INFERENCE ON THE INVERSE POWER BURR-HATKE DISTRIBUTION UNDER TYPE II CENSORING

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Abstract

There are many real-life situations, where data require probability distribution function which have decreasing or upside-down bathtub (UBT) shaped failure rate function. The inverse power burr hatke distribution consists both decreasing and UBT shaped failure rate functions. Here, we address the different estimation methods of the parameter and reliability characteristics of the inverse Pareto distribution from both classical and Bayesian approaches. We consider classical estimation procedures to estimate the unknown parameter of inverse power burr-hatke distribution, such as maximum likelihood. Also, we consider Bayesian estimation using squared error loss function based joint priors. The Monte Carlo simulations are performed to compare the performances of the obtained estimators in mean square error sense. Finally, the flexibility of the proposed distribution is illustrated empirically using one real-life datasets. The analyzed data shows that the introduced distribution provides a superior fit than some important competing distributions such as the Weibull, inverse Pareto and Burr-Hatke distributions.

Keywords: Burr-Hatke Distribution, Inverse Power Burr- Hatke Distribution, Type II censoring, Bayesian estimation, Lindley's Approximation technique.

I. Introduction

Statistical distributions can be used to model many real-life scenarios, such as reliability, actuarial science, survival analysis and lifetime data. Different lifetime distributions have been introduced in the statistical literature to provide greater flexibility in modelling data in these applied sciences. One of the important features of generalized distributions is their capability for providing superior fit for various life-time data encountered in the applied fields. Hence, the statisticians have been interested in constructing new families of distributions to model such data.
Recently, several new distributions and regression models to provide inferences on these distributions have been developed for modeling health and biomedical data, among other fields. Some distributions and classes of distributions developed include exponentiated Burr XII Poisson distribution by da Silva et al. [1], Weibull Burr XII (WBXII) distribution by Afify et al. [2], odd log logistic Topp–Leone G family of distributions by Alizadeh et al. [3], Burr-Hatke exponential (BHE)

distribution by Abouelmagd [4] and Yadav et al. [5], odd generalized gamma-G family of distributions by Nasir et al. [6], Chen-G family of distributions by Anzagra et al. [7], inverse-power Burr-Hatke distribution by Afify et al. [8], harmonic mixture Weibull-G family of distributions by Zamanah et al. [9], harmonic mixture G family of distributions by Kharazmi et al. [10] and Alshenawy R. [11] studied Progressive Type-II Censoring Schemes of Extended Odd Weibull Exponential Distribution with Applications in Medicine and Engineering. Ahmed et. al. [12] studied Bayesian and Classical Inference under Type-II Censored Samples of the Extended Inverse Gompertz Distribution with Engineering Applications. Hassan [13] studied Statistical Inference of Chen distribution Based on Two Progressive Type-II Censoring Schemes.

Burr Hatke model provides only a decreasing hazard rate (HR) shape; hence, its use will be limited to modelling the data that exhibits only increasing failure rate. IPBH model can accommodate right-skewed shape, symmetrical shape, reversed J shape and left-skewed shape densities. Its hazard rate (HR) can be an increasing shape, a unimodal shape, or a decreasing shape. IPBH distribution provides more accuracy and flexibility in fitting engineering and medicine data. The IPBH distribution was constructed using the inverse-power (IP) transformation. The aim of this article is to develop the classical and Bayesian estimation procedures for the parameters of the IPBH.

The rest of the article is organized as follows: IPBH is discussed in Section 2. Also, mathematical formulation is given for type II censoring with failure and censoring time distributions in this section. Section 3 deals with the maximum likelihood estimation and asymptotic confidence intervals of the parameters. Section 4 describes asymptotic confidence interval. Sections 5 describe the formulation of Bayes estimation procedure using Markov chain Monte Carlo (MCMC) methods under SELF loss function using gamma informative priors. Section 6 deals with a Monte Carlo simulation study to explore the properties of various estimates developed in this article. Real life dataset is analyzed for illustration purposes in Section 7. Finally, conclusive remarks are given in section 8. Also, it is essential to mention that the statistical software R 3.5.2, [R Core Team (2018)] is used for computation purposes throughout the article.

