# Weibull-Lindley Distribution: A Bathtub Shaped Failure Rate Model

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**Abstract**

The Lindley and Weibull are the two most commonly used distributions for analyzing lifetime data. These distributions have several desirable properties and nice physical interpretations. This paper introduces a new distribution, which generalizes the well-known Lindley and Weibull distribution, having Bathtub shaped failure rate. The Statistical properties of this distribution are discussed in this paper. Applications in reliability study are discussed. A real data set is analyzed and it is observed that the present distribution can provide a better than some other very well known distributions.

**Keywords:** Reliability, Bathtub shaped failure rate, Weibull distribution, Lindley distribution.

## I. Introduction

In order to apply suitable maintenance activities to a system or to apply reliability improvement procedures, one should know the dynamic behaviors of system reliability [2]. Increasing, decreasing and Bathtub curves are usually adopted to represent the failure rate of the system. Many statistical distributions are proposed in literature to model the Bathtub behavior of failure rate. The problem of getting optimal burn in time for the industrial burn in process is the major concern of industrial engineers. The failure rate of some engineering systems over time follows what is called the "bathtub" curve. There is a high rate of infant mortality initial failures. Then the failure rate drops, only to increase at the end of life due to wear out failures [3]. The reliability of a part can be enhanced by providing a burn-in at elevated temperatures prior to usage. This burn-in is typically done at pre specified time. It is also good to monitor the part performance during burning, so that the time point of failures can be detected. That data can be used to set the optimum burn-in length. A continuous distribution with a bathtub-shaped failure rate function with desirable characteristics is quite appropriate in this context, [9,7].

In analyzing lifetime data one often uses the Exponential, Generalized Lindley and Weibull distributions. It is well known that Exponential can have only constant hazard function, Generalized Lindley has a bathtub shape hazard function whereas Weibull can have constant or monotone (increasing/decreasing) hazard functions. Unfortunately, in practice often one needs to consider non-monotonic function such as bathtub shaped hazard function also. In this paper we present a new simple distribution which may have bathtub shaped hazard function, with high initial failure rate, which decreases rapidly and then slowly increases.

In this paper, we propose a new distribution whose failure rate function has monotone (increasing/decreasing) or bathtub shape. Section II discussed the definition of the Weibull-Lindley distribution (WLD). Section III discussed the statistical behaviours of the distribution. Section IV discussed the distribution of maximum and minimum. The maximum likelihood estimation of the parameters determined in section V. Section VI discussed three parameter Weibull-Lindley distribution (3WLD) and real data sets are analyzed in Section VII and the results are compared with existing distributions. Conclusions are given in Section VIII.

## II. The Weibull-Lindley Distribution

Let $X$ be a random variable with the following cumulative distribution function (CDF) for $\alpha, \beta, \lambda > 0$ as follows;

$$F(x;\alpha,\beta,\lambda) = 1 - e^{-\alpha\left((1+\lambda x)e^{(\lambda x)^{\beta}} - 1\right)}, x > 0 \tag{2.1}$$

Assume λ=1 and $\alpha, \beta > 0$. Then, the probability density function (PDF) corresponding to Eq. (2.1) is given by

$$f(x;\alpha,\beta) = \alpha\left(\beta x^{\beta-1}(1+x)e^{x^{\beta}} + e^{x^{\beta}}\right)e^{-\alpha\left((1+x)e^{x^{\beta}} - 1\right)}, x > 0, \alpha, \beta > 0 \tag{2.2}$$

Here $\beta$ is shape parameter. The distribution with PDF of form (2.2) is said to be Weibull-Lindley distribution with parameters $\alpha, \beta$ and will be denoted by $WLD(\alpha,\beta)$.

$$f(x;\alpha,\beta) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+j}\alpha^{i+1}\beta(i-j+1)}{j!k!(i-j-k+1)!} e^{x^{\beta(i-j+1)}} x^{\beta+k-1} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{i+j}\alpha^{i+1}}{j!m!(i-j-m)!} e^{x^{\beta(i-j+1)}} x^{m} \tag{2.3}$$

Figure 1 provide the PDFs of $WLD(\alpha,\beta)$ for different parameter values. From the below figures it is immediate that the PDFs are unimodal.

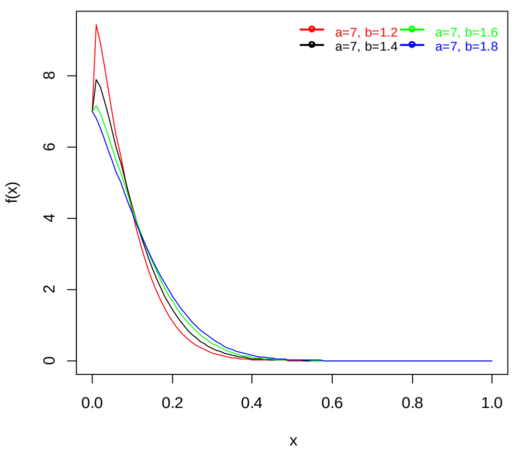


Figure 1: Probability density function of the $WLD(\alpha,\beta)$.

