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Guest Editors

With the rapid development of technology, complexity and economy have achieved unprecedented development and received extensive attention from the society and industry. Due to growing complexity, Reliability theory become important day by day. This special issue of the journal Reliability: Theory and Applications on Recent Communications in Reliability, Availability, Maintainability (RAM) & Associated Areas published high-quality articles of reliability engineering. We would like to express our gratitude to all the authors for submitting their work and to the reviewers for their efficient work in assessing the submissions. We are truly pleased by their excellent timely responses. The guest editors are also highly thankful to Managing Editor Prof. Alexander Bochkov for providing the continuous support and constructive suggestions during the review process and shaping the special issue. We hope that this special issue significant contributions in the field of reliability.

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Stocks' Data Mathematical Modeling using Differential Equations: The Case of Healthcare Companies in Athens Stock Exchange

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Abstract

Stock prices' prediction is fundamental for investment decision-making. In this research, a differential equations model is developed for stock prices prediction. More specifically, a 7×7 differential equations system based on Lanchester's combat models will be used. Data concerning the short-term stock's prices of healthcare firms listed in Athens Stock Exchange will be analyzed in order to develop and evaluate the stocks' prices predictive model. The obtained results revealed the differential equations model potential for stock prices' prediction in the short-term.

Keywords: stock price prediction, forecasting, differential equations, Lanchester's combat model

I. Introduction

Stock markets are formal, organized and regulated markets for securities whose prices are determined by the law of supply and demand. In these markets the opposite expectations of investors are met for the formation of stock prices at a given time. More specifically, there are always some investors who believe that the price of a stock is going to fall and others who believe that price of the same stock is going to rise. The former are trying to sell their stocks pushing their price to fall, while the latter are trying to buy these stocks, pushing their prices to rise. Investors see the stock markets as an alternative form of investing their capitals, in order to gain a satisfactory return, higher than that these other investments such as bank deposits or government bonds.

Stock markets are also parts of the financial systems and, like banks, they provide the means and services to transfer funds from investors' savings to firms. Based on stock markets, investors expect positive returns, which are achieved through the growth of the firms leading to a rise in the stock prices.

Several studies suggest a correlation between many factors such as political and financial stability and stock prices [1]. Macroeconomic and psychological factors can affect stock prices [2]. Invest decisions in stocks are found to be affected by factors such as optimism and pessimism as well [3]. Furthermore, stock prices can be affected by factors such as macroeconomic data, market circles, trade balance, firms' profits, technology and globalization [4].

Predicting stock prices would be really crucial for investment decision-making [5–6]. Despite the many factors that can affect stock prices, their prediction can be achieved even it is a difficult process. This is the main reason why stock prices prediction is in the spotlight of most of the investors and professional analysts [7].

Chang and Liu [7], propose a model to predict stocks' future prices using a first order Takagi–Sugeno model. Their model was tested on Taiwan Stock Exchange stocks and the model's output outperformed other approaches such regression analysis. Schöneburg [8], used neural networks to predict German stock prices by the aid of temporary and not-long lasting framework. The proposed model achieved a degree of accuracy up to 90%. Neural networks were used by Kohara *et al*. [9] as well. In their research, they used data from 330 days to estimate their model's coefficients. Adebiyi *et al*. [10], used the following ARIMA model to predict the future prices of stock prices:

$$Y_t = \varphi_0 + \varphi_1 Y_{t-1} + +\varphi_2 Y_{t-2} + \ldots + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \ldots - \theta_q \varepsilon_{t-q} \quad (1)$$

where Y_t is the real value and ε_t is the random error at t, φ_i and θ_j are the coefficients, p and q are the integers called autoregressive and moving average respectively. The same authors [11] are comparing the accuracy of ARIMA and neural networks models in predicting stock prices. Their results show that both the models achieve effective forecast for stock prices. Hafezi *et al*. [12], proposed a bat-neural network model based on a multi-agent framework to predict DAX stock prices in quarterly periods of eight years.

Katsouleas *et al*. [13] proposed a generalized differential equations model by the aid of the so-called Lanchester's combat approach to predict the healthcare firms of Athens Stock Exchange (ASE) stocks' prices. The model was based on the prior work of Chalikias and Skordoulis on Lanchester's combat approach concerning the case of a duopolistic market [14] as there is evidence that warfare models can be applied in business cases [14–17]. The primary differential equations model was the following one:

$$\begin{aligned} \frac{dx}{dt} &= -ay + f(t) \\ \frac{dy}{dt} &= -bx + g(t) \end{aligned} \quad (2)$$

where x(t) and y(t) refer to the amount of ready-for-use product items for sale of firm A and B correspondingly, f(t) and g(t) refer to their respective increase and decrease rates, while ay(t), bx(t) correspond to the handy product items' rates.

The principal objective of the present manuscript is to develop a differential equations model by the aid of Lanchester's combat approach for stock price prediction.

II. Methods

The data used in this research concern healthcare firms listed in ASE. In Greece, citizens receive health care from both public and private providers. The increasing problems on public health care system is the main factor which is responsible for the growth of private sector [18]. The private healthcare sector represents the 32.9% of health care market in Greece [19]. Greek private health services contain diagnostics centers and clinics as primary and secondary health care units respectively. The 5 largest groups of the private healthcare sector correspond to 53% all of the market's stocks [20]. This market contains seven firms in Athens stock exchange, Axon (AXON), Euromedica (EUROM), Iaso (IASO), Iatriko Athinon (IATR), Lavipharm (LAVI), Medicon Hellas (MENTI) and Hygeia (YGEIA).

The Athens Stock Exchange constitutes the only authorized stock market in Greece. Before 2002 comes to an end, almost three hundred seventy-five firms had been included, while their overall capitalization equal to € 85.5 billion. Only ASE affiliates may carry out purchase and sale requisitions for shares via the so-called Integrated Automatic Electronic Trading System (OASIS) of the market. The ASE is actually an order-driven market, since its affiliates can continually commence offer requisitions in the system from 11:00 a.m. to 4:00 p.m. [21].

The study used historical stock prices of ASE healthcare firms during a 12-month period. More specifically, the stock data were picked out daily data files of ASE containing for all of the months the closing stock prices from the first day of each month.

For developing the deferential equations approach, we utilized the random variables T, U, V, W, X, Y and Z which correspond to the stock prices of the market's 7 firms. Thus, the next 7×7 differential equations system was primarily concluded:

$$\begin{cases} \frac{dT}{dt} = \frac{b}{a}U + \frac{c}{a}V + \frac{d}{a}W + \frac{e}{a}X + \frac{f}{a}Y + \frac{g}{a}Z \\ \frac{dU}{dt} = \frac{a}{b}T + \frac{c}{b}V + \frac{d}{b}W + \frac{e}{b}X + \frac{f}{b}Y + \frac{g}{b}Z \\ \frac{dV}{dt} = \frac{a}{c}T + \frac{b}{c}U + \frac{d}{c}W + \frac{e}{c}X + \frac{f}{c}Y + \frac{g}{c}Z \\ \frac{dW}{dt} = \frac{a}{d}T + \frac{b}{d}U + \frac{c}{d}V + \frac{e}{d}X + \frac{f}{d}Y + \frac{g}{d}Z \\ \frac{dX}{dt} = \frac{a}{e}T + \frac{b}{e}U + \frac{c}{e}V + \frac{d}{e}W + \frac{f}{e}Y + \frac{g}{e}Z \\ \frac{dY}{dt} = \frac{a}{f}T + \frac{b}{f}U + \frac{c}{f}V + \frac{d}{f}W + \frac{e}{f}X + \frac{g}{f}Z \\ \frac{dZ}{dt} = \frac{a}{g}T + \frac{b}{g}U + \frac{c}{g}V + \frac{d}{g}W + \frac{e}{g}X + \frac{f}{g}Y \end{cases} \quad (3)$$

III. Results

Let us consider the following $n \times n$ system of differential equations:

$$\frac{dx}{dt} = Ax \quad (4)$$

where $x(t) = (x_1(t), \ldots, x_n(t))^T$ is an $n \times 1$ vector of functions of the variable t, the coefficient matrix $A \in \mathbb{R}^{n \times n}$ takes the following form:

$$A = \begin{bmatrix} 0 & \frac{a_2}{a_1} & \frac{a_3}{a_1} & \cdots & \frac{a_{n-1}}{a_1} & \frac{a_n}{a_1} \\ \frac{a_1}{a_2} & 0 & \frac{a_3}{a_2} & \cdots & \frac{a_{n-1}}{a_2} & \frac{a_n}{a_2} \\ \vdots & & \ddots & \ddots & & \\ \frac{a_1}{a_{n-1}} & \frac{a_2}{a_{n-1}} & \frac{a_3}{a_{n-1}} & \cdots & 0 & \frac{a_n}{a_{n-1}} \\ \frac{a_1}{a_n} & \frac{a_2}{a_n} & \frac{a_3}{a_n} & \cdots & \frac{a_{n-1}}{a_n} & 0 \end{bmatrix} \tag{5}$$

and $[a_j]_{j=1}^{n} \subset \mathbb{R}$ in its definition (2) satisfy $\sum_{j=1}^{n} a_j = 1$.

An interesting and somewhat surprising result is that the spectrum $\sigma(A)$ is in fact independent of these parameters $[a_j]_{j=1}^{n}$. More precisely, it may be shown that $\sigma(A)$ is intimately related to the order n of the matrix A, including simply the eigenvalue pair $\lambda_1 = n - 1$ with algebraic multiplicity (n-1) and $\lambda_2 = n - 1$. In this direction, we will prove its characteristic polynomial may be factored as follows:

$$\chi_A(\lambda) = \det(A - \lambda I_n) = (-1)^n(\lambda + 1)^{n-1}(\lambda - n + 1) \tag{6}$$

by induction on n. Indeed, this assertion is readily verified for $n = 2$ and $n = 3$.

To simplify our analysis for larger n, for an arbitrary $n \times n$ matrix X we introduce the notation $X_{i,j}$ for its $(n-1) \times (n-1)$ submatrix deduced by erasing its i-th row and j-th column. Our recursive assumption may then be stated as follows:

$$\chi_{A_{n,n}}(\lambda) = \det\left(A_{n,n} - \lambda I_{n-1}\right) = (-1)^{n-1}(\lambda + 1)^{n-2}(\lambda - n + 2) \tag{7}$$

since $A_{n,n}$ is simply the leading $(n-1) \times (n-1)$ submatrix of A. Laplace expansion along the last row of $A - \lambda I_n$ yields to:

$$\chi_A(\lambda) = \det(A - \lambda I_n) = \sum_{j=1}^{n-1} \frac{a_j}{a_n}(-1)^{n+j}\det\left((A - \lambda I_n)_{n,j}\right) + (-\lambda)(-1)^{n+n}\det\left((A - \lambda I_n)_{n,n}\right) = \sum_{j=1}^{n-1} \frac{a_j}{a_n}(-1)^{n+j}\det\left((A - \lambda I_n)_{n,j}\right) + (-1)^n(\lambda + 1)^{n-2}(\lambda - n + 2) \tag{8}$$

To continue, we turn our attention to $\left[\det\left((A - \lambda I_n)_{n,j}\right)\right]_{j=1}^{n-1}$ and note the following properties:

Lemma 1. (a) $\det\left((A - \lambda I_n)_{n,1}\right) = \frac{a_n}{a_1}(\lambda + 1)^{n-2}$. (b) $\det\left((A - \lambda I_n)_{n,j}\right) = (-1)^{j+1}\frac{a_n}{a_j}(\lambda + 1)^{n-2}$, for $j = 2, \ldots, n-1$.

Proof. (a) By induction on n. Indeed, denoting:

$$B \equiv (A - \lambda I_n)_{n,1} = \begin{bmatrix} \frac{a_2}{a_1} & \frac{a_3}{a_1} & \cdots & \frac{a_{n-1}}{a_1} & \frac{a_n}{a_1} \\ -\lambda & \frac{a_3}{a_2} & \cdots & \frac{a_{n-1}}{a_2} & \frac{a_n}{a_2} \\ \vdots & \ddots & & \ddots & \\ \frac{a_2}{a_{n-1}} & \frac{a_3}{a_{n-1}} & \cdots & -\lambda & \frac{a_n}{a_{n-1}} \end{bmatrix} \in \mathbb{C}^{(n-1)\times(n-1)} \tag{9}$$

A direct computation verifies the statement for $n = 3$, since $\begin{vmatrix} \frac{a_2}{a_1} & \frac{a_3}{a_1} \\ -\lambda & \frac{a_3}{a_2} \end{vmatrix} = \frac{a_3}{a_1}(\lambda + 1)$.

Proceeding further, we make the recursive assumption:

$$\left(\det\left(B_{n-1,n-1}\right)=\right)\begin{vmatrix} \frac{a_2}{a_1} & \frac{a_3}{a_1} & \cdots & \frac{a_{n-2}}{a_1} & \frac{a_{n-1}}{a_1} \\ -\lambda & \frac{a_3}{a_2} & \cdots & \frac{a_{n-2}}{a_2} & \frac{a_{n-1}}{a_2} \\ \vdots & \ddots & & \ddots & \\ \frac{a_2}{a_{n-2}} & \frac{a_3}{a_{n-2}} & \cdots & -\lambda & \frac{a_{n-1}}{a_{n-2}} \end{vmatrix} = \frac{a_{n-1}}{a_1}(\lambda+1)^{n-3}, \tag{10}$$

whereby this determinant is independent of $\left[a_j\right]_{j=2}^{n-2}$ and involves only a_{n-1} i.e. the nominator of the last column entries in (10), and a_1, the denominator of the first row in (10).

Since $B_{n-1,n-2} = \begin{bmatrix} \frac{a_2}{a_1} & \frac{a_3}{a_1} & \cdots & \frac{a_{n-2}}{a_1} & \frac{a_n}{a_1} \\ -\lambda & \frac{a_3}{a_2} & \cdots & \frac{a_{n-2}}{a_2} & \frac{a_n}{a_2} \\ \vdots & \ddots & & \ddots & \\ \frac{a_2}{a_{n-2}} & \frac{a_3}{a_{n-2}} & \cdots & -\lambda & \frac{a_n}{a_{n-2}} \end{bmatrix}$ has the same formulation as in (10), but with a_n instead of a_{n-1} in its last column, we conclude that:

$$\det\left(B_{n-1,n-2}\right) = \frac{a_n}{a_1}(\lambda+1)^{n-3} \tag{11}$$

On the other hand, it is immediately revealed that:

$$\det\left(B_{n-1,j}\right) = 0,\ \text{for j=1,2,}\ldots\text{, n-3} \tag{12}$$

since each of these minors includes a pair of linearly dependent rows; namely, rows 1 and j+1. Hence, Laplace expansion of (9) along its last row verifies the assertion:

$$\begin{aligned} \det(B) &= \sum_{j=1}^{n-3}\frac{a_{j+1}}{a_{n-1}}(-1)^{(n-1)+j}\det\left(B_{n-1,j}\right) + (-\lambda)(-1)^{(n-1)+(n-2)}\det\left(B_{n-1,n-2}\right) \\ &+ \frac{a_n}{a_{n-1}}(-1)^{(n-1)+(n-1)}\det\left(B_{n-1,n-1}\right) \overset{(9)}{=} 0 + \lambda\det\left(B_{n-1,n-2}\right) + \frac{a_n}{a_{n-1}}\det\left(B_{n-1,n-1}\right) \\ &= \lambda\frac{a_n}{a_1}(\lambda+1)^{n-3} + \frac{a_n}{a_{n-1}}\frac{a_{n-1}}{a_1}(\lambda+1)^{n-3} = \frac{a_n}{a_1}(\lambda+1)^{n-2}. \end{aligned}$$

(b) For $j = 2$ we have:

$$\begin{aligned} \text{et}\left((A-\lambda I_n)_{n,2}\right) &= \begin{vmatrix} -\lambda & \frac{a_3}{a_1} & \cdots & \frac{a_{n-1}}{a_1} & \frac{a_n}{a_1} \\ \frac{a_1}{a_2} & \frac{a_3}{a_2} & \cdots & \frac{a_{n-1}}{a_2} & \frac{a_n}{a_2} \\ \vdots & \ddots & & \ddots & \\ \frac{a_1}{a_{n-1}} & \frac{a_3}{a_{n-1}} & \cdots & -\lambda & \frac{a_n}{a_{n-1}} \end{vmatrix} = (-1)\begin{vmatrix} \frac{a_1}{a_2} & \frac{a_3}{a_1} & \cdots & \frac{a_{n-1}}{a_1} & \frac{a_n}{a_1} \\ -\lambda & \frac{a_3}{a_1} & \cdots & \frac{a_{n-1}}{a_1} & \frac{a_n}{a_1} \\ \vdots & \ddots & & \ddots & \\ \frac{a_1}{a_{n-1}} & \frac{a_3}{a_{n-1}} & \cdots & -\lambda & \frac{a_n}{a_{n-1}} \end{vmatrix} \\ &= (-1)\frac{a_1}{a_2}\begin{vmatrix} \frac{a_1}{a_1} & \frac{a_3}{a_1} & \cdots & \frac{a_{n-1}}{a_1} & \frac{a_n}{a_1} \\ -\lambda & \frac{a_3}{a_1} & \cdots & \frac{a_{n-1}}{a_1} & \frac{a_n}{a_1} \\ \vdots & \ddots & & \ddots & \\ \frac{a_1}{a_{n-1}} & \frac{a_3}{a_{n-1}} & \cdots & -\lambda & \frac{a_n}{a_{n-1}} \end{vmatrix} \end{aligned} \tag{13}$$

where we have used elementary properties to bring the matrix under consideration in the general form (9). Since the previous statement ensures the determinant of (13) is dependent only on the parameters a_n (appearing as nominator of the last column entries) and a_1 (denominator of first row entries), we obtain:

$$\det\big((A-\lambda I_n)_{n,2}\big) = (-1)\frac{a_1}{a_2}\frac{a_n}{a_1}(\lambda+1)^{n-2} = (-1)\frac{a_n}{a_2}(\lambda+1)^{n-2} \tag{14}$$

The remaining assertions for $2 < j \leq n-1$ are proved similarly.

The statements in Lemma 1 may be summarized together as:

$$\det\big((A-\lambda I_n)_{n,j}\big) = (-1)^{j+1}\tfrac{a_n}{a_j}(\lambda+1)^{n-2}, \text{ for } j = 1, \dots, n-1.$$

Hence, plugging (14) in (8) and after some algebraic manipulations, we reach:

$$\begin{aligned}\chi_A(\lambda) &= (-1)^n(\lambda+1)^{n-2}\left[\sum_{j=1}^{n-1}(-1)^{2j+1} + \lambda(\lambda-n+2)\right]\\ &= (-1)^n(\lambda+1)^{n-2}\big(\lambda^2-(n-2)\lambda-(n-1)\big)\\ &= (-1)^n(\lambda+1)^{n-2}(\lambda+1)(\lambda-n+1)\end{aligned}$$

whereby (6) and $\sigma(A) = \Big[\underbrace{-1, \dots -1}_{n-1}, n-1\Big]$ are immediate.

Denoting $[e_j]_{j=1}^n \subset \mathbb{R}^n$ the standard basis vectors, i.e. $e_j = \Big(\underbrace{0, \dots 0}_{j-1}, 1, \underbrace{0, \dots 0}_{n-j}\Big)^T$ it is straightforward to check that span $\left[-\frac{a_j}{a_1}e_1 + e_j\right]_{j=2}^n$ is the $(n-1)$ −dimensional eigenspace associated to $\lambda_1 = -1$, while $\left(\frac{a_n}{a_1}, \frac{a_n}{a_2}, \dots, \frac{a_n}{a_{n-1}}, 1\right)^T$ is an eigenvector corresponding to $\lambda_2 = n - 1$.

Hence, the general solution to (4) takes the form:

$$x(t) = e^{-t}\begin{bmatrix} -\frac{a_n}{a_1} & -\frac{a_{n-1}}{a_1} & -\frac{a_{n-2}}{a_1} & \cdots & -\frac{a_2}{a_1} \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & & & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \cdots & & \end{bmatrix}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1}\end{pmatrix} + c_n e^{(n-1)t}\begin{pmatrix} \frac{a_n}{a_1} \\ \frac{a_n}{a_2} \\ \vdots \\ \frac{a_n}{a_{n-1}} \\ 1 \end{pmatrix} \tag{15}$$

with arbitrary $[c_j]_{j=1}^n$ or, equivalently,

$$x_1(t) = -\left(\sum_{j=1}^{n-1} c_j \frac{a_{n-j+1}}{a_1}\right)e^{-t} + c_n\frac{a_n}{a_1}e^{(n-1)t}$$

$$x_j(t) = c_{n-j+1}e^{-t} + c_n\tfrac{a_n}{a_j}e^{(n-1)t} \quad (\text{for } j = 2, \dots, n-1),$$

$$x_n(t) = c_1e^{-t} + c_ne^{(n-1)t} \quad \blacksquare$$

We note that the case of (4) for $n = 4$ has been previously studied by Chalikias *et al.* [22] in relation to the issue of banking industry in Greece.

For the set of the examined stocks the following data were extracted. The function of time has been estimated in order to fit the above solution to the real data [22]. Because of the different monotony of every stock, different functions of time for every stock were used. More specifically the time functions of the following table were used.

Table 1. *Stocks' time functions.*

Stock symbol	Variable i	Time function
AXON	$i = -0.05$ where $j = 0, 0.083, 0.166, 0.025, 0.033, \dots, 1$	i^3
EUROM	$i = -0.05$ where $j = 0, 0.083, 0.166, 0.025, 0.033, \dots, 1$	i^4
IASO	$i = -0.05$ where $j = 0, 0.083, 0.166, 0.025, 0.033, \dots, 1$	i
IATR	$i = -0.05$ where $j = 0, 0.083, 0.166, 0.025, 0.033, \dots, 1$	i
LAVI	$i = -0.05$ where $j = 0, 0.083, 0.166, 0.025, 0.033, \dots, 1$	i^4
MENTI	$i = -0.05$ where $j = 0, 0.083, 0.166, 0.025, 0.033, \dots, 1$	i^3
YGEIA	$i = -0.05$ where $j = 0, 0.083, 0.166, 0.025, 0.033, \dots, 1$	i^4

If we change a_i coefficients with the stock percentages we take the c_i coefficients of the model.

Table 2. *Model's c_i coefficients.*

Coefficient	Value
C_1	–177.324
C_2	–1.93013
C_3	–1064.27
C_3	0.643501
C_5	0.258205
C_6	–151.673

In order to evaluate the good fit of the experimental results, the real stocks' data and the model's predicted data were compared with Wilcoxon Test as the precaution of normality weren't satisfied.

Table 3. *Wilcoxon Test results.*

	Real data – predicted data
Z	–1.574
Asymp. Sig. (2-tailed)	0.116

Based on the above table, we conclude that the real data and model's predicted have the equal distribution with the same median as Wilcoxon Test's null hypothesis is accepted $(\text{Asymp. Sig. } (2-\text{tailed}) = 0.116)$.

Furthermore, the same results are drawn by Sign Test as shown in the following table $(\text{Asymp. Sig. } (2-\text{tailed}) = 0.909)$.

Table 4. *Sign Test results.*

	Real data – predicted data
Z	–0.115
Asymp. Sig. (2-tailed)	0.909

The following figure shows the real data and the model's predicted values.