II. The Model

If a random variable X follows IPBH with parameter (λ, θ) the cdf is given by:

$$F(x:\lambda,\theta) = \frac{\exp(-\lambda\, x^{-\theta})}{x^{-\theta}+1}, \quad \lambda,\theta > 0 \tag{2.1}$$

Therefore, the corresponding probability density function is given by

$$f(x:\theta,\lambda) = \frac{\theta\, exp(-\lambda x^{-\theta})[\lambda + (1+\lambda)]x^{-\theta}}{x(x^{\theta}+1)^2}\ , \quad \lambda,\theta > 0 \tag{2.2}$$

Where θ and λ are shape parameters, respectively.

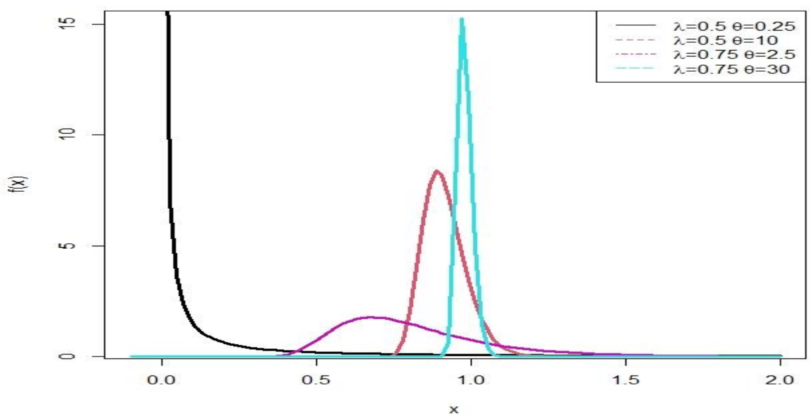

Figure 1. *Possible density shapes of the IPBH distribution for several values of λ and θ.*

The survival function (SF) and HR function of the IPBH distribution take the following forms, respectively:

$$S(x:\lambda,\theta) = 1 - \frac{x^{\theta}\exp(-\lambda\, x^{\theta})}{x^{-\theta}+1} \tag{2.3}$$

$$h(x:\lambda,\theta) = \frac{\theta[\lambda + (\lambda+1)x^{-\theta}]}{x(x^{\theta}+1)[(x^{\theta}+1)\exp(-\lambda x^{-\theta}) - x^{\theta}]} \tag{2.4}$$

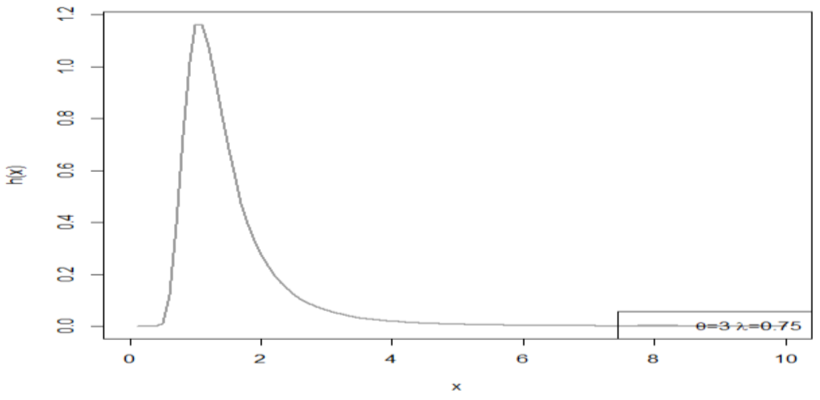

Figure 2. *Possible failure rate shape of the IPBH distribution for values of λ =0.75 and θ = 3*