The survival function $S(x)$, reversed failure rate function $r(x)$ and cumulative failure rate function $H(x)$ of $X$ are

$$S(x;\alpha,\beta) = 1 - F(x;\alpha,\beta) = e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}, x > 0 \quad (2.4)$$

$$r(x;\alpha,\beta) = \frac{\alpha e^{x^\beta}\left(1+\beta x^{\beta-1}(1+x)\right)e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}}{1-e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}}, x > 0 \quad (2.5)$$

and

$$H(x;\alpha,\beta) = \int_0^x h(t;\alpha,\beta)dt = \alpha(1+x)e^{t^\beta} \quad (2.6)$$

As a result, the hazard rate function of the WL distribution can exhibit monotonically increasing, monotonically decreasing and bathtub shapes. We can see from that

$$\lim_{x\to 0} h(x) = \begin{cases} \infty, & \beta < 1 \\ 2\alpha, & \beta = 1 \\ \alpha, & \beta > 1 \end{cases}$$

Figure 2 provide the failure rate functions of $WLD(\alpha,\beta)$ for different parameter values. From the below figures it is immediate that the failure rate function can be increasing, decreasing or bathtub shaped.

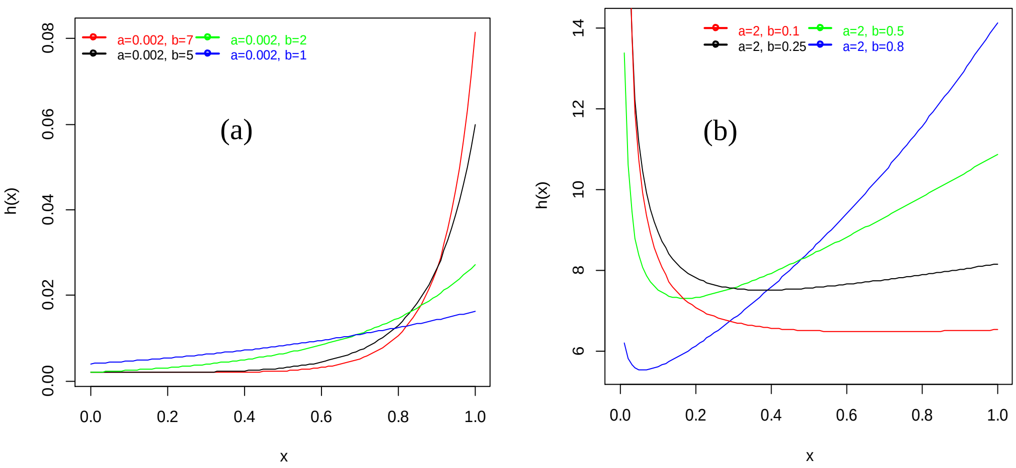


Figure 2: Failure rate function of the $WLD(\alpha,\beta)$.

It is clear that the PDF and the failure rate function have many different shapes, which allows this distribution to fit different types of lifetime data. For fixed $\alpha$, the failure rate function is (a) non-decreasing function if $\beta > 1$, and (b) non-increasing and bathtub function if $\beta < 1$.

## III. Statistical Properties

In this section, we study the statistical properties for the Weibull-Lindley distribution, specially Quantile function and Median, Mode, Moments etc

**Quantile and median:** We obtain the 100 $p^{th}$ percentile,

$$(1+x)e^{x^\beta} = -\frac{1}{\alpha}\log(1-p)+1 \quad (3.1)$$

Setting $p = 0.5$ in Eq. (3.1), we get the median of WLD as follows.

$$(1+x)e^{x^\beta} = \frac{1}{\alpha}\log\left(\frac{1}{1-0.5}\right)+1$$

$x_p$ is the solution of above monotone increasing function. Software can be used to obtain the Quantiles/Percentiles

**Mode:** Mode can be obtained as solution of

$$\frac{\partial}{\partial x}\left(\alpha\left(\beta x^{\beta-1}(1+x)e^{x^\beta}+e^{x^\beta}\right)e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}\right)=0$$

$$\frac{\partial}{\partial x}\left(h(x;\alpha,\beta)\cdot S(x;\alpha,\beta)\right)=0$$

$$h^{'}(x;\alpha,\beta)\cdot S(x;\alpha,\beta)+h(x;\alpha,\beta)\cdot S^{'}(x;\alpha,\beta)=0$$

Then
$$[h^{'}(x;\alpha,\beta)-\left(h(x;\alpha,\beta)\right)^2]\cdot S(x;\alpha,\beta)=0 \qquad (3.2)$$

It is not possible to get an analytic solution in $x$ to Eq. (3.3) in the general case. It has to be obtained numerically by using methods such as fixed-point or bisection method.