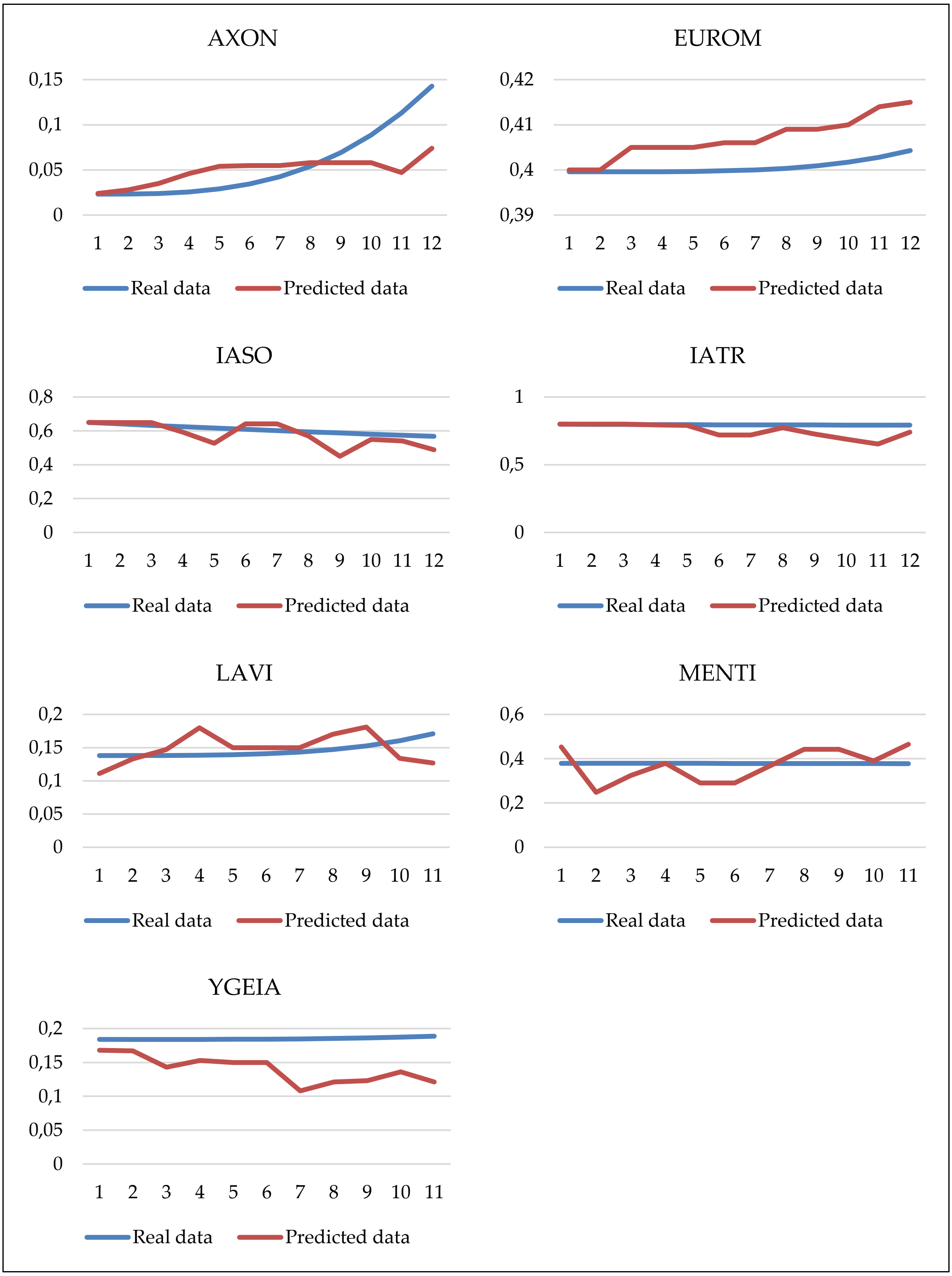

Figure 1. *Real data and the model's predicted values.*

IV. Discussion

In this paper we proposed a differential equations approach by the aid of Lanchester's combat model which can predict stock prices. Input variables concerned the 7 private healthcare firms listed in ASE. The data analysis was based on 7×7differential equations model. The experimental results were found to have equal distribution and same median as the real data. This demonstrated the model's good fit. Thus, the method used can be applied in cases of stock price prediction. Furthermore, another scope of Lanchester's combat models is found, as there is no other application of these models in such a case.

As already mentioned, the data used in this study concern a 12-month period. Similar models are used in various cases of long-term data. In these cases, the models' predictive capability is high [14-16]. However, such models have not be used in stock prices prediction cases. Thus, a long-term model could be analyzed in a future research. Furthermore, in future models, stock prices prediction could take into consideration various factors such as the economy of a country [23-24], the political structure of a country [22,25], or psychological factors [26]. Several modifications and extensions of the proposed methodological approach seem to be of some research interest for future work, since the specific topic is really contemporary and meets a variety of real data applications.

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Supply Chain Resilience Analysis using the Quality Function Deployment (QFD) Approach in a Freight Forwarding Company

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Abstract

The increasing volume of Indonesian exports each year has made freight forwarders should properly maintain their service quality performance to survive in this competitive supply chain situation. PT. Schenker Petrolog Utama Surabaya is one of the freight forwarders who experience an issue where the increasing export volume was not supported by the increase in the shipper's reorder numbers. This study aims to determine the resilience measures that can be implemented by freight forwarding in overcoming maritime risks to meet the needs of shippers and improve service quality performance. The method used in this study is QFD with a two-step house of quality design, which involved 75 respondents from various company representatives who used Schenker's ocean freight export service from 2014 to 2019. The results identified the eighth-highest attributes of customer requirements with the five most influential maritime risks, and in the end, were equipped by the five-best resilience measures as the compatible solution. Several strategies to survive in the maritime supply chain are arranging appropriate qualifications and providing training to employees, expanding cooperation with vendors and shipping liners, making appropriate backup plans, maintaining good coordination between key players, also carrying out maintenance and update of the IT systems regularly.

Keywords: Supply chain resilience, maritime risks, customer requirements, QFD, and freight forwarding.

I. Introduction

The times that are increasingly advanced without any limitations have stimulated the movement of export activities in Indonesia. Based on recorded data of the Central Statistics Agency (*Badan Pusat Statistika*), the last update on January 3, 2020, the volume of national export activities had increased continuously from 2016-2018. It was consistent with the increasing volume of annual export containers at a freight forwarding company in Surabaya, Indonesia. Seeing the increasing and dynamic pattern of customer needs, the ocean freight forwarders, as one of the export service providers, must follow this trend to avoid risks that will affect customer satisfaction. In the worst-case scenario, risks can also significantly influence the entire supply chain process.

When the supply chain is disrupted, its performance will be threatened in terms of profitability, namely the cost and inventory structure [1]. Not only does it threaten profitability, but supply chain disruption will also affect the overall service satisfaction level since forwarders are the key players in the supply chain of the maritime sector [2]. The instability causes of these services used do not only come from the internal party inside the freight forwarding company but also related to other external parties who take part in the process of sending export goods in the supply chain. For example, the ship's departure schedule is backward from the schedule listed on the shipping liner website. These are the example of accidents at sea such as burning ship, sinking [3], being shot, being hijacked [4], deviating from the route [5], and so on. Moreover, the risk of a labour strike at the port could hamper the export-import process due to the absence of activities at the port. The occurrence of natural disasters that cannot be predicted accurately can also disrupt the supply chain process, particularly in the maritime sector.

A reliable supply chain can assist the freight forwarders in surviving any possible risks that can harm the company in order to create excellent service quality. To create a unified supply chain from upstream to downstream within the scope of international trade, it is necessary to integrate between key players in order to create a resilient supply chain. Supply chain resilience is the ability of the supply chain to reinstate to its primary or more reliable state, after being affected by a disruption, and to avoid failure in the supply chain [6], especially in the maritime supply chain.

Therefore, to bring back and maintain the supply chain sustainability in freight forwarding companies, this study aims to determine the suitable resilience measures which can be implemented effectively to the current company's condition as an effort to provide better service performance. Further results will be analyzed through previous studies' results related and discussed with the company's stakeholders to assess its compatibility for future improvement. This study will expand the supply chain resilience literature in freight forwarding companies, particularly the ocean freight export business process, which is still infrequent and limited.

II. Methods

The flowchart in Figure 1 shows the step-by-step in conducting this study. This research is a typical case study done in one of the Indonesian freight forwarding companies. Delivering goods by the sea has been the most favourite pathway in sending customers' goods at the lowest price, yet quite fast. Thus, this study is focusing on the ocean freight export business process.

The primary data used in this study are collected through questionnaires aimed at the customers (shippers), as well as brainstorming activities and structured interviews with experts in positions of Manager and Supervisor at one of the freight forwarding companies in Surabaya, Indonesia, PT. Schenker Petrolog Utama. The QFD method used in this study is a two-step house of quality with three groups of attributes, they are customer requirements, maritime risks, and resilience measures. The first house of quality will identify the relationship between consumer needs and maritime risk. Meanwhile, the second house of quality is used to determine the relationship between maritime risk and resilience measures that forwarders can get in complex maritime supply chain processes. The construction stages of the two quality houses according to Lam and Bai [7] can be carried out as follows:

1) Identification of Customer Requirements (CR)

Conduct literature studies regarding customer need in maritime logistics and conduct brainstorming and interviews with industry professionals (experts) regarding customer needs.

2) Prioritizing Customer Requirements (CR)

Make a priority ranking of interests from the results of distributing questionnaires to customers, using a 4-score Likert scale. Followed by the calculation of the Weights in Equation 1 below:

$$W_i = \frac{I_i}{\sum_{i=1}^{n} I_i} \qquad i = 1, \ldots n \qquad (1)$$

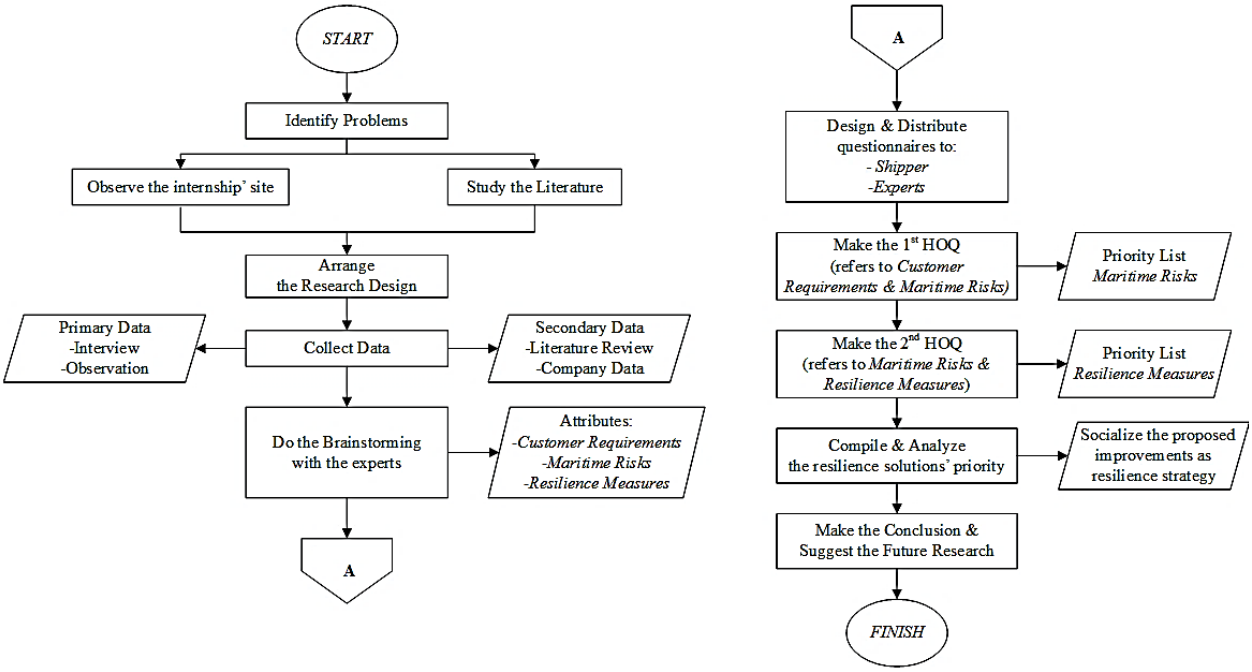


Figure 1: *Research Methodology Flowchart*

3) Identification and Assessment of Design Requirements (DR)

Conduct a literature study on the risks that exist in maritime logistics. In addition, it also conducted interviews with experts and assessed likelihood, impact, and effectiveness for potential risks using 1-4 scores of the Likert scale.

4) Determination of Technical Correlation

Determine the technical correlation between all "*Hows*" attributes (on the house of quality's roof), by implementing symbols that identify the different relationships between the attributes. Relationship correlations consist of strong positive, positive, negative, and strong negative correlations.

5) Determination of the Relationship Matrix

The relationship that is assessed is the impact that the potential of the "*Hows*" attribute group can have on the "*Whats*" attribute group.

6) Assessment of the Technical Matrix

Calculate Absolute Importance (AI) as in Equation 2, and Relative Importance (RI) according to Equation 3 below.

$$\mathrm{AI}_j = \sum_{i=1}^{n} W_i R_{ij} \qquad j = 1, \ldots, m \qquad (2)$$

$$\mathrm{RI}_j = \frac{\mathrm{AI}_j}{\sum_{j=1}^{m} \mathrm{AI}_j} \qquad j = 1, \ldots, m \qquad (3)$$

III. Results and Discussion

A) First House of Quality

In an effort to improve the quality of freight forwarding services, it is necessary to have a quality dimension for these services to determine the interests and needs of consumers as the main users.

Therefore, we need a list of quality dimensions to build the "*Whats*" section of the first house of quality as shown in Table 1, accompanied by its weight of importance.

From Table 1, the results show that the on-time delivery attribute has the highest value of importance along with the shipment attribute sent according to destination. The attribute of on-time delivery also has high importance, specifically 5 (1-5 scale), in the study conducted by Lam and Bai [7] with a similar research topic in the shipping liner industry, as well as in research based on Saraswati et al. [2] with topics and research objects similar to this current study. Delivery to the right destination, as well as accuracy in the creation and provision of the bill of lading (B/L) documents, are the essential support for the service performance of freight forwarding companies [8]. Loss and damaged records can also affect the shipper's assessment of the delivery service performance of the goods. Based on the results of a study conducted by Xu [9], the attribute of fast service, which is included in the dimension of "personnel service quality", is also a significant factor in determining service quality. Meanwhile, the loading-unloading process, which is in the "reliability" dimension group, is also the most important factor according to customers in an effort to achieve satisfaction, based on research conducted by Bottani and Rizzi [10]. Customers will also feel appreciated if the forwarder can help them when issues arise in the delivery process [11]. Moreover, when they also give their best solution in handling those issues.

Table 1: *Customer Requirements Attributes*

No	Attributes	Relative Weight	Rank
1	On-time delivery	12.86	1
2	*Shipment's delivered to the right destination*	12.86	1
3	Complete & suitable documents	12.81	2
4	Timely documents procurement	12.33	3
5	No goods were lost at the port/warehouse	12.33	3
6	Fast service	12.29	4
7	The smoothness of the loading-unloading process	12.29	4
8	The company always offers the best solution	12.24	5

The maritime risks which are selected as the cause of the problem, as well as composing the "*Hows*" section, are then correlated with one another so that they can fill the ceiling and roof of the house. After the "*Whats*" and "*Hows*" sections are filled, then a relationship matrix is arranged to determine the relationship between the two attribute groups, as the middle part of the house of quality. Furthermore, the calculation of absolute and relative importance is carried out for compiling the base of HOQ 1 as shown in Figure 2, with the yellow-coloured box as the five-highest ranking of importance rate.

The first of the top five maritime risks that must be prioritized according to Figure 2, with a score of 10.39, is the risk of the Customs Clearance process. The internal experts of the case company have admitted that this risk has been very influential in the activity of export goods delivery services. The assessment of the importance per attribute has shown that this risk has the highest total value with an average amount of 3.33 out of 4.00. The differences in customs regulations and operational procedures abroad are still counted as a major risk in the global shipping goods process [12]. Besides, the lack of responsiveness of the Customs in handling the export-import documents by sea can also result in serious effects within the process of goods shipping abroad. As the results of research conducted by Tseng et al. [13], when the staff of the transportation service does not complete the customs duties commensurate with the proper instructions, the process of goods delivery will not run smoothly, the worst case is, the goods will not reach to the consignee within the specified time. In the study conducted by Saraswati et al. [2], it was obtained that the risk of delays in handling documents by Customs was ranked in the top five most affected risks in a freight forwarding company.

As for the second rank, with a relative weight score of 9.74, the experts stated that the uncertainty of ship schedules had a big effect in affecting overall customer satisfaction. Supply chain risks can be derived from a less flexible scheduling and routing process, which can cause operational problems and result in delays of the goods delivery [14]. This statement is supported by Arif [11], who found the results in his research related to customer perception of the service quality in one of the freight forwarding companies in Bangladesh, that customers feel dissatisfied when the delivery of goods is delayed, and unfortunately, they tend to do not want to understand the cause of it.

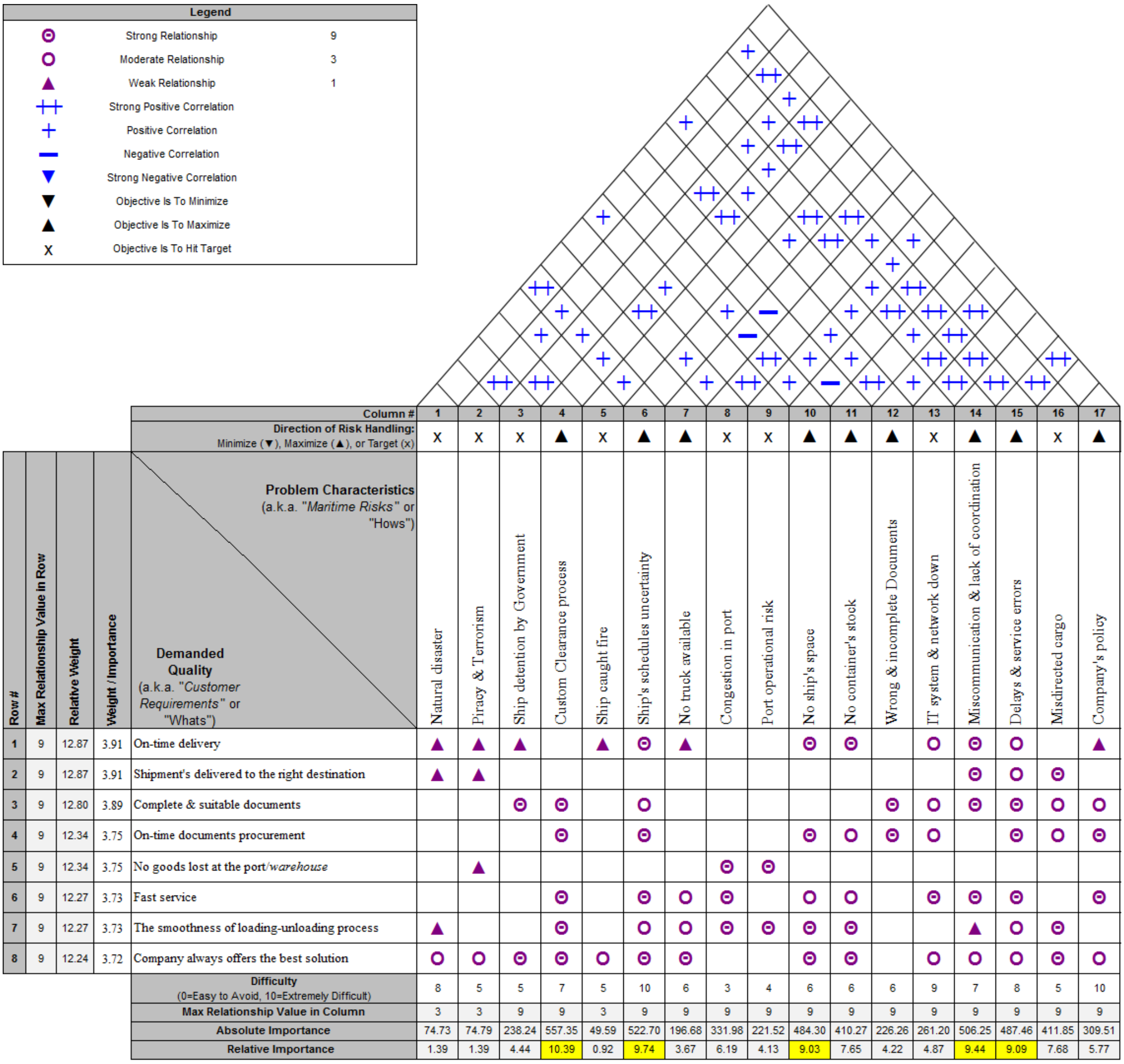


Figure 2: *House of Quality* 1

A study related to supply chain risk management highlights the collaborative aspects involving all key players, which can reduce vulnerabilities in the entirety of the supply chain [15]. On the contrary, Lam [5] argued that the cause of the increasing number of risks in maritime supply chains is the small, or even lack of coordination between key players, which causes miscommunication and other risks in the process of shipping goods from shippers to consignees. Poor communication and coordination between key players in maritime supply chain activities can have fatal consequences like the wrong destination of the delivery, which make the goods do not reach the hands of the consignee, as the recipient. The worst-case scenario is when the customers feel disappointed, they might cut off subscriptions.

In Table 2, it can be seen that the fourth place, with a relative weight score of 9.09, is the attribute "delay and service error". This risk is closely related to resource management, which is the ability of employees to serve the document needed by the customer promptly. According to the results of the ranking of customer requirements shows in Table 1, the rank of both attributes is in the second and third respectively, which indicates that it is important for employees to not make any mistakes and create a delay in the process of providing the export documents for shippers. In line with the results of the top eight customer requirements attributes in Table 1 in this study, delays and errors in the service process can significantly affect customer satisfaction. If the document is wrong or delivered over what has been promised, the goods cannot get into the port and to the container. Therefore, customer relationship management is an important element of the supply chain and logistics, to reduce the risk of such service errors [14].

Table 2: *Maritime Risks Attributes*

No	Attributes	Relative Weight/Importance	Rank
1	Custom Clearance process	10.39	1
2	Ship's schedules uncertainty	9.74	2
3	Miscommunication & lack of coordination	9.44	3
4	Delay & service errors	9.09	4
5	No ship's space	9.03	5

There is no space left in the ship, with a score of 9.03, has been identified as the fifth risk which going to influence the maritime supply chain of the ocean freight export scheme. One of the experts from the internal case company stated that the risks that have a major impact in influencing the shipper's assessment of the freight forwarding company's performance are the cargo space confirmation from the liner and the freight charges changes during the erratic turn of the month. The unavailability of space means that the transportation facilities to be used for freight delivery to the destination on the selected schedule are not currently available. Transportation availability is a critical factor that can influence maritime freight management decisions, apart from the type and characteristics of freight management, origin, and destination, also the cargo packaging systems [16]. Subhashini and Preetha [17] found a rating result where the availability of cargo space, which is included in the reliability dimension, ranked first on the determination of service quality factors in the ocean freight forwarding business scheme. This means that if there is a problem in providing cargo space on the ship that takes longer to be handled, the longer the goods delivery process will run. Thus, customers will feel harmed, which in the end, can make them become sick of the service performed within the forwarding company and decide not to repeat to use their service again. Besides, this unavailability of cargo space will also lead to delay in delivery [13]. This risk is not the most harmful one, yet the company should still take this risk into account.

B) Second House of Quality

The arrangement of "*Whats*" in the second house of quality is taken from the maritime risk attribute which was previously used as "*Hows*" in HOQ 1, complete with their absolute value and relative importance. Meanwhile, the "*Hows*" section of the second house of quality is compiled from a group of attributes of resilience measures. Using the same stages as the construction of the second house of quality, the final result of the second house of quality is obtained as shown in Figure 3.

From the second HOQ, it can be seen that there is a total of 12 solutions derived from other similar literature which are being cross-checked and discussed with the internal experts of the case company since its business process is unique. However, this study only thoroughly discusses the five best solutions to meet the compatible resiliency within the ocean freight export supply chain, as shown in Table 3. These five resilience measures are also being highlighted in a yellow-coloured box in Figure 3. These five attributes are expected to minimize and eliminate the maritime risks that appear unpredictably, to help the ocean freight export business process to survive in complex maritime supply chain activities.

The first highest resilience measure is the labour’s qualification skills. Its relative importance score is 15.31, which indicates that this attribute is highly recommended to be improved immediately. Based on the research carried out by Berle et al. [18], it is mentioned that the resilience of a third-party logistics company (freight forwarding) can be improved by the flexibility of the company in its operational activities, one of which is by the presence of a workforce that has more than one skill (multi-skills). The internal practitioners advise companies to provide certification programs, preventive education, and other programs for employees to develop their capabilities because according to Buyukozkan and Gizem [19], employees are the main resources that must be developed first.

