A 'One Parameter' Bathtub Shaped Failure Rate Distribution

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Abstract

Most real life system exhibit bathtub shapes for their failure rate functions. Generalized Lindley, Generalized Gamma, Exponentiated Weibull and x-Exponential distributions are proposed for modeling lifetime data having bathtub shaped failure rate model. This paper considered a simple model but exhibiting bathtub shaped failure rate and discuss the failure rate behavior. The proposed distribution has only one parameter. A Little works are available in literature with one parameter. Computation of moments requires software. Applications in reliability study is discussed.

Keywords: Bathtub failure rate, Reliability

I. Introduction

There are many distributions for modeling lifetime data. Among the known parametric models, the most popular are the Lindley, Gamma, log-Normal, Exponentiated Exponential and the Weibull distributions. These five distributions are suffer from a number of drawbacks. None of them exhibit bathtub shape for their failure rate functions. Most real life system exhibit bathtub shapes for their failure rate functions. Generalized Lindley (GL), Generalized Gamma (GG), Exponentiated Weibull (EW) and x-Exponential distributions are proposed for modeling lifetime data having bathtub shaped failue rate model. In this paper we consider a simple model but exhibiting bathtub shaped failure rate and discuss the failure rate behavior of the distribution. The inference procedure also become simple than GL, GG and EW distributions.

Section II, discussed new distribution and their properties, Maximum likelihood estimator is obtained in section III. Conclusions are given at the final section.

II. New Bathtub shaped failure rate model

In this section we consider failure rate function

$$h(x) = \frac{1 + ax + x^2}{1 + x + x^2}\ , x > 0, a > 0.$$

a is considered to be arbitrary

$$\int \frac{1+ax+x^2}{1+ax+x^2}dx = \int (\frac{(a-1)x}{1+x+x^2}+1)dx$$
$$= (a-1)\int \frac{x}{1+x+x^2}dx + \int 1dx$$
$$= (a-1)\int \frac{2x+1}{2(1+x+x^2)}dx - \int \frac{1}{2(1+x+x^2)}dx + \int 1dx$$
$$= \frac{(a-1)}{2}\int \frac{1}{w}dw - \frac{2}{2\sqrt{3}}\int \frac{1}{(1+u^2)}du + \int 1dx,$$

by substituting $w = \frac{1}{1+x+x^2}$, $u = \frac{1+2x}{\sqrt{3}}$.

$$= \frac{(a-1)}{2} ln(1+x+x^2) - (a-1)\frac{1}{\sqrt{3}} arctan\frac{(1+2x)}{\sqrt{3}} + x$$

Here, we consider a simplified form of distribution function,

$$F(x) = 1 - e^{-(x+(a-1)(\frac{\log(1+x+x2)}{2}-\frac{\arctan((1+2x)/\sqrt{3})}{\sqrt{3}})}, x > 0, -\infty < a < \infty \quad (1)$$

It is an alternative model GL, GG, EW distributions. Clearly F(0)=0, F(∞) = 1, F is non-decreasing and right continuous. More over F is absolutely continuous. The probability density function (pdf) is given by

$$f(x) = \frac{1+ax+x^2}{1+x+x^2}\, e^{-(x+(a-1)(\frac{\log(1+x+x2)}{2}-\frac{\arctan((1+2x)/\sqrt{3})}{\sqrt{3}})}, x > 0,\ -\infty < a < \infty.$$

It is positively skewed distribution.