**Moments:** If $X$ has WLD, we obtain the r[th] moment of WLD in the form

$$\mu_r^{'}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+\frac{\beta+r+k}{\beta(i-j+1)}}\alpha^{i+1}}{j!k!(i-j-k+1)!}\Gamma\left(\frac{\beta+r+k}{\beta(i-j+1)}\right)+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j+\frac{m+r+1}{\beta(i-j+1)}}\alpha^{i+1}}{j!m!(i-j-m)!}\frac{\Gamma\left(\frac{m+r+1}{\beta(i-j+1)}\right)}{\beta(i-j+1)} \qquad (3.3)$$

If (3.3) is a convergent series for any $r \geq 0$, therefore all the moments exist and for integer values of $\alpha$ and $\beta$ (3.4) can be represented as a finite series representation. Therefore putting $r=1$, we obtain the mean as

$$E(X)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+\frac{\beta+k+1}{\beta(i-j+1)}}\alpha^{i+1}}{j!k!(i-j-k+1)!}\Gamma\left(\frac{\beta+k+1}{\beta(i-j+1)}\right)+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j+\frac{m+2}{\beta(i-j+1)}}\alpha^{i+1}}{j!m!(i-j-m)!}\frac{\Gamma\left(\frac{m+2}{\beta(i-j+1)}\right)}{\beta(i-j+1)}$$

and putting $r=2$, we obtain the second moment as

$$E(X^2)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+\frac{\beta+k+2}{\beta(i-j+1)}}\alpha^{i+1}}{j!k!(i-j-k+1)!}\Gamma\left(\frac{\beta+k+2}{\beta(i-j+1)}\right)+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j+\frac{m+3}{\beta(i-j+1)}}\alpha^{i+1}}{j!m!(i-j-m)!}\frac{\Gamma\left(\frac{m+3}{\beta(i-j+1)}\right)}{\beta(i-j+1)}$$

which in turn can be used to obtained the higher central moments and variance.

**Moment Generating Function and Characteristic function**

The moment generating function, $M_X(t)$, is

$$M_X(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{(-1)^{i+j+\frac{\beta+r+k}{\beta(i-j+1)}}\alpha^{i+1}}{j!k!(i-j-k+1)!}\Gamma\left(\frac{\beta+r+k}{\beta(i-j+1)}\right)+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{(-1)^{i+j+\frac{m+r+1}{\beta(i-j+1)}}\alpha^{i+1}}{j!m!(i-j-m)!}\frac{\Gamma\left(\frac{m+r+1}{\beta(i-j+1)}\right)}{\beta(i-j+1)}$$

The characteristic function is

$$\phi_X(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\frac{(-1)^{i+j+\frac{\beta+r+k}{\beta(i-j+1)}}\alpha^{i+1}}{j!k!(i-j-k+1)!}\Gamma\left(\frac{\beta+r+k}{\beta(i-j+1)}\right)+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\frac{(-1)^{i+j+\frac{m+r+1}{\beta(i-j+1)}}\alpha^{i+1}}{j!m!(i-j-m)!}\frac{\Gamma\left(\frac{m+r+1}{\beta(i-j+1)}\right)}{\beta(i-j+1)}$$

## IV. Distribution of Maximum and Minimum

Series, Parallel, Series-Parallel and Parallel-Series systems are general system structure of many engineering systems. The theory of order statistics provides a use-full tool for analysing life time data of such systems. Let $X_1, X_2,...,X_n$ be a simple random sample from WLD with CDF and PDF as in (2.1) and (2.2), respectively. Let $X_{(1)}, X_{(2)},\ldots, X_{(n)}$ denote the order statistics obtained from this sample. The CDF of $X_{(r)}$ is given by,

$$F_{r:n}(x)=\sum_{j=r}^{n}\binom{n}{j}\left[1-e^{-\alpha\left((1+x)e^{x^{\beta}}-1\right)}\right]^{j}\left[e^{-\alpha\left((1+x)e^{x^{\beta}}-1\right)}\right]^{n-j} \tag{4.1}$$

Reliability of a series system having n components with independent and identically distributed (iid) WLD distribution is $R(x)=\left[e^{-\alpha\left((1+x)e^{x^{\beta}}-1\right)}\right]^{n}$. Reliability of a parallel system having n components with iid WLD distribution is $R(x)=1-\left[1-e^{-\alpha\left((1+x)e^{x^{\beta}}-1\right)}\right]^{n}$

## V. Parameter Estimation

In this section, point estimation of the unknown parameters of the WLD are derived by using the method of maximum likelihood based on a complete sample data. First partial derivatives of the log-likelihood function with respect to the two-parameters are

$$\frac{\partial L}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}(1+x_i)e^{x_i^{\beta}}+n \tag{5.1}$$

and

$$\frac{\partial L}{\partial\beta}=\sum_{i=1}^{n}x_i^{\beta}\log x_i-\alpha\sum_{i=1}^{n}\left((1+x_i)e^{x_i^{\beta}}x_i^{\beta}\log x_i\right)+\sum_{i=1}^{n}\frac{(1+x_i)\left[x_i^{\beta-1}+\beta x_i^{\beta-1}\log x_i\right]}{\beta x_i^{\beta-1}(1+x_i)+1} \tag{5.2}$$

Setting the left side of the above two equations to zero, we get the likelihood equations as a system of two nonlinear equations in $\alpha$ and $\beta$. Solving this system in $\alpha$ and $\beta$ gives the maximum likelihood estimates (MLE) of $\alpha$ and $\beta$. It is very easy to obtain estimates using R software by numerical methods.

