ON φ-CONHARMONICALLY FLAT LORENTZIAN PARA-KENMOTSU MANIFOLDS

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Abstract

The present paper deals with a class of Lorentzian almost paracontact metric manifolds namely Lorentzian para-Kenmotsu (briefly LP-Kenmotsu) manifolds. We study and have shown that a quasi-conformally flat Lorentzian para-Kenmotsu manifold is locally isomorphic with a unit sphere $S^n(1)$. Further it is shown that an LP-Kenmotsu manifold which is φ-conharmonically flat is an η-Einstein manifold with the zero scalar curvature. At the end, we have shown that a φ-projectively flat LP-Kenmotsu manifold is an Einstein manifold with the scalar curvature $r = n(n-1)$.

Keywords: Lorentzian para-Kenmotsu manifold, Weyl-projective curvature tensor, conformal curvature tensor, Einstein manifold.

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I. Introduction

In 1989, K. Matsumoto [3] introduced the notion of Lorentzian paracontact and in particular, Lorentzian para- Sasakian (briefly LP-Sasakian) manifolds. Later, these manifolds have been widely studied by many geometers such as Matsumoto and Mihai [5], Mihai and Rosca [6], Mihai, Shaikh and De [7], Venkatesha and Bagewadi [16], Venkatesha, Pradeep Kumar and Bagewadi [17, 18] and obtained several results of these manifolds.

In 1995, Sinha and Sai Prasad [14] defined a class of almost paracontact metric manifolds namely para-Kenmotsu (briefly P-Kenmotsu) and Special Para-Kenmotsu (briefly SP- Kenmotsu) manifolds in similar to P-Sasakian and SP- Sasakian manifolds. In 2018, Abdul Haseeb and Rajendra Prasad defined a class of Lorentzian almost paracontact metric manifolds namely Lorentzian para-Kenmotsu (briefly LP- Kenmotsu) manifolds [1]. As an extension, Rajendra Prasad *et al.,* [10] have studied φ-semisymmetric LP-Kenmotsu manifolds with a quarter-symmetric non-metric connection admitting Ricci solitons.

On the other hand, In 1970, Pokhariyal and Mishra [9] introduced new tensor fields, called the Weyl-projective curvature tensor $P(X, Y)Z$ of type (1, 3) and the tensor field E on a Riemannian manifold. Further many geometers have studied the properties of these tensor fields [2, 4, 8, 11, 12, 13, 15] as they play an important role in the theory of projective transformations of connections.

The projective curvature tensor $P(X,Y)Z$, with respect to the Riemannian connection on a Riemannian manifold (M_n, g), is given by:

$$P(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX],$$

where $QX = (n-1)X$, and the Riemannian Christoffel curvature tensor R of type $(1, 3)$ is given by:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1}$$

Here ∇ is said to be the Levi-Civita connection.

In the present work, we study a class of LP-Kenmotsu manifolds and it is organized as follows. Section 2 is equipped with some prerequisites about Lorentzian para-Kenmotsu manifolds. In section 3, we study the quasi-conformally flat Lorentzian para-Kenmotsu manifolds. Sections 4 and 5 respectively deals with φ-conharmonically flat and φ-projectively flat LP-Kenmotsu manifolds.

II. Preliminaries

An n-dimensional differentiable manifold M_n admitting a (1, 1) tensor field ϕ, contravariant vector field ξ, a 1-form η and the Lorentzian metric $g(X, Y)$ satisfying

$$\eta(\xi) = -1, \tag{2}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{3}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{4}$$

$$g(X, \xi) = \eta(X), \tag{5}$$

$$\phi\xi = 0, \ \eta(\phi X) = 0, \ rank\, \phi = n-1; \tag{6}$$

is called Lorentzian almost paracontact manifold [3].

In a Lorentzian almost paracontact manifold, we have

$$\Phi(X,Y) = \Phi(Y,X) \quad where \quad \Phi(X,Y) = g(\phi X, Y). \tag{7}$$

A Lorentzian almost paracontact manifold M_n is called Lorentzian para-Kenmotsu manifold if [1]

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{8}$$

for any vector fields X and Y on M_n, and ∇ is the operator of covariant differentiation with respect to the Lorentzian metric g.

