} + \mu_w P_{1,1}, \tag{1}$$

$$[\lambda_w(1+\delta) + \eta' + n(\mu_{wv} + \gamma_w)]P_{n,0} = \lambda_w(1+\delta)P_{n-1,0} + (n+1)((\mu_{wv} + \gamma_w)P_{n+1,0}, \quad if n \geq 1, \tag{2}$$

$$(\lambda_w(1+\delta) + \mu_w)P_{1,1} = \eta' P_{1,0} + 2\mu_w P_{2,1}, \tag{3}$$

$$(\lambda_w(1+\delta) + n\mu_w)P_{n,1} = \lambda_w(1+\delta)P_{n-1,1} + (n+1)\mu_w P_{n+1,1} + \eta' P_{n,0}, \quad if 2 \leq n \leq c-1, \tag{4}$$

$$(\lambda_w(1+\delta) + c\mu_w)P_{n,1} = \lambda_w(1+\delta)P_{n-1,1} + c\mu_w P_{n+1,1} + \eta' P_{n,0} \quad if n \geq c. \tag{5}$$

By letting the probability generating functions as

$$P_0(z) = \sum_{n=0}^{\infty} z^n P_{n,0},$$

$$P_1(z) = \sum_{n=1}^{\infty} z^n P_{n,1}.$$

with $P_0(1) + P_1(1) = 1$ and $P_0'(z) = \sum_{n=1}^{\infty} nz^{n-1}P_{n,0}$.

Now, By Multiplying Eq(2) with z^n and adding over 'n' and rearranging the terms, the differential equation is attained as :

$$(\mu_{wv} + \gamma_w)(1-z)P_0'(z) = [\lambda_w(1+\delta)(1-z) + \eta']P_0(z) - (\eta' P_{0,0} + \mu_w P_{1,1}). \quad (6)$$

Likely multiplying Eq(4) and Eq(5) by z^n and adding over 'n', the following equation is obtained,

$$(1-z)(\lambda_w(1+\delta)z - c\mu_w)P_1(z) = \eta' z P_0(z) - (\eta' P_{0,0} + \mu_w P_{1,1})z + \mu_w(1-z)\sum_{n=1}^{c}(n-c)z^n P_{n,1}. \quad (7)$$

Let us consider ,

$$A = \eta' P_{0,0} + \mu_w P_{1,1}. \quad (8)$$

Then, for z $\neq 1$,

$$P_0'(z) - [\frac{\lambda_w(1+\delta)}{(\mu_{wv}+\gamma_w)} + \frac{\eta'}{(\mu_{wv}+\gamma_w)(1-z)}]P_0(z) = -\frac{A}{(\mu_{wv}+\gamma_w)(1-z)}. \quad (9)$$

Eq(9) is an ordinary linear differential equation with constant coefficients To solve the equation, an integrating factor can be considered as

$$I.F = e^{-\int [\frac{\lambda_w(1+\delta)}{(\mu_{wv}+\gamma_w)} + \frac{\eta'}{(\mu_{wv}+\gamma_w)(1-z)}]dz} = e^{-\frac{\lambda_w(1+\delta)z}{(\mu_{wv}+\gamma_w)}}(1-z)^{\frac{\eta'}{(\mu_{wv}+\gamma_w)}}$$

The General solution to Eq(9) is given by:

$$\frac{d}{dz}[e^{-\frac{\lambda_w(1+\delta)z}{(\mu_{wv}+\gamma_w)}}(1-z)^{\frac{\eta'}{(\mu_{wv}+\gamma_w)}}]P_0(z) = [\frac{-A}{(\mu_{wv}+\gamma_w)(1-z)}]e^{-\frac{\lambda_w(1+\delta)z}{(\mu_{wv}+\gamma_w)}}(1-z)^{\frac{\eta'}{(\mu_{wv}+\gamma_w)}}. \quad (10)$$