#### Asymptotic Confidence bounds

In this section, we derive the asymptotic confidence intervals of these parameters when $\alpha>0$ and $\beta>0$ as the MLEs of the unknown parameters $\alpha>0$ and $\beta>0$ cannot be obtained in closed forms, by using variance covariance matrix $I^{-1}$, where $I^{-1}$ is the inverse of the observed information matrix which defined as follows

$$I^{-1}=\begin{pmatrix}-\frac{\partial^2 L}{\partial\alpha^2} & -\frac{\partial^2 L}{\partial\alpha\partial\beta}\\ -\frac{\partial^2 L}{\partial\beta\partial\alpha} & -\frac{\partial^2 L}{\partial\beta^2}\end{pmatrix}^{-1}=\begin{pmatrix}Var(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\beta})\\ Cov(\hat{\beta},\hat{\alpha}) & Var(\hat{\beta})\end{pmatrix} \tag{5.3}$$

The second partial derivatives as follows

$$\frac{\partial^2 L}{\partial\alpha^2}=-\frac{n}{\alpha}$$

$$\frac{\partial^2 L}{\partial \beta^2} = \sum_{i=1}^{n} x_i^{\beta} (\log x_i)^2 - \alpha \sum_{i=1}^{n} (1+x_i)\left[e^{x_i^{\beta}} \left(x_i^{\beta} \log x_i\right)^2 + e^{x_i^{\beta}} x_i^{\beta} (\log x_i)^2\right] +$$

$$\sum_{i=1}^{n} \frac{\left(\beta x_i^{\beta-1}(1+x_i)+1\right)\left((1+x_i)\beta x_i^{\beta-1}(\log x_i)^2 + 2(1+x_i)x_i^{\beta-1}\right) - \left((1+x_i)\left[x_i^{\beta-1} + \beta x_i^{\beta-1}\log x_i\right]\right)^2}{\left(\beta x_i^{\beta-1}(1+x_i)+1\right)^2}$$

$$\frac{\partial^2 L}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} (1+x_i) e^{x_i^{\beta}} x_i^{\beta} \log x_i$$

We can derive the $(1-\delta)100\%$ confidence intervals of the parameters $\alpha$ and $\beta$ by using variance matrix as in the following forms

$$\hat{\alpha} \pm Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\alpha})}, \; \hat{\beta} \pm Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\beta})}$$

where $Z_{\frac{\delta}{2}}$ is the upper $\left(\frac{\delta}{2}\right)^{th}$ percentile of the standard normal distribution.

## VI. Three parameter Weibull-Lindley Distribution

In order to address scaling problem, as given in (2.1), this section considered the CDF of Three parameter Weibull-Lindley Distribution (3WLD), for $\alpha, \beta, \lambda > 0$ as follows;

$$F(x;\alpha,\beta,\lambda) = 1 - e^{-\alpha\left((1+\lambda x)e^{(\lambda x)^{\beta}} - 1\right)}, x > 0 \quad (6.1)$$

The probability density function (PDF) corresponding to Eq. (6.1) is given by

$$f(x:\alpha,\beta,\lambda) = \alpha\left(\beta\lambda(\lambda x)^{\beta-1}(1+\lambda x)e^{(\lambda x)^{\beta}} + \lambda e^{(\lambda x)^{\beta}}\right)e^{-\alpha\left((1+\lambda x)e^{(\lambda x)^{\beta}} - 1\right)}, \; x > 0, \; \alpha,\beta,\lambda > 0 \quad (6.2)$$

Here $\beta$ is shape parameter and $\lambda$ is scale parameter. The distribution of this form with parameters $\alpha, \beta,$ and $\lambda$ and will be denoted by $3WLD(\alpha,\beta,\lambda)$.

The survival function $S(x;\alpha,\beta,\lambda)$, failure rate function $h(x;\alpha,\beta,\lambda)$, reversed failure rate function $r(x;\alpha,\beta,\lambda)$ and cumulative failure rate function $H(x;\alpha,\beta,\lambda)$ of $X$ are

$$S(x;\alpha,\beta,\lambda) = 1 - F(x;\alpha,\beta,\lambda) = e^{-\alpha\left((1+\lambda x)e^{(\lambda x)^{\beta}} - 1\right)}, x > 0 \quad (6.3)$$

$$h(x;\alpha,\beta,\lambda) = \alpha\left(\beta\lambda(\lambda x)^{\beta-1}(1+\lambda x)e^{(\lambda x)^{\beta}} + \lambda e^{(\lambda x)^{\beta}}\right), \; x > 0 \quad (6.4)$$

$$r(x;\alpha,\beta) = \frac{\alpha\left(\beta\lambda(\lambda x)^{\beta-1}(1+\lambda x)e^{(\lambda x)^{\beta}} + \lambda e^{(\lambda x)^{\beta}}\right)e^{-\alpha\left((1+\lambda x)e^{(\lambda x)^{\beta}} - 1\right)}}{1 - e^{-\alpha\left((1+\lambda x)e^{(\lambda x)^{\beta}} - 1\right)}}, x > 0 \quad (6.5)$$

and

$$H(x;\alpha,\beta,\lambda) = \int_0^x h(t;\alpha,\beta,\lambda)dt = \alpha(1+\lambda x)e^{(\lambda x)^{\beta}} \quad (6.6)$$

respectively.

