SEQUENTIAL TESTING PROCEDURE FOR THE PARAMETERS OF INVERSE DISTRIBUTION FAMILY

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Abstract

The sequential probability ratio test is a powerful statistical tool that is frequently employed for hypothesis testing, parameter estimation, and statistical inference. The aspect of robustness is of utmost importance when employing SPRTS in practical applications. Past studies have investigated the robustness of SPRTS for specific distributions. We have developed SPRTS for a family of inverse distributions that includes eleven distinct distributions. The primary objective of this study is to investigate and evaluate the robustness of SPRTS under various conditions and distributions, focusing on the parameters of the inverse distribution family. SPRTS efficacy is measured using OC and ASN functions. This study comprehensively covers the construction and rigorous evaluation of SPRTS, particularly in testing simple null hypotheses against simple alternative hypotheses. Additionally, we investigate the robustness of SPRTS under various factors, including the presence of other parameters and specified coefficients of variation. Conclusive results, graphic representations, tables, and acceptance and rejection regions add clarity to the findings.

Keywords: Inverse Distributions Family, Sequential Probability Ratio Tests (SPRT), Operating Characteristics (OC), Average Sample Number (ASN).

1. Introduction

Sequential Probability Ratio Tests (SPRT) are innovative methodologies that prove highly effective for both hypothesis testing and parameter estimation in statistical inference. The foundational work by [21] introduced the concept of SPRT for analyzing simple null hypotheses against simple alternatives. To assess the effectiveness of SPRT, operating characteristic (OC) and average sample number (ASN) functions were developed as performance measures. Sequential probability ratio tests (SPRTS) have long been recognized as valuable tools for making efficient and prompt decisions in various statistical applications. These tests play a crucial role in scenarios where data are collected sequentially over time, and the goal is to make a conclusive determination about a specific hypothesis. Robustness, which ensures the validity and reliability of these tests under varying conditions, is an essential aspect to consider when employing SPRTS in real-world situations. Multiple studies have scrutinized the robustness of SPRTS in disparate scenarios, enhancing our understanding of their performance and versatility. For instance, [1] examined Wald's SPRT for Levy processes, while [3] explored the robustness of sequential testing procedures for generalized life distributions. Other research, such as that by [6] studied the robustness of sequential testing procedures for parameters of zero-truncated negative binomial, binomial and Poisson distributions. Previous works have also assessed the robustness of SPRTS in specific settings, such as [8], considered sequential life tests in the exponential case. [9] examined the robustness of sequential probability ratio tests in the presence of nuisance parameters.[11] evaluated exponential and Weibull test plans, whereas [12] concentrated on investigating the robustness of the SPRT for a negative binomial distribution in cases where the shape parameter

is not specified. Additionally, [13] investigated the robustness of the exponential SPRT when failures from a Weibull distribution were transformed using a known shape parameter. Other relevant research includes [17]discusses the performance analysis of the Sequential Probability Ratio Test (SPRT) under various conditions and [14] explored robustifying the SPRT for a discrete model under "contamination." In contrast, [15] analyzed the performance and robustness of an SPRT for non-identically distributed observations. The robustness of SPRTS has also been examined in the context of exponential life-testing procedures [18] and the scale parameter of gamma and exponential distributions [19].[16] discusses the use of sequential probability ratio tests (SPRTS) for the statistical analysis of simulation outputs generated by computers. The type I and type II errors exponents of sequential probability ratio tests, when the actual distributions differ from the test distributions analyzed by [2]. In light of these studies, this research aims to investigate further and evaluate the robustness of sequential probability ratio tests under various conditions and distributions. In this study, we aim to extend the existing research and contribute to the robustness analysis of SPRTS for parameters of inverse distribution family suggested by [7]. Our focus will be on thoroughly examining the robustness of these tests using OC and ASN functions. We will develop and rigorously evaluate the SPRTS, with specific attention given to their robustness about the OC and ASN functions. Sections 3 and 5 will cover the essential elements of constructing and evaluating the SPRTS, including testing simple null hypotheses against simple alternatives, sequential analyses of composite hypotheses, and comprehensively examining their robustness. Section 4 shall analyze simple null hypotheses established on the parameter γ, taking into account the presence of the illustrious δ. Furthermore, in Section 6, we shall investigate comparable hypotheses founded on the parameter δ, factoring in the existence of γ. In Section 7, we will further investigate the robustness of the SPRTS in the presence of a specified coefficient of variation. Section 8 presents the regions of acceptance and rejection deduced for the null hypothesis H_0 compared to the alternative hypothesis H_1. Finally, Section 9 will effectively explain the synthesized data and provide conclusive findings using a combination of tables and graphics.
Through this comprehensive analysis, we aim to gain valuable insights into the robustness, performance, and limitations of SPRTS in the inverse family of distributions.

2. Inverse Distributions Family

Suppose a random variable (rv) x having p.d.f.

$$f\left(x;a^{-1},\gamma,\delta,\underline{\theta}\right)=\frac{\gamma^{\delta}g^{\delta-1}\left(x^{-1};\underline{\theta}\right)g'\left(x^{-1};\underline{\theta}\right)}{x^{2}\Gamma(\delta)}\exp\left(-\gamma g\left(x^{-1};\underline{\theta}\right)\right); \quad 0<x<a^{-1},\quad \gamma>0,\delta>0. \tag{1}$$