Now, integrating from 0 to z, following equation is attained,

$$P_0(z) = [e^{\frac{\lambda_w(1+\delta)z}{(\mu_{wv}+\gamma_w)}}(1-z)^{-\frac{\eta'}{(\mu_{wv}+\gamma_w)}}[P_o(0) - \frac{A}{(\mu_{wv}+\gamma_w)}\int_0^z e^{-\frac{\lambda_w(1+\delta)z}{(\mu_{wv}+\gamma_w)}}(1-x)^{\frac{\eta'}{(\mu_{wv}+\gamma_w)}-1}dx]. \quad (11)$$

then,

$$P_0(1) = e^{\frac{\lambda_w(1+\delta)}{(\mu_{wv}+\gamma_w)}}[P_o(0) - \frac{A}{(\mu_{wv}+\gamma_w)}\int_0^1 e^{-\frac{\lambda_w(1+\delta)z}{(\mu_{wv}+\gamma_w)}}(1-x)^{\frac{\eta'}{(\mu_{wv}+\gamma_w)}-1}dx]\lim_{z\to 1}(1-z)^{\frac{-\eta'}{(\mu_{wv}+\gamma_w)}}. \quad (12)$$

Since $0 \leq P_0(1) = \sum_{n=0}^{\infty} z^n P_{n,0} \leq 1$and $\lim_{z\to 1}(1-z)^{-\frac{\eta'}{(\mu_{wv}+\gamma_w)}} = \infty$,and thus the existing term is as follows

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$$P_{0,0} = P_0(0) = \frac{A}{(\mu_{wv} + \gamma_w)} L \tag{13}$$

$$\text{Where} \quad L = \int_0^1 e^{-\frac{\lambda_w(1+\delta)z}{(\mu_{wv}+\gamma_w)}} (1-x)^{\frac{\eta'}{(\mu_{wv}+\gamma_w)}-1} dx. \tag{14}$$

Defin $$Z(\lambda_w(1+\delta), \eta') = -\lambda_w(1+\delta)^{-\eta'} e^{-\lambda_w(1+\delta)} (-\Gamma(\eta', -\lambda_w(1+\delta)) + \Gamma(\eta')) \tag{15}$$

where $\Gamma(z)$ is the Γ function which is represented as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{16}$$

$$\text{and} \qquad \Gamma(y, z) = \int_z^\infty e^{-t} t^{y-1} dt. \tag{17}$$

some calculations give

$$L = Z\left(\frac{\lambda_w(1+\delta)}{(\mu_{wv} + \gamma_w)}, \frac{\eta'}{(\mu_{wv} + \gamma_w)}\right). \tag{18}$$

By Eq(8) and Eq (13), it is observed that

$$P_{0,0} = \frac{\eta' P_{0,0} + \mu_w P_{1,1}}{(\mu_{wv} + \gamma_w)} L = \frac{L\mu_w}{\mu_{wv} + \gamma_w - \eta' L} P_{1,1}. \tag{19}$$

Now, using the value of A from Eq(13) in Eq(11), $P_0(z)$ is obtined as

$$P_0(z) = \frac{e^{\frac{\lambda_w(1+\delta)z}{(\mu_{wv}+\gamma_w)}}}{(1-z)^{\frac{\eta'}{(\mu_{wv}+\gamma_w)}}} \left[1 - \frac{1}{L} \int_0^z e^{-\frac{\lambda_w(1+\delta)z}{(\mu_{wv}+\gamma_w)}} (1-x)^{\frac{\eta'}{(\mu_{wv}+\gamma_w)}-1} dx\right] P_{0,0}. \tag{20}$$

By applying L'Hospital's rule to Eq(20), we get

$$P_0(1) = \frac{(\mu_{wv} + \gamma_w)}{\eta' L} P_{0,0} \tag{21}$$

and now substituting the value of $P_{0,0}$ from Eq(19), the following relation is obtained

$$\eta' P_0(1) = \eta' P_{0,0} + \mu_w P_{1,1}. \tag{22}$$

From Eq(7), $P_1(z)$ is attained as,

$$P_1(z) = \frac{[\eta' P_0(z) - A]z}{(\lambda_w(1+\delta)z - c\mu_w)(1-z)} - \frac{\mu_w}{(\lambda_w(1+\delta)z - c\mu_w)} F(z), \tag{23}$$

where,

$$F(z) = \sum_{n=1}^{c} (n-c) z^n P_{n,1}. \tag{24}$$

It is clear from Eq(20) that $P_0(z)$ is a function of $P_{0,0}$ and the ratio between the time of the servers on working vacation and the system is empty. Similarly from Eq(23), $P_1(z)$ is a function of $P_0(z)$, A and F(z). Hence, if $P_{0,0}$ and $P_{j,1}$(j=1,2,...c) are obtained, $P_0(z)$ and $P_1(z)$ can be determined completely .