Figure 3 and Figure 4 provide the PDFs and the failure rate functions of $GoED(\alpha,\beta,\lambda)$ for different parameter values. From the below figures it is immediate that the PDFs can be unimodel and the failure rate function can be increasing, decreasing or bathtub shaped

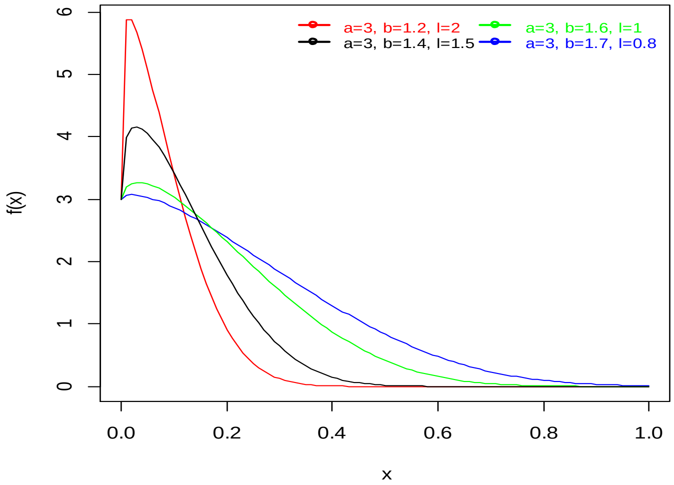


Figure 3: Probability density function of the $3WLD(\alpha, \beta, \lambda)$.

It is clear that the PDF and the failure rate function have many different shapes, which allows this distribution to fit different types of lifetime data.

(a) (b)

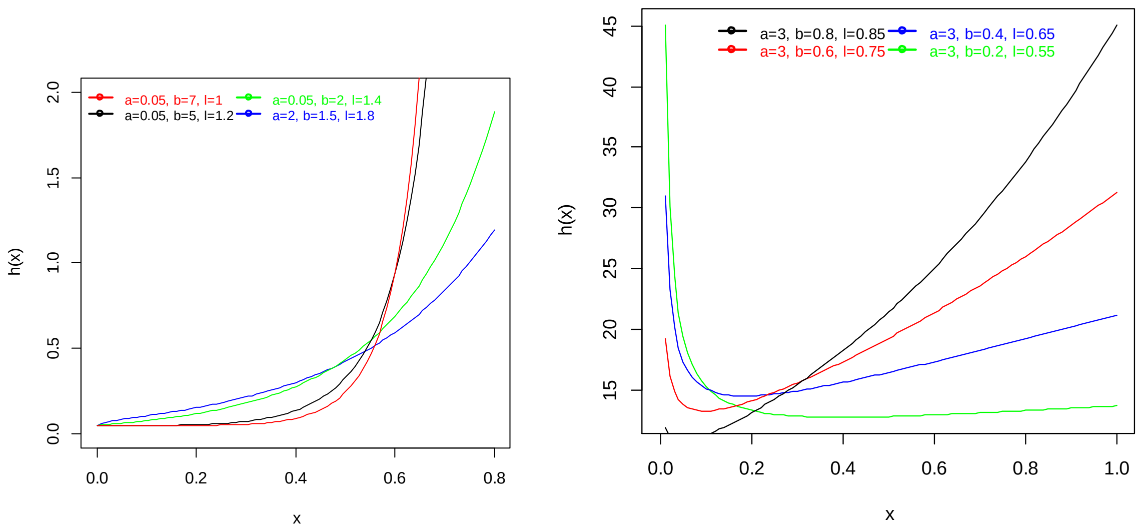


Figure 4: Failure rate function of the $3WLD(\alpha, \beta, \lambda)$.

For fixed $\alpha$, the failure rate function is (a) non-decreasing function (IFR) if $\beta > 1$ and $\lambda > 1$, and (b) non-increasing (DFR) and bathtub function if $\beta < 1$and $\lambda < 1$.

### Parameter Estimation

In this section, point estimation of the unknown parameters of the 3WLD are derived by using the method of maximum likelihood based on a complete sample data. The first partial derivatives of the log-likelihood function with respect to the three-parameters are

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} (1 + \lambda x_i) e^{(\lambda x_i)^\beta} + n \tag{6.7}$$

and

$$\frac{\partial L}{\partial \beta}=\sum_{i=1}^{n}(\lambda x_i)^{\beta}\log(\lambda x_i)-\alpha\sum_{i=1}^{n}\left((1+\lambda x_i)e^{(\lambda x_i)^{\beta}}(\lambda x_i)^{\beta}\log(\lambda x_i)\right)+\sum_{i=1}^{n}\frac{(1+\lambda x_i)\left((\lambda x_i)^{\beta-1}+\beta(\lambda x_i)^{\beta-1}\log(\lambda x_i)\right)}{\beta(\lambda x_i)^{\beta-1}(1+\lambda x_i)+1} \tag{6.8}$$

$$\frac{\partial L}{\partial \lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}x_i^{\beta}\beta\lambda^{\beta-1}-\alpha\sum_{i=1}^{n}\left((1+\lambda x_i)e^{(\lambda x_i)^{\beta}}x_i\beta(\lambda x_i)^{\beta}+x_i e^{(\lambda x_i)^{\beta}}\right)+\sum_{i=1}^{n}\frac{\beta x_i^{\beta}\left((\beta-1)\lambda^{(\beta-2)}+x_i\beta\lambda^{\beta-1}\right)}{\left(\beta(\lambda x_i)^{\beta-1}(1+\lambda x_i)+1\right)} \tag{6.9}$$

Setting the left side of the above three equations to zero, we get the likelihood equations as a system of three nonlinear equations in $\alpha,\beta$ and $\lambda$. Solving this system in $\alpha,\beta$ and $\lambda$ gives the maximum likelihood estimates (MLE) of $\alpha,\beta$ and $\lambda$. It is very easy to obtain estimates using R software by numerical methods.