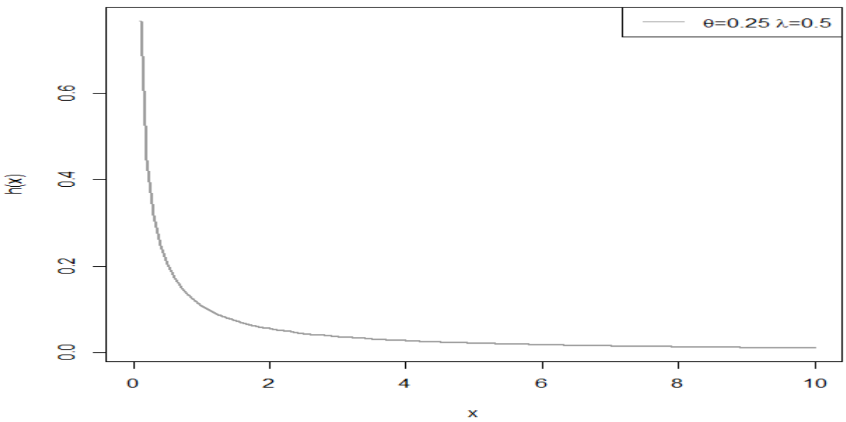

Figure 3. *Possible failure rate shape of the IPBH distribution for values of λ =0.5 and θ = 0.25*

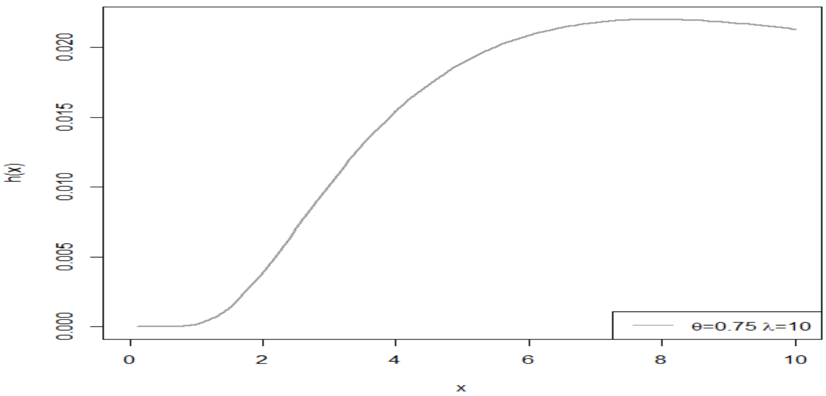

Figure 4. *Possible failure rate shape of the IPBH distribution for values of λ = 10 and θ = 0.75*

III. Maximum Likelihood Estimation

In the literature, Several censoring schemes have been discussed. Even though, Type-I and Type- II censoring schemes are most popular censoring.Consider a life test where n independent units taken from a IPBH distribution are placed under observation and failure time of each unit is recorded.Suppose that the test is terminated when rth, (1 ≤ r ≤ n), r is prefixed unit fails.These observed failure times, say $(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots\ldots, \boldsymbol{x_r})$ is a Type-II censored sample of size r. In this censoring scheme n-r units remain unobserved and survive beyond the time of termination.In Type-II censoring the time of termination is a random variable nad the likelihood function based on $(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_r})$ is given by Cohen [14].

$$L(\lambda,\theta|\underline{x}) = \frac{n}{(n-r)}\prod_{i=0}^{r} f(x_i)\left[1-F(x_{(r)})\right]^{n-r} \tag{2.5}$$