Labor’s skill qualifications have an explanation that the company must improve and determine the best qualifications for employee-specific abilities that will communicate directly with customers. One of the examples is employees who work as customer service staff, which required interpersonal skills to communicate with two or more people, such as in negotiation activities, cooperation, and so forth. In the process of interpersonal communication that requires the interlocutor, employees are required to be able to send, understand, and receive messages delivered by the interlocutor, which not every human being is able to do so. Therefore, in the process of employee procurement for the customer service section, the company needs to establish the right qualifications from appropriate candidates to support the service performance in order to obtain a high level of customer satisfaction. In the survey that has been conducted to experts from the internal case company, it was obtained that the capabilities of employees are very important in efforts to increase the resilience of freight forwarding business with the relative importance score of 3.67, from a maximum scale of 4.00.

Table 3: *Resilience Measures Attributes*

No	Attributes	Relative weight	Rank
1	Labor’s qualification skills	15.31	1
2	Cooperation with vendor & shipping liner	14.40	2
3	Backup plan	12.73	3
4	Key players coordination	9.11	4
5	Advanced IT system (real-time tracking)	8.19	5

Resilience measures attribute with the second-highest score of relative importance in 14.40 is the "cooperation with vendors & shipping liners". Vendors in this study refer to the truck and warehouse rental companies’ representatives. This attribute is considered important, because it concerns the smoothness of the initial process in the supply chain from shipper to consignee, before transporting it abroad by ship. If the forwarding company only cooperates with both one vendor and shipping liner locally, when they run out of trucks stock, have full warehouse capacity, run out of container and ship space, or even abruptly stop their business, then inevitably, the forwarder has to immediately find another company to transport their goods. As a result, more and more time will be wasted in looking for other suitable companies. Furthermore, due to the limitations of facilities and infrastructure, not all vendors and shipping liners can meet every need of the forwarding company.

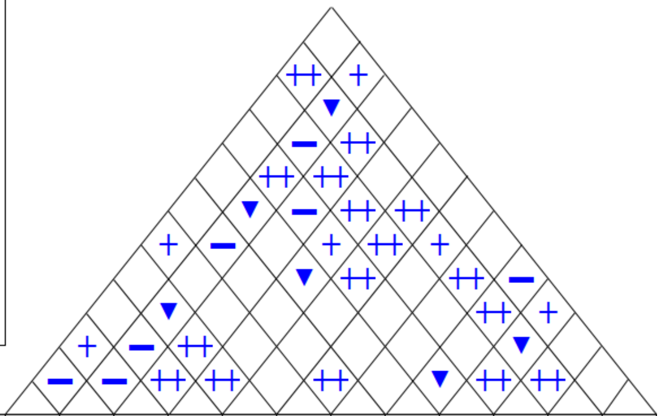
Therefore, sea freight forwarding companies need to cooperate with more than one vendor and shipping liners, in order to have a smooth supply chain. Huang et al. [20] found the high importance level of "service points and networks" to achieve customer satisfaction and avoid risks that can affect the forwarding’s performance. Thus, companies are advised to expand their scope of services and networks through partnerships with major operators and local logistics service providers. In the case of ocean freight forwarding, it requires cooperation with shipping liners and truck rental companies massively and globally, yet the forwarder also has to consider the distance to avoid overspending time. Besides, it is important for freight forwarders to conduct supervision and maintenance of their performances, to sustain the smooth supply chain and avoid any possible risks in the future [19]. In addition to supervision, the company needs to determine the right type of vessel according to the size, equipment, and certification needed for various types of cargo, to avoid vessel failure mode [18]. If there is no continuous monitoring and relying solely on trust, when suddenly their performance is out of control, the impact will arise afterwards and will also harm the forwarder itself.

The backup plan is determined as the third-best solution in reaching ocean freight forwarder's supply chain resilience. To avoid problems that may occur in the maritime supply chain, the manager, as the executive in charge of the division underneath, must play an active role in planning and preparation [21]. The manager needs information from employees in his circle who are directly involved in every stage of service in the field. Information collection can be done through regular communication, thus creating alternative planning that is on target, which can later be implemented by the company to maintain the activities of the maritime supply chain that has been running. It is suggested to make preparation and planning at all major risks that had been identified earlier through the help of overall evaluation [22], [18]. With the creation of backup plans, it means that the company has taken preventive measures to maintain the sustainability of its business before those risks can harm the company's situation.

Lack of coordination can lead to errors in the document interpretation process, therefore, by managing regular communications with maritime supply chain key players, such risks can be reduced [7]. Maritime risk management can be done by carrying out friendly supply chain activities through collaboration with all relevant parties [5]. Following the results in Table 3, with a relative importance score of 9.11, coordination between key players (freight forwarders, shipping liners, port/terminal operators, and Customs) needs to be done by the company as a resilience solution to the overall maritime supply chain. Visibility also becomes very important in building supply chain resilience [23]. Without transparency of information in supply chain activities, disruptions will arise, which leads to failure in reaching the goals. When a risk arises at one stage in the supply chain process, the consequences will spread to other stages, like the domino effect. Therefore, with the collaboration between key players, the supply chain relationship will develop and become more sustainable.

The resilience measures attribute in the fifth rank, with a score of 8.22, is "real-time tracking". In an earlier study, Lu [24] suggested implementing a tactical knowledge-based scheduling system for forwarding companies to support the scheduling process of delivery plans. The provision of real-time control is also a key step [25] for freight forwarding companies in expanding load consolidation, reducing empty vehicle travel, and addressing dynamic disruptions. In 2014, Schenker began to build its information system, it is varied from a system to create a more structured and easily-filled document, a system for tracking employee attendance and performance, and a special website made for customers to track their shipments in real-time. On this website named "*Trackingmore*", customers can check the status of their shipments just by entering the tracking number, then click "search", then they will get the latest real-time location of the goods. Moreover, it has also provided a form that can be filled by customers who want to convey their testimony and complaints to the services that have been provided by the company in any mode of transportation. Nevertheless, not all systems can work properly at any time. The EDI (Electronic Data Interchange) systems, which are commonly used in the process of disseminating information between several companies, do not always show better results for the company's performance [26]. Therefore, particularly in this study, the real-time tracking system has not been the best choice of solutions that must be rushed in implementation by the company in the short-term improvement process, yet it is still prioritized as the five-best rankings out of 12 resilience measures that have been proposed.

Legend		
Θ	Strong Relationship	9
O	Moderate Relationship	3
▲	Weak Relationship	1
++	Strong Positive Correlation	
+	Positive Correlation	
−	Negative Correlation	
▼	Strong Negative Correlation	
▼	Objective Is To Minimize	
▲	Objective Is To Maximize	
x	Objective Is To Hit Target	



Row #	Max Relationship Value in Row	Relative Importance	Absolute Importance	Problem Solving (a.k.a. "Resilience Measures" or "Hows") / Problem Characteristics (a.k.a. "Maritime Risks" or "Whats")	Wrong document handling	Delivery reschedule	Supply chain relationship management	Strategic alliance	Key players coordination	Supervision & Maintenance	Advanced IT system (real-time tracking)	Forecast accuracy	Backup plan	Cooperation with vendor & shipping liner	Labor's qualification skills	Insurance offers to customers
				Column #	1	2	3	4	5	6	7	8	9	10	11	12
				Direction of Improvement: Minimize (▼), Maximize (▲), or Target (x)	▲	▼	▲	▲	▲	▲	▲	x	▲	▲	▲	x
1	9	1.39	74.73	Natural disaster												Θ
2	9	1.39	74.79	Piracy & Terrorism				▲		Θ	Θ					Θ
3	9	4.44	238.24	Ship detention by Government			O	▲			Θ					O
4	3	10.39	557.35	Custom Clearance process			O									O
5	9	0.92	49.59	Ship caught fire			▲			Θ	Θ					Θ
6	9	9.74	522.70	Ship's schedules uncertainty		Θ	Θ	▲	Θ	O		▲	Θ	Θ		
7	9	3.67	196.68	No truck available		O		O	O			Θ	Θ	Θ	Θ	
8	3	6.19	331.98	Congestion in port				O					▲			
9	9	4.13	221.52	Port operational risk						Θ			▲			▲
10	9	9.03	484.30	No ship's space		Θ		Θ	Θ			▲	Θ	Θ	Θ	
11	9	7.65	410.27	No container's stock		O		O	O			Θ	Θ	Θ	Θ	
12	9	4.22	226.26	Wrong & incomplete Documents	Θ									O	Θ	
13	9	4.87	261.20	IT system & network down						Θ	Θ	▲			Θ	
14	9	9.44	506.25	Miscommunication & lack of coordination	Θ	▲		Θ	Θ	O	Θ	▲		Θ	Θ	
15	9	9.09	487.46	Delays & service errors	Θ	▲				▲					Θ	
16	9	7.68	411.85	Misdirected cargo							Θ		Θ	Θ		O
17	9	5.77	309.51	Company's policy								Θ	Θ	O	Θ	O
				Difficulty (0=Easy to Accomplish, 10=Extremely Difficult)	5	7	5	5	5	3	8	2	6	7	8	10
				Max Relationship Value in Column	9	9	9	9	9	9	9	9	9	9	9	9
				Absolute Importance	204.67	221.40	133.11	234.26	287.81	168.48	258.68	186.82	402.10	454.75	483.48	122.36
				Relative Importance	6.48	7.01	4.21	7.42	9.11	5.34	8.19	5.92	12.73	14.40	15.31	3.87

Figure 3: *House of Quality 2*

Based on the experts, improving the information and technology (IT) system based on real-time tracking will make it easier for customers to confirm the location of their shipments in "real-time", i.e., the actual location at the specified time. It can also lighten the employees' duty in the freight forwarding company since it will avoid excessive contact and questions from customers who want to confirm the position of their shipments continuously. Therefore, the company must carry out continuous development on the website to make it more user-friendly [8]. Furthermore, maintenance and regular updates to the information system need to be done by the company to achieve competitive services globally.

IV. Implications

This study has substantially contributed to both theoretical and practical implications. Since there is limited literature that focuses on the freight forwarding case, this study will build on updated literature on this field. In addition, this case study was conducted in one of the countries which prioritize freight delivery through the ocean by utilizing ships. Therefore, this study is applicable for any other forwarders who deliver their freight using ships. As for the practical implication, understanding the results of this research will help managers from related companies with similar risks in searching for any other substitute solutions to maintain their supply chain resiliency, most importantly in the maritime scope. In this case, if certain risk strikes, the company will be ready to take the right action. Not only does this paper provide some actual risks in the maritime supply chain, but it also will help freight forwarding companies in achieving their customer needs. By knowing the customer's needs, freight forwarding companies will create high value in delivering the best service to their customers.

V. Conclusion

Freight forwarding companies, as distributors in the process of moving goods from sender to recipient, have the responsibility for the delivery of the goods properly, safely, and quickly. Shipper, as a user in the process of shipping export goods, has several points to determine which freight forwarding company meets the requirements. Some of the identification points that are considered very important by the shipper are on-time delivery, delivery of shipments to the right destination, complete documents as needed, provision of documents on time, never lost goods at the port or warehouse, fast service, smooth loading-unloading process, and the company that always offers the best solution for the occurred problems.

The risks that can disrupt the customer satisfaction of the freight forwarding company come from three different categories, such as the internal business operations (included all the possible internal sources whether from the people, service, or system), the external environment (included unexpected third party and natural phenomenon), and lastly is any possible issues come from within the whole maritime supply chain process which engages other recognized third parties. While the top five risks which going to highly affect the freight forwarder's business process are the customs clearance process, the uncertainty of ship schedules, miscommunication and lack of coordination, delays and errors in service, also not getting any ship spaces.

Not all risks can be dodged or even eliminated, consequently, the company needs to forecast and predict any risks that may harm the overall company's service performance by making immediate actions. The five resilience measures found in this study to minimize the maritime risks are included fulfilling the qualifications of human resources skills, expanding cooperation with vendors and shipping liners, creating appropriate backup plans, maintaining good coordination between key players, and frequently developing real-time tracking systems.

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On Reliability Structures with Two Common Failure Criteria Under Age-Based Maintenance Policy

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Abstract

In this work we study reliability structures with two common failure criteria under specific age-based preventive maintenance models. The aforementioned systems, which consist of n independent components, fail upon the occurrence of two different scenarios. The theoretical framework for delivering the expected cost rate of such structures is presented in detail, while a variety of numerical outcomes for different choices of the design parameters are also provided and discussed.

Keywords: reliability systems with two failure criteria; preventive maintenance policy; Samaniego's signature; cost rate.

I. Introduction

In the area of Reliability Engineering, an enthralling quest calls for the design of appropriate structures, which are related to real-life applications or existing devices and contrivances. A particular group of reliability models, which seems to reel in the scientists during the last decades, is the family of consecutive-type systems with two stopping rules. Due to the abundance of their applications in Engineering and Statistical Modelling, the aforementioned structures comprise an engrossing scope of research activity.

The general framework of constructing a consecutive-type system with two common stopping rules requires *n* linearly or circularly ordered units. The resulting system fails, whenever a pre-specified condition (out of two ones) is satisfied or both of them are met. In such a framework, several structures have been already introduced in the literature. For example, the (n, f, k) structure proposed in [1], fails if, and only if, there exist at least *f* failed units or at least *k* consecutive failed units. Several reliability characteristics of the (n, f, k) systems are studied in detail in [2] or [3]. Among others, the $<n, f, k>$ structure (see, e.g. [4] or [5]), the constrained (*k*, *d*)–out–of–*n*: *F* system (see, e.g. [6] or [7]) are well-known consecutive-type reliability systems with two failure criteria. For a detailed and up-to-date survey on the consecutive-type systems, we refer to the detailed reviews offered [8] or [9] and the well-documented monographs devised by [10] or [11]. Some recent advances on the topic can be found in the works [12], [13] or [14]. A survey of reliability approaches in various fields of Engineering and Physical Sciences is also provided in [15].

In the present work, we study the (n, f, k) and $<n, f, k>$ structures under specific age-based preventive maintenance policy. In Section 2, the general framework for obtaining the expected cost rate of the underlying reliability systems is presented in detail. In Section 3, extensive numerical experimentation is accomplished in order to shed light on the behavior of the cost rate of the aforementioned reliability structures under different choices for their design parameters. Finally, the Discussion section summarizes the contribution of the present manuscript, while some interesting conclusions based on previous sections are also highlighted.

II. The theoretical framework for delivering the expected cost rate of (*n,f,k*) and <*n,f,k*> structures

Let us first consider the (n,f,k) structure consisting of *n* independent and identically distributed (*i.i.d.*) components $X_1, X_2, ..., X_n$ ordered in a line ($f > k$). The particular system fails if, and only if, there exist at least *f* failed components or at least *k* consecutive failed components (see, e.g. [1]). We next denote by *T* the lifetime of the (n,f,k) system with distribution *F*, while the lifetimes $T_1, T_2, ..., T_n$ of its components share a common exponential distribution *G* with parameter *λ*, namely

$$G(t) = P(T_i \le t) = 1 - e^{-\lambda t},\ i = 1,2,..,n. \tag{1}$$

If we denote by $T_{1:n}, T_{2:n}, ..., T_{n:n}$ the corresponding ordered lifetimes of the components, the signature of the system is defined as the probability vector $(s_1(n), s_2(n), ..., s_n(n))$ with

$$s_i(n) = P(T = T_{i:n}),\ i = 1,2,..., n. \tag{2}$$

Following the so-called age-based maintenance policy, the system is replaced either at a pre-specified time t_0 with cost c_0 or at its failure with cost c_1 (whichever comes first). It is common to assume that $c_0 < c_1$. Denoting by *c* the corresponding replacement cost of a single component, we next provide the so-called expected cost rate of the (n,f,k) structure with exponentially distributed components, namely its expected cost per time unit. The proposed procedure is based on the signature vector of the underlying system.

Proposition 1. Let us consider the (n,f,k) structure consisting of *n* independent components sharing a common Exponential distribution with mean *λ*. The expected cost rate of the underlying system under the age-based preventive maintenance policy is given by

$$ECR(t_0) = \frac{c_0 + (c_1 - c_0)\sum_{i=k}^{n} s_i(n)\left(1 - \sum_{j=n-i+1}^{n}\binom{n}{j}e^{-\lambda j t_0}(1-e^{-\lambda t_0})^{n-j}\right) + c\sum_{i=k}^{n}\left(\sum_{m=1}^{i} s_m(n)\right)\left(1 - \sum_{j=n-i+1}^{n}\binom{n}{j}e^{-\lambda j t_0}(1-e^{-\lambda t_0})^{n-j}\right)}{\sum_{i=k}^{n} s_i(n)\int_0^{t_0}\sum_{j=n-i+1}^{n}\binom{n}{j}e^{-\lambda j v}(1-e^{-\lambda v})^{n-j}dv} \tag{3}$$

Proof. For a coherent system consisting of *n* components with lifetime *T* and signature $p_i(n), i = 1,2,...,n$, the expected cost rate for the age-based preventive maintenance policy is determined via the following (see [16])

$$ECR(t_0) = \frac{c_0 + (c_1 - c_0)\sum_{i=1}^{n} s_i(n)P(T_{i:n} \le t_0) + c\sum_{i=1}^{n}\left(\sum_{m=1}^{i} s_m(n)\right)P(T_{i:n} \le t_0)}{\sum_{i=1}^{n} s_i(n)\int_0^{t_0} P(T_{i:n} > v)dv}. \tag{4}$$

Lifetimes $T_1, T_2, ..., T_n$ of the components are random variables with exponential distribution with parameter *λ*, namely $T_i \sim G, i = 1,2,...,n$, where *G* is defined in (1). Consequently, the following holds true (see, e.g. [17])

$$P(T_{i:n} > t) = \sum_{j=n-i+1}^{n} \binom{n}{j} e^{-\lambda jt} (1-e^{-\lambda t})^{n-j} \tag{5}$$

Since the first (k-1) coordinates of the signature vector of the (n, f, k) structure with n independent components ($f > k$) are equal to zero (see, e.g. [3]), the desired result is readily obtained by the aid of equations (4) and (5).

□

We next consider the $< n, f, k >$ structure with *i.i.d.* linearly ordered units $X_1, X_2, ..., X_n$ ($f > k$). The particular structure has been introduced in [4] and involves two common stopping rules. More precisely, the $< n, f, k >$ structure consists of n components and fails if, and only if, there exist at least f failed components and at least k consecutive failed components. Denoting by T^* the lifetime of the $< n, f, k >$ system with distribution F and by $T_1^*, T_2^*, ..., T_n^*$ the lifetimes of its components with Exponential distribution G as defined in (1), we next determine its expected cost rate under preventive maintenance strategy. More precisely, if we assume that the signature vector of the $< n, f, k >$ system is given by $(s_1^*(n), s_2^*(n), ..., s_n^*(n))$, the following proposition sheds light on the aforementioned issue.

Proposition 2. Let us consider the $< n, f, k >$ structure consisting of n independent components sharing a common Exponential distribution with mean λ. The expected cost rate of the underlying system under the age-based preventive maintenance policy is given by

$$ECR^*(t_0) = \frac{c_0 + (c_1 - c_0)\sum_{i=f}^{n} s_i^*(n) \left(1 - \sum_{j=n-i+1}^{n} \binom{n}{j} e^{-\lambda j t_0} (1-e^{-\lambda t_0})^{n-j}\right) + c\sum_{i=f}^{n}\left(\sum_{m=1}^{i} s_m^*(n)\right)\left(1 - \sum_{j=n-i+1}^{n} \binom{n}{j} e^{-\lambda j t_0} (1-e^{-\lambda t_0})^{n-j}\right)}{\sum_{i=f}^{n} s_i^*(n) \int_0^{t_0} \sum_{j=n-i+1}^{n} \binom{n}{j} e^{-\lambda j v} (1-e^{-\lambda v})^{n-j} dv} \tag{6}$$

Proof. We first take into account that the first (f-1) coordinates of the signature vector of the $< n, f, k >$ structure with n independent components ($f > k$) are equal to zero (see, e.g. [5]). Following a parallel argumentation as the one presented in the proof of Proposition 1, the outcome is readily deduced.

□

III. Numerical Results

In the present section, we run through extensive numerical experimentation in order to study the cost behavior of the (n, f, k) and $< n, f, k >$ structures with exponentially distributed components. Based on the theoretical results proved previously, we compute the expected cost rate of the aforementioned reliability systems for several choices of their design parameters under preventive maintenance policy.

Table 1 displays several numerical results referring to the *ECR* function defined earlier for the (n, f, k) structure with exponentially distributed components under a pre-specified preventive maintenance policy with constants c, c_0, c_1.

Table 1: *The ECR-values of the (n,f,k) system with exponentially distributed components under preventive maintenance policy (c=0.1, c₀=1, c₁=3).*

n	(f,k)	t_0	λ=1	λ=0.8	λ=0.6	λ=0.4	λ=0.2
5	(3,2)	0.01	90.6672	92.5294	94.3938	96.2603	98.1291

		0.02	40.6946	42.5469	44.4036	46.2647	48.1302
		0.03	24.0554	25.8978	27.7468	29.6024	31.4646
		0.04	15.7497	17.5821	19.4233	21.2734	23.1324
		0.05	10.7774	12.5997	14.4332	16.2778	18.3340
		0.06	7.4719	9.2842	11.1098	12.9489	14.8012
		0.07	5.1191	6.9210	8.7389	10.5723	12.4213
		0.08	3.3615	5.1532	6.9631	8.7910	10.6367
5	(4,2)	0.01	90.7073	92.5615	94.4178	96.2764	98.1371
		0.02	40.7343	42.5788	44.4276	46.2807	48.1382
		0.03	24.0944	25.9293	27.7707	29.6184	31.4726
		0.04	15.7876	17.6131	19.447	21.2894	23.1404
		0.05	10.814	12.6301	14.4566	16.2937	18.1415
		0.06	7.5069	9.3136	11.1329	12.9647	14.8092
		0.07	5.1521	6.9495	8.7614	10.588	12.4293
		0.08	3.3924	5.1805	6.9852	8.8065	10.6447
6	(3,2)	0.01	87.7976	90.2437	92.687	95.1275	97.5652
		0.02	37.7633	40.2215	42.6744	45.1218	47.5638
		0.03	21.0636	23.5333	25.9954	28.4496	30.8957
		0.04	12.6984	15.179	17.6499	20.1107	22.5609
		0.05	7.66759	10.1587	12.6381	15.1053	17.5595
		0.06	4.30444	6.8056	9.29326	11.7666	14.2248
		0.07	1.89456	4.40538	6.901	9.38041	11.8425
		0.08	0.08076	2.60088	5.1042	7.58951	10.0554
6	(4,2)	0.01	87.898	90.3241	92.7473	95.1677	97.5853
		0.02	37.8624	40.3013	42.7345	45.162	47.5838
		0.03	21.16	23.6119	26.0551	28.4897	30.9158
		0.04	12.7908	15.2556	17.7089	20.1507	22.581
		0.05	7.7548	10.2327	12.696	15.145	17.5796
		0.06	4.38558	6.8763	9.34977	11.8059	14.2448
		0.07	1.9686	4.4723	6.95589	9.41925	11.8625
		0.08	0.14711	2.66363	5.15724	7.62781	10.0753
7	(3,2)	0.01	90.6864	92.5489	94.4115	96.2743	98.1371
		0.02	40.6889	42.5502	44.412	46.2744	48.1371
		0.03	24.0266	25.8857	27.7463	29.608	31.4705
		0.04	15.6993	17.5555	19.4143	21.2751	23.1372
		0.05	10.7069	12.5594	14.416	16.2756	18.1373
		0.06	7.38259	9.23085	11.0847	12.9429	14.804
		0.07	5.01207	6.85542	8.70612	10.5626	12.4231
		0.08	3.2381	5.07595	6.92314	8.7777	10.6375

Based on Table 1, we next deduce some concluding remarks. More precisely, the *ECR* function of the (n,f,k) structure with exponentially distributed components (with mean λ), under the preventive maintenance policy with cost constants c, c_0, c_1 seems to:

- increases as λ decreases (for pre-fixed values of t_0,n,f,k,c,c_0,c_1)
- decreases as t_0 increases (for pre-fixed values of λ,n,f,k,c,c_0,c_1)
- decreases as n increases (for pre-fixed values of $t_0,\lambda,f,k,c,c_0,c_1$)
- increases as f increases (for pre-fixed values of $t_0,\lambda,n,k,c,c_0,c_1$)

We next investigate the impact of parameters c, c_0, c_1 on the *ECR*-behavior of the resulting (n,f,k)

structure. Table 2 provides several numerical results referring to the *ECR* function of the (n, f, k) structure with exponentially distributed components for pre-specified design parameters *n*, *f*, *k*.