2.2. Performance Measures

By using L'Hospital's rule in Eq(23), we get

$$P_1(1) = \frac{[\eta' P_0(1) - A] + \eta' P_0'(1)}{c\mu_w - \lambda_w(1+\delta)} + \frac{\mu_w}{c\mu_w - \lambda_w(1+\delta)} F(1), \tag{25}$$

where

$$F(1) = \sum_{n=1}^{c} (c-n) P_{n,1}. \tag{26}$$

Using Eq(22) and Eq(8) in Eq(25),we get,

$$P_1(1) = \frac{\eta'}{c\mu_w - \lambda_w(1+\delta)} E(L_0) + \frac{\mu_w}{c\mu_w - \lambda_w(1+\delta)} F(1). \tag{27}$$

Now, by applying L'hospital's rule to Eq(6), we have

$$E(L_0) = \lim_{z\to 1} P_0'(z) = \frac{-\lambda_w(1+\delta)P_0(1) + \eta' P_0'(1)}{-(\mu_{wv}+\gamma_w)} = \frac{-\lambda_w(1+\delta)P_0(1) - E(L_0)}{(\mu_{wv}+\gamma_w)} \text{which implies} \tag{28}$$

$$P_0(1) = \frac{\eta' + \mu_{wv} + \gamma_w}{\lambda_w(1+\delta)} E(L_0). \tag{29}$$

As $P_0(0) + P_0(1) = 1$, from Eq(27) and Eq(29),the expected number of customers during working vacation period is obtained as

$$E(L_0) = \frac{\lambda_w(1+\delta)(1-\rho)}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)} - \frac{\frac{\lambda_w(1+\delta)}{c}}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)} F(1). \tag{30}$$

On substituting Eq(30) in Eq(29), the probability that the system in working vacation period is as

$$P(J=0) = P_0(1) = \frac{(1-\rho)(\eta' + \mu_{wv} + \gamma_w)}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)} - \frac{\frac{\eta' + \mu_{wv} + \gamma_w}{c}}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)} F(1) \tag{31}$$

and the probability that the system is in busy period is found as

$$P(J=1) = P_1(1) = 1 - P_0(1) = \frac{(\eta'\rho)}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)} + \frac{\frac{\eta' + \mu_{wv} + \gamma_w}{c}}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)} F(1). \tag{32}$$

$E(L_1)$ can be obtained by differentiating Eq(23) and using L'Hospital's rule,

$$i.e., E(L_1) = \lim_{z\to 1} P_1'(z)$$

$$= \lim_{z\to 1} \Big\{ \frac{-\lambda_w(1+\delta)[z(-A + \eta' P_0(z))}{(1-z)(\lambda_w(1+\delta)z - c\mu_w)^2} + \frac{-A + \eta' P_0(z) + z\eta' P_0'(z)]}{(1-z)(\lambda_w(1+\delta)z - c\mu_w)}$$
$$+ \frac{z(-A + \eta' P_0(z)}{(1-z)^2(\lambda_w(1+\delta)z - c\mu_w)} + \mu_w \frac{[(c\mu_w - \lambda_w(1+\delta)z)F'(z) + \lambda_w(1+\delta)F(z)]}{(c\mu_w - \lambda_w(1+\delta)z)^2} \Big\} \tag{33}$$

$$= \frac{\eta'(c\mu_w - \lambda_w(1+\delta)E(L_0(L_0-1)) + 2c\mu_w\eta' E(L_0)}{2(c\mu_w - \lambda_w(1+\delta)z)^2} + \frac{F'(1)}{c(1-\rho)} + \frac{\rho F(1)}{(c(1-\rho)^2} \tag{34}$$