Assume that n independent observed values taken of IPBH distribution as presented in (2) are put on a test.Using the Type-II censoring, we obtained the ordered r failures. If the ordered r failures are then $(\mathbf{x_1}, \mathbf{x_2}, \ldots\ldots, \mathbf{x_r})$ the likelihood function of (λ,θ) under Type-II censored data drawn of an IPBD distribution, is obtained as follows:

$$L(\lambda,\theta|\underline{x}) = \frac{n}{(n-r)}\prod_{i=0}^{r} f(x_i)\left[1-F(x_{(r)})\right]^{n-r}$$

$$L(\lambda,\theta|\underline{x}) = rlog(\theta) - \lambda\sum_{i=1}^{r} x_i^{-\theta} + \sum_{i=1}^{r}\log\left[\lambda+(\lambda+1)x_i^{\theta}\right] - 2\sum_{i=1}^{r}\log\left(x_i^{\theta}+1\right) - \sum_{i=1}^{r}\log(x_i) - \eta$$

MLEs of λ and θ is a solution of equation (2.5) accomplished by addressing the first partial derivatives of the total log-likelihood to be zero.So,we consider the equation as follows,

$$\frac{dlogL}{d\lambda} = \sum_{i=1}^{r}\frac{x_i^{\theta}+1}{\lambda+(\lambda+1)x_i^{\theta}} + \sum_{i=1}^{r} x_i^{\theta} + \frac{(n-r)}{\left(x_{(r)}^{\theta}+1\right)e^{\lambda x_{(r)}^{-\theta}} - x_{(r)}^{\theta}}$$

$$\frac{dlogL}{d\theta} = -\lambda\sum_{i=1}^{r} x_i^{-\theta}\, log(x_i) + \sum_{i=1}^{r}\frac{(\lambda+1)x_i^{\theta}\log(x_i)}{\lambda+(\lambda+1)x_i^{\theta}} - 2\sum_{i=1}^{r}\frac{x_i^{\theta}\log(x_i)}{x_i^{\theta}+1} + \eta_1(x)$$

The closed form solutions to the nonlinear Equations are difficult to reach and a numerical method must be applied to solve these simultaneous equation for obtaining the MLE of λ and θ.

IV. Asymptotic Confidence Intervals

The maximum likelihood estimators of the unknown parameters are not in closed form, it is not easy to drive the exact distributions of the MLEs. Thus, we use the asymptotic distribution of MLEs for the constructions of asymptotic confidence intervals of the parameters based on observed Fisher information matrix. Let $\hat{\alpha}$= ($\hat{\lambda}, \hat{\theta}$), be the MLE of $\alpha = (\lambda, \theta)$.The observed Fisher information matrix is given by:

$$I(\alpha) = \begin{bmatrix} \frac{\partial \ln L(\theta,\lambda)}{\partial\lambda^2} & \frac{\partial \ln L(\theta,\lambda)}{\partial\lambda\,\partial\theta} \\ \frac{\partial \ln L(\theta,\lambda)}{\partial\theta\,\partial\lambda} & \frac{\partial \ln L(\theta,\lambda)}{\partial\theta^2} \end{bmatrix}$$

$$\frac{\partial \ln L(\theta,\lambda)}{\partial\lambda^2} = \sum_{i=1}^{r}\frac{\left(x_i^{\theta}+1\right)^2}{\left((\lambda+(\lambda+1)x_i^{\theta})\right)^2} + \frac{(n-r)\left(x_{(r)}^{\theta}+1\right)x_{(r)}^{\theta}e^{\lambda x_{(r)}^{-\theta}}}{\left(x_{(r)}^{\theta} - (x_{(r)}^{\theta}+1)e^{\lambda x_{(r)}^{-\theta}}\right)^2}$$

$$\frac{\partial \ln L(\theta,\lambda)}{\partial\theta\,\partial\lambda} = \sum_{i=1}^{r}\left(\frac{x_i^{-\theta}\log(x_i)}{\lambda+(\lambda+1)x_i^{\theta}} - \frac{(\lambda+1)x_i^{\theta}(x_i^{\theta}+1)\log(x_i)}{\left(\lambda+(\lambda+1)x_i^{\theta}\right)^2}\right) - \sum_{i=1}^{r} x_i^{-\theta}(-\log(x_i))$$