Table 2: *The ECR-values of the (n,f,k) system with exponentially distributed units (λ=0.5) under pre-specified design (n=5, f=4, k=2, t_0=0.02).*

	Parameter c_1			
c_0	***c_1=2***	***c_1=3***	***c_1=4***	***c_1=5***
0.5	21.3479 20.6849 20.022	19.3529 18.6899 18.027	17.3579 16.6949 16.032	15.3629 14.6999 14.037
0.75	34.3482 33.6853 33.0224	32.3532 31.6903 31.0274	30.3582 29.6953 29.0324	28.3632 27.7003 27.0374
1	47.3486 46.6857 46.0228	45.3536 44.6907 44.0278	43.3586 42.6957 42.0328	41.3636 40.7007 40.0378
1.25	60.349 59.6861 59.0232	58.354 57.6911 57.0282	56.359 55.6961 55.0332	54.364 53.7011 53.0382
1.5	73.3494 72.6865 72.0235	71.3544 70.6915 70.0285	69.3594 68.6965 68.0335	67.3644 66.7015 66.0386
1.75	86.3498 85.6869 85.0239	84.3548 83.6919 83.0289	82.3598 81.3598 81.0339	80.3648 79.7019 79.0389

Each cell contains the *ECR*-values for *c*=0.1(upper entry), *c*=0.2 (middle entry), *c*=0.3 (lower entry)

Based on Table 2, we deduce that the *ECR* function of the (n, f, k) structure with exponentially distributed components, under a pre-specified design, seems to:

- decreases as c_1 increases (for pre-fixed values of remaining parameters)
- increases as c_0 increases (for pre-fixed values of remaining parameters)
- decreases as *c* increases (for pre-fixed values of remaining parameters).

On the other hand, it is of some interest to shed light on the *ECR* behavior of the $< n, f, k >$ structure with *i.i.d.* components. Table 3 displays several numerical results referring to the *ECR* function defined earlier for the $< n, f, k >$ structure with exponentially distributed components under a pre-specified preventive maintenance policy with constants *c*, c_0, c_1.

Table 3: *The ECR-values of the <n,f,k> system with exponentially distributed components under preventive maintenance policy (c=0.1, c_0=1, c_1=3).*

n	***(f,k)***	***t_0***	***λ=1***	***λ=0.8***	***λ=0.6***	***λ=0.4***	***λ=0.2***
5	(3,2)	0.01	90.7744	92.6296	94.4799	96.3252	98.1653
		0.02	40.7146	42.5902	44.4572	46.3148	48.1626
		0.03	23.9937	25.8871	27.769	29.6381	31.4933
		0.04	15.6114	17.5201	19.4153	21.2951	23.1574
		0.05	10.5674	12.4891	14.3962	16.2858	18.1549
		0.06	7.19479	9.12734	11.0449	12.9436	14.8191
		0.07	4.77896	6.72036	8.64706	10.554	12.4357
5	(4,2)	0.01	91.135	92.9183	94.6964	96.4695	98.2374
		0.02	41.0719	42.8775	44.6733	46.4591	48.2348
		0.03	24.3443	26.1712	27.984	29.7822	31.5655
		0.04	15.9522	17.7992	19.6284	21.4388	23.2296

		0.05	10.8956	12.7617	14.6066	16.4288	18.227
		0.06	7.50793	9.39205	11.2519	13.0856	14.8911
		0.07	5.07495	6.97589	8.85013	10.6949	12.5076
6	(3,2)	0.01	87.9497	90.3833	92.8053	95.2155	97.6138
		0.02	37.8079	40.2909	42.7524	45.1916	47.6078
		0.03	21.0074	23.5361	26.0347	28.5016	30.9351
		0.04	12.5478	15.1186	17.6521	20.1455	22.5958
		0.05	7.4285	10.0382	12.6045	15.1232	17.5899
		0.06	3.9823	6.6281	9.2251	11.7681	14.2508
		0.07	1.4949	4.1738	6.7997	9.36582	11.8641
6	(4,2)	0.01	88.3513	90.7047	93.0463	95.3761	97.694
		0.02	38.2043	40.6103	42.993	45.3524	47.6881
		0.03	21.3927	23.8502	26.2735	28.662	31.0154
		0.04	12.9166	15.4245	17.8876	20.3051	22.6762
		0.05	7.77636	10.3333	12.8356	15.2817	17.6703
		0.06	4.3054	6.91004	9.45072	11.9251	14.331
		0.07	1.78962	4.44048	7.01874	9.52098	11.9442

Based on Table 3, we next deduce some concluding remarks. More precisely, the *ECR* function of the $<n, f, k>$ structure with exponentially distributed components (with mean λ), under the preventive maintenance policy with cost constants c, c_0, c_1 seems to:

- increases as λ decreases (for pre-fixed values of $t_0, n, f, k, c, c_0, c_1$)
- decreases as t_0 increases (for pre-fixed values of $\lambda, n, f, k, c, c_0, c_1$)
- decreases as n increases (for pre-fixed values of $t_0, \lambda, f, k, c, c_0, c_1$)
- increases as f increases (for pre-fixed values of $t_0, \lambda, n, k, c, c_0, c_1$).

We next investigate the impact of parameters c, c_0, c_1 on the *ECR*-behavior of the resulting $<n, f, k>$ structure. Table 4 provides several numerical results referring to the *ECR* function of the $<n, f, k>$ structure with exponentially distributed components for pre-specified design parameters n, f, k.

Table 4: *The ECR-values of the <n,f,k> system with exponentially distributed components (λ=0.5) under pre-specified design (n=6, f=4, k=2, t_0=0.02).*

	Parameter c_1			
c_0	***c_1=2***	***c_1=3***	***c_1=4***	***c_1=5***
0.5	21.5749 21.1719 20.769	19.5601 19.1571 18.7542	17.5453 17.1424 16.7394	15.5305 15.1276 14.7246
0.75	34.5786 34.1756 33.7727	32.5638 32.1608 31.7579	30.549 30.1461 29.7431	28.5342 28.1313 27.7283
1	47.5823 47.1793 46.7764	45.5675 45.1645 44.7616	43.5527 43.1497 42.7468	41.5379 41.135 40.732
1.25	60.586 60.183 59.78	58.5712 58.1682 57.7653	56.5564 56.1534 55.7505	54.5416 54.1387 53.7357
1.5	73.5897	71.5749	69.5601	67.5453

	73.1867 72.7837	71.1719 70.769	697542 68.7542	67.1424 66.7394
1.75	86.5933 86.1904 85.7874	84.5786 84.1756 83.7727	82.5638 82.1608 81.7579	80.549 80.1461 79.7431

Each cell contains the *ECR*-values for c=0.1(upper entry), c=0.2 (middle entry), c=0.3 (lower entry)

Based on Table 4, we deduce that the *ECR* function of the $<n, f, k>$ structure with exponentially distributed components, under a pre-specified design, seems to:

- decreases as c_1 increases (for pre-fixed values of remaining parameters)
- increases as c_0 increases (for pre-fixed values of remaining parameters)
- decreases as c increases (for pre-fixed values of remaining parameters).

IV. Discussion

In the present paper, we focus on reliability structures with two common failure criteria. More precisely, the so-called (n, f, k) and $<n, f, k>$ structures are considered, while the corresponding expected cost rate under preventive maintenance policy is determined. The numerical results are produced by the aid of the theoretical framework given in Section 2 of the present manuscript. The influence of the design parameters of (n, f, k) and $<n, f, k>$ structures on their cost behavior is studied and some concluding remarks are also provided. All design parameters seem to influence the cost behavior of the underlying structures. Finally, a parallel reliability study of different consecutive-type structures is among future plans for the authors.

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Intelligent Valve Fault Diagnosis Approach for Reciprocating Compressor Based on Acoustic Signals

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Abstract

Reciprocating compressor is one of the critical components in petrochemical, process and gas storage /transportation industry. Due to the dynamic operating conditions, valve failures of the compressors are one of the most frequent failures that causes around 36% of the unplanned shutdowns for reciprocating compressors. For reliable operations of the compressor, a condition-based fault diagnosis model is proposed in this study. Majority of the existing studies are based on vibration, pressure and other intrusive sensors, which interferes with the system dynamics. Hence, non-intrusive acoustic sensor-based signal analysis technique is tested in this study. Due to the non-stationary nature of the acoustic signal obtained from the working compressor, it is quite difficult to extract fault-specific information. Hence, Minimum Entropy Deconvolution Adjusted (MEDA)-Empirical Mode Decomposition (EMD) approach is proposed for valve-fault diagnosis of reciprocating compressor. After enhancing the impulsiveness of the raw signals with the help of MEDA, a set of IMFs is obtained through EMD method. From this IMF-set, information rich IMF is selected based on the kurtosis value. For extraction of fault related information, two time-domain features (RMS and sample entropy) are used. The results show that MEDA can enhance the signal features effectively which is critical for fault detection. Compared to EMD method, the proposed methodology has shown promising results for inlet and outlet valve-fault diagnosis of reciprocating compressor.

Keywords: *Condition based health monitoring, Fault diagnosis, Reciprocating compressor, Minimum Entropy Deconvolution Adjusted (MEDA), Empirical Mode Decomposition (EMD)*

I. Introduction

In petrochemical, gas transportation and process industries, reciprocating compressor are considered to be one of the most heavily used equipment for fluid compression. In order to fulfil industrial demands economically, reciprocating compressors are run at its maximum capacity and without any backup in many cases. Hence, its reliability is critical to avoid unscheduled production shutdowns. However, due to its complex structure and dynamic nature of operation, even with advanced designs and materials, failure is inevitable. With the advent of industry 4.0 associated technologies, organizations are interested in adopting automated technical systems to monitor critical equipment. Such automated condition monitoring systems can help in reducing the on-site inspection risks as well as labor costs.

Condition based health monitoring and fault diagnosis has attracted much attention due to its unparallel benefits with Internet of Things (IoT) and sensor technology. This helps in understanding system behavior and taking preventive steps to avoid progression of early faults to system failure. For monitoring system health state various physical properties, such as vibration, pressure, current, temperature, are measured via different types of sensors. Selection of appropriate sensor is of importance for accurate diagnosis respective fault. In case of reciprocating compressor, around 36% of unscheduled maintenance are caused due to suction and discharge valve failure, which is the most common reason [1]. According to [2], observations from real case study of reciprocating compressor, likely reasons for valve are dynamic operating conditions, improper valve installation, incorrect choice of valve parameters and pressure pulsations in transportation tubes. It is crucial to diagnose a valve failure as it can cause exhaust pressure drop which can eventually lead to functional failure.

Several condition-based health monitoring studies addressing the valve failure detection have been proposed. Due to the non-stationary nature of the signals collected from reciprocating compressor [3], many signal processing techniques employing vibration analysis [4–7], temperature [8], pressure-volume analysis [9], angular speed based analysis [10] and acoustic analysis [11–13] have been employed for fault diagnosis. In [14], a basis pursuit based denoising approach and wave-matching based feature extraction approach is proposed to diagnose reciprocating compressor faults. Vibration, current and pressure signals of reciprocating compressor were analyzed through Teager-Kaizer Energy operator (TKEO) based feature extraction approach in [15] for valve fault diagnosis. Furthermore, for non-stationary signal analysis, Empirical Mode Decomposition (EMD) [3,16], Short-Time Fourier Transform [17,18], wavelet transform [19–22], and Wigner-Ville distribution [23,24] are some of the most popular techniques for feature extraction. For fault diagnosis of reciprocating compressor used in refrigerators, Yang et al. [25] employed wavelet transform for vibration signal denoising and feature extraction. Although wavelet analysis can extract useful features from the non-stationary signals, for effective results optimal wavelet basis and number of layers for signal decomposition needs to be selected beforehand [26]. These parameter values are critical as quality of decomposition depends on it [27]. However, the problem of parameter selection can be avoided in EMD technique as it is self-adaptive analysis method that uses intrinsic characteristics of signal [28]. It has been successfully applied for compressor fault detection in combination with autocorrelation function [29]. Later, EMD coherence based fault diagnosis approach was proposed by the same authors in [30] for eliminating the cylinder signal interference. In [31], proposed rational Hermite interpolation based EMD technique to diagnose valve failure.

Most of these studies requires the sensors to be mounted in contact with the system for data acquisition. Due to such intrusive nature of these sensors, researchers find it inconvenient. To avoid this problem, several studies have employed non-intrusive acoustic sensors for fault diagnosis of reciprocating compressor. In [11], statistical features are extracted to diagnose valve failure. For varying pressure conditions, fault diagnosis approach is proposed in [32] based on acoustic emission signals. Effects of gas and fluid leakage through types of valve are studied based on acoustic emission variations in [33,34]. Moreover, for pipeline leakage detection also acoustic analysis has found to be effective [35,36]. Unlike rotary systems such as bearings and gearboxes, reciprocating compressors has rotary as well as reciprocating components which resonates huge background noise in acoustic signals [37]. Hence, extraction of efficient and characteristic features for fault diagnosis of reciprocating compressor poses a great challenge. Verma et al. [13] has conducted an extensive study on reciprocating compressor fault diagnosis based on acoustic signals. This study has employed wavelet-based feature extraction approach and used different feature selection approach (such as mutual information, Bhattacharya distance and principal component analysis) to input optimal features for SVM variant-based fault diagnosis approach. However, it is to be noted that the acoustic signals collected from reciprocating compressors have periodic impulses generated by valve opening and closing. In case of valve failure, non-linearity and randomized periodic impulses generates harmonic components in the signal. The traditional feature extraction techniques (such as wavelet transform) are not efficient enough to extract characteristic features from the signal containing high background noise. Hence, it is imperative to propose an advanced fault diagnosis approach for identifying valve faults of reciprocating compressor.

In this paper, Minimum Entropy Deconvolution Adjusted (MEDA)-EMD based fault diagnosis approach is proposed for detection of valve fault in reciprocating compressor. First, raw signal is preprocessed through MEDA technique. Then the deconvolved signal is decomposed into intrinsic mode functions (IMFs) with the help of EMD to extract significant frequency components for fault diagnosis. Condition Indicators (CI) are extracted from the selected IMF to separate the healthy and fault state of the system. The rest of the paper is structured as: Section II briefly discusses the theoretical background of MEDA and EMD techniques and proposed methodology. The following section presents the proposed methodology. The adopted dataset of reciprocating compressor is discussed the next section. Section IV provides discussion on the obtained results and analyses it. In the following conclusion section, summary of the paper is presented.

II. Methods

It is critical to figure out fault features from the acquired raw signal. In order to preprocess the non-stationary acoustic signals, MEDA and EMD techniques are employed in this paper. This section discusses both the signal processing techniques, MEDA and EMD, and CIs used for fault detection in brief.

I. Minimum Entropy Deconvolution Adjusted (MEDA)

MED technique is designed to enhance the spike-like features and localize the impulse response frequencies closer to the cause of original impulse [38]. This technique has yielded promising results for seismic signal- [39], machinery fault diagnosis [40–42]. However, convolution definition assumption in MED technique generated discontinuity in output results for rotating machinery [43]. To mitigate the discontinuity effects, Auto Regressive model based preprocessing was proposed by Endo H. et al. [41] before applying MED to rotary machine signals. Another, solution, by adjusting the convolution definition, was proposed in [43] to tackle this limitation. The MEDA approach is as follows:

For measured raw signal $\vec{x}$, finite length filter $\vec{f}$ is found out in order to enhance the source fault impulse (having higher kurtosis values) and undermine the noise and system dynamics (having lower kurtosis values). $\vec{y}$ is filtered output signal of length N datapoints.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \qquad \vec{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix} \tag{1}$$

$$\vec{y} = \vec{f} * \vec{x} \tag{2}$$

$$Y_k = \sum_l^L f_l x_{k+L-l}, \qquad k=1, 2, \dots, N-L+1 \tag{3}$$

Matrix form of above expressions (1-3) is presented in expression (4):

$$\vec{y} = X_0^T \vec{f} \tag{4}$$

$$\text{Where, } X_0 = \begin{bmatrix} x_1 & x_{L+1} & x_{L+2} & \cdots & x_N \\ x_{L-1} & x_L & x_{L+1} & \cdots & x_{N-1} \\ x_{L-2} & x_{L-1} & x_L & \cdots & x_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_{N-L+1} \end{bmatrix}_{L\ by\ N-L+1}$$

Iterative selection of MEDA results into expression (5).

$$\vec{f} = \frac{\sum_{n=1}^{N-L} y_n^2}{\sum_{n=1}^{N-L} y_n^4} (X_0 X_0^T)^{-1} X_0 \, [y_1^3 \; y_2^3 \; y_3^3 \; \ldots \; y_{N-L}^3]^T \quad (5)$$

Initially $\vec{f}$ is assigned [0, …, 0, 1, -1, 0, …, 0] and iteratively find out $\vec{y}$ from the above equations for new filters in every iteration.

In previous studies, MED based approaches have been successfully applied for early fault detection of bearing [44] and compressor system [42]. Inspired from the efficacy of the technique, MEDA is employed in this study for signal processing. The performance of MEDA technique is presented in the results section IV.

II. Empirical Mode Decomposition (EMD)

EMD is an adaptive signal decomposition technique that works on the basis of oscillatory components [45]. For identifying the fault information, EMD does not necessitates signal stationarity, periodicity or linearity information beforehand. EMD is used to decompose multi-component signal into mono-component Intrinsic Mode Functions (IMFs) [46]. IMFs are a set of frequency bands that constitute the multi-component signal. As per the assumptions on the basis of which EMD is developed, any non-stationary signal can be expressed as sum of IMFs and residual component. It can be expressed in mathematical form as following (equation (6)):

$$x(t) = \sum_{n=1}^{N} IMF_n(t) + res(t) \quad (6)$$

Where, $IMF_n(t)$ and res(t) denotes n^{th} IMF and residual component. It is to be noted that, to qualify as an IMF, a constituent function (extracted from raw signal) must satisfy the following two conditions: (i) the count of extrema and zero-crossings should be equal or defer by one at most. (ii) mean value of envelope developed by maxima and minima must be zero at any point. The steps of EMD technique are as follows:

(1) Detect all minima and maxima points of original signal x(t).

(2) Build lower envelope (E_{lower}(t)) and upper envelope (E_{upper}(t)) by using cubic spline interpolation technique for minima and maxima data points.

(3) Calculate mean envelope by using the equation: $E_{mean}(t) = (E_{lower}(t)+ E_{upper}(t))/2$.

(4) Compute $s(t) = x(t) – E_{mean}(t)$.

(5) Verify the two conditions for s(t) if is qualified to be an IMF.

(6) If s(t) is qualified to be an IMF, then $IMF_n(t) = s(t)$ and repeat the procedure from step (1) for residual signal $res(t) = x(t) – s(t)$.

Else, replace x(t) with s(t) and repeat the procedure.

(7) Repeat steps (1-6) until monotonic res(t) is obtained.

The set of $IMF_n(t)$ contains frequency components such that $IMF_1(t)$ represents highest frequency component and subsequent IMFs express lower frequency components.

III. Condition Indicators

In order to extract fault information from the IMFs, various feature extraction approaches have been developed. Root mean square (RMS) and entropy are one of the most commonly used condition indicators (features) for compressor faults [42,47]. A short description on these condition indicators, used in this study, is presented below.

(1) Root Mean Square (RMS):

RMS is the square root value of the mean of the sum of squares of signal. It is generally used to evaluate the signal amplitude and signal energy in time domain [48]. Formula to calculate is presented as equation (7).

$$\text{RMS} = \sqrt{\frac{1}{N}\sum_{t=1}^{N}(x(t))^2} \tag{7}$$

(2) Sample entropy:

Sample entropy is used for evaluation of signal complexity that reflects the probability of novel pattern generation in accordance with the signal variation [49]. This method was proposed by Richman and Moorman [50] to overcome the limitations of approximate entropy and it has been quite popular since [27]. Sample entropy is independent of data length (which adversely affected the performance of approximate entropy) and it could tackle the signal noise effectively. Steps to calculate sample entropy are as follows:

For a given time series signal {z(t), t=1, 2, ..., N}, at time i and dimensional vector m can be expressed as (equation (8)):

$$Z_t^m = \{z(t), z(t+1\}, \ldots, z(t+(m-1))\},\ \ t = 1, 2, \ldots, N-(m-1) \tag{8}$$

Where, Z_t^m represents new time series, τ denotes time delay and m is embedding dimension.

(1) Compute $d_{ij} = \max_{0\le k\le m-1} |x(t+k) - x(j+k)|, \quad 1 \le t, j \le N-m+1;$

(2) Calculate $C_t^m(r) = \frac{Num(d_{ij} \le r)}{N-m-1}$ for all x(t). Here, r is tolerance value and $Num(d_{ij} \le r)$ is count of distances qualifying the $d_{ij} \le$ r constraint.

(3) Compute $B^m(r) = \frac{1}{N-m}\sum_{t=1}^{N-m} \ln(C_t^m(r))$

(4) Take embedded dimension as m+1, and repeat steps (1-3) to compute $B^{m+1}(r)$.

(5) Obtain sample entropy: SampE (m, r, N) = $-\ln\frac{B^{m+1}(r)}{B^m(r)}$.

III. Proposed Methodology

Flowchart of the proposed fault diagnosis algorithm is illustrated below in Figure 1. First, machinery signals are extracted through sensitive and accurate sensors. The details of sensor type and data acquisition is discussed in detail with experimental setup in Section 4. After acquiring raw signal dataset, MEDA is applied to the raw signal for enhancement of signal. It is to be noted that selection of filter size is reflected on the results obtained by MEDA. Hence, after careful investigations of different filter size cases, it is selected as 10 for this study. The filtered signal is decomposed by EMD method and IMFs are obtained. From the set of IMFs, IMF with rich fault information is selected on the basis of kurtosis value. Next, for capturing the fault dynamics condition indicators, RMS and sample entropy, are extracted from the selected IMFs. The difference in these condition indicators is used for fault diagnosis of the system.

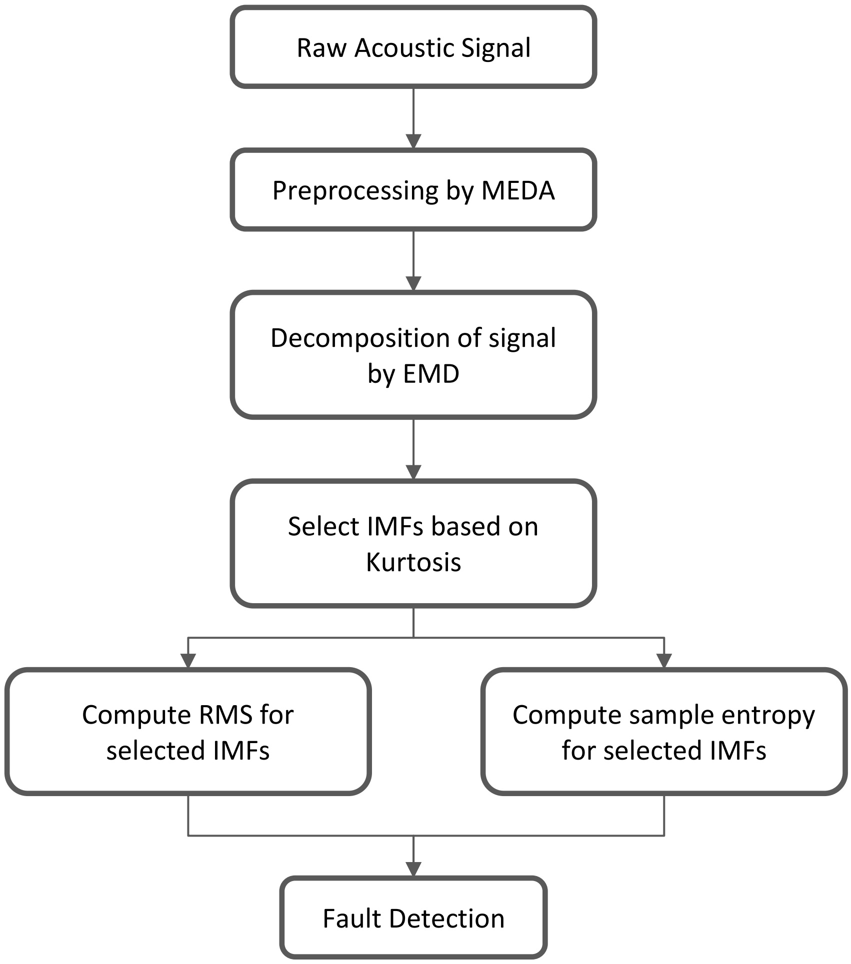

Figure 1: *Steps of Proposed Methodology*

VI. Experimental results and discussion

In order to validate to test the proposed methodology, a single stage reciprocating compressor dataset is used. This study is validated with the same dataset used in [13]. The compressor used for the experiments was run by 5HP-1440 rpm (415V, 50 Hz, 5 amp) induction motor and had air pressure range of 0-35 kg/cm^2. The pressure switch used in the experiment was of Type PR-15 with pressure range of 100-213 psi. Acoustic sensor readings for healthy state and two faults (Leakage in Inlet Valve (LIV) and Leakage in Outlet Valve (LOV)) are acquired. The dataset was collected at sampling frequency of 50 kHz.