Where, $g\left(x^{-1};\underline{\theta}\right)$, is a function of $\underline{\theta}$ and x. Moreover, $g\left(x^{-1};\underline{\theta}\right)$ real-valued, Strict decreasing the function of x with $g(\infty;\underline{\theta})=\infty$ and $g'\left(x^{-1};\underline{\theta}\right)$ stances for the derivative of $g(x;\underline{\theta})$ by x^{-1}.
the equation (1) shows that the above distribution can be converted in the following distributions as special cases: If $g(x;\underline{\theta})=x^2,\delta=k+1(k\geq 0),\left(k=\frac{-1}{2}\right)$ provide the inverse Half-normal distribution and (k = 0) the inverse Rayleigh distribution. If $g(x;\underline{\theta})=\log\left(1+\frac{x^b}{v^b}\right),b>0,v>0,\delta=1$, provide the inverse log-logistic model. If $g(x;\underline{\theta})=\log\left(1+\frac{x^b}{v^b}\right),b>0,v=1,\delta>1$, provide the inverse Burr distribution. If $g(x;\underline{\theta})=\log\left(1+\frac{x^b}{v^b}\right),b=1,v>1,\delta>1$, provide the inverse Lomax distribution. If $g(x;\underline{\theta})=\frac{x^2}{2},\delta=\frac{h}{2}(h>0)$, it becomes inverse Chi-distribution. If $g(x;\underline{\theta})=\log\left(\frac{x}{a}\right)$ and $\delta=1$, obtain inverse Pareto distribution. If $g(x;\underline{\theta})=x^r\exp(ax),r>0,a>0,\delta=1$, obtain inverse modified Weibull distribution. If $g(x;\underline{\theta})=\mu x+\frac{vx^2}{2},\gamma=\delta=1$, obtain inverse linear exponential distribution. If $g(x;\underline{\theta})=\log x$, obtain the inverse of the log-gamma distribution. If $g(x;\underline{\theta})=x^p,p>0,\delta>0$, obtained the inverse generalized gamma distribution.

3. SPRT FOR EVALUATING THE HYPOTHESES OF γ

Let a series $X_1, X_2, \ldots$ from (1), assume one needs to assess the simple hypotheses $H_0 : \gamma = \gamma_0$ as opposed to $H_1 : \gamma = \gamma_1\,(> \gamma_0)$. The analysis of SPRT on behalf of H_0, expressed in this manner

$$Z_i = ln\left\{\frac{f\left(X_i; a, \gamma_1, \delta, \underline{\theta}\right)}{f\left(X_i; a, \gamma_0, \delta, \underline{\theta}\right)}\right\} = \delta.ln\left(\frac{\gamma_1}{\gamma_0}\right) - g\left(x_i^{-1}; \underline{\theta}\right)(\gamma_1 - \gamma_0) \tag{2}$$

Admit H_o if $\sum_{i=1}^{n} Z_i \leq lnB$, refuse H_0 if $\sum_{i=1}^{n} Z_i \geq ln\,A$, or else, carry on sampling using the value of $(n+1))^{\text{th}}$. If α and β belong to the interval $(0, 1)$ and represent type I and type II errors sequentially, the work by [21] provides definitions for A and B that are specified as

$$A \cong \frac{(1-\beta)}{\alpha}$$

and

$$B \cong \frac{\beta}{(1-\alpha)}$$

Where $0 < B < 1 < A$
The OC function is almost specified as

$$L(\gamma) \cong \frac{\left(A^{t_0} - 1\right)}{\left(A^{t_0} - B^{t_0}\right)}$$

Where t_0 is the non-zero result for equation

$$E\left(e^{t_0 z_i}\right) = 1 \tag{3}$$

Note 1: Use the statement that $g\left(x^{-1}; \theta\right)$ follows gamma distribution
Using (1) with (3), we find

$$\left(\frac{\gamma_1}{\gamma_0}\right)^{\delta t_o}\left\{\frac{t_0\left(\gamma_1 - \gamma_0\right) + \gamma}{\gamma}\right\}^{-\delta} = 1$$

or,

$$\gamma = \frac{t_0\left(\gamma_1 - \gamma_0\right)}{\left(\frac{\gamma_1}{\gamma_0}\right)^{t_0} - 1} \tag{4}$$

To find the values of OC and ASN functions, evaluate (4) as

$$t_0 \ln\left(\frac{\gamma_1}{\gamma_0}\right) = \ln\left[1 + t_0\left(\frac{\gamma_1 - \gamma_0}{\gamma}\right)\right] \tag{5}$$

By utilizing the natural logarithm function of $(1+x)$, which is defined for $1 < x < 1$, in (5).we can achieve the desired outcome from (6).

$$\left\{\frac{1}{3}\left(\frac{\gamma_1 - \gamma_0}{\gamma}\right)^3\right\} t_0^2 - \left\{\frac{1}{2}\left(\frac{\gamma_1 - \gamma_0}{\gamma}\right)^2\right\} t_0 + \left\{\left(\frac{\gamma_1 - \gamma_0}{\gamma}\right) - \ln\left(\frac{\gamma_1}{\gamma_0}\right)\right\} = 0 \tag{6}$$

Using (2), provides that

$$E\left(Z_i \mid \gamma\right) = \delta\left[ln\left(\frac{\gamma_1}{\gamma_0}\right) - \left(\frac{\gamma_1 - \gamma_0}{\gamma}\right)\right] \tag{7}$$

Using (7), we get, the ASN function

$$E(N \mid \gamma) \cong \frac{L(\gamma) ln\,B + \{1 - L(\gamma)\} ln\,A}{\delta\left[ln\left(\frac{\gamma_1}{\gamma_0}\right) - \left(\frac{\gamma_1 - \gamma_0}{\gamma}\right)\right]} \tag{8}$$

Using (8) the ASN function for H_0 along with H_1 specified as

$$E_0(N) \cong \frac{(1-\alpha) ln\, B + \alpha\, ln\, A}{\delta \left[ln \left(\frac{\gamma_1}{\gamma_0}\right) - \left(\frac{\gamma_1-\gamma_0}{\gamma}\right)\right]}$$

and

$$E_1(N) \cong \frac{\beta\, ln\, B + (1-\beta) ln\, A}{\delta \left[ln \left(\frac{\gamma_1}{\gamma_0}\right) - \left(\frac{\gamma_1-\gamma_0}{\gamma}\right)\right]}$$

4. SPRT FOR EVALUATING THE HYPOTHESES OF γ ALTHOUGH δ IS CHANGING

Using section (3), The maximum value of ASN gets on behalf of $\gamma = \widetilde{\gamma}$ where $\widetilde{\gamma}$ is getting from $E\left(Z_i \mid \gamma\right) = 0$ and the maximum value is specified as

$$E_{\widetilde{\gamma}}(N) \cong -\frac{(ln\, A * ln\, B)}{E\left(z_i^2 \mid \widetilde{\gamma}\right)} \tag{9}$$