Thus, the observed variance-covariance matrix becomes $I^{-}(\hat{\alpha})$.The asymptotic distribution of MLE

$\hat{\alpha}$ is a bivariate normal distribution as $\hat{\alpha}$N $(0, I^{-}(\hat{\alpha}))$. Consequently, two sided equal tailed 100(1−η) asymptotic confidence intervals for the parameters λ and θ are given by $\left[\hat{\lambda} + Z_{\frac{\eta}{2}}\sqrt{var(\hat{\lambda})}\right]$ and $\left[\hat{\theta} + Z_{\frac{\eta}{2}}\sqrt{var(\hat{\theta})}\right]$ respectively. Here, Var $(\hat{\lambda})$ and Var $(\hat{\theta})$ are diagonal elements of the observed variance-covariance matrix $I^{-}(\hat{\alpha})$ and $Z_{\frac{\eta}{2}}$ is the upper $\left(Z_{\frac{\eta}{2}}\right)^{th}$ percentile of the standard normal distribution.

V. The Bayesian Estimation

In this section, we discuss the Bayes estimators of the unknown parameters of the model in (2) under square error loss function (SELF). In order to select the best decision in decision theory, an appropriate loss function must be specified. SELF is generally used for this purpose. The use of the SELF is well justified when over estimation and under estimation of equal magnitude has the same consequences. When the true loss is not symmetric with respect to over estimation and under estimation, asymmetric loss functions are used to represent the consequences of different errors. If all parameters of the model are unknown, a joint conjugate prior for the parameters does not exist. In such conditions there are numerous ways to choose the priors. Hence, we choose to consider the piecewise independent priors. The proposed priors for the parameters λ and θ may be taken as:

$$g_1(\lambda) = \lambda^{a_1-1}e^{-\lambda b_1}, \qquad a_1, b_1 > 0$$
$$g_2(\theta) = \lambda^{a_1-1}e^{-\lambda b_1}, \qquad a_2, b_2 > 0$$

Thus, the joint prior distribution of λ and θ can be written as:

$$g(\lambda, \theta) = \lambda^{a_1-1}\theta^{a_1-1}\, e^{-(\lambda b_1+\theta b_1)} \tag{4.1}$$

Now we derive the Bayes estimators for the unknown parameters λ and θ under squared error loss function. If μ is the parameter to be estimated by an estimator $\hat{\mu}$ then the squared error loss function is defined as $L_s(\mu, \hat{\mu}) = (\mu - \hat{\mu})^2$. The joint posterior distribution of λ and θ after simplification is:

$$\Pi(\lambda, \theta|\underline{x}) = \frac{\frac{n}{(n-r)}\lambda^{a_1-1}\theta^{a_2-1}e(\lambda b_1 + \theta b_2)\prod_{i=0}^{r} f(x_i)(1-F(x))^{n-r}}{\int_0^\infty\int_0^\infty \frac{n}{(n-r)}\lambda^{a_1-1}\theta^{a_2-1}e(\lambda b_1 + \theta b_2)\prod_{i=0}^{r} f(x_i)(1-F(x))^{n-r}\,\partial\lambda\partial\theta} \tag{4.2}$$

Therefore, the Bayes estimator of any function of λ and θ, say $\alpha(\hat{\lambda},\hat{\theta})$ under squared error loss function is.