For testing the effectiveness of the proposed algorithm, traditional EMD method and proposed method are applied to the raw acoustic signal. In order to visualise the effectiveness of signal enhancement, the raw signals and MEDA-filtered signals are illustrated in Figure 2.

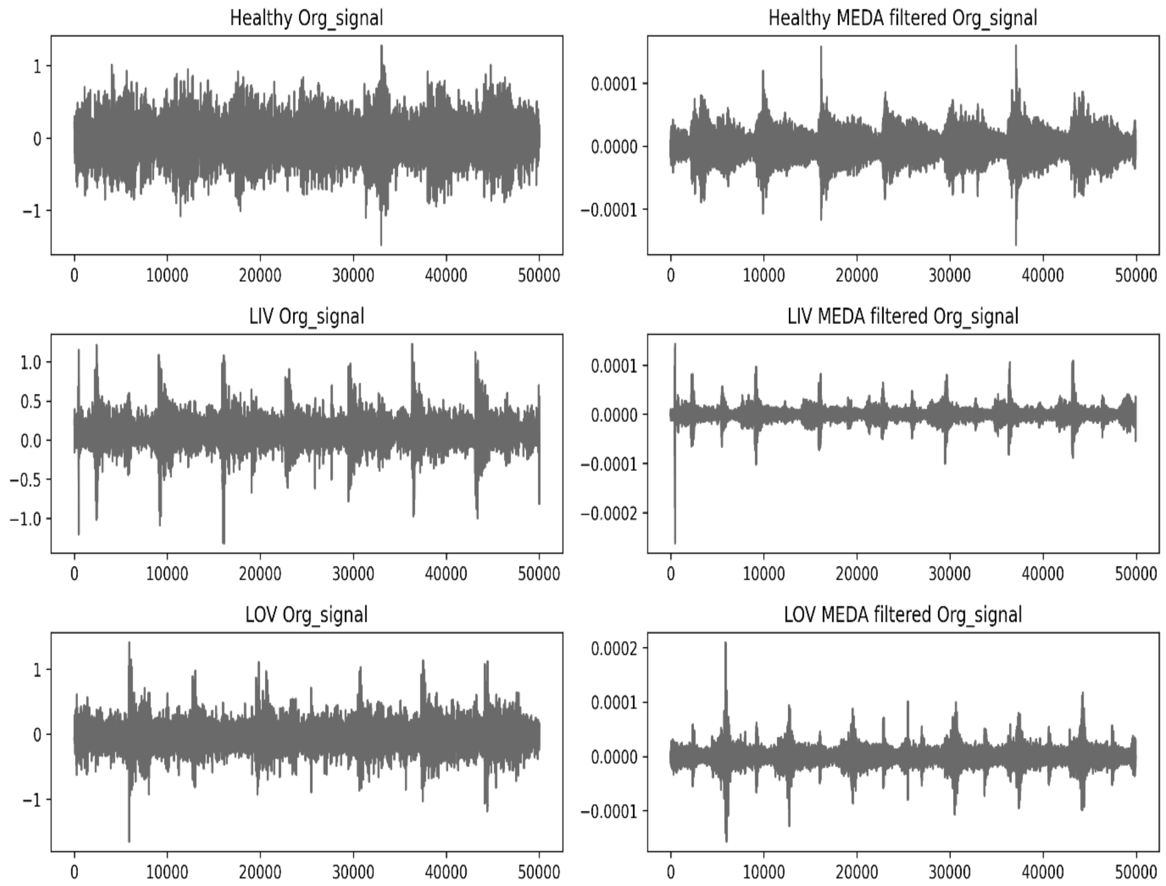

Figure 2: *Raw and MEDA filtered acoustic signals for healthy, LIV and LOV states*

From the Figure 2, it can be observed that the transient spikes (impulsiveness) for each state are clearly enhanced, as compared to the raw noisy signal, by MEDA method. Next, the filtered signals are used as input signal for EMD method and decomposed into set of IMFs. From the set of IMFs, fault-information rich components are selected on the basis of higher kurtosis value. Table 1, depicts kurtosis value for first 6 IMFs of EMD and MEDA-EMD signals. It can be observed that IMF2 has higher frequency of highest kurtosis values. Hence, for acoustic signal analysis, the condition indicators (RMS and sample entropy) are extracted from IMF2 for healthy as well as fault signals.

Table 1: *Kurtosis value of IMF 1-7 extracted with EMD and MEDA-EMD method for healthy, LIV and LOV states*

Kurtosis	Healthy		LIV		LOV	
	EMD	MEDA_EMD	EMD	MEDA_EMD	EMD	MEDA_EMD
IMF1	1.10885	1220.009	8.957949	**804.0393**	4.750699	9.917119
IMF2	1.305557	**1832.79**	**21.37063**	759.5153	**6.21626**	6.779707
IMF3	**2.50234**	249.2484	6.87206	408.9886	1.95156	15.593

	1		6		7	
IMF4	1.023 75 9	57.10541	4.540 99 2	203.2668	2.275 16	3.154513
IMF5	0.418 79 5	27.10442	1.869 75 9	153.5696	1.589 10 3	9.239799
IMF6	1.266 58 6	49.32549	4.829 07 4	21.73766	5.573 46 8	10.38324
IMF7	1.205 46 1	73.24571	3.114 34 5	30.11287	0.659 16 2	14.50046

Figure 3 depicts the RMS values of IMF2 obtained by EMD and MEDA-EMD method for healthy, LIV fault and LOV fault signal. From the Figure 3(a), it can be noticed that healthy state of compressor is clearly distinguished from fault signals. However, this condition indicator can not separate LIV and LOV fault by EMD approach. Figure 3(b), shows that MEDA-EMD method can differentiate each state of compressor. It is to be noted that RMS value for EMD processed IMF is quite higher than that for MEDA-EMD processed IMF.

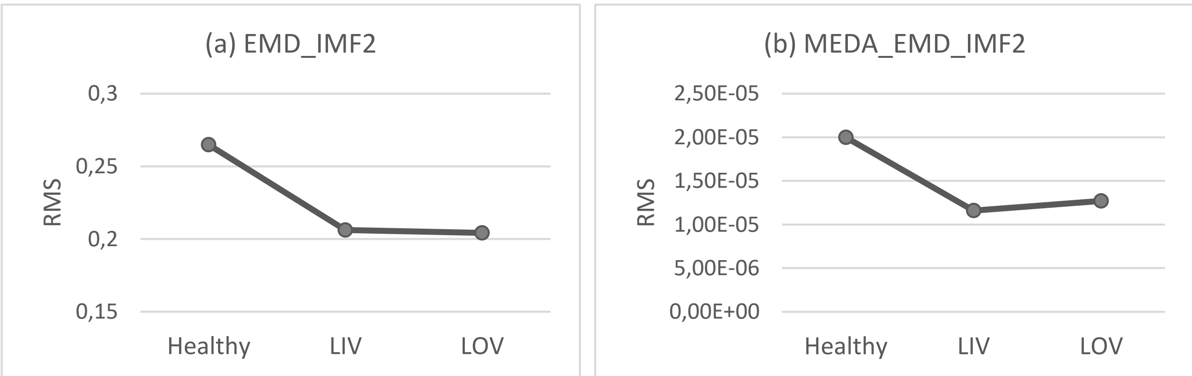

Figure 2: *RMS value of IMF2 extracted with (a) EMD and (b) MEDA-EMD method for healthy, LIV and LOV states*

Figure 4 and Figure 5 illustrates the sample entropy feature for IMF2 component obtained through EMD and MEDA-EMD technique respectively. Figure 4 shows that sample entropy values for LIV and LOV fault conditions are almost overlapping each other. Hence, it can be noted that EMD-sample entropy approach is not as effective in differentiating the LIV and LOV faults from each other as it is for differentiating the healthy from valve-fault (LIV and LOV) conditions of reciprocating compressor.

However, as shown in Figure 5, the MEDA-EMD-sample entropy approach has clearly

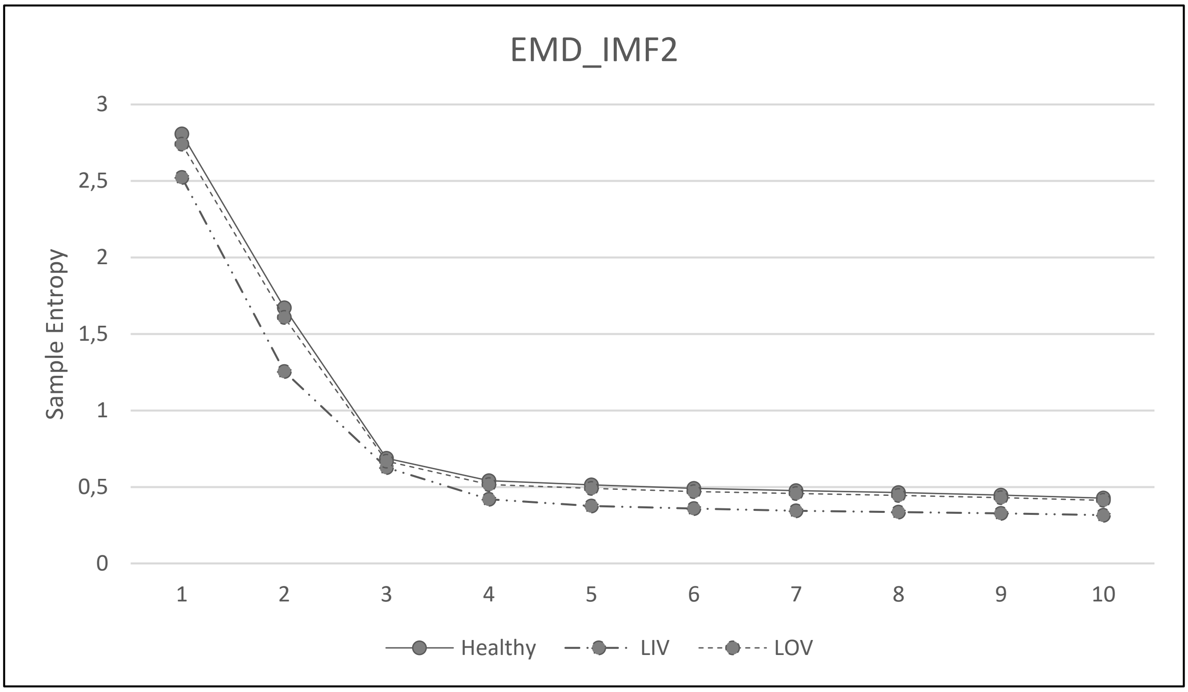

Figure 4: *Sample entropy of IMF2 extracted with EMD method for healthy, LIV and LOV states*

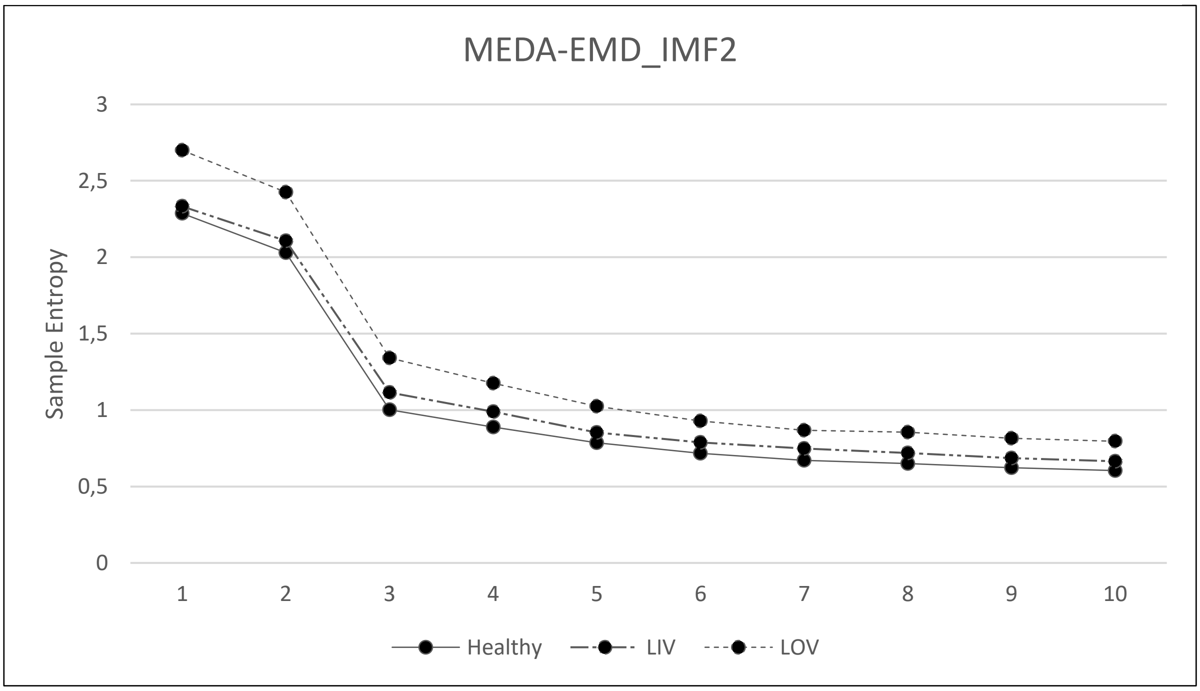

Figure 3: *Sample entropy of IMF2 extracted with MEDA-EMD method for healthy, LIV and LOV states*

identified the healthy, LIV and LOV states of the compressor. The results demonstrate the effectiveness of the MEDA-EMD approach for valve-fault diagnosis of reciprocating compressor based on acoustic signals.

V. Conclusion

In petrochemical, process and gas storage and transportation industries, reciprocating compressors plays a crucial role. For reliable operation of the compressor, it is imperative to develop a condition-based health monitoring and fault diagnosis model that can accurately detect faults. Among compressor faults, valve failure is critical as it causes most frequent unscheduled maintenance shutdowns. Hence, to address the problem, a MEDA-EMD based fault diagnosis approach is proposed in this study. For this purpose, unlike other signal analysis techniques that are commonly based on signals obtained from vibration, pressure and other intrusive sensors, non-intrusive acoustic signal-based signal analysis technique is employed in this study. After applying the MEDA-EMD technique, IMF with highest kurtosis value is selected as it contains higher fault-information compared to others. In order to quantify the fault information, time-domain based RMS and sample entropy features are extracted from the selected IMF. To validate the proposed approach, acoustic signals of healthy and inlet-outlet valve faults (LIV and LOV) are analysed. From the presented results, it can be concluded that compared to self-adaptive EMD method, MEDA-EMD based approach is more effective. Although EMD can separate the healthy and valve-fault conditions, it fails to differentiate the LIV and LOV. However, RMS and sample entropy features obtained after applying the MEDA-EMD method can clearly identify healthy, LIV and LOV states of the compressor from each other. It is to be noted that the applicability of the proposed methodology is validated, in this study, for acoustic signals of the mentioned states of the compressor only. Nonetheless, the results of this study shows that potential of this approach is promising as fault diagnosis model for other faults of compressor as well as other rotary or reciprocating systems.

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Optimal Resource Allocation for Software Development under Agile Framework

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Abstract

Software development techniques have always been in-focus while creating high-quality software. Off late, most software firms have adopted the latest approach of the market- "Agile Software Development" to create a superior product. Agile software development allows an incremental and iterative development of the product with close interaction of all the teams on deck. This allows for the efficient utilization of the available resources to satisfy the demands of the end consumer. In the current proposal, a mathematical model has been created which allows optimal allocation of the limited developmental resources among the various sprint cycles and modules. Optimization tools have been used to determine this optimal distribution of resources. The proposed model has been validated on the resource allocation data with promising results.

Keywords: Agile Software; Budgeting; Optimization Method; Resource Allocation; Software Development.

I. Introduction

The rapidly progressing world of technology forces us, individuals, to keep moving ahead with the flow and one is left with no other option than evolve and keep up with the times. The same has happened with the field of software. Till a decade or so ago, there were a handful of software development approaches available. But with time not only has the number of approaches increased, but so many different models have evolved out of the basic few which cater to the diverse environments. Agile based software development is no different. It incorporates the basics of the waterfall model by using the various phases like the requirements analysis, design, coding, testing, and implementation. The add-on in agile based methodologies is that each of these phases can be re-visited repeatedly to incorporate the changing requirements. The core idea of building the basic prototype and enhancing it in various developmental cycles is the main principle in the prototype model and in the iterative and incremental model. The iteration or cyclical development approach is achieved in agile based methodologies, specifically in scrum, with the help of sprints. Each sprint is a time-bound (alternatively task-bound event) wherein a pre-decided number of tasks are performed in a given time frame. The incremental approach is implemented by developing the most essential and most characteristic features of the software in the first sprint. Each subsequent sprint adds-on to this basic version by developing more items from the requirements list. The Verification and Validation model also called the V-model, designs test cases for each development phase thus making testing an essential step of each phase. The same principle is implemented in scrum wherein various types of testing take place throughout a particular sprint to ensure that the right product is being developed by building the product right. Continuous testing is achieved through Unit testing, Component Testing, Functional Testing, Exploratory Testing, User Acceptance Testing (Alpha, Beta, and Gamma Testing), Performance Testing, Load Testing, Security Testing, etc. performed throughout the project. What was missing in other approaches was the flexibility to change and deal with unforeseen circumstances and ever-changing requirements. This indispensable feature was inculcated in the agile principles which made it a go-to option for software development. Thus, it can be safely concluded that agile manifesto creators [1] were visionaries who not only identified the strengths and weaknesses of the existing models and inculcated them in their proposed methodology but also created a dynamic product that is still relevant two decades later. The principles proposed by them can be used to make any model, project, or sector agile. Frameworks like Scrum, Kanban, Lean, Test Driven Development, Feature Driven Development, Crystal, RAD, etc. are the popular ones based on the agile principles. The agile frameworks have gained popularity in recent times with them being utilized in numerous sectors besides software development like product development, distribution, healthcare, education, governance, construction, management, research, etc. [2-4]

The scrum methodology is the most used among the various agile based methodologies. The name itself draws focus to one of the key aspects of agile i.e. the highly interactive teams. The team consists of three classes of people: the Product Owner, the Scrum Master, and the Development Team. The product owner is the person who represents and conveys the wishes of the various stakeholders to the development team. The development team consists of the various members like the coders, the testers, the researchers, the analysts, etc. who build the product. The scrum master acts as a link between the product owner and the development team and provides them with the logistics to perform their job [5]. The team size is kept small and each member is multi-talented, cross-functional, and works in very close association with others. Daily scrum meetings, face-to-face interactions, open channels of communication, lack of hierarchy system within a team, retrospective meetings, etc. all promote a healthy work environment and an easy flow of information. All the team members are valuable resources within an organization. Hence, providing them with the right inputs and helping them learn and grow within an organization is essential.

Efficient project management helps in optimally allocating the tasks, assigning resources, planning the process, deciding the timeline, effort estimation, cost estimation, and resource estimation. Three main areas which require skillful planning are product backlog planning, sprint planning, and release planning. All of these activities require resources such as man-power, data, hardware components, software, office supplies, infrastructure, work amenities, etc. Most of these factors like manpower, technical aids, etc. are the direct contributors towards the success of the project while others like infrastructure and work amenities contribute to a comfortable and safe work environment which are essential for higher productivity. Software development process specifically focusing on software testing and software maintenance has been widely studied [6-9]. In the current proposal, the focus is on studying how the project money or budget can be used to achieve the best possible output. All the varied resources have been normalized and studied with a common denominator of money spent in acquiring, using, and maintaining them. Utilizing these resources optimally is essential to garnering profits without compromising on the product.

The resource allocation problem in the software testing and operational phases has been studied widely in the literature [10]. Huang and Lyu [11] have discussed optimal resource allocation throughout the software development cycle. Huang and Lo [12] focused on assigning testing resources in modular software. Inoue and Yamada [13] discussed an optimal testing effort allocation problem. Anand et al. [14] allocated testing budget to various versions of the software. Bhatt et al. [15] assigned optimal resources to detect software vulnerabilities of different severity levels. Anand et al. [16] designed an optimization problem to optimally allocate resources to various software updates to remove software vulnerabilities.

Most resource allocation problems were defined keeping in mind the waterfall approach wherein the resources were assigned to the various phases of the development cycle. Considering the prevalence of agile based models, an alternate resource allocation method is required which considers the characteristic features of sprints in agile based models to make an optimal allocation. In the context of agile software development projects, not much work has been done with respect to resource allocation. The topic has been dealt indirectly through effort and budget estimation. Mahnic [17] discussed a case study on the learning patterns observed in effort estimation made by teams implementing scrum for the first time. Morris et al. [18] patented a method of resource allocation in an agile environment. Colomo-Palacios et al. [19] suggested a recommendation system to assign the human resource to the team using fuzzy and rough sets. Cao et al. [20] discussed the project funding and budgeting in an agile based project. Mansor et al. [21] discussed the budgeting challenges faced in agile based projects. Thom-Manuel et al. [22] discussed a budget model to incorporate risk management in agile software development. Ploder et al. [23] gave a Continuous Forecasting Framework to handle project budgeting in an agile project.

The approach to determine the effort consumed in an agile based project is to first breakdown the project into user stories and then determine the time required to successfully complete each user story. The story points, ideal days, etc. are outputs used to measure them [24]. Techniques such as analogous estimation, parametric estimation, Delphi method, Poker method, 3 Point Estimate, Expert Judgment, Published Data Estimates, Vendor Bid Analysis, Reserve Analysis, Bottom-Up Analysis, and Simulation are used to determine the same. Litoriya and Kothari [25] discussed cost and effort estimation for web-based projects through their proposed AgileMOW. Owais and Ramakishore [26] discussed cost, effort, and duration in agile software development. A systemic review of cost estimation in agile software development has been given by Bilgaiyan [27]. Usman and Petersen [28] compiled the available knowledge to create a taxonomy for effort estimation in agile based software development. Gultekin and Kalipsiz [29] used Machine Learning Algorithms to conduct story point-based effort estimation. Fernández-Diego et al. [30] provided extended literature on the work done in effort estimation in agile software development.

In the proposed work, a model has been discussed which allocates the available resources to the different sprints depending on the number of tasks that need to be performed in it. The current work has been organized in the following way: Sec. II discusses the notations used in model development. Section III proposes a model to allocate the resources in the development project. Section IV demonstrates the working of the model through an illustration and discusses the findings. Section V concludes the work and is followed by a list of the references used herein.

II. Notations

The notations used in the development of the model are as follows:

n : Total number of sprints required to complete the project

i : A counter variable representing a particular sprint, $i = 1,2,...,n$

$k_i(R_i)$: Cumulative number of tasks dealt using R_i resources in the i^{th} sprint of software

$g_i(t)$: Probability Distribution function

$G_i(t)$: Cumulative Distribution function, $F(t) = \int f(t)\,dt$

a_i : Number of tasks in the i^{th} sprint of the software

b_i, β_i : Model parameters

R_i : Resources allocated to sprint i

q_i : Aspiration level for sprint i

c_B : Project development budget

III. Model Development

The resource allocation has been achieved through an optimization problem. The problem considers that there are a certain number of user stories defined in the product backlog which are to be developed in the various sprint cycles. Before the start of each sprint, the sprint backlog is decided in the sprint meeting which defines the sprint's task list.