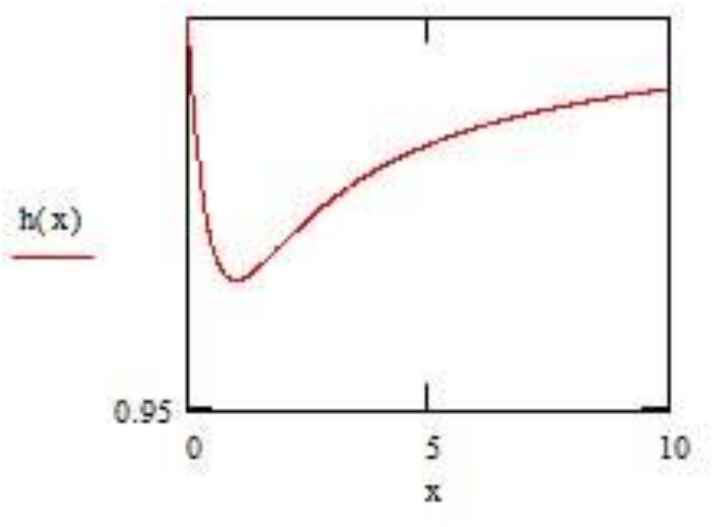

Figure 1. Failure rate function for a=0.9

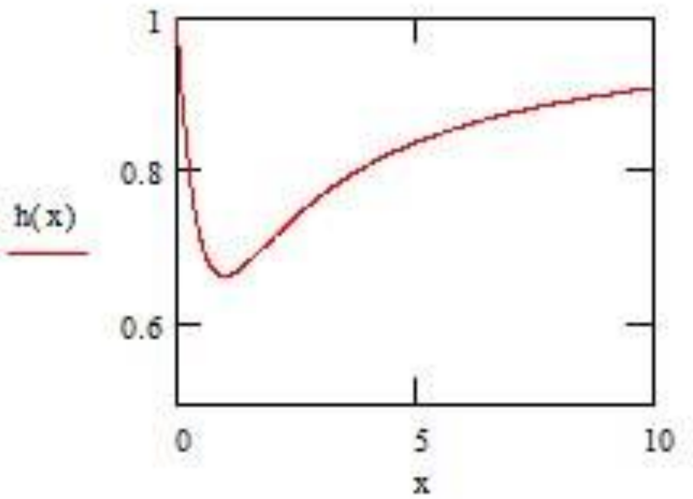

Figure 2. Failure rate function for a=0.001

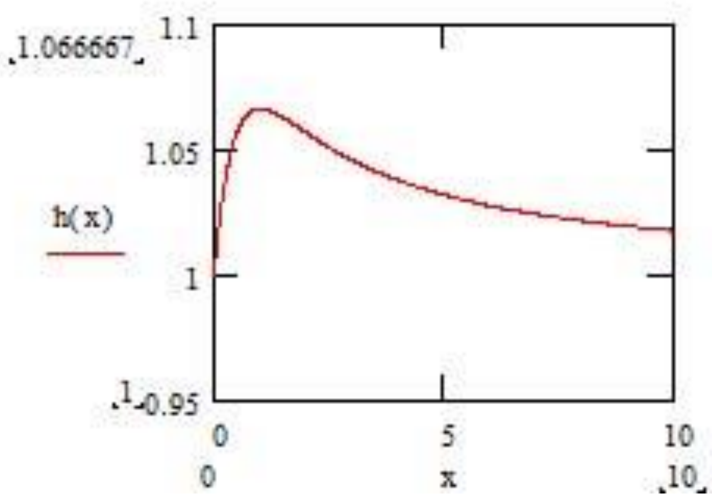

Figure 3. Failure rate function for a=0.001

From Figure 1 and 2, the shape of the hazard rate function appears monotonically decreasing or to initially decrease and then increase, a bathtub shape. The proposed distribution allows only bathtub shapes for its hazard rate function. Fig. 3 shows Upside down bathtub shape in its failure rate function, for a=1.2. When a=1, it becomes constant failure rate model.

Proposition 1: The proposed distribution is a generalization of Exponential distribution.
When a=1, it becomes exponential distribution $f(x) = e^{-x}, x > 0.$

A generalization to Two parameter distribution

Here, we consider a simplified form of distribution function,

$$F(x) = 1 - e^{-\lambda(x+(a-1)(\frac{\log(1+x+x2)}{2} - \frac{\arctan((1+2x)/\sqrt{3})}{\sqrt{3}})}, x > 0, \lambda > 0 - \infty < a < \infty \quad (1)$$

It is an alternative model GL, GG, EW distributions. Clearly F(0)=0, F(∞) = 1, F is non-decreasing and right continuous. More over F is absolutely continuous. The probability density function (pdf) with scale parameter λ is given by

$$f(x) = \frac{\lambda(1+ax+x^2)}{1+x+x^2}\; e^{-\lambda(x+(a-1)(\frac{\log(1+x+x2)}{2}-\frac{\arctan((1+2x)/\sqrt{3})}{\sqrt{3}})}, x > 0,\; \lambda > 0, -\infty < a < \infty.$$

It is positively skewed distribution.

The failure rate function is

$$h(x) = \frac{\lambda(1 + ax + x^2)}{1 + x + x^2}\;, x > 0\;, \lambda > 0.$$

Moments

All the raw and central moments, moment generating functions etc exist, since the function f(x) is having countable number of discontinuities, and integrable but the resulting function require more mathematical treatment. It can be done by softwares like Matlab. It left to reader.

Estimation

Here, we consider estimation by the method maximum likelihood.

$$L(a, x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} f(x_i)$$

$$L(a, x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} \frac{1 + ax_i + x_i^2}{1 + x_i + x_i^2}\; e^{-(x_i+(a-1)(\frac{\log(1+x_i+x_i^2)}{2}-\frac{\arctan((1+2x_i)/\sqrt{3})}{\sqrt{3}})}$$

$$logL = \sum_{i=1}^{n}[\log(1 + ax_i + x_i^2) - \log(1 + x_i + x_i^2)]$$

$$- \sum_{i=1}^{n}[x_i + (a - 1)(\frac{\log(1 + x_i + x_i^2)}{2} - \frac{arctan\;(\frac{1 + 2x_i}{\sqrt{3}})}{\sqrt{3}})]$$