I. Subsection One

Lindley's Approximation

It is difficult to compute Eq. (4.2) analytically. Lindley's [15] approximation is used to compute the ratio of integrals of the form Eq. (4.3). Based on Lindley's approximation, the approximate Bayes estimator of λ under the squared error loss function is:

$$\hat{\lambda}_{\text{lindley}} = \hat{\lambda} + \frac{1}{2}[\mu_1(2\rho_1\sigma_{11} + 2\rho_2\sigma_{21} + \sigma_{11}^2 L_{111} + 2\sigma_{12}\sigma_{21}L_{111} + \sigma_{11}\sigma_{22}L_{211} + \sigma_{12}\sigma_{22}L_{222})] \tag{4.4}$$

$$\hat{\theta}_{\text{lindley}} = \hat{\theta} + \frac{1}{2}[\mu_2(2\rho_2\sigma_{22} + 2\rho_1\sigma_{21} + \sigma_{22}^2 L_{222} + 2\sigma_{12}\sigma_{11}L_{111} + 3\sigma_{12}\sigma_{22}L_{122})] \tag{4.5}$$

Here L(λ, θ) is the log-likelihood and $\rho(\lambda, \theta)$ is the log of prior distribution $\pi(\lambda, \theta)$, $\hat{\lambda}$ and $\hat{\theta}$ are the MLEs of λ and θ respectively.

VI. Simulation Study

This section deals with a Monte Carlo simulation study. Here, we compare various estimators developed in the previous sections with the help of Monte Carlo simulation study. Six different sample sizes n = 50, 60, 70, 80 and 90 are considered in the simulation study. Following combination of the true values of the parameters (λ, θ) = (0.5, 1) and (λ, θ) = (1.5, 1) are taken. In each case the ML and Bayes estimates of the unknown parameters are computed. The whole process is simulated 1000 times. Tables 1–2 report the simulation results including Average Estimate (AE), MSE of the IPBH parameters.

Table 1: *Bayes estimate of the parameter λ and θ when θ = 1 and λ = 0.5*

n	r	Prior1		Prior2		Prior1		Prior2	
		$\hat{\lambda}$				$\hat{\theta}$			
		AE	MSE	AE	MSE	AE	MSE	AE	MSE
50	46	0.5332	0.018	0.5714	0.0152	1.0862	0.0082	1.0778	0.2459
50	48	0.5321	0.0137	0.5318	0.0142	1.0571	0.0715	1.0754	0.0821
60	56	0.5263	0.0124	0.5268	0.0122	1.0655	0.00615	1.0553	0.00567
60	58	0.5195	0.0102	0.5257	0.0099	1.0525	0.0516	1.0529	0.0588
70	66	0.5173	0.0089	0.5224	0.0091	1.0491	0.0506	1.0551	0.0511
70	68	0.5171	0.0084	0.5223	0.0092	1.0468	0.0485	1.0492	0.0492
80	76	0.5168	0.0071	0.5152	0.007	1.0423	0.0447	1.0468	0.0429
80	78	0.5156	0.0061	0.5187	0.0074	1.0387	0.0394	1.0271	0.0366
90	86	0.5078	0.0054	0.5162	0.0063	1.0311	0.0343	1.0327	0.0364
90	88	0.5115	0.0049	0.511	0.0053	1.0296	0.0316	1.0329	0.0349

Table 2: *Bayes estimate of the parameter λ and θ when θ = 1 and λ = 1.5*

n	r	Prior1		Prior2		Prior1		Prior2	
		$\hat{\lambda}$				$\hat{\theta}$			
		AE	MSE	AE	MSE	AE	MSE	AE	MSE
50	46	1.7311	0.464	1.7088	0.394	1.0504	0.0528	1.0498	0.0492
50	48	1.6536	0.2447	1.6543	0.2456	1.0369	0.0407	1.0439	0.0423
60	56	1.6038	0.1654	1.6382	0.1643	1.0295	0.0313	1.0453	0.0338
60	58	1.5855	0.1197	1.5290	0.1250	1.0244	0.0286	1.0306	0.0298
70	66	1.5841	0.1195	1.5771	0.1181	1.0258	0.0248	1.0221	0.0258
70	68	1.5723	0.0956	0.1181	0.0922	1.0243	0.0244	1.0202	0.0231
80	76	1.5636	0.0958	1.5771	0.1127	1.0143	0.0199	1.0285	0.0237
80	78	1.573	0.0856	1.5639	0.0827	1.0228	0.0198	1.0191	0.002
90	86	1.5614	0.0821	1.5587	0.0792	1.0186	0.0182	1.0162	0.0198
90	88	1.5534	0.0712	1.5489	0.0710	1.0199	0.0171	1.0276	0.0175