To describe the task performance/completion phenomena in an agile environment, an analogy from the fault debugging phenomenon observed in software reliability literature has been drawn [10, 31-32]. The number of tasks being performed in a sprint can be treated as a counting process. A counting process is used to describe the occurrence of an event of time (here the completion of a task).

Let $N(T)$ represent the number of occurrences of an event in the interval $[0,t]$ with a mean value function $k(t)$. Let $v(t)=\frac{g(t)}{1-G(t)}$ denote the instantaneous rate at which tasks are performed, also referred to as the intensity function. Then, $N(T)$ is considered to be a Non-Homogenous Poisson Process (NHPP) if it satisfies the following properties:

- At the start of the process no tasks have been completed, i.e. at $t=0$, $N(0)=0$.
- $\{N(t), t>0\}$ has independent increments.
- $P\{N(t+\Delta t)-N(\Delta t)\geq 2\}=o(\Delta t)$, i.e. the probability of completing two or more tasks simultaneously is negligible.
- $P\{N(t+\Delta t)-N(\Delta t)=1\}=v(t)+o(\Delta t)$ denotes the probability of completing a single task.

where $o(\Delta t)$ denotes a quantity that tends to zero for a small value of t.

Let $k(t)$ represent the mean number of tasks completed by time t, i.e. $k(t)=\int_0^t v(x)\,dx$, $t>0$ then it can be shown that:

$$\Pr\left[N(t)=j\right]=\frac{\left(k(t)\right)^j e^{-k(t)}}{j!}, \qquad j=0,1,2,\ldots \tag{1}$$

i.e. $N(t)$ has a Poisson distribution with a mean value function $k(t)$. Consider the situation where the time scale is very large as compared to the total number of tasks to be completed and the mean number of tasks are completed at random time points. Hence, the cumulative number of tasks that are completed by a given time *t* can be defined using the following differential equation:

$$\frac{dk(t)}{dt}=v(t)\left(A-k(t)\right) \tag{2}$$

Equation (2) defines a direct relationship such that the number of tasks that will be performed by time *t* is dependent on the number of tasks that have not yet been performed and the intensity function. On solving equation (2) with the initial condition $t=0, k(t)=0$ i.e. at the start of the sprint, no tasks have been completed, the following expression is obtained:

$$k(t)=AG(t) \tag{3}$$

Here, the cumulative distribution function $G(t)$ can take different functional forms depending on the nature of the problem. In the agile framework, it is observed that with passing time, the velocity or the efficiency of the team increases due to factors like improved project knowledge, better communication, commitment towards the project, etc. Hence, a logistic rate has been considered that denotes a learning pattern or an increased task completion rate with passing time. It is represented as:

$$v(t)=\left(\frac{b}{1+\beta e^{-bt}}\right) \tag{4}$$

Using eq. (4) in eq. (3), the following expression is obtained:

$$k(t)=\frac{A\left(1-e^{-bt}\right)}{\left(1+\beta e^{-bt}\right)} \tag{5}$$

The above expression can be generalized for i sprints.

Using the work of Kapur et al. [10], the number of tasks completed in the sprint i by using resources R_i can be obtained as:

$$k_i(R_i)=\frac{A_i(1-\mathrm{e}^{-b_i R_i})}{1+\beta_i e^{-b_i R_i}},\ i=1,2,...,n \tag{6}$$

I. Basic resource allocation assigning resources to various sprints

The simple resource allocation problem is designed which assigns the resources to each sprint while trying to maximize the number of tasks performed within a sprint. The problem is constrained such that the total budget is not exceeded and the non-negativity condition holds. Thus, problem P1 is defined as:

$$\begin{aligned} &Maximize \sum_{i=1}^{n} k_i(R_i)=\sum_{i=1}^{n}\frac{A_i(1-\mathrm{e}^{-b_i R_i})}{1+\beta_i e^{-b_i R_i}} \\ &Subject\ to \quad \sum_{i=1}^{n} R_i \le C_B \\ &\qquad R_i \ge 0, \quad i=1,2,...,n \end{aligned} \tag{P1}$$

Since the objective function is a fraction hence P1 is a fractional programming problem and to solve it, the following procedure is used:

$$\text{Let } g_i(R_i)=A_i(1-\mathrm{e}^{-b_iR_i}) \text{ and } h_i(R_i)=1+\beta_i e^{-b_iR_i} \tag{7}$$

$$\text{Let } H_i(R_i)=g_i(R_i)/h_i(R_i) \tag{8}$$

Using the work of Dur et al. [33] and Geoffrion's scalarization [34] formulation with fixed weights λ_i for the objective function, the problem can be reframed as:

$$\begin{aligned} &Maximize \sum_{i=1}^{n} \lambda_i\left(g_i(R_i)-h_i(R_i)\right) \\ &Subject\ to \quad \sum_{i=1}^{n} R_i \le C_B \\ &\qquad R_i \ge 0, \quad i=1,2,...,n \\ &\qquad \lambda \in \Omega=(\lambda \in R^N / \sum \lambda_i=1, \lambda_i \ge 0, i=1,2,...,n) \end{aligned} \tag{P2}$$

II. Modified problem to include aspiration constraint in each sprint

The number of tasks that can be performed using the assigned resources may vary from sprint to sprint. Hence, an aspiration is assigned to each sprint such that at least a certain proportion of the task list is completed. The proportion can be varied from sprint to sprint depending on the sprint goal. Hence, the new optimization problem P3 is:

$$\begin{aligned} &Maximize \sum_{i=1}^{n} k_i(R_i) = \sum_{i=1}^{n} \frac{A_i(1-\mathrm{e}^{-b_i R_i})}{1+\beta_i e^{-b_i R_i}} \\ &Subject\,to \quad k_i \geq A_{io} \\ &\sum_{i=1}^{n} R_i \leq C_B \\ &R_i \geq 0, \quad i = 1,2,...,n \end{aligned} \tag{P3}$$

where $A_{io} = q_i A_i$ denotes the number of tasks that need to be performed to meet the sprint aspiration.

Since $k_i \geq A_{io}$

$$\Rightarrow \frac{A_i(1-\mathrm{e}^{-b_i R_i})}{1+\beta_i e^{-b_i R_i}} \geq A_{io} \tag{9}$$

$$\Rightarrow R_i \geq -\frac{1}{b_i}\log\left[\frac{1-\frac{A_{io}}{A_i}}{1+\frac{A_{io}}{A_i}\beta_i}\right] = Z_i \tag{10}$$

where Z_i denotes the minimum amount of resources required to perform A_{io} tasks. Using the approach discussed in subsection I, the fractional programming problem (P3) is transformed to obtain (P4) as:

$$\begin{aligned} &Maximize \sum_{i=1}^{n}\left[(A_i-1)-(A_i+\beta_i)\mathrm{e}^{-b_i R_i}\right] \\ &Subject\,to \quad R_i \geq Z_i \\ &\sum_{i=1}^{n} R_i \leq C_B \\ &R_i \geq 0, \quad i = 1,2,...,n \end{aligned} \tag{P4}$$

The number of tasks that could be completed by satisfying the sprint aspiration is obtained by solving the problem (P4) as:

$$k_i(Z_i) = \frac{A_i(1-\mathrm{e}^{-b_i Z_i})}{1+\beta_i e^{-b_i Z_i}}, \; i = 1,2,...,n \tag{11}$$

III. Resource allocation to meet the overall project aspiration constraint

The sprint aspiration will help achieve the sprint goal but to meet the project goal another aspiration is required. Thus, the resources left over after meeting the sprint aspiration will be reallocated to meet the project aspiration. For this, the problem can be modified in the following way:

$$\text{Let } S_i = R_i - Z_i; \; \overline{Z} = C_B - \sum_{i=1}^{n} Z_i \text{ and } \overline{A_i} = A_i - A_{io} \; i = 1,2,...,n \tag{12}$$

Then,

$$\begin{aligned} & Maximize \sum_{i=1}^{n}\left[(\overline{A_i}-1)-(\overline{A_i}+\beta_i)\,\mathrm{e}^{-b_i S_i} \right] \\ & Subject\ to \quad \sum_{i=1}^{n} S_i \le \overline{Z} \\ & \qquad S_i \ge 0, \quad i=1,2,\ldots,n \end{aligned} \tag{P5}$$

The optimal number of tasks that could be completed in meeting the project goal is obtained by solving the problem (P5) and is given as:

$$k_i(S_i^*)=\frac{\overline{A_i}(1-\mathrm{e}^{-b_i S_i^*})}{1+\beta_i e^{-b_i S_i^*}},\ i=1,2,\ldots,n-1 \tag{13}$$

Hence, the total optimal number of tasks that could be completed in a particular sprint i of the project is given as:

$$k_i(R_i)=k_i(S_i^*+Z_i),\ i=1,2,\ldots,n \tag{14}$$

IV. Model Illustration

For numerical illustration, a simulated agile based software development dataset has been considered. Let the project contain 175 tasks to be performed in 8 sprints of 1 month each. The project budget to be allocated is assumed to be $200,000. The minimum aspiration of task completion from each sprint is 65% i.e. $q_i = 0.65$. The minimum aspiration of task completion from the whole project is 85%. SAS software [35] has been used to estimate the model parameters for each sprint (using eq. 5) and has been demonstrated in Table 1. LINGO software [36] has been used to solve optimization problems (P1-5).

The aim was to utilize the minimum possible resources, hence in the initial round of assignment, 80% of the available budget was allocated i.e. $C_B = \$160,000$. On solving the optimization problem P2, the basic allocation was obtained and the number of tasks completed in each sprint and the overall project was determined. As can be seen in Table 1, a sequential distribution of resources was occurring which results in lesser resources being available for the last few sprints and a low task completion rate. In sprint 1, almost all the tasks could be completed but in the last sprint, only 32% (approx.) tasks could be performed. Despite this, the overall project task completion was quite high at 84% (approx.).

Table 1: *Parameter Estimates and initial resource allocation for sprints*

Sprint No.	A_i	b_i	β_i	R_i	$k_i(R_i)$	% of Tasks Performed	% of Tasks Remaining
Sprint 1	45	0.000412932	0.170795673	10921.64	44.42	**98.71**	1.29
Sprint 2	13	0.000319987	0.129642265	9435.735	12.29	**94.52**	5.48
Sprint 3	16	0.000264216	0.124115883	11480.16	15.14	**94.62**	5.38
Sprint 4	35	0.000150112	0.26584682	21653.54	33.30	**95.14**	4.86
Sprint 5	14	8.95707E-05	0.256627137	20413.03	11.29	**80.61**	19.39

Sprint 6	21	5.87323E-05	0.324640162	30800.98	16.67	**79.40**	20.60
Sprint 7	20	3.20165E-05	0.69718776	36618.9	11.36	**56.78**	43.22
Sprint 8	11	3.15115E-05	0.728208159	18676.01	3.48	**31.68**	68.32
	175			**159999.995**	147.95	**84.54**	15.46

Thus, the need for evaluating problem P4 arises which satisfies the sprint aspiration. As can be seen in Table 2, out of the $160,000 budget, $156,880 was spent in meeting the sprint aspiration of 65%. The least task completion percentage was observed in sprint 7. The leftover budget of $3,120 was allocated by solving problem P5. But since the leftover budget is quite low, the project aspiration of 85% product backlog task completion could not be met and the rate fell to 74% (approx.).

Table 2: *Minimum aspiration of 65% task completion in each sprint*

Sprint No.	A_i	A_{i0}	Z_i	$k_i(Z_i)$	$\overline{A_i}$	S_i	$k_i(S_i)$	R_i	$k_i(R_i)$	% of Tasks Performed	% of Tasks Remaining
Sprint 1	45	30	2921.6606	30	15	3006.7590	10.1645	5928.4196	40.5109	**90.02**	9.98
Sprint 2	13	9	3952.0526	9	4	0.0000	0.0000	3952.0526	9.0000	**69.23**	30.77
Sprint 3	16	11	4712.1847	11	5	0.0000	0.0000	4712.1847	11.0000	**68.75**	31.25
Sprint 4	35	23	8203.5656	23	12	114.0340	0.1615	8317.5996	23.1892	**66.26**	33.75
Sprint 5	14	10	15865.3937	10	4	0.0000	0.0000	15865.3937	10.0000	**71.43**	28.57
Sprint 6	21	14	22041.1933	14	7	0.0000	0.0000	22041.1933	14.0000	**66.67**	33.33
Sprint 7	20	13	44463.6643	13	7	0.0000	0.0000	44463.6643	13.0000	**65.00**	35.00
Sprint 8	11	8	54719.4874	8	3	0.0000	0.0000	54719.4874	8.0000	**72.73**	27.27
	175	118	**156879.2022**	118	57	**3120.7930**	10.3260	**159999.9952**	128.7001	**73.54**	26.46

After running simulations it was realized that an additional $25,000 was required to meet the 85% project completion rate while maintaining the 65% aspiration level from each sprint. Table 3 shows the results obtained by evaluating problem P5 with this increased budget. It can be seen that here both the aspirations were fulfilled.

Table 3: *Minimum aspiration of 65% task completion in each sprint and 85% in the whole project*

Sprint No.	A_i	A_{i0}	Z_i	$k_i(Z_i)$	$\overline{A_i}$	S_i	$k_i(S_i)$	R_i	$k_i(R_i)$	% of Tasks Performed	% of Tasks Remaining
Sprint 1	45	30	2921.6606	30	15	6785.121	13.9449	9706.7816	44	**97.78**	2.22
Sprint 2	13	9	3952.0526	9	4	3892.634	2.7465	7844.6866	12	**92.31**	7.69
Sprint 3	16	11	4712.1847	11	5	4806.122	3.4745	9518.3067	15	**93.75**	6.25
Sprint 4	35	23	8203.5656	23	12	10507.74	9.0261	18711.3056	32	**91.43**	8.57
Sprint 5	14	10	15865.3937	10	4	29.3288	0.0084	15894.7225	10	**71.43**	28.57
Sprint 6	21	14	22041.1933	14	7	2099.849	0.6311	24141.0423	15	**71.43**	28.57
Sprint 7	20	13	44463.6643	13	7	0	0.0000	44463.6643	13	**65.00**	35.00
Sprint 8	11	8	54719.4874	8	3	0	0.0000	54719.4874	8	**72.73**	27.27
	175	118	**156879.2022**	118	57	**28120.7948**	29.8315	184999.997	149	**85.1429**	14.8571

Thus, the two goals of project and sprint task completion aspiration could be achieved with $185,000 and net savings of 7.5% of the allocated budget of $200,000. It will be a managerial decision to either invest the leftover resources in achieving an even higher output or let it be the project cost saving. Hence, it is a trade-off between how much one wants to invest to get a higher completion rate. A rational choice needs to be made here and the agile principle of developing only the most essential features should be applied in determining the level of output desired. In this model illustration, the aspiration level was considered to be the same for each sprint. But this can be varied for different sprints. By running different simulations of the model and varying the input and the desired level of output, the best fit for a particular project/firm can be obtained.

V. Conclusion

Agile based methodologies have gained prominence and are the most used software development approach in the current time. Since all the developmental resources are usually limited hence there optimal utilization is necessary. In the current proposal, a resource allocation problem in agile based software has been discussed through optimization problems. Three different problems were discussed wherein the best fit can be chosen as per the managerial decisions. A basic problem determined how much of the available resources need to be spent in each sprint. The second problem was able to determine how many resources would be required in each sprint to complete a minimum proportion of sprint backlog. The third problem further extended the second problem and was able to meet the product backlog aspiration along with the sprint aspiration. Hence, the current proposal provides a simple yet effective mathematical tool to determine the optimal allocation of resources for maximum yield. The management can decide on how much to invest in the project by correlating with the gains achieved in terms of the tasks completed.

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Signature Analysis of Bleaching and 2-out-of-4 Systems

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Abstract

A real-life bleaching system is proposed, and its reliability function is estimated using universal generating function technique. The units are connected to each other in series and parallel arrangements. Further, a 2-out-of-4 system is described and taken as an example where out of 4 units, if 2 are working the system will perform its task. Universal Generating Function technique is applied to both the systems and by using Owen's and Boland method signature, tail signature, Barlow-Proschan Index and expected time of the proposed complex system is calculated.

Keywords: Universal generating function; signature; tail signature; 2-out-of-4 system; independent identically distributed; Barlow-Proschan index.

I. Introduction

In past few decades, many efforts had been done in the field of reliability to evaluate the reliability characteristics of various systems and models. Engineering and mathematics connect with reliability theory perfectly. When any system is allowed to work in definite mode and under specified environment, the representation of the characteristics of a working of the system is reliability engineering. Reliability engineering tells us regarding the performance of the framework and therefore the operating time period of it. Reliability theory have helped to establish finest designs or models for various systems. When any system performs a given task in a specified time limit with different efficiency and performance rates, called multi-state systems (MSS). Eryilmaz et al. [16] had thought about a k-out-of-n structure with n independent units having three states, where state "2" was assumed to be the perfectly working state, state "1" is assumed to be partially working state and state "0" is failed state. This research creates formulae for the endurance capacities comparing to the two diverse framework's states depicted previously. For representation purposes, mathematical model which expects that the debasement happens as per a Markov cycle is introduced. Eryilmaz et al. [19] created formulae for the endurance capacities relating to the two distinctive framework's states depicted previously. For representation purposes, a mathematical model which accepts that the degradation happens as per a Markov process is introduced. There are many techniques to estimate the reliability of MSS and UGF technique is one of them. In the past, many researchers had shown interest in UGF technique and also to evaluate the signature of the complex system [1,12,23]. Levitin [25] used the UGF technique to seek out the reliability of a consecutive *k*-out-of-*r*-from-*n*:F system in multi-state case. Author had taken a linear multi-state sliding window system in which system had *n* elements and condition for failure of the system was the failure of k elements out of r consecutive elements. Levitin [24] considered a framework comprises of *n* straightly requested multi-state components. Every component can have different states: from complete-disappointment up to consummate working. A performance rate is related with each state. The sliding window framework fails if the amount of the performance rates of any *r* sequential multi-state components is lower than a base permissible level. Author proposes another model that sums up the sequential *k*-out-of-*r*-from-*n*:F framework to the multi-state case. Levitin [26] describes the use of UGF technique in different type of systems i.e., if the system is a parallel then what would be the algorithms, if the system is a *k*-out-of-*n* type then the algorithms will change etc. Levitin and Ben-Haim [27] that proposed the linear consecutive sliding window system and the model had *n* multi state elements which were linearly ordered. The condition for failure of the system was when in out of *m* consecutive groups of *r* consecutive elements the sum of the performance rates of units was lesser then demand *W*. Levitin and Dai [28] in which the system consists of *n* linearly ordered elements and the condition for failure was defined when at least *k* elements failed in the *m* groups which were consecutively overlapped and had *r* units in each group where the units in the group are also consecutive. Levitin and Dai [29] proposed a model in which there were m independent linearly ordered multistate elements in the system, and the conditions for failure were if the sum of performance rates was lower than the minimum allowable rates in at least *k* number of groups in which each group consists of *r* elements arranged consecutively. Jafary and Fiondella [2] had developed a model which was discrete and continuous for a multistate system and based on series and parallel UGF where the elements, consists of multistate units, were identical but correlated.

da Costa Bueno [3] discussed how to calculate the importance of every unit for a given system by using its signature representation. The definition of signature was given in sense of compensator transform and in the context of system signature. Eryilmaz [23] deliberate the reliability characteristics of *m*-consecutive-*k*-out-of-*n*:F system with overlapping runs through signature and used it to evaluate different reliability characteristics of the system. Harish et al. [4] assessment the reliability with the help of vague lambda-tau methodology for industrial system in which the collected information about the units of the system was uncertain and the nature of the information was also inaccurate. Also, rather than fuzzy set theory author had used intuitionistic fuzzy set theory to control the uncertainty in the data. Da and Hu [5] given the definition of the bivariate signature in terms of order statistics of lifetime elements and then establish the formula, for 3-state system and calculating the bivariate variables. Kumar and Singh [6, 7] estimated the signature reliability characteristics of the complex and binary sliding window coherent system, with the assistance of UGF, reliability function, signature and minimal signature of the sliding window system were also assessed. Kumar and Singh [8] had proposed the A-within-B-from-D/G sliding window coherent system, by considering the case of multiple failures. Authors used UGF technique and Owen's method to obtain the signature of the system. Kumar and Ram [9] considered a 2-out-of-5-linear consecutive system and discussed the signature of the system. Some other parameters for example tail signature, expected lifetime and expected cost from reliability function were also evaluated. Kumar and Ram [10] had proved that in spite of having number of techniques and methods for evaluation such as fuzzy, supplementary and Markov chain technique, UGF technique gave more amended results in case of interval-valued. Kumar and Singh [11] had revisited the sliding window system and estimated the interval-valued reliability using UGF technique.

In the above discussion about signature and UGF, a bleaching system has been considered which the mixture of series and parallel arrangements and reliability is defined with the assistance of state performance. A 2-out-of-4 system is also considered in which out of 4 elements at least 2 elements should work for the whole system to be in working condition. The elements of the system are independent identically distributed. A brief discussion of UGF technique and signature is given in Section 2. The description of the considered complex model is given in Section 3. Then a k-out-of-n system is considered and solved with the help of UGF technique. The description of the model is given in Section 4. In Section 5, algorithms and used formulas are discussed for the complex system and in Section 6 procedures for calculating the reliability of the k-out-of-n arrangement is given. An example of complex system is considered and solved according to the algorithms and formulas in Section 7 and in Section 8 an example of 2-out-of-4 system is also considered and solved with the help of algorithms given in the 5 and 6 Sections. At last, in Section 9 the conclusion is drawn and the results are discussed.

II. Universal Generating Function and Signature

UGF was first introduced by Ushakov in 1986, this technique is said to be effective within the analysis of function of reliability for the system. With this technique one can optimize the various kind of complex and multi-state devices. UGF technique includes the contribution of the performance of all the units existing in an entire system and then calculate the performance distribution of the system based on the performance distribution of its units. To reduce the problem, it joins different units of the device by using configuration operators.

The UGF of an independent discrete random variable Y is expressed in a polynomial form as

$$U(Z) = \sum_{m=1}^{M} q_m z^{y_m},$$

where, the possible values of the variable y are m and the possibility of the system in working state is q_m.

In 1985 Samaniego first introduced the concept of signature. To characterize the system reliability, signature is used and it is a very useful tool. Signature tells us mainly about the probability of the failure time of the system’s unit, i.e., at what time the particular unit of the system will fail and which is the last unit whose failure will cause the failure of the whole system. The system signature is given by the n-tuples whose kth coordinate s_k is the probability that the kth unit failure causes the system to fail. That is,

$$S_k = P_r(T_s = T_{k:n}),$$

where, $T_{k:n}$ denotes the kth smallest lifetime, i.e., the k^{th} order statistics computed by rearranging the variables $T_1, \dots, T_n$ in ascending order of magnitude.

III. Description of Bleaching System

A bleaching system is considered which comprises of both series and parallel structure. The proposed system is a type of complex system that can’t be simplified as pure series structure or pure parallel structure, this is the combination of both. The system is said to be in series configuration when on the failure of one element the whole system fails and the system is said to be in parallel configuration, when till the last working element of the system will work i.e.; the system fails when the last working element fails.

In the proposed model, the system is simplified and then the reliability function is evaluated with the help of UGF technique. In the given case, 2 and 3 is in parallel and 4 and 5 are also in parallel with each other. 1 is in series with both of the above.

Hence UGF of the bleaching system is defined as follows

$$U_6(z) = \max\left(U_2(z), U_3(z)\right) \tag{1}$$

$$U_7(z) = \max\left(U_4(z), U_5(z)\right) \tag{2}$$

$$U_8(z) = \min\left(U_1(z), U_6(z), U_7(z)\right) \tag{3}$$

Reliability of the system will be calculated (Levitin, 2003) as

$R = U'(z)$ at $(z = 1)$.

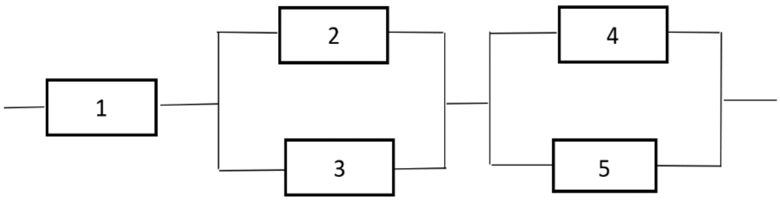

Figure 1. *Block diagram of bleaching system*

IV. Description of k-out-of-n System

Considered a k-out-of-n arrangement system where total n number of identical units are connected with each other and the system will perform its desired task if at least k units out of n will work perfectly. These type of systems are called k-out-of-n:G systems. If k elements of the system fail and that lead to the failure of the whole system i.e., the system will fail if at least k elements fail, that kind of systems are known as k-out-of-n:F system.