where

$$F'(1) = \frac{dF(z)}{dz} \text{at z=1}$$

$$= \sum_{j=1}^{c}(c-j)P_{j,1} \tag{35}$$

Now, the value of $P_0''(1)$ is obtained on differentiating Eq(6) twice on both sides as

$$(\eta' + \gamma_w)(1-z)P_0''(z) + 2\lambda_w(1+\delta)P_0'(z) = [\lambda_w(1+\delta)(1-z) + \eta' + 2(\mu_{wv} + \gamma_w]P_0''(z) \tag{36}$$

where

$$P_0'''(z) = \frac{d^3P_0(z)}{dz^3}$$

By letting z=1 in Eq(36),we get $P_0''(1) = \dfrac{2\lambda_w(1+\delta)}{\eta' + 2(\mu_{wv} + \gamma_w)}P_0'(1)$ (37)

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or it can also be denoted as, $E(L_0(L_0-1)) = \dfrac{2\lambda_w(1+\delta)EL_0}{\eta' + 2(\mu_{wv} + \gamma_w)}.$ (38)

Now, substituting, Eq(38) into Eq(34),the Mean number of customers, when the system in regular busy period is obtained as

$$E[L_1] = \frac{\rho\eta'}{(1-\rho)}[\frac{1}{\eta' + 2(\mu_{wv} + \gamma_w)} + \frac{1}{\lambda_w(1+\delta)(1-\rho)}]E[L_0] + \frac{1}{c(1-\rho)}F'(1) + \frac{\rho}{c(1-\rho)^2}F(1) \tag{39}$$

Hence, $E[L] = E[L_0] + E[L_1]$

$$= 1 + \frac{\rho\eta'}{(1-\rho)}[\frac{1}{\eta' + 2(\mu_{wv} + \gamma_w)} + \frac{1}{\lambda_w(1+\delta)(1-\rho)}][\frac{\lambda_w(1+\delta)(1-\rho) - \frac{\lambda_w(1+\delta)}{c}F(1)}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)}]$$

$$+ \frac{1}{c(1-\rho}F'(1) + \frac{\rho}{c(1-\rho)^2}F(1) \tag{40}$$

Substituting Eq(31) in Eq(21) results in $P_{(0,0)} = \frac{\eta' k}{(\mu_{wv}+\gamma_w)}P_0(1)$

$$= \frac{\eta' k}{(\mu_{wv} + \gamma_w)}[\frac{(1-\rho)((\eta' + \mu_{wv} + \gamma_w)}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)} - \frac{\frac{(\eta'+\mu_{wv}+\gamma_w)}{c}}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)}]F(1). \tag{41}$$

Suppose, the state of the system is (n,1),then the service rates of the servers are $n\mu_w$ for $n \leq c$ and $c\mu_w$ for $n > c$ respectively.

In this manner, the expected number of customers served per unit of time is given by

$$N_s = \sum_{n=1}^{c} n\mu_w P_{n,1} + \sum_{n=c+1}^{\infty} c\mu_w P_{n,1} = \mu_w[cP_1(1) - F(1)] \tag{42}$$

and the proportion of customers served per unit of time is given by

$$P_s = \frac{N_s}{\lambda_w(1+\delta)} = \frac{1}{c\rho}[cP_1(1) - F(1)] \tag{43}$$

where $P_1(1) is given by Eq(32)$.

If the state of the system is (n,1), n$\geq$ 1, the rate of customer abandonment of a customer due to impatience is nγ_w. Thus the mean rate of the customer abandonment due to impatience is given by

$$R_a = \sum_{n=1}^{\infty} n\gamma_w P_{n,0} = \gamma_w E[L_0]. \tag{44}$$

Thus an M/M/c Queuing model with Multiple Working Vacations under encouraged arrival with impatient behaviour is evaluated.

2.3. Single Working Vacation Model:

A Single working Vacation policy define that the server(s) in the queuing system takes vacation immediately , when he find no customers waiting in the queue. At the end of the working vacation, if the server find the system non empty, then he starts his regular busy period by shifting his service rate from μ_{wv} to μ_w. If not,the server will remain idle in the system itself than going for vacation and waits until the customer arrives for the new busy period. To construct this system, we defin a Markov chain as $\{(M(t), N(t)), t \geq 0\}$ with state space as in Multiple Working Vacations for Single Working Vacation also. s = $\{(n, j)\}, n \geq 0, j = 0, 1\}$

Where M(t) denotes the total number of customers in the system at time t and N(t) denotes the state of the system at time t with

N(t) = {1 when the servers are in non-vacation period at time t} and

N(t) = {0 when the servers are in working vacation period at time t}.