VII. Real-Life Applications

In this section, we illustrate estimation procedures discussed in the previous sections with the help of one real datasets. Here, we consider a real dataset namely the strengths of glass fibres The Data I, respectively are given below:

Data set:

This dataset consists of 63 observations which are generated to simulate the strengths of glass fibres [18].The 63 observations of the dataset are as follows: "1.014, 1.081, 1.082, 1.185, 1.223, 1.248, 1.267, 1.271, 1.272, 1.275, 1.276, 1.278, 1.286, 1.288, 1.292, 1.304, 1.306, 1.355, 1.361, 1.364, 1.379, 1.409, 1.426, 1.459, 1.460, 1.476, 1.481, 1.484, 1.501, 1.506, 1.524, 1.526, 1.535, 1.541, 1.568, 1.579, 1.581, 1.591, 1.593, 1.602, 1.666, 1.670, 1.684, 1.691, 1.704, 1.731, 1.735, 1.747, 1.748, 1.757, 1.800, 1.806, 1.867, 1.876, 1.878, 1.910, 1.916, 1.972, 2.012, 2.456, 2.592, 3.197, and 4.121".

We calculate MLEs of the unknown parameters together with some useful measure of goodness-of fit tests for one dataset, namely, the negative log likelihood function −lnL, the Akaike information criterion denoted by AIC = 2k–2lnL, proposed by Akaike [16] and Bayesian information criterion denoted by BIC = kln(n)–2lnL, proposed by Schwarz [17], where k is the number of parameters in the model, n is the number of observations in the given datasets, L is the maximized value of the likelihood function for the estimated model and Kolmogorov-Smirnov (K-S) statistic with its p-value. The best distribution corresponds to the lowest –lnL, AIC, BIC and K-S statistic and the highest p values. The K-S statistic with its p-value is obtained using ks test function in statistical software R. The results of the MLEs and measures of goodness-of-fit tests are reported in Tables 3 and 4, respectively. These results show that IPBH distribution is the best choice for the considered datasets. However, for Data I, according to K-S test IPBH is better than the BH.

Table 3: *Data Summary for the Data Set*

Min	1st Qu.	Median	mean	3rd Qu.	Max
1.014	1.305	1.526	1.616	1.741	4.121

Table 4: *Goodness of Fit criterions on the data set*

Distribution	Estimates	-logL	AIC	BIC	K-S (stat)	P-value
IPBD	$\hat{\theta}$ = 5.7408 $\hat{\lambda}$ =6.03415	15.403	26.8066	39.0929	0.08507	0.7197
BR	$\hat{\theta}$ =0.2325 0	113.364	222.7286	235.0148	0.77052	< 0.001
Weibull	$\hat{\theta}$ = 3.0521 $\hat{\lambda}$ =1.7873	43.254	96.7345	101.12	0.2051	0.009
Exponential	$\hat{\lambda}$ =0.6189	93.222	187.432	190.523	0.4721	< 0.001

VIII. Conclusion

This article deals with the classical and Bayesian estimation procedures for parameters of inverse power Burr-Hatke distribution using second type censoring. The maximum likelihood estimators and corresponding asymptotic confidence intervals based on observed Fisher information matrix of the unknown parameters were derived. The Bayes estimates of the parameters under square error loss function were approximated using Lindley's approximation. The performance of these estimators was examined by extensive Monte Carlo simulation study, which indicated that the MLEs can be obtained easily and quickly with satisfactory estimates. For more efficient estimators, Bayes estimation method with available prior information or convenient non-informative priors in the absence of prior information is recommended.

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