Also

$$\widetilde{\gamma} = \left\{ \frac{\gamma_1-\gamma_0}{ln\left(\frac{\gamma_1}{\gamma_0}\right)} \right\} \tag{10}$$

Also, using (7) we get

$$E\left(Z_i^2 \mid \widetilde{\gamma}\right) = \delta \left[ln \left(\frac{\gamma_1}{\gamma_0}\right) - \left(\frac{\gamma_1-\gamma_0}{\gamma}\right)\right]^2 + \frac{\left(\gamma_1-\gamma_0\right)^2 \delta}{\widetilde{\gamma}^2} \tag{11}$$

Utilizing (9) and (11), we find that

$$E_{\widetilde{\gamma}}(N) \cong \frac{-(ln\, A * ln\, B)}{\left\{ \delta ln \left(\frac{\gamma_1}{\gamma_0}\right) - \frac{(\gamma_1-\gamma_0)\delta}{\widetilde{\gamma}} \right\}^2 + \frac{(\gamma_1-\gamma_0)^2 \delta}{\widetilde{\gamma}^2}}$$

Assuming that there has been a modification to the parameter δ and that (1) has transformed into $f(x; a, \gamma, d, \theta)$, this can be attained by replacing δ with d. To analyze the robustness of SPRT, suggest t_0 as the result of the equation

$$\int_0^{a^{-1}} \left\{ \frac{f\left(x_i; a, \gamma_1, \delta, \theta\right)}{f\left(x_i; a, \gamma_o, \delta, \underline{\theta}\right)} \right\}^{t_0} f\left(x_i; a, \gamma, d, \underline{\theta}\right) dx_i = 1 \tag{12}$$

We achieve from (12) and put $\phi_1 = \left(\frac{\delta}{d}\right)$

$$\left(\frac{\gamma_1}{\gamma_o}\right)^{\delta t_0} \frac{\gamma^d}{\Gamma(d)} \int_0^{a^{-1}} exp \left[- \left\{ \left(\gamma_1 - \gamma_0\right) t_0 + \gamma \right\} g\left(x_i^{-1}; \underline{\theta}\right)\right] \frac{g^{d-1}\left(x_i^{-1}; \theta\right) g'\left(x_i^{-1}; \underline{\theta}\right)}{x_i^2} dx_i = 1,$$

or,

$$\left(\gamma_1 - \gamma_0\right) \frac{t_0}{\gamma} + 1 = \left(\frac{\gamma_1}{\gamma_o}\right)^{\frac{\delta t_0}{d}}$$

or,

$$\gamma = \frac{\left(\gamma_1 - \gamma_0\right) t_0}{\left(\frac{\gamma_1}{\gamma_0}\right)^{\phi_1 t_0} - 1} \tag{13}$$

To find the values of OC functions, evaluate (13) as

$$\phi_1 t_0 \ln \left(\frac{\gamma_1}{\gamma_0}\right) = \ln \left[1 + t_0 \left(\frac{\gamma_1 - \gamma_0}{\gamma}\right)\right] \tag{14}$$

Equation (14), Solve as (5) and find the roots of t_0 from (15)

$$\left\{\frac{1}{3}\left(\frac{\gamma_1-\gamma_0}{\gamma}\right)^3\right\}t_0^2-\left\{\frac{1}{2}\left(\frac{\gamma_1-\gamma_0}{\gamma}\right)^2\right\}t_0+\left\{\left(\frac{\gamma_1-\gamma_0}{\gamma}\right)-\phi_1\ln\left(\frac{\gamma_1}{\gamma_0}\right)\right\}=0 \tag{15}$$

where $\phi_1=\left(\frac{\delta}{d}\right)$. The ASN function coincides with (8)

$$E\left(Z_i\mid\gamma\right)=\phi_1\left[ln\left(\frac{\gamma_1}{\gamma_0}\right)-\left(\frac{\gamma_1-\gamma_0}{\gamma}\right)\right] \tag{16}$$

5. SPRT for Evaluating the Hypotheses of δ

Suppose taking a sequence $X_1, X_2, \ldots$ from (1) are independently and identically distributed. To analyze the simple null hypotheses in contradiction of the simple alternative hypotheses when γ is identified. $H_0:\delta=\delta_0$ as opposed to $H_1:\delta=\delta_1\ (>\delta_0)$.
We suggest the resulting SPRT

$$Z_i\ =\left(\delta_1-\delta_0\right)ln\,\gamma+\left(\delta_1-\delta_0\right)ln\left\{g\left(x_i^{-1};\theta\right)\right\}+ln\left(\frac{\Gamma\left(\delta_0\right)}{\Gamma\left(\delta_1\right)}\right) \tag{17}$$

Admit H_0 on the n^{th} step, if

$$\sum_{i=1}^{n}ln\,\left\{g\left(x_i^{-1};\underline{\theta}\right)\right\}\leq\left\{ln\,B-n\,\left(\delta_1-\delta_0\right)ln\,\gamma-n\,ln\,\left(\frac{\Gamma\left(\delta_o\right)}{\Gamma\left(\delta_1\right)}\right)\right\}/\left(\delta_1-\delta_o\right) \tag{18}$$

Reject H_0 if

$$\sum_{i=1}^{n}ln\,\left\{g\left(x_i^{-1};\underline{\theta}\right)\right\}\geq\left\{ln\,A-n\,\left(\delta_1-\delta_0\right)ln\,\gamma-n\,ln\,\left(\frac{\Gamma\left(\delta_0\right)}{\Gamma\left(\delta_1\right)}\right)\right\}/\left(\delta_1-\delta_o\right) \tag{19}$$

Then using the $(n+1)^{\text{th}}$ value carry on sampling if

$$\frac{\left\{ln\,B-n\left(\delta_1-\delta_0\right)ln\,\gamma-n\,ln\left(\frac{\Gamma(\delta_0)}{\Gamma(\delta_1)}\right)\right\}}{\left(\delta_1-\delta_0\right)}<\sum_{i=1}^{n}ln\left\{g\left(x_i^{-1};\underline{\theta}\right)\right\}<\frac{\left\{ln\,A-n\left(\delta_1-\delta_0\right)ln\,\gamma-n\,ln\left(\frac{\Gamma(\delta_0)}{\Gamma(\delta_1)}\right)\right\}}{\left(\delta_1-\delta_0\right)} \tag{20}$$