## VII. Application

In this section, we present the analysis of a real data set using the $WLD(\alpha,\beta)$ and $3WLD(\alpha,\beta)$ model and compare it with the other bathtub models such as Generalized Lindley distributions (GLD), [7], Exponentiated Weibull distribution (EW), [9], using Kolmogorov-Smirnov (K-S) statistic. We considered the data sets are obtained strengths of 1.5 cm glass fibres data [10] and infection for AIDS data [4] to estimate the parameter values.

**Data Set 1:** The data are the strengths of 1.5 cm glass fibres [10], measured at the National Physical Laboratory, England. Unfortunately, the units of measurement are not given in the paper. This data set 1 is in Table 1.

| |
|---|
| 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78 and 1.89. |

Table 2 gives MLEs of parameters of the WLD, GLD, EW and 3WLD and goodness of fit statistics.

Table 2: MLEs of parameters, Log-likelihood.

| **Model** | **MLEs of parameters** | log **L** | **K-S** | **p-value** |
|---|---|---|---|---|
| WLD | $\hat{\alpha}$ = 0.02852<br>$\hat{\beta}$= 1.8927 | -16.63882 | 0.13681 | 0.189 |
| GLD | $\hat{\alpha}$ = 26.17181<br>$\hat{\lambda}$ = 2.990087 | -30.61986 | 0.22639 | 0.003136 |
| EW | $\hat{\alpha}$ = 7.2847<br>$\hat{\beta}$ = 0.67122<br>$\hat{\lambda}$ =0.58203 | -14.67552 | 0.14623 | 0.1352 |
| 3WLD | $\hat{\alpha}$ = 0.000212<br>$\hat{\beta}$=0.83783<br>$\hat{\lambda}$ =5.32574 | -14.42277 | 0.12564 | 0.273 |

3WLD gives the largest Log-likelihood value and largest p value based on the KS statistic. The second largest Log-likelihood value and p value based on the KS statistic is given by the EW distribution. The third largest Log-likelihood value and p value based on the KS statistic is given by the WL distribution.

Figure 5 gives the form of the failure rate for the WLD and 3WLD which are used to fit the data after replacing the unknown parameters.

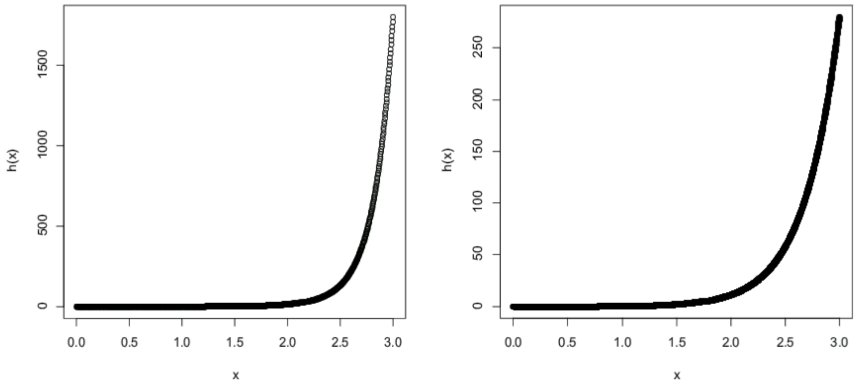


Figure 5: Failure rate function for WLD and 3WLD

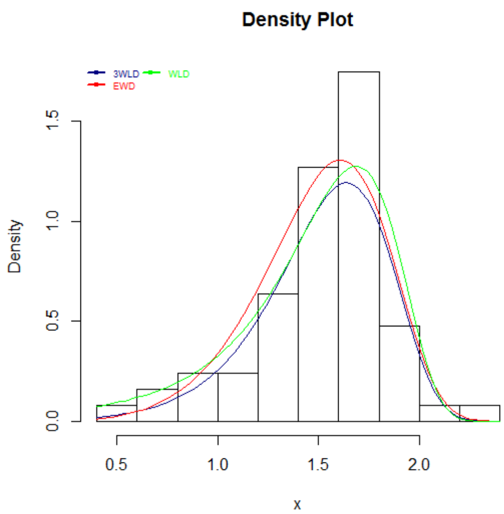


Figure 6: Fitted pdfs of the three best fitting distributions for data set 1.