2.4. Steady State Equations and its Solutions for Single Working Vacation Model:

Now, the set of balance equations as

$$(\lambda_w(1+\delta)+\eta')P_{00} = (\mu_{wv}+\gamma_w)P_{1,0} + \mu_w P_{1,1}, \tag{45}$$

$$[\lambda_w(1+\delta)+\eta'+n(\mu_{wv}+\gamma_w)]P_{n,0} = \lambda_w(1+\delta)P_{n-1,0} + (n+1)((\mu_{wv}+\gamma_w)P_{n+1,0}, \quad if n \geq 1, \tag{46}$$

$$(\lambda_w(1+\delta))P_{0,1} = \eta' P_{0,0}, \tag{47}$$

$$(\lambda_w(1+\delta)+n\mu_w)P_{n,1} = \lambda_w(1+\delta)P_{n-1,1} + (n+1)\mu_w P_{n+1,1} + \eta' P_{n,0}, \quad if 1 \leq n \leq c-1, \tag{48}$$

$$(\lambda_w(1+\delta)+c\mu_w)P_{n,1} = \lambda_w(1+\delta)P_{n-1,1} + c\mu_w P_{n+1,1} + \eta' P_{n,0} \quad if n \geq c. \tag{49}$$

By letting the probability generating functions as

$$R_0(z) = \sum_{n=0}^{\infty} z^n P_{n,0},$$

$$R_1(z) = \sum_{n=1}^{\infty} z^n P_{n,1}.$$

with $R_0(1) + R_1(1) = 1$ and $R_0^{'}(z) = \sum_{n=1}^{\infty} n z^{n-1} P_{n,0}$.

Now, By Multiplying Eq(46) with z^n and adding over 'n' and rearranging the terms, the differential equation is attained as :

$$(\mu_{wv}+\gamma_w)(1-z)R_0^{'}(z) = [\lambda_w(1+\delta)(1-z)+\eta']R_0(z) - (\mu_w P_{1,1}). \tag{50}$$

Likely multiplying Eq(48) and Eq(49) by z^n and adding over 'n', the following equation is obtained,

$$(1-z)(\lambda_w(1+\delta)z - c\mu_w)R_1(z) = \eta' z R_0(z) - (\eta' P_{0,0} + \mu_w P_{1,1})z + z^2 \eta' P_{0,0} + \mu_w(1-z)\sum_{n=1}^{c}(n-c)z^n P_{n,1}. \tag{51}$$

Then, for z $\neq 1$,

$$R_0^{'}(z) - [\frac{\lambda_w(1+\delta)}{(\mu_{wv}+\gamma_w)} + \frac{\eta'}{(\mu_{wv}+\gamma_w)(1-z)}]R_0(z) = -\frac{\mu_w P_{1,1}}{(\mu_{wv}+\gamma_w)(1-z)}. \tag{52}$$

Solving the differential equation, as in Multiple Working Vacations Model we get,

$$R_0(z)=\frac{e^{\frac{\lambda_w(1+\delta)z}{(\mu_{wv}+\gamma_w)}}}{(1-z)^{\frac{\eta'}{(\mu_{wv}+\gamma_w)}}}[1-\frac{1}{L}\int_0^z e^{-\frac{\lambda_w(1+\delta)z}{(\mu_{wv}+\gamma_w)}}(1-x)^{\frac{\eta'}{(\mu_{wv}+\gamma_w)}-1}dx]P_{0,0}. \quad (53)$$