The *OC* function, *A* and *B* same as previously.

$$L(\delta)\cong\frac{\left(A^{t_0}-1\right)}{\left(A^{t_0}-B^{t_0}\right)} \tag{21}$$

Here t_o is the positive as well as negative but not zero

$$E\left\{e^{t_oZ_i}\right\}=1. \tag{22}$$

Using Note 1 with (22), we get

$$\left\{\frac{\Gamma\left(t_0\left(\delta_1-\delta_0\right)+\delta\right)}{\Gamma(\delta)}\right\}=\left(\frac{\Gamma\left(\delta_1\right)}{\Gamma\left(\delta_0\right)}\right)^{t_0}. \tag{23}$$

Taking the logarithm of both sides of (23), with $\ln(1+x); -1<x<1$

$$\ln\Gamma(x)=\ln\sqrt{2\pi}-x+\left(x-\frac{1}{2}\right)\ln x \tag{24}$$

By using the equation (24) of approximation, we get

$$\frac{{t_0}^2}{6}\left(\frac{\delta_1-\delta_0}{\delta}\right)^3(\delta+1)-\frac{t_0}{4}\left(\frac{\delta_1-\delta_0}{\delta}\right)^2(2\delta+1)-\left(\delta_0-\frac{1}{2}\right)\ln\delta_0+\left(\delta_1-\frac{1}{2}\right)\ln\delta_1 \\ -\left(1+\ln\delta-\frac{1}{2\delta}\right)(\delta_1-\delta_0)=0 \tag{25}$$

Simplifying terms up to the third degree in t_0, we get the roots of t_0 from (25).

$$E\left\{ln(g\left(X_i^{-1};\theta\right))\right\}=\frac{\gamma^\delta}{\Gamma(\delta)}\int_0^\infty(ln\,x)x^{\delta-1}e^{-\gamma x}dx \tag{26}$$

We achieved, using [10], that

$$E\left\{ln(g\left(X_i^{-1};\theta\right))\right\}=\{\psi(\delta)-ln\gamma\}, \tag{27}$$

And $\psi(\delta)$ is specified as

$$\psi(\delta)=\frac{d}{d(\delta)}ln\Gamma(\delta)$$

Using (7) and (26), we find

$$E\left(Z_i\mid\delta\right)=\left[ln\left\{\Gamma\left(\delta_o\right)\right\}-ln\left\{\Gamma\left(\delta_1\right)\right\}\right]+\left(\delta_1-\delta_0\right)\psi(\delta) \tag{28}$$

The ASN function for H_o and H_1 using (22) and (27) are specified as

$$E_0(N)\cong\frac{(1-\alpha)ln\,B+\alpha\,ln\,A}{\{ln\left(\Gamma\left(\delta_0\right)\right)-ln\left(\Gamma\left(\delta_1\right)\right)\}+\left(\delta_1-\delta_0\right)\psi\left(\delta\right)} \tag{29}$$

and

$$E_1(N)\cong\frac{\beta\,ln\,B+(1-\beta)ln\,A}{\{ln\left(\Gamma\left(\delta_0\right)\right)-ln\left(\Gamma\left(\delta_1\right)\right)\}+\left(\delta_1-\delta_0\right)\psi\left(\delta\right)} \tag{30}$$

6. SPRT FOR EVALUATING THE HYPOTHESES OF δ ALTHOUGH γ IS CHANGING

Using Section (5), The greatest value of ASN attained for $\delta=\widetilde{\delta}$, where $\widetilde{\delta}$ is the result of $E\left(Z_i\mid\delta\right)=0$

$$\psi(\widetilde{\delta})=\frac{\{ln\Gamma\left(\delta_1\right)-ln\Gamma\left(\delta_0\right)\}}{\left(\delta_1-\delta_0\right)}$$

This gives the highest worth as

$$E_\delta(N)\cong-\frac{(ln\,A*ln\,B)}{E\left(Z_i^2\mid\widehat{\delta}\right)}$$

Using (17) and [10], we get

$$E\left(Z_i^2\mid\widetilde{\delta}\right)=\left\{ln\left(\Gamma\left(\delta_0\right)/\Gamma\left(\delta_1\right)\right)\right\}^2+\left(\delta_1-\delta_0\right)^2\left\{(\psi(\widetilde{\delta}))^2+\xi(2,\widetilde{\delta}-1)\right\}$$

Where $\xi(z,q)$ is specified as

$$\xi(z,q)=\sum_{n=0}^{\infty}\left(\frac{1}{(q+n)^2}\right)$$

Where t_0 is the solution of the equation

$$\int_0^{a^{-1}}\left\{\frac{f\left(x_i;a,\gamma,\delta_1,\underline{\theta}\right)}{f\left(x_i;a,\gamma,\delta_0\underline{\theta}\right)}\right\}^{t_0}f\left(x_i;a,\eta,\delta,\underline{\theta}\right)dx_i=1. \tag{31}$$

We achieve this using (17) and (31),

$$\gamma^{(\delta_1-\delta_0)t_0}\left\{\frac{\Gamma(\delta_0)}{\Gamma(\delta_1)}\right\}^{t_0}\frac{\eta^{\delta}}{\Gamma(\delta)}\int_0^{a^{-1}}\frac{g^{(\delta_1-\delta_0)h+\delta-1}\left(x_i^{-1};\theta\right)g'\left(x_i^{-1}:\theta\right)exp\left(-\eta g\left(x_i^{-1}:\underline{\theta}\right)\right)dx_i}{x_i^2}=1$$

or,

$$\phi_2{}^{(\delta_1-\delta_0)t_0}\left\{\frac{\Gamma(\delta_0)}{\Gamma(\delta_1)}\right\}^{t_0}\frac{\Gamma\left((\delta_1-\delta_0)\,t_0+\delta\right)}{\Gamma(\delta)}=1 \tag{32}$$