Pure series and parallel system are also be the special cases of k-out-of-n systems. As it is known that in pure series system, the failure of any one unit will lead to the failure of the whole system. So, this case can be taken as 1-out-of-n systems where out of n identical units if even one unit fails the whole system will go to fail state, and in pure parallel system, when all the units fail then the system is considered to be fail i.e., the failure of the last working unit will lead to the failure of the whole system. So, this case can be considered as n-out-of-n systems, where out of n identical units the system will go to the failed state only when all n units will fail. The k-out-of-n system also have a wide application in the technical field.

V. Algorithm for Computing Various Measures

1. Algorithm for computing the signature with the assistance of reliability function.

Calculate signature of the considered system from reliability function in the following manner. Firstly, by using Boland's formula [13] the signature of the system will be evaluated. The formula for the estimation of signature is

$$B_a = \frac{1}{\binom{s}{s-a+1}} \sum_{\substack{k \subseteq [s] \\ |k| = s-a+1}} \varphi(K) - \frac{1}{\binom{s}{s-1}} \sum_{\substack{K \subseteq [s] \\ |K| = s-1}} \varphi(K) \tag{4}$$

Compute the reliability polynomial for above complex structure using

$$K(P) = \sum_{j=1}^{s} e_j \binom{s}{j} P^j q^{n-j} \tag{5}$$

where, $e_i = \sum_{i-s-j+1}^{s} w_i,\ j = 1,2,\ldots s$.

The reliability function of the system is to be obtained with the help of Taylor evolution at $w = 1$ from the polynomial form, which we get from the above algorithm for the calculation of reliability function. So, the formula for reliability function is

$$P(w) = w^s K\left(\frac{1}{w}\right). \tag{6}$$

Then the estimation of the values of Tail signature for the complex system with $(p+1)$ -tuple $W=(W_0,...,W_s)$ is done with the following formula [20,21,22].

$$W_a = \sum_{i=a+1}^{s} w_i = \frac{1}{\binom{s}{s-a}} \sum_{|H|=s-a} \varphi(K). \tag{7}$$

Now by using Marichal and Mathonet [14] and equation (7), the Tail signature of the system is calculated by using the formula given below.

$$W_a = \frac{(s-1)!}{s!} d^a P(1),\ a = 0,1,\dots,s. \tag{8}$$

With the help of Tail signature of the system, finally the signature of the system is calculated as follows

$$w = W_{a-1} - W_a,\ a = 1,2,\dots,s. \tag{9}$$

2. The Algorithm to estimate the expected lifetime of the system by using minimum signature.

Calculation of the expected lifetime for the system is done using minimal signature. The elements of the system are considered to be independent and identically distributed and the mean time to failure (MTTF) is calculated for the elements which have exponentially distributed element with mean $\otimes$.

Then by using the formula given in [15] estimate expected lifetime $E(T)$ of the system with the help of given formula.

$$E(T) = \mu \sum_{i=1}^{n} \frac{e_i}{i}, \tag{10}$$

where, $e=(e_1,e_2,...,e_n)$ is a vector coefficient obtained with the help of minimal signature.

3. Algorithm for obtaining the Barlow-Proschan index for system.

Using [16,17,18] the Barlow-Proschan Index will be calculated from reliability function of the complex system where the elements of the system are independent and identically distributed. So, the following formula is used to estimate the Barlow-Proschan Index

$$I_{BP}^{(a)} = \int_0^1 (\partial_a K)(w)dw,\ a = 1,2,...,n \tag{11}$$

where, k are reliability functions of system.

4. Algorithm to determine the expected value of the system [23].

Next the expected value is to be calculated for the elements of the proposed system. The formula used for the expected value for the elements is given below.

$$E(X) = \sum_{i=1}^{n} i w_i,\ i = 1,2,...,n\ . \tag{12}$$

Then two measures are to be calculated at last. One $E(X)$ and second is $\frac{E(X)}{E(T)}$ for the proposed bleaching system.

VI. Algorithm for Computing Various Measures

First it is to note that the operator $\otimes_+$contains the associative property and the following procedure can be defined by using the structure function mathematically. With this procedure one can get the reliability function of k-out-of-n system as follows:
First, determine the u-function of each element in the following form
$u_i(z) = p_i z^1 + (1 - p_i) z^0$.
Assign $U_1(z) = u_1(z)$.
Obtain $U_i(z) = U_{i-1}(z) \otimes_+ u_i(z)$
For all $i = 2,3, \ldots, n$.
Probability mass function (p.m.f.) of the random variable X is represented by the last u-function $U_n(z)$ calculated for the system.
Calculate the u-function $U(z)$ which represents the p.m.f of structure function

$$\emptyset(X_1, \ldots, X_n) = 1(\sum_{i=1}^{n} X_i \geq k)$$

as $U(z) = U_n(z) \otimes$ k, where $\varphi(X, k) = 1(X \geq k)$.
Calculate the system's reliability with the help of the following formula
$E(\varphi(X, k)) = U'(1)$ (differentiate the final function at $z = 1$).

VII. Example of the Bleaching Structure

Consider a complex bleaching structure as shown in Figure 1, in which unit 1 is in series with the combination of unit (2, 3) and (4, 5). Unit 2, 3 and 4, 5 are connected in parallel with each other. The reliability function of corresponding system using UGF can be calculated as follows:
UGF of all elements is defined as

$$U_1(\mathrm{z}) = P_1 z^1 + (1 - P_1) z^0 \quad (13)$$

$$U_2(\mathrm{z}) = P_2 z^1 + (1 - P_2) z^0 \quad (14)$$

$$U_3(z) = P_3 z^1 + (1 - P_3) z^0 \quad (15)$$

$$U_4(z) = P_4 z^1 + (1 - P_4) z^0 \quad (16)$$

$$U_5(z) = P_5 z^1 + (1 - P_5) z^0 \quad (17)$$

Hence in Figure 1, $U_2(z), U_3(z)$ and $U_4(z), U_5(z)$ are in parallel with each other, the reliability of the units in the UGF form is as follows:

$$U_6(z) = \max(U_2(z), U_3(z)) \quad (18)$$

$$U_7(z) = \max(U_4(z), U_5(z)) \quad (19)$$

Now,$U_1(z)$ is in series with $U_6(z)$ and $U_7(z)$. So, the reliability function of the unit in UGF form is as follows:

$$U_8(z) = \min(U_1(z), U_6(z), U_7(z)) \quad (20)$$

After simplifying the proposed complex system, the reliability function of the system can be estimated from$U_8(z)$ as
Now evaluate UGF of the above
$U_6(z) = \max(U_2(z), U_3(z))$
$\Rightarrow P_2 z^1 + (1 - P_2) z^0 \otimes P_3 z^1 + (1 - P_3) z^0$
$\Rightarrow (P_2 + P_3 - P_2 P_3) z^1 + (1 - P_2)(1 - P_3) z^0$.
$U_7(z) = \max(U_4(z), U_5(z))$
$\Rightarrow P_4 z^1 + (1 - P_4) z^0 \otimes P_5 z^1 + (1 - P_5) z^0$
$\Rightarrow (P_4 + P_5 - P_4 P_5) z^1 + (1 - P_4)(1 - P_5) z^0$.
$U_8(z) = \min(U_1(z), U_6(z), U_7(z))$
$\Rightarrow P_1 z^1 + (1 - P_1) z^0 \otimes (P_2 + P_3 - P_2 P_3) z^1 + (1 - P_2)(1 - P_3) z^0 \otimes (P_4 + P_5 - P_4 P_5) z^1 + (1 - P_4)(1 - P_5) z^0$
$\Rightarrow (P_1 P_2 + P_1 P_3 - P_1 P_2 P_3)(P_4 + P_5 - P_4 P_5) z^1 + [(P_2 + P_3 - P_2 P_3 - P_1 P_2 - P_1 P_3 + P_1 P_2 P_3)(P_4 + P_5 - P_4 P_5) + (P_1 - P_1 P_3 - P_1 P_2 + P_1 P_2 P_3)(P_4 + P_5 - P_4 P_5) + (1 - P_3 - P_2 + P_2 P_3 - P_1 + P_1 P_3 + P_1 P_2 -$

$P_1P_2P_3)(P_4 + P_5 - P_4P_5) + (P_1P_2 + P_1P_3 - P_1P_2P_3)(1 - P_4 - P_5 + P_4P_5) + (P_2 + P_3 - P_2P_3 - P_1P_2 - P_1P_3 + P_1P_2P_3)(1 - P_4 - P_5 + P_4P_5) + (P_1 - P_1P_3 - P_1P_2 + P_1P_2P_3)(1 - P_4 - P_5 + P_4P_5) + (1 - P_3 - P_1 + P_1P_3 - P_2 + P_2P_3 + P_1P_2 - P_1P_2P_3)(1 - P_4 - P_5 + P_4P_5)]z^0$.

Hence, reliability of the system is (Levitin, [25]) as

$R = P_1P_2P_4 + P_1P_2P_5 - P_1P_2P_4P_5 + P_1P_3P_4 + P_1P_3P_5 - P_1P_3P_4P_5 - P_1P_2P_3P_4 - P_1P_2P_3P_5 + P_1P_2P_3P_4P_5$.

Now, if all the elements are independent and identically distributed i.e.

$P_1 = P_2 = P_3 = P_4 = P_5 = P$, then the reliability function of the complex system is

$R = 4P^3 - 4P^4 + P^5$ (21)

1. Signature of the Bleaching Structure

Using Owen's method, the reliability function of the bleaching system is obtained in the form of p as follows

$H(p) = 4P^3 - 4P^4 + P^5$.

Now from equation (6), the polynomial function is

$P(v) = P^5 H\left(\frac{1}{P}\right) = 1 - 4P + 4P^2$.

To obtain the tail signature P of the complex bleaching system, procedure given in section 5.1 is followed

$P = \left(1, \frac{4}{5}, \frac{2}{5}, 0,0,0\right)$.

Now, to estimate the signature of the considered bleaching system equation (9) from section 5.1 is used

$P = \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}, 0,0\right)$.

2. Barlow-Proschan Index of Bleaching System

Now with the help of equation (11) calculate the Barlow-Proschan index for the considered bleaching system as follows

$I_{BP}^{(1)} = \int_0^1 (4P^2 - 4P^3 + P^4)dp = \frac{8}{15}$.

Similarly, Barlow-Proschan index $I_{BP}^{(K)}$ for $K = (2, \dots ,5)$ of all elements is

$I_{BP} = (\frac{8}{15}, \frac{7}{60}, \frac{7}{60}, \frac{7}{60}, \frac{7}{60})$.

3. Expected Lifetime of Bleaching System

Using Equation (5) from above the minimal signature M of the bleaching system is determined as

Minimal signature $(0,0,4,-4,1)$

Using minimal signature, expected $E(t)$ is obtained as

$E(t) = 0.534$.

4. Expected Cost Rate

Using equation (12), the expected value of the considered bleaching system is

$E(X) = 2.2$.

Expected cost rate $= E(X)/E(t)$,

$= 4.1199$

VIII. Example of the 2-out-of-4 Structure

The reliability of the considered 2-out-of-4 system can be calculated by using the algorithms as discussed above in section 6 such as

$U_1(z) = P_1 z^1 + (1 - P_1)z^0$ (22)

$U_2(z) = P_2 z^1 + (1 - P_2)z^0$ (23)

$U_3(z) = P_3 z^1 + (1 - P_3)z^0$ (24)

$U_4(z) = P_4z^1 + (1 - P_4)z^0$ (25)

Now following the second step from the above algorithm

$U_1(z) = u_1(z) = P_1z^1 + (1 - P_1)z^0,$

Following the third step from the algorithm and obtaining the other u-functions,

$U_2(z) = (P_1z^1 + (1 - P_1)z^0)(P_2z^1 + (1 - P_2)z^0)$

$U_2(z) = P_1P_2z^2 + (P_1 + P_2 - 1P_1P_2)z^1 + (1 - P_1 - P_2 + P_1P_2)z^0,$

$U_3(z) = P_1P_2z^2 + (P_1 + P_2 - 1P_1P_2)z^1 + (1 - P_1 - P_2 + P_1P_2)z^0(P_3z^1 + (1 - P_3)z^0),$

$U_3(z) = P_1P_2P_3z^3 + (P_1P_2 + P_1P_3 + P_2P_3 - 3P_1P_2P_3)z^2 + (P_1 + P_2 + P_3 - 2P_1P_2 - 2P_1P_3 - 2P_2P_3 + 3P_1P_2P_3)z^1 + (1 - P_1 - P_2 - P_3 + P_1P_2 + P_2P_3 + P_1P_3 - P_1P_2P_3)z^0,$

$U_4(z) = P_1P_2P_3z^3 + (P_1P_2 + P_1P_3 + P_2P_3 - 3P_1P_2P_3)z^2 + (P_1 + P_2 + P_3 - 2P_1P_2 - 2P_1P_3 - 2P_2P_3 + 3P_1P_2P_3)z^1 + (1 - P_1 - P_2 - P_3 + P_1P_2 + P_2P_3 + P_1P_3 - P_1P_2P_3)z^0(P_4z^1 + (1 - P_4)z^0,$

$U_4(z) = P_1P_2P_3P_4z^4 + (P_1P_2P_4 + P_1P_3P_4 + P_2P_3P_4 + P_1P_2P_3 - 4P_1P_2P_3P_4)z^3 + (P_1P_2 + P_1P_3 + P_1P_4 + P_2P_3 + P_2P_4 + P_3P_4 - 3P_1P_2P_3 - 3P_1P_2P_4 - 3P_1P_3P_4 - 3P_2P_3P_4 + 6P_1P_2P_3P_4)z^2 + (P_1 + P_2 + P_3 + P_4 - 2P_1P_4 - 2P_2P_4 - 2P_3P_4 - 2P_1P_2 - 2P_1P_3 - 2P_2P_3 + 3P_1P_2P_4 + 3P_2P_3P_4 + 3P_1P_3P_4 + 3P_1P_2P_3 - 4P_1P_2P_3P_4)z^1 + (1 - P_1 - P_2 - P_3 - P_4 + P_1P_2 + P_2P_3 + P_1P_3 + P_1P_4 + P_2P_4 + P_3P_4 - P_1P_2P_3 - P_1P_2P_4 - P_2P_3P_4 - P_1P_3P_4 + P_1P_2P_3P_4)z^0.$

Hence, from the above u-function, the expression for the reliability function of the considered 2-out-of-4 system is

$R = 3P_1P_2P_3P_4 - 2P_1P_2P_4 - 2P_1P_3P_4 - 2P_1P_2P_3 - 2P_2P_3P_4 + P_1P_2 + P_1P_3 + P_1P_4 + P_2P_3 + P_2P_4 + P_3P_4.$

Let all the elements of 2-out-of-4 system are independent and identically distributed i.e.,

$$P_1 = P_2 = P_3 = P_4 = P$$

After, applying above condition, the reliability function of the 2-out-of-4 system will become

$R = 3P^4 - 8P^3 + 6P^2$ (26)

1. Signature of the 2-out-of-4 System

Now, the reliability function of the considered system in the form of p by using Owen's method is as follows

$H(p) = 6P^2 - 8P^3 + 3P^4.$

Now from Equations (6) (calculation of signature) polynomial function is

$P(v) = P^5H\left(\frac{1}{P}\right) = 3 - 8P + 6P^2.$

With the help of equation (8) in section 5.1 obtain the tail signature P of 2-out-of-4 structure

$P = (1, 1,1,0,0).$

Now, using equation (9) estimate the signature of the considered 2-out-of-4 structure,

$P = (0,0,1,0).$

2. Barlow-Proschan index of 2-out-of-4 structure system

Now with the help of equation (11) calculate the Barlow-Proschan index of the 2-out-of-4 structure as

$I_{BP}^{(1)} = \int_0^1(3P^3 - 6P^2 + 3P)dp = \frac{1}{4}.$

Similarly, for $K = (2, \dots, 4)$ the Barlow-Proschan index $I_{BP}^{(K)}$ for all elements of the considered 2-out-of-4 system is as follows

$I_{BP} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4},).$

3. Expected Lifetime of the System

Now, using Equation (5) from above we get the minimal signature M of the 2-out-of-4 system as Minimal signature $(0,6,-8,3)$.

Using minimal signature, we obtain expected $E(t)$ such as

$E(t) = 0.4167.$

4. Expected Cost Rate

Using equation (12) of algorithm 5.4, the expected value of the 2-out-of-4 structure is determined as $E(X) = 3$,

Calculating the expected cost rate for the system using the formula

Expected cost rate = $E(X)/E(t)$,

$= 7.19943$.

IX. Conclusion

The signature reliability characteristics of bleaching system have been studied. This study mainly aims to evaluate the reliability function, signature, minimal signature, Barlow-Proschan Index, expected cost and expected lifetime of the considered system with UGF technique and also system have equal reliability with independent and identically distributed elements. This research discusses about the series parallel arrangement of the system and the failure of the system as a whole due to the failure of units. Some fundamental results concluded in this paper are as follows: Signature $P = \frac{1}{5}, \frac{2}{5}, \frac{2}{5}, 0, 0$, Barlow-Proschan index $I_{BP} = \frac{8}{15}, \frac{7}{60}, \frac{7}{60}, \frac{7}{60}, \frac{7}{60}$, and expected cost rate of the system by using Owen's method is 4.1199. One can extend his/her work in the area of complex system with the help of this paper.

Also, 2-out-of-4 system is considered, where out of 4 units in the system at least 2 elements should work for the whole system to work properly. The following system is described in brief and is solved to get the reliability of the system. The system is solved with the help of the algorithm discussed above and results are drawn.

Signature $P = 0,0,1,0$ Barlow-Proschan index $I_{BP} = \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$, and expected cost rate of the system by using Owen's method is 7.19943. The area of any analysis within the field of reliability and its characteristics is very beneficial for designing more reliable with low-cost systems. This study is abundantly helpful for the engineers, designers, researchers and students of various fields. Also, signature of the designed systems can be calculated with the help of order statistics for the future research purpose.

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Reliability Analysis of Periodically Inspected Systems under Imperfect Preventive Maintenance

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Abstract

In this study a periodically inspected system is analyzed which is subject to imperfect maintenance policy. The considered system is inspected and maintained periodically and passes through a fixed number of imperfect repairs before being replaced. The imperfect effect of the preventive maintenance is modelled on the basis of the increasing failure rate of the system. Since the preventive maintenance is imperfect, it brings the considered system to an operating state which lies between two states, namely the "as bad as old state" and "as good as new state". The distribution of the failure time of the system and that of the repair times are assumed to be arbitrary. The times required for the preventive maintenance of the system and its replacement are further assumed to be negligible. Incorporating the given facts, the reliability of the periodically inspected system is evaluated. Also, various reliability measures like availability, steady-state availability are estimated. An optimal interval of inspection is estimated so that the total cost in minimized. With the help of a case study on Nuclear Power Plant System (NPPS) the derived results are illustrated.

Keywords: reliability, imperfect preventive maintenance, steady-state availability.

I. Introduction

Maintenance systems are of utmost importance for various equipments to operate as desired, to facilitate them to function as required etc. Maintenance has a major impact on delivery, quality and cost of various systems. If maintenance is optimized, it can play a very significant role to improve the ability of an organization to fulfil its objectives. Maintenance includes preventive and corrective actions so as to reduce the chance of failure which not only further enhances reliability of the system but also reduces operation cost. There are four major categories of maintenance namely, perfect, minimal, imperfect, worst repair. Last few decades have seen a considerable amount of attention being paid to the area of maintenance modelling in various fields of reliability. For instance, Levitin and Lisnianski [10] considered a multi-state system and generalized a preventive maintenance policy to optimize it. In order to estimate the reliability of the system many techniques including universal generating function have been incorporated. Also, a genetic algorithm has been used for the optimization purpose. El-Ferik and Ben-Daya [7] investigated an age-based model (hybrid) subject to imperfect preventive maintenance. The model involves two types of failure modes namely, maintainable and non-maintainable. Also, both the hazard rate and the age (effective) have been incorporated in the study. The optimal number of maintenance actions have been found out so that the expected cost is minimized. Castro [2] investigated a system which can fail due to two possible failure modes namely, maintainable mode and non-maintainable mode. Also, at the instance of failure a minimal repair is carried out. Preventive maintenances are conducted at regular intervals with the system being replaced after a fixed number of maintenance cycles. The basic problem was to find a feasible length between consecutive maintenance cycles so that cost rate is minimized. Bartholomew et al. [1] in their study dealt with the notion of scheduling the imperfect preventive maintenance of the system. The model was based on Kijima where each application of maintenance reduces the age of the system. Also, a numerical study illustrates the optimum schedule for different measures of the age-based model. Soro et al. [17] in their paper examined a multi-state system which is subject to preventive maintenance (imperfect) and estimated its availability, reliability etc. The minimal repair brought back the system to the previous state without impacting the rate of failure. The system has been modeled as a Markov process to evaluate the various performance measures. Liu et al. [11] in their paper proposed a model for the maintenance of mission-oriented systems which are exposed to continuous shocks and degradation. The degradation of the system has been modelled as a Weiner process. Also, an optimal policy for the maintenance of the system has been developed. Mercier and Castro [13] considered a degrading system and compared two models subject to imperfect repair. The level of degradation has been modelled by a Gamma process (non-homogeneous). A comparison has been made between both the models based on stochastic property. Furthermore, optimal strategies for maintenance have been explored.

All the aforementioned models have been studied in one or the other way to improve the availability and reliability of the systems. There has been an extensive study on the reliability models in the past. Liu and Kapur [12] studied a multistate system and developed various reliability measures. It was assumed that the considered system follows Markov process. Also, it was focused on evaluating considered systems for system designing. Singh [16] considered a system which consists of two subsystems. Subsystem-1 is a k-out of n subsystem while subsystem-2 consists of two units arranged in a parallel configuration. The failure rates have been assumed to be constant while the repair is governed by general and exponential distributions. Various performance measures like reliability, cost have been evaluated with the technique of supplementary variable, Laplace and copula methodology. Fan et al. [9] developed a model for estimating the reliability dependent of a system subject to failure processes which often results in the shock dependence. The method of Monte Carlo simulation has been used to calculate the various reliability measures. A real-time application has been demonstrated to study the dependent behavior of the failures. Song et al. [18] in their study considered a multi-state system whose components are dependent. In the proposed study stochastic multi valued models have been presented and a comparative analysis has been done to illustrate the model. Cha et al. [4] studied a model in consideration to the systems which operate in a given environment. The system is subject to a Poisson process of shocks where each of the shocks has a double effect on the system. Hence, the considered system is bivariate and the study is illustrated with numerical examples. Dong et al. [6] investigated a parallel system with redundance comprising of various components which are non-identical. Also, a self-healing mechanism is developed with respect to the damage load and numbers of shock arrivals. In this study reliability model is proposed to evaluate the closed form expressions for reliability and preventive maintenance. Furthermore, a Nelder-Mead simplex method has been applied to find the feasible age of replacement of the system. Nautiyal et al. [14] in their study considered a k out of n network and focused on evaluating reliability and its other measures. The method of Gumbel- Hougaard has been applied to obtain the various performance measures. Eloy and Dawabsha [8] considered a multistate system with warm standby. There are two major states in which the system has been partitioned namely, internal degradation and shocks. The process of repair comprises of more than one repairperson. The proposed model is presented and built in an algorithm form which makes its implementation to be

easier. Cao et al. [3] examined a multi-state system having multiple components and explored how the process of ageing affects the process of failure in the considered system. The analysis of reliability and its various measures is done on the basis of a semi-Markov chain. With the help of an example of a transformer the proposed method has been demonstrated. Pan et al. [15] in their study desired to model a method for the evaluation of reliability. To model the parameter for obtaining the reliability the Wiener process has been incorporated. Then after Copula function has been effectively used to demonstrate the proposed method. Chen et al. [5] investigated about multi-layer systems and studied the behavior of the failure in the system. Further, a simulation method has been proposed in order to estimate the reliability of the system along with a maintenance cost model.
In this paper, we have restricted our study to imperfect preventive maintenance in which the health of the considered system lies between "as good as new" and "as bad as old." Also, we have considered a system which is inspected at regular constant intervals and is subject to a fixed number of maintenance cycles before being replaced. The motivation for performing inspections periodically arises from the fact that it may reduce the cost which is involved in continuous monitoring of the system. Some real-life instances where systems are inspected periodically are gas detectors, safety valves etc. Also, a perfect repair might require high costs, thus various imperfect repairs are done before the system is actually replaced. The failure rate model has been used to model the effect of imperfect maintenance system. Also, corrective maintenance is carried out whenever there is a failure in the system. Hence, the proposed model extends the contribution to the existing literature since it allows the effect of imperfect maintenance on the periodically inspected system which is followed by a corrective repair at the occurrence of every failure.