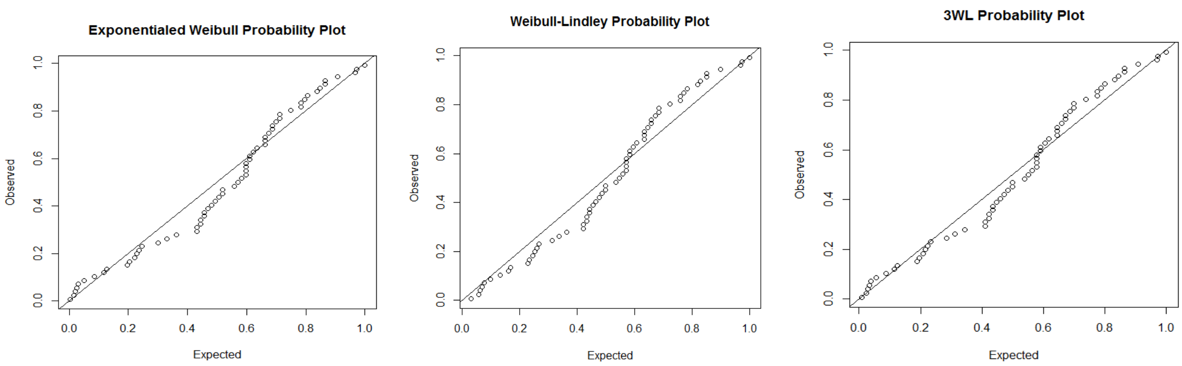


Figure 7: PP plots of the three best fitting distributions for data set 1.

**Data Set 2:** The second data set are times to infection for AIDS for two hundred and ninety five patients. The data were taken from Section 1.19 of Klein and Moeschberger [4]. The two distributions were fitted to this data. The parameter estimates and the goodness of fit statistics are given in Table 3. Table 3 gives MLEs of parameters of the WLD, GLD, EW and 3WLD and goodness of fit statistics.

Table 3: MLEs of parameters, Log-likelihood

| Model | MLEs of parameters | log **L** | K-S | p-value |
|---|---|---|---|---|
| WLD | $\hat{\alpha}$ = 0.03552611<br>$\hat{\beta}$= 0.57122324 | -457.3015 | 0.077618 | 0.08931 |
| GLD | $\hat{\alpha}$ = 2.4144951<br>$\hat{\lambda}$ = 0.8924887 | -453.523 | 0.71652 | $2.22\times10^{-16}$ |
| EW | $\hat{\alpha}$ = 1.9565778<br>$\hat{\beta}$ = 0.9598033<br>$\hat{\lambda}$ = 0.3212501 | -450.1305 | 0.063912 | 0.2426 |
| 3WLD | $\hat{\alpha}$ = $8.751896\times10^{-04}$<br>$\hat{\beta}$ = 0.2994<br>$\hat{\lambda}$ = 15.0999 | -451.8749 | 0.061941 | 0.2755 |

Here, EW gives the largest Log-likelihood value and largest p value based on the KS statistic. The second largest Log-likelihood value and p value based on the KS statistic is given by the 3WL distribution. The third largest Log-likelihood value and p value based on the KS statistic is given by the WL distribution.

Figure 8 and 9 gives the form of the failure rate for the WLD and 3WLD which are used to fit the data after replacing the unknown parameters.