Thus a similar expression for $R_0(z)$ as in Multiple Working Vacations Model and here we arrive at,

$$R_0(0)=P_{0,0}=\frac{(\mu_w P_{1,1})}{(\mu_{wv}+\gamma_w)}L \quad (54)$$

$$R_0(1)=\frac{(\mu_{wv}+\gamma_w)}{\eta' L}P_{0,0} \quad (55)$$

and from Eq(54) and Eq(55), the following relation is obtained

$$\eta' R_0(1)=\mu_w P_{1,1}. \quad (56)$$

From Eq(51), $R_1(z)$ is attained as,

$$R_1(z)=\frac{[\eta' R_0(z)-A]z+z^2\eta' P_{0,0}}{(\lambda_w(1+\delta)z-c\mu_w)(1-z)}-\frac{\mu_w}{(\lambda_w(1+\delta)z-c\mu_w)}F(z), \quad (57)$$

where,

$$F(z)=\sum_{n=1}^{c}(n-c)z^n P_{n,1}. \quad (58)$$

It is clear from Eq(53) that $R_0(z)$ is a function of $P_{0,0}$ and the ratio between the time of the servers on working vacation and the system is empty. Similarly from Eq(57), $R_1(z)$ is a function of $R_0(z)$, A and F(z). Hence, if $P_{0,0}$ and $P_{j,1}$(j=1,2,...c) are obtained, $P_0(z)$ and $P_1(z)$ can be determined completely .

2.5. Performance Measures

By using L'Hospital's rule in Eq(57), we get

$$R_1(1)=\frac{[\eta' E(L)_0]+B}{c\mu_w-\lambda_w(1+\delta)}+\frac{\mu_w}{c\mu_w-\lambda_w(1+\delta)}F(1) \quad (59)$$

where

$$B=\eta'(2-c)P_{0,0} \quad and \quad F(1)=\sum_{n=1}^{c}(n-c)P_{n,1}. \quad (60)$$

Using Eq(22) and Eq(8) in Eq(25), the following equation is obtained,

$$P_1(1)=\frac{\eta'}{c\mu_w-\lambda_w(1+\delta)}E(L_0)+\frac{\mu_w}{c\mu_w-\lambda_w(1+\delta)}F(1). \quad (61)$$

Now, applying L'hospital's rule to Eq(6), we have

$$E(L_0)=\lim_{z\to 1}P_0'(z)=\frac{-\lambda_w(1+\delta)P_0(1)+\eta' P_0'(1)}{-(\mu_{wv}+\gamma_w)}=\frac{-\lambda_w(1+\delta)P_0(1)-E(L_0)}{\mu_{wv}+\gamma_w}\text{which implies} \quad (62)$$

$$P_0(1)=\frac{\eta'+\mu_{wv}+\gamma_w}{\lambda_w(1+\delta)}E(L_0). \quad (63)$$

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As $P_0(0) + P_0(1) = 1$, from Eq(27) and Eq(29), is the expected number of customers during working vacation period is obtained as

$$E(L_0) = \frac{\lambda_w(1+\delta)(1-\rho)}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)} - \frac{-\rho\eta'(2-c)P_{0,0}}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)} - \frac{\frac{\lambda_w(1+\delta)}{c}}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)}F(1). \tag{64}$$

On substituting Eq(30) in Eq(29), the probability that the system in working vacation period is as

$$P(J=0) = R_0(1) = \frac{(1-\rho)(\eta'+\mu_{wv}+\gamma_w)}{\eta'+\mu_{wv}(1-\rho)+\gamma_w(1-\rho)} - \frac{\rho\eta'(\eta'+\mu_{wv}+\gamma_w)(2-c)P_{0,0}}{\lambda_w(1+\delta)[\eta'+\mu_{wv}(1-\rho)+\gamma_w(1-\rho)]} - \frac{\frac{\eta'+\mu_{wv}+\gamma_w}{c}}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)}F(1)$$

where

$$X = \frac{\rho\eta'(\eta'+\mu_{wv}+\gamma_w)(2-c)P_{0,0}}{\lambda_w(1+\delta)[\eta'+\mu_{wv}(1-\rho)+\gamma_w(1-\rho)]}, \quad \text{then}$$

$$P(J=0) = R_0(1) = \frac{(1-\rho)(\eta'+\mu_{wv}+\gamma_w)}{\eta'+\mu_{wv}(1-\rho)+\gamma_w(1-\rho)} - X - \frac{\frac{\eta'+\mu_{wv}+\gamma_w}{c}}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)}F(1) \tag{65}$$