It can be easily seen that in a LP-Kenmotsu manifold M_n, the following relations hold [1]:

$$\nabla_X \xi = -\phi^2 X = -X - \eta(X)\xi, \tag{9}$$

$$(\nabla_X \eta)Y = -g(X, Y)\xi - \eta(X)\eta(Y), \tag{10}$$

for any vector fields X and Y on M_n.

Also, in an LP-Kenmotsu manifold, the following relations hold [1]:

$$g(R(X,Y)Z, \xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \tag{11}$$

$$R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X, \tag{12}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{13}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{14}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{15}$$

$$S(X, Y) = ag(X, Y) + b\,\eta(X)\,\eta(Y); \tag{16}$$

for any vector fields X, Y and Z, where R is the Riemannian curvature tensor and S is the Ricci tensor of M_n.

III. LP-Kenmotsu manifolds with $\tilde{C}\ (X,Y)\,Z=0$

The quasi-conformal curvature tensor $\tilde{C}$ is defined as

$$\begin{aligned}\tilde{C}(X,Y)Z=&aR(X,Y)Z+b\{S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX\\&-g(X,Z)QY\}-\frac{r}{n}\left(\frac{a}{n-1}+2b\right)\{g(Y,Z)X-g(X,Z)Y\}\end{aligned} \tag{17}$$

where a,b are constants such that $ab\neq 0$ and

$$S(Y,Z)=g(QY,Z).$$

From (17), we get

$$\begin{aligned}R(X,Y)Z= & -\tfrac{b}{a}\{S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX\\ & -g(X,Z)QY\}+\tfrac{r}{n}\left(\tfrac{a}{n-1}+2b\right)\{g(Y,Z)X-g(X,Z)Y\}.\end{aligned} \tag{18}$$

Taking $Z=\xi$ in (18) and on using (5), (13), (14), we get

$$\eta(Y)X-\eta(X)Y=-\frac{b}{a}\{\eta(Y)QX-\eta(X)QY\}\left\{\frac{r}{an}\left(\frac{a}{n-1}+2b\right)-\frac{b}{a}(n-1)\right\}\{\eta(Y)X-\eta(X)Y\}. \tag{19}$$

Taking $Y=\xi$ and applying (2) we have

$$\begin{aligned}QX= & \left\{\tfrac{r}{bn}\left(\tfrac{a}{n-1}+2b\right)-(n-1)-\tfrac{a}{b}\right\}X\\ & +\left\{\tfrac{r}{bn}\left(\tfrac{a}{n-1}+2b\right)-\tfrac{a}{b}-2(n-1)\right\}\eta(X)\xi.\end{aligned} \tag{20}$$

Contracting (20), we get after a few steps

$$r=n(n-1). \tag{21}$$

Using (21) in (20), we get

$$QX=(n-1)X. \tag{22}$$

Finally, using (22), we find from (18)

$$R(X,Y)Z=g(Y,Z)X-g(X,Z)Y.$$

Thus, we state

Theorem 3.1: A quasi-conformally flat LP-Kenmotsu manifold is locally isometric with a unit sphere $S^n(1)$.

IV. LP-Kenmotsu manifolds with φ-conharmonically flat curvature tensor

The conharmonic curvature tensor K is defined as

$$K(X,Y)Z=R(X,Y)Z-\frac{1}{n-2}\left[S(Y,Z)X-S(X,Z)Y+g(Y,Z)\mathcal{S}X-g(X,Z)\mathcal{S}Y\right].$$

A differentiable manifold $(M_n, g), n > 3$, satisfying the condition

$$\varphi^2 K(\varphi X, \varphi Y)\varphi Z = 0 \tag{23}$$

is called φ-conharmonically flat.

In this section, we study LP-Kenmotsu manifolds with the condition (23).