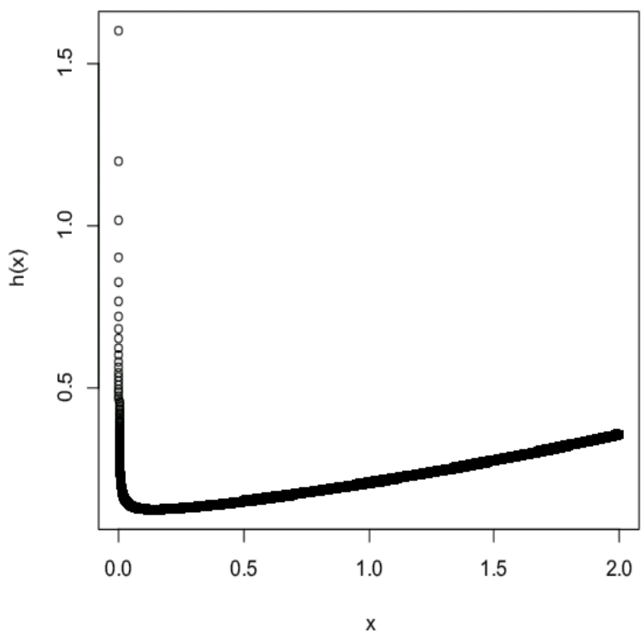


Figure 8: Failure rate function for WLD

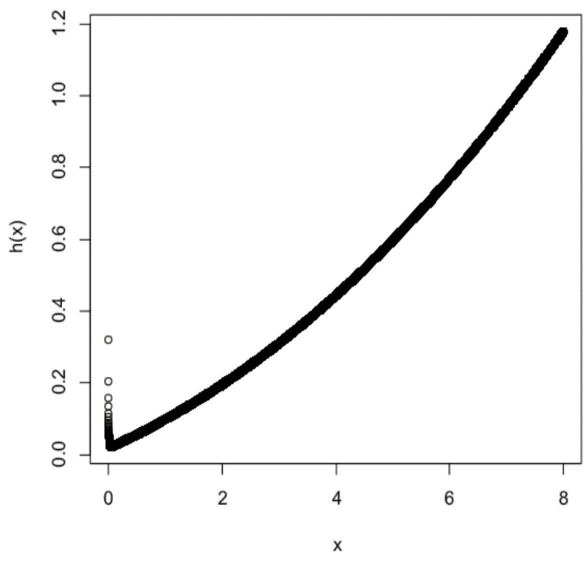


Figure 9: Failure rate function for 3WLD

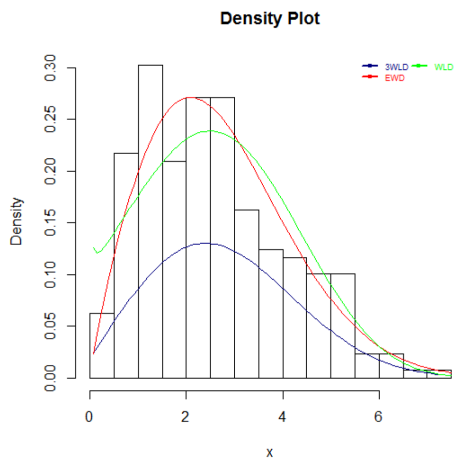


Figure 10: Fitted pdfs of the three best fitting distributions for data set 2.

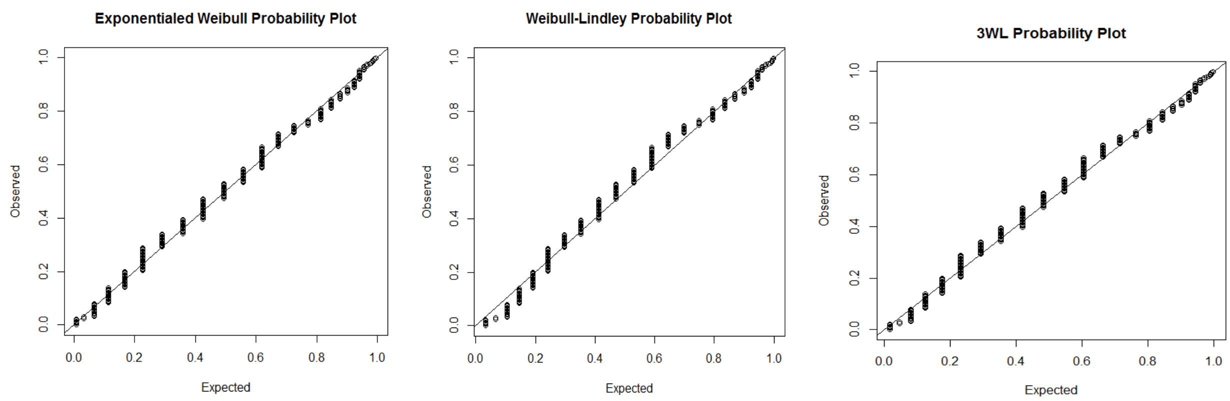


Figure 11: PP plots of the three best fitting distributions for data set 2.

It is observed that 3WLD fits the best in the first data set whereas EW fits the best in the second data in terms of likelihood and in terms of KS Statistic. Therefore, it is not guaranteed the 3WLD will behave always better than WLD or EW or GLD but at least it can be said in certain circumstances 3WLD might work better than WLD or EW or GLD.

## VIII. Conclusion

A new distribution, Weibull-Lindley distribution (WLD), has been proposed and its properties studied. Three parameter Weibull-Lindley distribution (3WLD) is introduced for avoid scale problem. We have studied maximum likelihood estimators and the parameters estimation is carried out in the presence of real data. We present two real life data sets, where in one data set it is observed that 3WLD has a better fit compare to EW or WLD or GLD but in the other the EW has a better fit than 3WLD or WLD or GLD.

## References

[1] Ammar M. Sarhan.: A two-parameter discrete distribution with a bathtub hazard shape. Com. for Stat. Appl. & Meth. 24(1), 15–27 (2017).

[2] Chacko, V.M.: A new bathtub shaped failure rate model, Reliability: Theo. & Appl. 3(42), Vol.11 (2016).

[3] Chacko, V.M.: X-Exponential bathtub failure rate model. Reliability: Theo & Appl. 4 (43), 55-66 (2016).

[4] Klein, J.P., Moesch Berger, M.L. (ed.): Survival Analysis Techniques for Censored and Truncated Data. Springer Verlag, New York (1997).

[5] Pal, M., Ali, M.M., Woo, J.: A Generalization of the Weibull Distribution with Application to the Analysis of Survival Data. STATISTICA. Anno LXVI, n. 2 (2006).

[6] Mustafa.A., Beih. S. El-Desouky, AL-Garash, S.: Weibull Generalized Exponential Distribution.

[7] Nadarajah, S., Bakouch, H.S., Tahmasbi, R.: A generalized Lindley distribution. Sankhya B, 73, 331-359 (2011).

[8] Nadarajah, S., Gupta, A.K.: The exponentiated gamma distribution with application to drought data. Calcutta Stat. Asso. Bulletin. 59, 233–234 (2007).

[9] Pal, M., Ali, M.M., Woo, J.: Exponentiated Weibull Distribution, Statistica, Anno LXVI, n.2 (2006).

[10] Smith, R. L. and Naylor, J. C.: A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. Appli. Stat. 36, 358–369 (1987).