Where $\phi_2=\frac{\gamma}{\eta}$.
By applying the logarithm function to both sides of the equation (32), and employing the approximation (24), the solutions for the variable t_0 are obtained from the following equation,

$$\begin{aligned}\frac{t_0{}^2}{6}\left(\frac{\delta_1-\delta_0}{\delta}\right)^3(\delta+1)-\frac{t_0}{4}\left(\frac{\delta_1-\delta_0}{\delta}\right)^2(2\delta+1)-\left(\delta_0-\frac{1}{2}\right)\ln\delta_0+\left(\delta_1-\frac{1}{2}\right)\ln\delta_1\\ -(\delta_1-\delta_0)\ln\phi_2-\left(1+\ln\delta-\frac{1}{2\delta}\right)(\delta_1-\delta_0)=0\end{aligned} \tag{33}$$

The ASN function coincides with (8),

$$E\left(Z_i\mid\delta\right)=ln\left\{\frac{\Gamma(\delta_0)}{\Gamma(\delta_1)}\right\}+(\delta_1-\delta_0)\,\Gamma(\delta)+(\delta_1-\delta_0)\,ln\,\phi_2. \tag{34}$$

7. SPRT ROBUSTNESS FOR γ WITH INDICATED COEFFICIENT OF VARIATION

If $g(x;\underline{\theta})=\frac{x^2}{2},\delta=\frac{h}{2}(h>0)$ in (1), the values of $\mu=\frac{h}{h-2},\quad$ for $h>2$ and $\sigma^2=\frac{2h^2}{(h-2)^2(h-4)},\quad$ for $h>4$. Then, the coefficient of variation (CV)

$$C=\sqrt{\frac{2}{(h-4)}} \tag{35}$$

Assume that the value of the coefficient of variation alters from to c to c^*, then δ becomes

$$\delta^*=\frac{1}{C^{*2}}+2 \tag{36}$$

The OC function is

$$\psi_1 t_0\ln\left(\frac{\gamma_1}{\gamma_0}\right)=\ln\left[1+t_0\left(\frac{\gamma_1-\gamma_0}{\gamma}\right)\right] \tag{37}$$

Solve (37) as (5) up to the third degree in t_0 and find the roots of t_0 from (39)

$$\left\{\frac{1}{3}\left(\frac{\gamma_1-\gamma_0}{\gamma}\right)^3\right\}t_0^2-\left\{\frac{1}{2}\left(\frac{\gamma_1-\gamma_0}{\gamma}\right)^2\right\}t_0+\left\{\left(\frac{\gamma_1-\gamma_0}{\gamma}\right)-\psi_1\ln\left(\frac{\gamma_1}{\gamma_0}\right)\right\}=0 \tag{38}$$

where $\psi_1=\left(\frac{\delta}{\delta^*}\right)$.
The ASN function coincides with (8)

$$E\left(Z_i\mid\gamma\right)=\psi_1\left[ln\left(\frac{\gamma_1}{\gamma_0}\right)-\left(\frac{\gamma_1-\gamma_0}{\gamma}\right)\right] \tag{39}$$

8. ACCEPTANCE AND REJECTION REGION

we need to assess the simple hypotheses $H_0 : \gamma = \gamma_0$ as opposed to $H_1 : \gamma = \gamma_1\,(> \gamma_0)$ having preassigned $0 < \alpha$ and $\beta < 1$ then Z_i is

$$Z_i = \delta.ln\left(\frac{\gamma_1}{\gamma_0}\right) - g\left(x_i^{-1};\underline{\theta}\right)(\gamma_1 - \gamma_0) \tag{40}$$

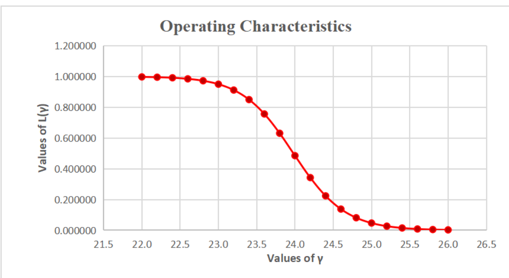
Define, $Z(N) = \sum_{i=1}^{n} X_i$ and $N =$ initial integer $n(\geq 1)$, so that the inequality is defined as $Z(N) \leq c_1 + dn$ or $Z(N) \geq c_2 + dn$ valid among the parameters.

$$c_1 = \frac{\ln B}{(\gamma_1 - \gamma_0)}, c_2 = \frac{\ln A}{(\gamma_1 - \gamma_0)} \text{ and } d = \frac{\delta \ln\left(\frac{\gamma_0}{\gamma_1}\right)}{(\gamma_1 - \gamma_0)}$$

9. RESULT AND DISCUSSION

Table 1: $H_0 : \gamma_0 = 22, H_1 : \gamma_1 = 26 \quad H_0 : \delta_0 = 22, H_1 : \delta_1 = 26$

γ	$L(\gamma)$	$E[N]$	δ	$L(\delta)$	$E[N]$
22.0	0.997848	396.3	22.0	0.997500	16.82
22.2	0.995846	442.9	22.2	0.995382	18.70
22.4	0.992101	499.5	22.4	0.991517	20.98
22.6	0.985191	568.6	22.6	0.984524	23.77
22.8	0.972657	653.2	22.8	0.972019	27.16
23.0	0.950427	755.6	23.0	0.950054	31.27
23.2	0.912296	875.9	23.2	0.912590	36.08
23.4	0.850178	1008.3	23.4	0.851663	41.38
23.6	0.756664	1136.4	23.6	0.759744	46.52
23.8	0.631008	1232.9	23.8	0.635534	50.40
24.0	0.485370	1268.9	24.0	0.490420	51.83
24.2	0.342685	1233.2	24.2	0.347054	50.32
24.4	0.224024	1140.9	24.4	0.227029	46.47
24.6	0.138008	1021.1	24.6	0.139693	41.46
24.8	0.081636	898.8	24.8	0.082409	36.35
25.0	0.047077	788.0	25.0	0.047344	31.74
25.2	0.026743	693.5	25.2	0.026777	27.81
25.4	0.015062	615.1	25.4	0.015011	24.55
25.6	0.008442	550.6	25.6	0.008374	21.88
25.8	0.004719	497.5	25.8	0.004660	19.69
26.0	0.002634	453.5	26.0	0.002590	17.87