$$= \sum_{i=1}^{n} \log(1 + ax_i + x_i^2) - \sum_{i=1}^{n} \log(1 + x_i + x_i^2) -$$

$$-\sum_{i=1}^{n} x_i + (a - 1)\sum_{i=1}^{n}(\frac{\log(1 + x_i + x_i^2)}{2} - \frac{arctan\;(\frac{1 + 2x_i}{\sqrt{3}})}{\sqrt{3}})]$$

$$\frac{\partial}{\partial a}\; log\; L\; = 0\; \Rightarrow \sum_{i=1}^{n} \frac{x_i}{(1 + ax_i + x_i^2)} - \sum_{i=1}^{n}\left[\frac{\log(1 + x_i + x_i^2)}{2} - \frac{arctan\;\left(\frac{1 + 2x_i}{\sqrt{3}}\right)}{\sqrt{3}}\right] = 0$$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{(n + a\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i^2)} = \sum_{i=1}^{n}\left[\frac{\log(1 + x_i + x_i^2)}{2} - \frac{arctan\;\left(\frac{1 + 2x_i}{\sqrt{3}}\right)}{\sqrt{3}}\right]$$

$$\Longrightarrow \frac{(n + a\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i^2)}{\sum_{i=1}^{n} x_i} = \frac{1}{\sum_{i=1}^{n}\left[\frac{\log(1 + x_i + x_i^2)}{2} - \frac{tan^{-1}\left(\frac{1 + 2x_i}{\sqrt{3}}\right)}{\sqrt{3}}\right]}$$

$$\Rightarrow \left(n + a\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i^2\right) = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n}\left[\frac{\log(1+x_i+x_i^2)}{2} - \frac{tan^{-1}\left(\frac{1+2x_i}{\sqrt{3}}\right)}{\sqrt{3}}\right]}$$

$$\Rightarrow a\sum_{i=1}^{n} x_i = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n}\left[\frac{\log(1+x_i+x_i^2)}{2} - \frac{tan^{-1}\left(\frac{1+2x_i}{\sqrt{3}}\right)}{\sqrt{3}}\right]} - n - \sum_{i=1}^{n} x_i^2$$

$$\Rightarrow \hat{a} = \frac{1}{\sum_{i=1}^{n}\left[\frac{\log(1+x_i+x_i^2)}{2} - \frac{tan^{-1}\left(\frac{1+2x_i}{\sqrt{3}}\right)}{\sqrt{3}}\right]} - \frac{(n - \sum_{i=1}^{n} x_i^2)}{\sum_{i=1}^{n} x_i}$$

Thus we obtained Maximum likelihood estimator for the parameter.

III. Applications and Conclusions

Identifying the failure rate model is crucial to the maintenance and replacement policies. The optimal burn in time can be computed for the Bathtub shaped failure rate models. The model suggested here provide Bathtub shaped failure rate distributions which is more flexible and simple than many existing distributions, in the sense of estimation. We considered Arset data [5] parameter is a=0.813225

Table 1. Aarset Data

0.1 0.2 1 1 1 1 1 2 3 6 7 11 12 18 18 18 18 18 21 32 36 40 45 46 47 50 55 60 63 63 67 67 67 67 72 75 79 82 82 83 84 84 84 85 85 85 85 85 86 86

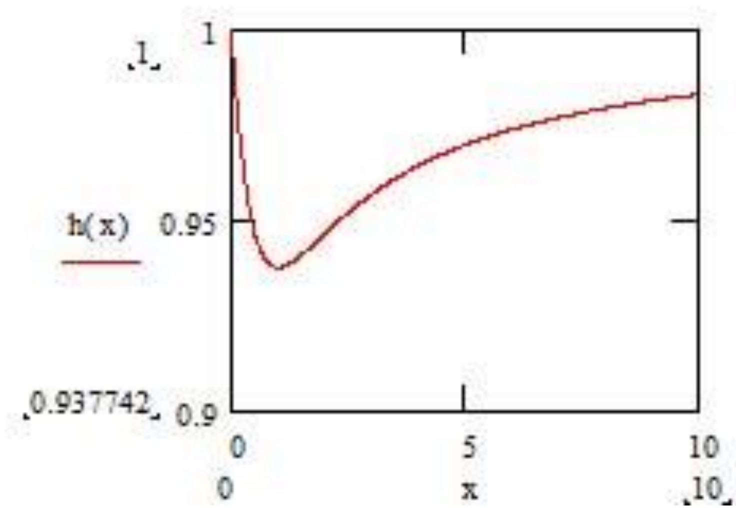

Figure 4. Failure rate function a=0.813225

We obtained bathtub shaped curve for the Aarset data as in figure 4.

IV. Discussion

There are many distributions in reliability which exhibit Bathtub shaped failure rate model, but most of them are complicated in finding estimators. The complication in using GL,GG,GE distributions is reduced in the proposed model. Any way the problem of computing Moments, characteristic functions etc still remains.

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