Theorem 4.1: Let M_n be an n-dimensional, $(n > 3)$,φ-conharmonically flat LP-Kenmotsu manifold. Then M_n is an η-Einstein manifold with the zero-scalar curvature.

Proof: Assume that $(M_n, g), n > 3$, is a φ-conformally flat LP-Kenmotsu manifold. It can be easily seen that $\varphi^2 K(\varphi X, \varphi Y)\varphi Z = 0$ holds if and only if

$$g(K(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0,$$

for any $X, Y, Z, W \in \chi(M_n)$.

$$\begin{aligned} g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) = \tfrac{1}{n-2}[g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W) \\ + g(\varphi X, \varphi W)S(\varphi Y, \varphi Z) - g(\varphi Y, \varphi W)S(\varphi X, \varphi Z)]. \end{aligned} \tag{24}$$

We suppose that $\{e_1, \ldots, e_{n-1}, \xi\}$ is a local orthonormal basis of vector fields in M_n. By using the fact that $\{\varphi e_1, \ldots, \varphi e_{2n}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (23) and sum up with respect to i, then

$$\begin{aligned} \textstyle\sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) = \tfrac{1}{n-2}\textstyle\sum_{i=1}^{n-1} [g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) \\ - g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i) + g(\varphi e_i, \varphi e_i)S(\varphi Y, \varphi Z) - g(\varphi Y, \varphi e_i)S(\varphi e_i, \varphi Z)], \end{aligned} \tag{25}$$

where

$$\sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) = S(\varphi Y, \varphi Z) + g(\varphi Y, \varphi Z), \tag{26}$$

$$\sum_{i=1}^{n-1} S(\varphi e_i, \varphi e_i) = r + n - 1, \tag{27}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i) = S(\varphi Y, \varphi Z), \tag{28}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi e_i) = n + 1. \tag{29}$$

So, by the use of (26)-(29) the equation (25) turns into

$$-S(\varphi Y, \varphi Z) = (r+1)g(\varphi Y, \varphi Z). \tag{30}$$

Then by using (4) and (15), from equation (30) we get

$$S(Y, Z) = -(r+1)g(Y, Z) - (n+r)\eta(Y)\eta(Z), \tag{31}$$

which gives us, from (16), M_n is an η-Einstein manifold. Hence on contracting (31) we obtain $nr = 0$, which implies the scalar curvature $r = 0$, which proves the theorem.

V. LP-Kenmotsu manifolds with φ-projectively flat curvature tensor

A differentiable manifold (M_n, g), $n > 3$, satisfying the condition

$$\varphi^2 P(\varphi X, \varphi Y)\varphi Z = 0 \tag{32}$$

is called φ-projectively flat, where $P(X, Y)Z$ is the Weyl-projective curvature tensor of (M_n, g).

Theorem 5.1: Let M_n be an n-dimensional, $(n > 3)$, φ-projectively flat LP-Kenmotsu manifold. Then M_n is an Einstein manifold with the scalar curvature $r = n(n-1)$.

Proof: It can be easily seen that $\varphi^2 P(\varphi X, \varphi Y)\varphi Z = 0$ holds if and

$$g(P(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0,$$

for any $X, Y, Z, W \in \chi(M_n)$.

$$g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) = \frac{1}{n-2}[g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W). \tag{33}$$

By choosing $\{e_1, \ldots, e_{n-1}, \xi\}$ as a local orthonormal basis of vector fields in M_n and using the fact that $\{\varphi e_1, \ldots, \varphi e_{2n}, \xi\}$ as a local orthonormal basis, on putting $X = W = e_i$ in (33) and summing up with respect to i, we have

$$\sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) = \frac{1}{n-2}\sum_{i=1}^{n-1} [g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i)]. \tag{34}$$

Therefore, by using (26)-(29) into (34) we get

$$nS(\varphi Y, \varphi Z) = rg(\varphi Y, \varphi Z).$$

Hence by virtue of (4) and (15) we obtain

$$S(Y, Z) = \frac{r}{n}g(Y, Z) + \left(\frac{r}{n} - (n-1)\right)\eta(Y)\eta(Z). \tag{35}$$

Therefore from (35), by contraction, we obtain

$$r = n(n-1). \tag{36}$$

Then by substituting (36) into (35) we get

$$S(Y, Z) = (n-1)g(Y, Z),$$

which implies M_n is an Einstein manifold with the scalar curvature $r = n(n-1)$.
This completes the proof of the theorem.