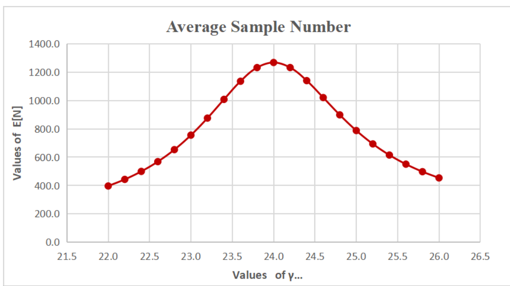


Figure 1: *OC and ASN Curve for section 3.*

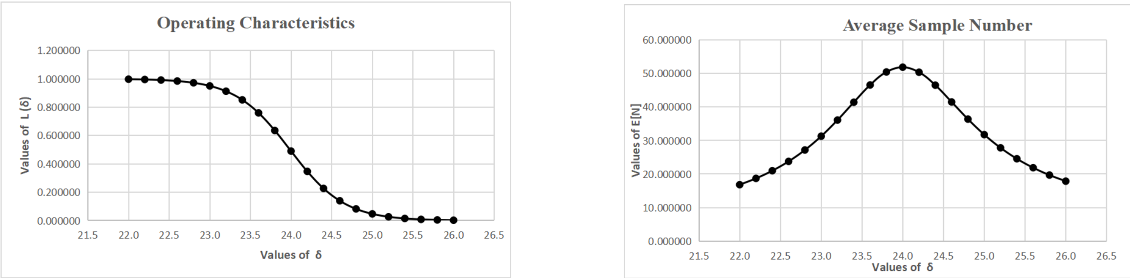


Figure 2: *OC and ASN Curve for section 5.*

I. The values denoted by the OC and ASN functions for sections 3 and 5 under $\alpha = \beta = 0.05$, corresponding to the parameters γ and δ can be found in Table 1, while the visuals representing these values are illustrated in Figures 1 and 2. The table mentioned above and curves yield outcomes that are deemed acceptable.

Table 2: *OC and ASN Functions for section 4, under $\alpha = \beta = 0.05$, where $H_0 : \gamma_0 = 22, H_1 : \gamma_1 = 26$*

	$\phi_1 = 0.95$		$\phi_1 = 0.98$		$\phi_1 = 1$		$\phi_1 = 1.02$		$\phi_1 = 1.05$	
γ	$L(\gamma)$	$E[N]$	$L(\gamma)$	$E[N]$	$L(\gamma)$	$E[N]$	$L(\gamma)$	$E[N]$	$L(\gamma)$	$E[N]$
22.0	0.999977	256.109	0.999593	325.533	0.997848	396.275	0.990174	501.269	0.925563	760.351
22.2	0.999949	275.377	0.999182	357.072	0.995846	442.881	0.981647	572.664	0.871228	878.478
22.4	0.999891	297.332	0.998388	394.382	0.992101	499.451	0.966278	659.846	0.787253	998.243
22.6	0.999773	322.565	0.996875	439.007	0.985191	568.593	0.939285	764.602	0.670219	1097.251
22.8	0.999539	351.837	0.994031	492.942	0.972657	653.184	0.893689	885.460	0.528271	1148.101
23.0	0.999084	386.143	0.988758	558.689	0.950427	755.616	0.821178	1013.745	0.382212	1134.421
23.2	0.998208	426.784	0.979124	639.166	0.912296	875.930	0.715853	1129.741	0.255118	1063.405
23.4	0.996548	475.461	0.961857	737.241	0.850178	1008.278	0.581040	1204.816	0.159600	959.524
23.6	0.993440	534.365	0.931752	854.390	0.756664	1136.436	0.433599	1215.202	0.095382	847.741
23.8	0.987698	606.223	0.881433	987.837	0.631008	1232.898	0.297492	1159.170	0.055358	743.688
24.0	0.977243	694.173	0.802653	1126.039	0.485370	1268.878	0.190055	1058.028	0.031567	653.709
24.2	0.958580	801.169	0.690763	1245.161	0.342685	1233.182	0.115203	939.908	0.017819	578.588
24.4	0.926211	928.388	0.551685	1313.657	0.224024	1140.910	0.067448	825.361	0.010001	516.709
24.6	0.872518	1071.916	0.404611	1309.361	0.138008	1021.076	0.038658	724.251	0.005595	465.802
24.8	0.789395	1217.829	0.273264	1236.136	0.081636	898.796	0.021884	639.054	0.003124	423.701
25.0	0.673200	1339.075	0.172445	1120.253	0.047077	788.025	0.012302	568.673	0.001742	388.586
25.2	0.531691	1402.143	0.103641	991.440	0.026743	693.530	0.006888	510.809	0.000971	359.005
25.4	0.385428	1386.471	0.060349	869.573	0.015062	615.103	0.003848	463.075	0.000540	333.831
25.6	0.257639	1300.211	0.034475	763.352	0.008442	550.600	0.002147	423.395	0.000301	312.195
25.8	0.161310	1173.099	0.019477	674.378	0.004719	497.491	0.001197	390.091	0.000167	293.426
26.0	0.096429	1035.896	0.010936	601.031	0.002634	453.477	0.000666	361.853	0.000093	277.004

II. Figure 3 illustrates the numerical values of the OC and ASN curves extracted from Table 2, corresponding to different ϕ_1 values. When $\phi_1 < 1 (\phi_1 > 1)$, the OC curve shifts either towards the right or left direction, while the ASN curve shifts towards the upper right or lower left direction. Both curves demonstrate that the SPRT exhibits a high degree of sensitivity towards alterations in δ.