and the probability that the system is in busy period is as follows

$$P(J=1) = R_1(1) = 1 - R_0(1) = \frac{(\eta'\rho)}{\eta'+\mu_{wv}(1-\rho)+\gamma_w(1-\rho)} + \frac{\rho\eta'(\eta'+\mu_{wv}+\gamma_w)(2-c)P_{0,0}}{\lambda_w(1+\delta)[\eta'+\mu_{wv}(1-\rho)+\gamma_w(1-\rho)]} + \frac{\frac{\eta'+\mu_{wv}+\gamma_w}{c}}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)}F(1).$$

since we know that,

$$X = \frac{\rho\eta'(\eta'+\mu_{wv}+\gamma_w)(2-c)P_{0,0}}{\lambda_w(1+\delta)[\eta'+\mu_{wv}(1-\rho)+\gamma_w(1-\rho)]},$$

we get

$$P(J=1) = R_1(1) = 1 - R_0(1) = \frac{(\eta'\rho)}{\eta'+\mu_{wv}(1-\rho)+\gamma_w(1-\rho)} + X + \frac{\frac{\eta'+\mu_{wv}+\gamma_w}{c}}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)}F(1). \tag{66}$$

Now, $E(L_1)$ can be obtained by differentiating Eq(58) and using L'Hospital's rule,

$$E(L_1) = \lim_{z\to 1} R_1'(z)$$

$$= \lim_{z\to 1}\Big\{\frac{-\lambda_w(1+\delta)[z(-A+\eta' R_0(z)) + z^2\eta' P_{0,0}]}{(1-z)(\lambda_w(1+\delta)z - c\mu_w)^2} + \frac{-A+\eta' R_0(z)) + 2z\eta' P_0'(z) + z\eta' R_0'(z)}{(1-z)(\lambda_w(1+\delta)z - c\mu_w)} + \frac{z(-A+\eta' R_0(z) + z^2\eta' P_{0,0}}{(1-z)^2(\lambda_w(1+\delta)z - c\mu_w)} + \mu_w\frac{[(c\mu_w - \lambda_w(1+\delta)z)F'(z) + \lambda_w(1+\delta)F(z)]}{(c\mu_w - \lambda_w(1+\delta)z)^2}\Big\} \tag{67}$$

$$= \frac{\eta'(c\mu_w - \lambda_w(1+\delta)E(L_0(L_0-1)) + 2c\mu_w\eta' E(L_0) + 2\eta'[(2(c\mu_w - \lambda_w(1+\delta) - c\lambda_w(1+\delta)]P_{0,0}}{2(c\mu_w - \lambda_w(1+\delta)z)^2} + \frac{F'(1)}{c(1-\rho)} + \frac{\rho F(1)}{(c(1-\rho)^2} \tag{68}$$

where

$$F'(1) = \frac{dF(z)}{dz} at z = 1$$

$$= \sum_{j=1}^{c}(c-j)P_{j,1} \tag{69}$$

Now, the value of $R_0''(1)$ is obtained on differentiating Eq(50) twice on both sides and proceeding similarly as in Multiple Working Vacations, We get

$$E(L_0(L_0-1)) = \frac{2\lambda_w(1+\delta)}{\eta' + 2(\mu_{wv}+\gamma_w)}E(L_0). \tag{70}$$

Now, substituting, Eq(70) into Eq(69),the Mean number of customers, when the system in regular busy period is obtained as

$$E[L_1] = \frac{\rho\eta'}{(1-\rho)}\{[\frac{1}{\eta'+2(\mu_{wv}+\gamma_w)} + \frac{1}{\lambda_w(1+\delta)(1-\rho)}]E[L_0] + [\frac{1}{\lambda_w(1+\delta)} - \frac{1}{\mu_w(1-\rho)}]P_{0,0}\}$$
$$+\frac{1}{c(1-\rho)}F^1 + \frac{\rho}{c(1-\rho)^2}F(1). \tag{71}$$