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References

[1] Abdul Haseeb and Rajendra Prasad. Certain results on Lorentzian para-Kenmotsu manifolds. Bulletin of Parana's Mathematical Society, (2018), 201-220.

[2] U. C. De and Avijit Sarkar. On a type of P-Sasakian manifolds. Math. Reports, 11(2009), no. 2, 139-144.

[3] K. Matsumoto. On Lorentzian Paracontact manifolds. Bulletin of the Yamagata University Natural Science, 12 (2)(1989), 151-156.

[4] K. Matsumoto, S. Ianus and I. Mihai. On P-Sasakian manifolds which admit certain tensor fields. Publ. Math. Debrece, 33(1986), 61-65.

[5] K. Matsumoto and I. Mihai. On a certain transformation in a Lorentzian para-Sasakian manifold. Tensor, N.S., 47(1988), 189-197.

[6] I. Mihai and R. Rosca. On Lorentzian P-Sasakian manifolds. Classical Analysis, World Scientific Publ., Singapore,(1992), 155-169.

[7] I. Mihai, A. A. Shaikh, U.C. De. On Lorentzian para-Sasakian manifolds. Rendicontidel Seminario Mathematico de Messina, Serie II (1999), 75-77.

[8] G. P. Pokharial. Study of a new curvature tensor in a Sasakian manifold. Tensor (N.S.), 36(1982), 222-225.

[9] G. P. Pokhariyal and R. S. Mishra. The curvature tensors and their relativistic significance. Yokohoma Math. J., 18(1970), 105-108.

[10] Rajendra Prasad, Abdul Haseeb and Umesh Kumar Gautam. On ϕ-semisymmetric LP Kenmotsu manifolds with a QSNM-connection admitting Ricci solitons. Kragujevac Journal of Mathematics, 45(5)(2021), 815-827.

[11] K. L. Sai Prasad and T. Satyanarayana. Some curvature properties on a Special paracontact Kenmotsu manifold with respect to Semi-symmetric connection. Turkish Journal of Analysis and Number Theory, 3(2015), no. 4, 94-96.

[12] K. L. Sai Prasad, S. Sunitha Devi and G. V. S. R. Deekshitulu. On a class of Lorentzian para Kenmotsu manifolds admitting the Weyl-projective curvature tensor of type (1,3). Italian Journal of Pure and Applied Mathematics, 45(2021), 990-1001.

[13] K. L. Sai Prasad, S. Sunitha Devi and G. V. S. R. Deekshitulu. On a class of Lorentzian paracontact metric manifolds. Italian Journal of Pure and Applied Mathematics, 49(2023), 514-527.

[14] B. B. Sinha and K. L. Sai Prasad. A class of almost paracontact metric Manifold. Bulletin of the Calcutta Mathematical Society, 87(1995), 307-312.

[15] S. Sunitha Devi, K. L. Sai Prasad and T. Satyanarayana. Certain curvature conditions on Lorentzian para-Kenmotsu Manifolds. Reliability Theory & Applications, 17(2022), no. 2(68), 413-421.

[16] Venkatesha and C. S. Bagewadi, On concircular Œ-recurrent LP-Sasakian manifolds. Differ. Geom. Dyn. Syst., 10(2008), 312-319.

[17] Venkatesha, K. T. Pradeep Kumar and C. S. Bagewadi. On Lorentzian para-Sasakian manifolds satisfying W_2 curvature tensor. IOSR J. of Mathematics, 9(2014), no. 6, 124-127.

[18] Venkatesha, K. T. Pradeep Kumar and C. S. Bagewadi. On quarter-symmetric metric connection in a Lorentzian para-Sasakian manifold. Ajerbaijan Journal of Mathematics, 5(2015), no. 1, 3-12.