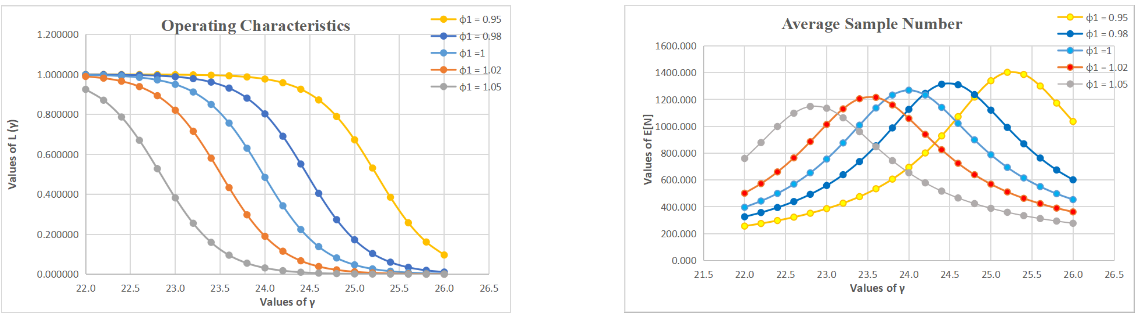


Figure 3: *OC and ASN Curve for section 4.*

Table 3: *OC and ASN Functions for section 6, under* $\alpha = \beta = 0.05$ *where* $H_0 : H_0 : \delta_0 = 22, H_1 : \delta_1 = 26$

	$\phi_2 = 0.95$		$\phi_2 = 0.99$		$\phi_2 = 1$		$\phi_2 = 1.02$		$\phi_2 = 1.05$	
δ	$L(\delta)$	$E[N]$	$L(\delta)$	$E[N]$	$L(\delta)$	$E[N]$	$L(\delta)$	$E[N]$	$L(\delta)$	$E[N]$
22.0	0.999949	12.805	0.998800	16.277	0.997500	16.821	0.989824	25.063	0.930141	38.018
22.2	0.999903	13.769	0.997769	17.854	0.995382	18.704	0.981471	28.633	0.879707	43.924
22.4	0.999816	14.867	0.995875	19.719	0.991517	20.984	0.966597	32.992	0.800964	49.912
22.6	0.999653	16.128	0.992418	21.950	0.984524	23.765	0.940663	38.230	0.689215	54.863
22.8	0.999350	17.592	0.986152	24.647	0.972019	27.161	0.896956	44.273	0.550252	57.405
23.0	0.998787	19.307	0.974920	27.934	0.950054	31.265	0.827248	50.687	0.403178	56.721
23.2	0.997751	21.339	0.955114	31.958	0.912590	36.081	0.725144	56.487	0.271799	53.170
23.4	0.995848	23.773	0.921109	36.862	0.851663	41.381	0.592699	60.241	0.171045	47.976
23.6	0.992379	26.718	0.865216	42.719	0.759744	46.521	0.445466	60.760	0.102434	42.387
23.8	0.986097	30.311	0.779487	49.392	0.635534	50.398	0.307359	57.959	0.059402	37.184
24.0	0.974842	34.709	0.660892	56.302	0.490420	51.829	0.196953	52.901	0.033783	32.685
24.2	0.955006	40.058	0.518186	62.258	0.347054	50.320	0.119421	46.995	0.018999	28.929
24.4	0.920963	46.419	0.372612	65.683	0.227029	46.466	0.069790	41.268	0.010619	25.835
24.6	0.865024	53.596	0.247071	65.468	0.139693	41.460	0.039871	36.213	0.005915	23.290
24.8	0.779245	60.891	0.153542	61.807	0.082409	36.355	0.022479	31.953	0.003290	21.185
25.0	0.660606	66.954	0.091170	56.013	0.047344	31.737	0.012580	28.434	0.001828	19.429
25.2	0.517881	70.107	0.052585	49.572	0.026777	27.807	0.007012	25.540	0.001015	17.950
25.4	0.372321	69.324	0.029808	43.479	0.015011	24.553	0.003901	23.154	0.000564	16.692
25.6	0.246826	65.011	0.016731	38.168	0.008374	21.882	0.002167	21.170	0.000313	15.610
25.8	0.153354	58.655	0.009340	33.719	0.004660	19.688	0.001204	19.505	0.000174	14.671
26.0	0.091038	51.795	0.005199	30.052	0.002590	17.872	0.000669	18.093	0.000097	13.850

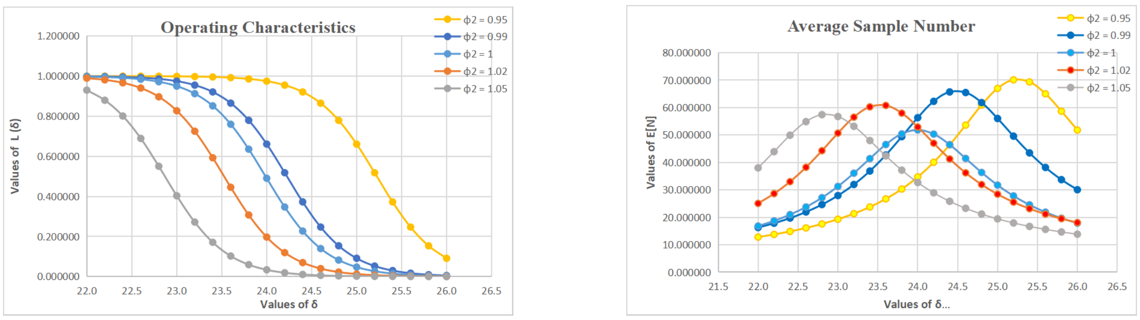


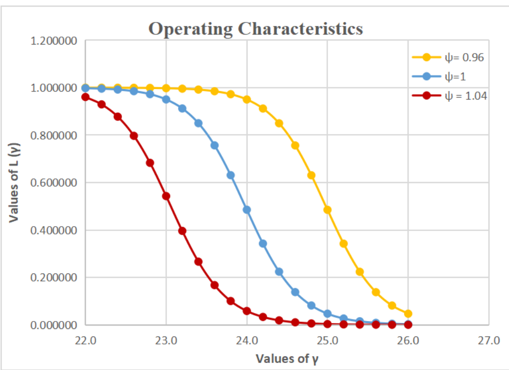
Figure 4: *OC and ASN Curve for section 6.*

III. Figure 4 portrays the values of the operational characteristic (OC) and average sample num-

ber (ASN) curves derived from Table 3 across various magnitudes of ϕ_2. When $\phi_2 < 1(\phi_2 > 1)$, the OC curve experiences a rightward (leftward) shift, while the ASN curve undergoes an upward rightward (downward leftward) shift. Both curves demonstrate the considerable sensitivity of the sequential probability ratio test (SPRT) to parameter γ alterations.
IV. Figure 5 illustrates the plotted values of the OC and ASN curves obtained from Table 4 while considering different values of $'\psi'$. When $\psi < 1(\psi > 1)$ is taken into account, the OC curve experiences a shift towards the right (left), while the ASN curve shifts upwards (downwards) towards the right. It is evident from both curves that the SPRT demonstrates a considerable level of sensitivity towards variations in $'\psi'$.