$E[L] = E[L_0] + E[L_1]$

$$= \{1 + \frac{\rho\eta'}{(1-\rho)}[\frac{1}{\eta'+2(\mu_{wv}+\gamma_w)} + \frac{1}{\lambda_w(1+\delta)(1-\rho)}][\frac{\lambda_w(1+\delta)(1-\rho) - \rho B - \frac{\lambda_w(1+\delta)}{c}F(1)}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho)}]\}$$

$$+Y + \frac{1}{c(1-\rho)}F'(1) + \frac{\rho}{c(1-\rho)^2}F(1) \tag{72}$$

$$\text{where } Y = \frac{\rho\eta'}{(1-\rho)}[\frac{1}{\lambda_w(1+\delta)} - \frac{1}{\mu_w(1-\rho)}]P_{0,0}$$

Substituting Eq(65) in Eq(55) results in $P_{(0,0)} = \frac{\eta' k}{(\mu_{wv}+\gamma_w)}R_0 1$

$$= \frac{\eta' k}{(\mu_{wv}+\gamma_w)}[\frac{\lambda_w(1+\delta)(1-\rho)((\eta'+\mu_{wv}+\gamma_w) - \frac{(\lambda_w(1+\delta)\eta'+\mu_{wv}+\gamma_w)}{c}}{\eta' + \mu_{wv}(1-\rho) + \gamma_w(1-\rho) + \frac{k\eta'^2\rho(2-c)(\eta'+\mu_{wv}+\gamma_w)}{(\mu_{wv}+\gamma_w)}}]. \tag{73}$$

Suppose, the state of the system is (n,1),then the service rates of the servers are $n\mu_w$ for $n \le c$ and $c\mu_w$ for $n > c$ respectively.

Thus, the expected number of customers served per unit of time is given by

$$N_s = \sum_{n=1}^{c} n\mu_w P_{n,1} + \sum_{n=c+1}^{\infty} c\mu_w P_{n,1} = \mu_w[cP_1(1) - F(1)] \tag{74}$$

and the proportion of customers served per unit of time is given by

$$P_s = \frac{N_s}{\lambda_w(1+\delta)} = \frac{1}{c\rho}[cP_1(1) - F(1)] \tag{75}$$

where $P_1(1)$ is given by Eq(66).

If the state of the system is (n,1), n $\ge$ 1, the rate of customer abandonment of a customer due to impatience is nγ_w. Thus the mean rate of the customer abandonment due to impatience is given by

$$R_a = \sum_{n=1}^{\infty} n\gamma_w P_{n,0} = \gamma_w E[L_0]. \tag{76}$$

Hence, an M/M/c Queuing model with Multiple Working Vacations under encouraged arrival with impatient behaviour is evaluated.

3. Results

In this paper ,an M/M/C Queuing model under Multiple working vacations and single working vacation with impatient behaviour of the customer during encouraged arrival are analysed. It is observed that for the system of steady state equations, performance measures like Mean Queue length (E[L]), Probability that the system is in working vacation period (P[J=0]), Probability that the system is in regular busy period (P[J=1]) are evaluated for the two different vacation policies separately .

4. Discussion

On comparing the performance measures between the two vacation policies, from Eq (65) and Eq(31),it is observed that the difference between the probability of the system (P[J=0]) in single working vacation and that during multiple working vacations, we notice that by reducing the term "X" from the probability of the system in multiple working vacations, we attain the probability of the system in single working vacation . Likely, from Eq (66) and Eq(32),it is clear that the probability of the system in regular busy period during single working vacation is obtained by adding the term "X" to the probability of the system in regular busy period during multiple working vacation. Moreover, while comparing the mean queue length during the two different vacation policies, we observe that from Eq (72) and Eq(40), E(L) in single working vacation is the addition of the term "Y" and the term ρB to the existing mean queue length of multiple working vacations.

5. Conclusion

As the M/M/c Queuing model with Multiple and single working vacation with impatient behaviour of the customers during encouraged arrival is analysed, apart from deriving the explicit formulations, some of the characteristic measures are also discussed. It can be concluded that, with the impact of the terms "X","Y" and"B" in multiple working vacations an M/M/C Queuing model with impatient behaviour of the customer during encouraged arrival can be shifted to Single working vacation.However,for an efficient functioning of the queue a single working vacation can be suggested. In future work, numerical examples may be evaluated to evident the obtained result.

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