Table 4: *OC and ASN Functions for section 7, under $\alpha = \beta = 0.05$ where $H_0 : H_0 : \gamma_0 = 22, H_1 : \gamma_1 = 26$*

	$\psi = 0.96$		$\psi = 1$		$\psi = 1.04$	
γ	$L(\gamma)$	$E[N]$	$L(\gamma)$	$E[N]$	$L(\gamma)$	$E[N]$
22.0	0.999936	275.746	0.997848	396.275	0.960659	658.849
22.2	0.999864	298.188	0.995846	442.881	0.929568	764.064
22.4	0.999719	324.055	0.992101	499.451	0.877735	883.456
22.6	0.999432	354.157	0.985191	568.593	0.796964	1006.162
22.8	0.998873	389.555	0.972657	653.184	0.683060	1110.363
23.0	0.997803	431.636	0.950427	755.616	0.542805	1168.026
23.2	0.995782	482.209	0.912296	875.930	0.396046	1160.562
23.4	0.992010	543.586	0.850178	1008.278	0.266293	1092.897
23.6	0.985068	618.594	0.756664	1136.436	0.167506	988.999
23.8	0.972493	710.346	0.631008	1232.898	0.100472	874.913
24.0	0.950216	821.423	0.485370	1268.878	0.058441	767.625
24.2	0.912031	951.852	0.342685	1233.182	0.033369	674.376
24.4	0.849857	1095.264	0.224024	1140.910	0.018849	596.352
24.6	0.756289	1234.025	0.138008	1021.076	0.010583	532.042
24.8	0.630591	1338.272	0.081636	898.796	0.005922	479.150
25.0	0.484941	1376.797	0.047077	788.025	0.003307	435.440
25.2	0.342284	1337.532	0.026743	693.530	0.001844	399.016
25.4	0.223688	1236.956	0.015062	615.103	0.001028	368.364
25.6	0.137752	1106.603	0.008442	550.600	0.000572	342.305
25.8	0.081457	973.726	0.004719	497.491	0.000318	319.929
26.0	0.046959	853.432	0.002634	453.477	0.000177	300.536



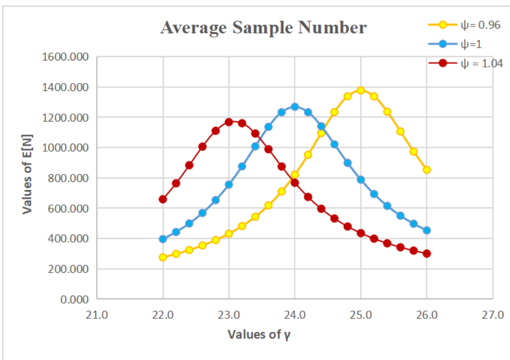


Figure 5: *OC and ASN Curve for section 7.*

V. The acceptance and rejection zones for the null hypothesis H_0, with $H_0 : \gamma_0 = 22$ and the alternative hypothesis $H_0 : \gamma_0 = 26$. Both the α and β significance levels are set to 0.05, and the degrees of freedom δ are set to 2. The values of the constants c_1, c_2, and d are -287.0828, 287.0828, and -27.90466, respectively. As a result, if the observed value $Z(n)$ is less than or equal to $-27.90466N + 287.0828$, we accept the null hypothesis H_0, and we accept the alternative hypothesis H_1 if $Z(n)$ is higher than or equal to $-27.90466N - 287.0828$. In the intermediate stages, the sampling procedure continues.

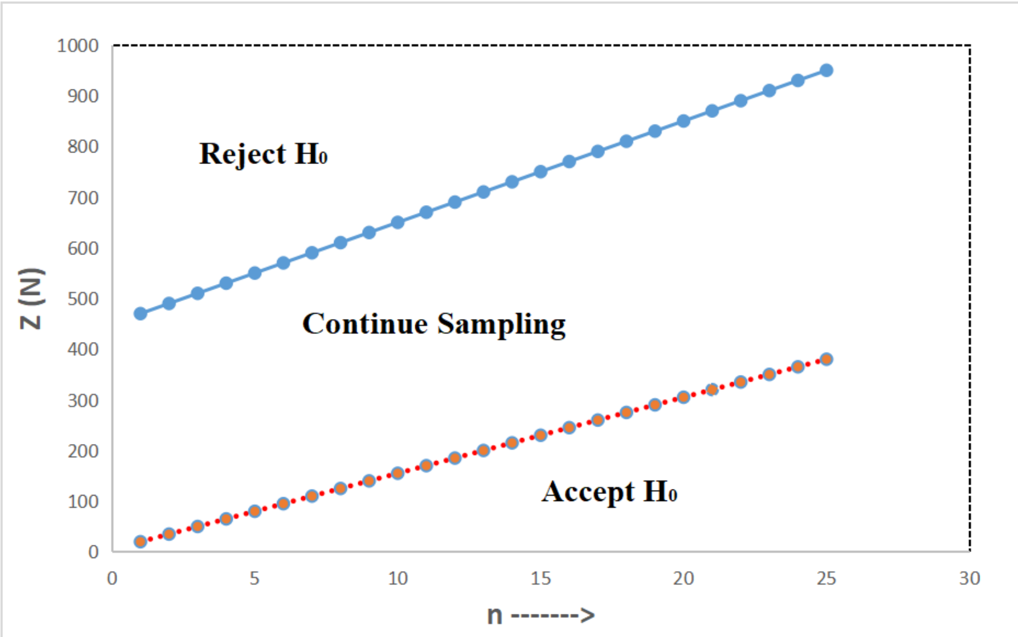


Figure 6: *The Acceptance and Rejection zones for H_0*

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