II. Model description and notations

II.I System Description

In this section we discuss the failure rate model for a periodically inspected system which has been used as a modelling framework of imperfect maintenance. For this we assume that the lifetime of the considered system follows general distribution with the corresponding failure rate function being denoted by $\sigma(t)$. Based on these assumptions, following observations are made:

- The failure rate $\sigma(t)$ of the system increases strictly after every mth imperfect preventive maintenance and becomes $l_m\sigma(t)$ when it was $\sigma(t)$ before the preventive maintenance took place.
- Thus, $L_m\,\sigma(t)$ is the failure rate in the mth preventive maintenance where $L_m = l_0\, l_1{*}l_2{*}\ldots\ldots{*}l_{m-1}$ and $1 = l_0 < l_1 < l_2 < \ldots\ldots\ldots < l_{Q-1}$

II.II Assumptions

1. At time $t = 0$ the system is operational.
2. The system is inspected at regular intervals and is preventively maintained.
3. The failure rate $\sigma_m(t)$ is strictly increasing.
4. The failure is detected only at the time of inspection and the system is subject to corrective repair at the instance of a failure.
5. The times required for preventive maintenance and replacement of the considered system is assumed to be negligible.
6. The system becomes as good as new after every replacement.
7. The probability density function of service times θ_m corresponding to the mth imperfect maintenance obeys general distribution $Y(t)$.
8. The cost rate associated with every inspection, preventive maintenance is denoted by I_{FC} and C_{CPM}

respectively. The cost associated with every system replacement is given by C_{REPL}. The cost rate during the system downtime is denoted by D_C. The CM expense (cost) rate corresponding to the failure of the system is represented by C_{RC}.

II.III Notations

$A(t)$	System's instantaneous availability
$R(t)$	System's reliability
CM	Corrective Maintenance
$\sigma(t)$	Failure Rate of the system.
i.i.d.	Identically and independently distributed
Q	Number of Preventive maintenance cycles
η_m	Failure times corresponding to the mth preventive maintenance
$B_m(t)$	Distribution function of η_m, m =1, 2, ..., Q-1.
$b_m(t)$	Probability density function of η_m, m =1, 2, 3…., Q-1.
θ_m	Repair times during the mth preventive maintenance, m = 1, 2…, Q-1.
$Y(t)$	cdf of repair times.
$y(t)$	Probability density function for repair time
H	Periodicity of inspection
N_T	Number of inspections performed in every system restoration
LC	Total length of cycle requires for restoration of the system
I_{FC}	Fixed cost of the inspection
C_{CPM}	Preventive maintenance cost
C_{REPL}	Cost associated with the replacement of the system
D_C	Cost rate(penalty) while system is in downtimes
TC_R	Total cost in a cycle (renewal) when the system is restored

III. Reliability of the periodically inspected system

The current section develops expressions to evaluate the reliability and availability of the periodically inspected system with imperfect maintenance modelled in the previous section. Thus, the function for the estimation of the reliability is derived as follows:

$$R(t) = \prod_{m=1}^{Q-1} P(\Omega_m > t) = \prod_{m=1}^{Q-1} R_m(t)$$

Thus, proposition-1 which is given below helps to calculate the instantaneous/point availability of the periodically inspected system based on the renewal theorem.

III.1 Instantaneous Availability Analysis

Proposition-1: A periodically inspected system is maintained such that the failure rate of the system is modelled by an imperfect preventive maintenance policy. The failure times η_m are governed by a general distribution function $B_m(t)$ in the mth preventive maintenance (m =1, 2, …., Q-1) respectively. Service times θ_m are assumed to follow general distribution$Y(t)$. Thus, the expression for the instantaneous availability of the considered periodically inspected system can be derived as follows:

$$A(t) = \prod_{m=1}^{Q-1} R_m(t) + \sum_{j=0}^{[t/H]-1} \sum_{m=1}^{Q-1} \int_{jH}^{(j+1)H} l_m \sigma(t) R(t) \int_{0}^{t-(j+1)H} A(t-(j+1)H-z)\, y(z)dz$$

Proof. There can be two possible states for the considered periodically inspected system, namely the down-state, which corresponds to Ω (t) = 0, or in the up-state, which implies Ω (t) = 1.
Based on the assumption that the system is operable at t = 0 and no inspection and preventive maintenance has been performed in the first interval, i.e. [0, H).
Thus, from the fundamental definition of availability we have the following:
Since the system is replaced at Qth cycle, hence, Q -1= J (say) will be the total number of inspections

$$A(t) = P \text{ (the system is available at an instant } t) = P\,(\Omega(t) = 1) \tag{1}$$

performed till the time system gets replaced. Furthermore, the time of failure for the periodically inspected system Ω are related as, (J −1) $H < \Omega < J$ H. Consequently, the probability mass function of J is given as

$$P(J = j) = R(jH) - R((j+1)H) \tag{2}$$

Therefore, equation (1) becomes

$$A(t) = \sum_{j=0}^{[t/H]-1} P(\Omega(t) = 1, J = j) + \sum_{j=[t/H]}^{\infty} P(\Omega(t) = 1, J = j) \tag{3}$$

The probability that the first failure of the system happens to be in [(j − 1) H, jH] is expressed by the first half of the above equation and the system gets repaired/restored at time jH, where j = 2, 3, …while the second half of (3) directs toward the event that system survives before time t and is represented as given below:

$$P(\Omega(t) = 1, J \geq [t/H]) = P(\Omega > t) \tag{4}$$

Since there are Q maintainable cycles, the failure rate in mth preventive maintenance cycle shall be $\sigma_m(t)$, m = 1, 2, …, (Q -1), hence the first half of equation (3) becomes

$$\sum_{j=0}^{[t/H]-1} P(\Omega(t) = 1, J = j) = \sum_{j=0}^{[t/H]-1} \sum_{m=1}^{Q-1} P(\Omega(t) = 1, J = j, M = m) P(J = j, M = m) \tag{5}$$

The event {J = j, M = m} implies no failure has occurred in [0, jH] and it is only in the consequent interval, i.e. [jH, (j +1) H] there is a probability of the failure of the considered system.
Thus, we have

$$P\,(J = j, M = m) = P\,\{jH < \Omega < (j+1)\,H,\ \Omega_m = \min\,(\Omega_1, \Omega_2, \Omega_3, \ldots, \Omega_{Q-1})\}$$

$$= \int_{jH}^{(j+1)H} R(t) l_m \sigma(t) dt \tag{6}$$

In the considered model, it is assumed that the failure rate of the periodically inspected system becomes $l_m\sigma(t)$ after the mth preventive maintenance when it was $\sigma(t)$ before the mth preventive maintenance. The repair rate of the system remains unchanged. Thus, the probability associated with the failure of the considered system in the mth imperfect preventive maintenance cycle can be obtained from equation (6) as

$$P(\Omega_m = \min(\Omega_1, \Omega_2, \ldots\ldots\ldots\Omega_{Q-1})) = \int_0^{\infty} R(t) l_m \sigma(t) dt \tag{7}$$

The total sojourn repair time θ_m elapsed in the restoration of the system due to system failure in the mth preventive maintenance is governed by the distribution function $Y(z)$. After the sojourn time to repair is elapsed the operating unit is functional again and the process repeats itself. Furthermore, the same process is repeated every time. Then, we have,

$$P(\Omega(t)=1 \mid J=j, M=m) = \int_0^{t-(j+1)V} P(\Omega(t)=1 \mid J=j, M=m, \theta_m = z) y(z) dz$$

$$= \int_0^{t-(j+1)H} A(t-(j+1)H-z) y(z) dz \tag{8}$$

Using the equations (7) and (8), the expression (5) becomes

$$\sum_{j=0}^{[t/H]-1} P(\Omega(t)=1), J=j) = \sum_{j=0}^{[t/H]-1} \sum_{m=1}^{Q-1} \int_{jH}^{(j+1)H} R(t) l_m \sigma(t) \int_0^{t-(j+1)H} A(t-(j+1)H-z) y(z) dz \tag{9}$$

Substituting the equations (8) and (9) in (3), the instantaneous availability of the periodically inspected system is obtained as

$$A(t) = \prod_{m=1}^{Q-1} R_m(t) + \sum_{j=0}^{[t/H]-1} \sum_{m=1}^{Q-1} \int_{jH}^{(j+1)H} l_m \sigma(t) R(t) \int_0^{t-(j+1)H} A(t-(j+1)H-z) y(z) dz \tag{10}$$

III.II Long-run availability

The following proposition gives the expression for estimating the steady-state availability of the periodically inspected system subject to imperfect maintenance.

Proposition-2: The availability of the proposed periodically inspected system in the steady state is obtained as follows:

$$A = \frac{\int_0^{\infty} R(t) dt}{\sum_{j=0}^{\infty} (j+1)H(R(jH) - R(j+1)H) + \sum_{m=1}^{Q-1} E(\theta_m) \int_0^{\infty} R(t) l_m \sigma(t) dt}$$

Proof: With the help of the renewal theorem the long-run availability of the considered periodically inspected system is derived as the ratio of expectation of the uptime of the system and the sum of the expectations of the system uptimes and downtimes.

Thus, based on Proposition 1, the expectation of the uptimes of the considered periodically inspected system is obtained as follows:

$$\text{E (Up times)} = \text{E}(\Omega) \tag{11}$$

the length(expected) of each restoration cycle (E(*LC*)) is calculated on the basis of the probability decomposition method which is obtained as given below:

$$E(LC) = \sum_{j=0}^{\infty} (j+1)H(R(jH) - R(j+1)H) + \sum_{m=1}^{Q-1} E(\theta_m) \int_0^{\infty} R(t) l_m \sigma(t) dt \tag{12}$$

The first half of equation (12) corresponds to the expected time between any unexpected/sudden failures to the time it has been restored again. The second half of equation (12) denotes the expected time for restoration.

On combining equations (11) and (12) the expression for the availability (in the steady state) of the considered periodically inspected system is derived as

$$A = \frac{\int_0^\infty R(t)dt}{\sum_{j=0}^{\infty}(j+1)H(R(jH)-R(j+1)H)+\sum_{m=1}^{Q-1}\theta_m\int_0^\infty R(t)l_m\sigma(t)dt} \tag{13}$$

Corollary 2.1: Here the bounds for the steady state availability, i.e., the upper bound and lower bound availability denoted by A_U and A_L respectively for the risk system are obtained as follows:

$$A_U = \frac{\int_0^\infty R(t)dt}{E(\Omega)+\sum_{m=1}^{Q-1}E(\theta_m)\int_0^\infty l_m\sigma(t)R(t)dt} \tag{14}$$

$$A_L = \frac{\int_0^\infty R(t)dt}{E(\Omega)+L+\sum_{m=1}^{Q-1}E(\theta_m)\int_0^\infty l_m\sigma(t)R(t)dt} \tag{15}$$

Clearly, equations (15) and (14) signify that the lower bound of availability is less than the availability of the considered system which is further less than the upper bound of availability.

III. III Maintenance Analysis

As per the maintenance policy implemented in the current study, the system may experience *Q* preventive maintenance cycles. Corrective maintenance is done whenever there is a system failure while preventive maintenance is carried out whenever an inspection is performed. Towards the end of every *Q*th cycle, the considered system is replaced with a new one. The expected duration of system replacement is assumed to be negligible. The optimal maintenance policy consists on finding a feasible value of the interval of inspection and the replacement period. Thus, the average maintenance cost rate is then a function of the both variables *H* and *Q*, and hereafter denoted by *W* (*H*, *Q*) and is defined as:

$$W(H,Q) = \frac{E(TC_R)}{E(LC)} \tag{16}$$

where $E(TC_R)$ is the expected total cost.

Proposition 3: For a periodically inspected system which is inspected and preventively maintained at regular intervals, the average rate of cost of maintenance is obtained as given below:

$$W(H,Q) = \frac{I_{FC}\sum_{j=0}^{\infty}(j+1)(R(jH)-R(j+1)H)+D_C\left(\sum_{j=0}^{\infty}(j+1)(R(jH)-R(j+1)H)-\int_0^\infty R(t)dt\right)+\sum_{m=1}^{Q-1}C_{RCm}E(\theta_m)\int_0^\infty R(t)l_m\sigma(t)dt+(Q-1)C_{CPM}+C_{REPL}}{\sum_{j=0}^{\infty}(j+1)(R(jH)-R(j+1)H)+\sum_{m=1}^{Q-1}E(\theta_m)\int_0^\infty R(t)l_m\sigma(t)dt}$$

Proof: In every renewal cycle when the considered system is restored then the expected(total) maintenance cost rate involved is given as follows:

$$E(TCR) = IFC\ E(NT) + E(C_{RC}) + DC\ E(Downtimes) + CCPM\ E(Q-1) + E(CREPL) \tag{17}$$

Hence, the mean cost involved in the corrective maintenance (CM) in every restoration cycle is given as

$$E(C_{RC}) = \sum_{m=1}^{Q-1} C_{RCm} \int_0^{\infty} l_m \sigma(t) R(t) dt \tag{18}$$

where $N_T = J + 1$ is the total count of inspections.
Also, the expectation of the system downtime is obtained as:

$$E(Downtimes) = \sum_{j=0}^{\infty} (j+1)(R(jH) - R(j+1)H) - \int_0^{\infty} R(t)dt \tag{19}$$

Therefore, the expectation of the cost (total) for every cycle when the periodically inspected system is restored is

$$E(TC_R) = I_{FC} \sum_{j=0}^{\infty} (j+1)(R(jH) - R(j+1)H) + D_C \left(\sum_{j=0}^{\infty} (j+1)(R(jH) - R(j+1)H) - \int_0^{\infty} R(t)dt \right) + \sum_{m=1}^{Q-1} C_{RCm} E(\theta_m) \int_0^{\infty} R(t) l_m \sigma(t) dt + (Q-1) C_{CPM} + C_{REPL}$$

Combining the results of equation (20) and equation (11) into equation (16), the mean cost rate for maintenance with respect to the imperfect preventive maintenance is obtained as given below:

$$W(H,Q) = \frac{I_{FC} \sum_{j=0}^{\infty} (j+1)(R(jH) - R(j+1)H) + D_C \left(\sum_{j=0}^{\infty} (j+1)(R(jH) - R(j+1)H) - \int_0^{\infty} R(t)dt \right) + \sum_{m=1}^{Q-1} C_{RCm} E(\theta_m) \int_0^{\infty} R(t) l_m \sigma(t) dt + (Q-1) C_{CPM} + C_{REPL}}{\sum_{j=0}^{\infty} (j+1)(R(jH) - R(j+1)H) + \sum_{m=1}^{Q-1} E(\theta_m) \int_0^{\infty} R(t) l_m \sigma(t) dt}$$

IV. Evaluation of Reliability Characteristics

Numerical Example: An Application of Nuclear Power Plant System

Background

Nuclear power plant systems (NPPS) need unremitting and careful monitoring from time to time. Whenever there is a chance of unsafe conditions in the NPPS corrective measures must be taken to ensure proper functioning of the system. In order to control such circumstances appropriate knowledge of the prevailing conditions and the ones to be changed should be known at prior. The diagnosis of any fault in the plant should be known very soon. Thus, preventive maintenance plays a major role in this direction. Through such maintenance policies the plant operators and engineers can perform the required preventive and corrective actions, if needed. The reactor coolant pump is an integral part of the NPPS. It prevents and safeguards the vibrations which may lead to trippage of the reactor. To illustrate the various obtained results, we assume that there is a fault in the reactor coolant of the NPPS. The failure time of the reactor coolant is governed by the distribution $B_m(t)$ = 1-exp(-mt), m > 0. Thus, as per the assumptions since l_0 =1, then the distribution for the failure time $B_1(t)$ =1-exp (-t) and the failure rate remains 1 in the interval [1,2]. Let us now suppose that l_1 = 2. Hence the rate of failure becomes 2 in the interval [2,3]. Also, let the distribution for repair, i.e., $Y(z)$ = 1- exp(-3z). We further assume that the system is replaced at 3rd cycle. Furthermore, the values of the relevant system parameters used to evaluate the desired results are listed below in Table 1.

Table 1: *System Parameter Values*

Parameter	
Parameter	Values

H	1 time unit
I_{FC}	1 cost unit
C_{RC1}=C_{RC2}	2 cost unit
C_{REPL}	1 cost unit
D_C	10 cost unit

IV.1. Reliability and Availability Evaluation

With respect to the imperfect maintenance policy, the results of Section 4 are well illustrated through the graphs plotted in Fig. 1 and Fig. 2. It is clear from the Fig. 1 that the reliability of the considered system decreases with the increase in time and approaches to zero over a long period of time. The Fig. 2 corresponds to the instantaneous availability of the system with respect to the inspection period H =1. It can be seen from the Fig. 2 that instantaneous availability is same as the system's reliability in the time [0,1). In the consequent interval which is followed by a preventive repair at H =1 the system availability increases till the time t = 1.6 and then further decreases till t = 2. The failure rate now further increases in the interval [2,3] as the system is subject to imperfect maintenance. Thus, in the interval [2,3] failure is detected and repaired which increases the availability of the system. From this figure the steady state availability is found to be 0.2405 %. Finally, the system is replaced at the end of the interval [2,3] after the preventive maintenance. Fig. 3 shows the sensitivity of the considered periodically inspected system with respect to the inspection period (H). It illustrates the availability of the system when inspection period varies for H = 1, H = 1.5, H = 2. As anticipated, Fig. 3 validates that when there is a longer gap between two inspection periods i.e., the value of H increases the system availability decreases.

IV.II. Optimal Inspection Policy

Fig. 4 reflects the steady state availability of the system with respect to different periodicity of inspections. As seen from the figure that it is a strictly decreasing curve, i.e., larger the periodicity lesser is the steady-state availability of the system. The current study seeks, however, to obtain an optimal inspection period which will minimize the average cost rate. Fig. 5 represents the average long run maintenance cost rate when periodicity of inspection is H =1. It is quite evident from the graph that the maintenance cost rate decreases initially and eventually becomes constant. From this figure the minimum cost rate is attained when H = 0.8 and the cost rate is found to be 9.5327. Fig. 6 shows the effect of varying replacement periods on the long run maintenance cost rate of the system. It demonstrates that as the value of Q increases the cost rate also increases. It is quite intuitive also since a larger gap between consecutive replacement periods will result in performing inspections more frequently thereby increasing the frequency of preventive maintenances. Hence, the cost rate of the system increases substantially.

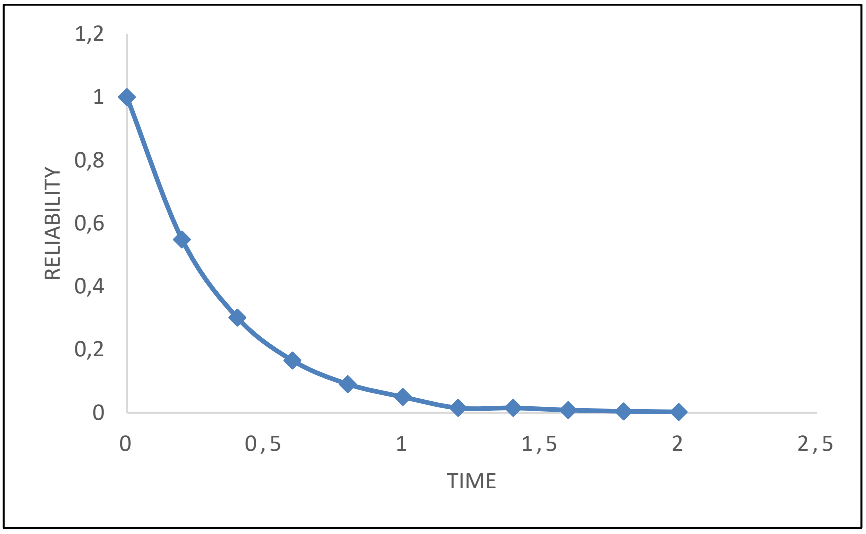

Fig. 1: *Reliability of the system.*

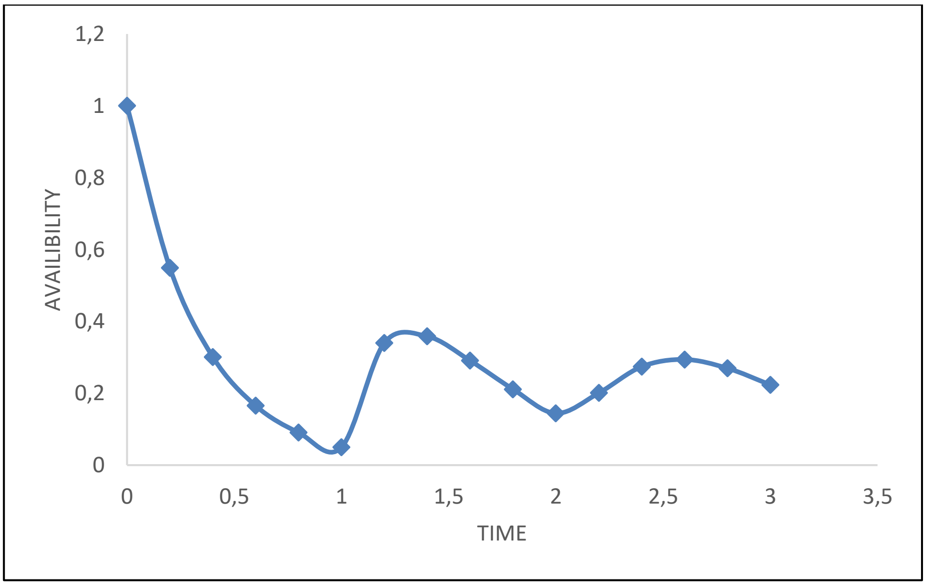

Fig. 2: *Availability when two imperfect repairs are followed by a replacement: H =1.*

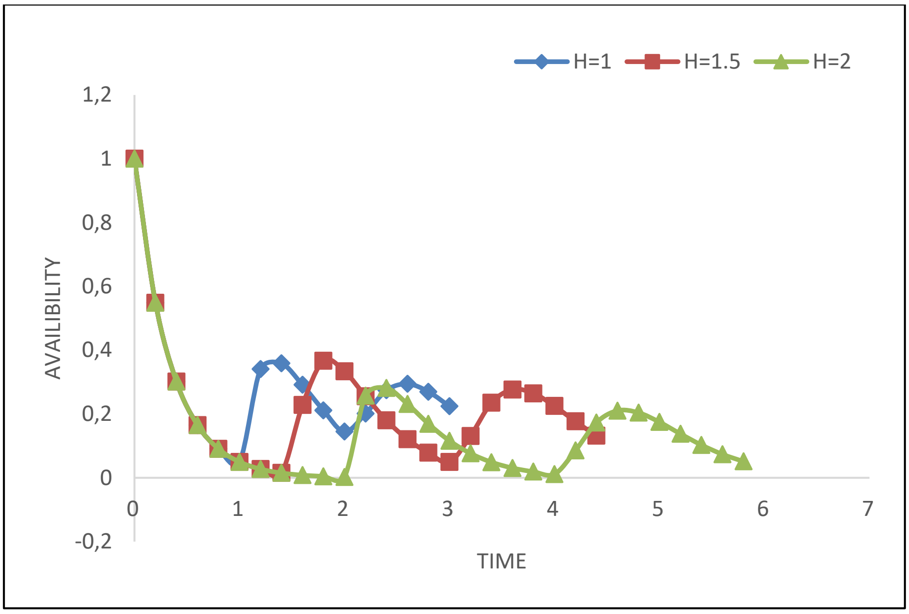

Fig. 3: *Sensitivity of the NPPS system*

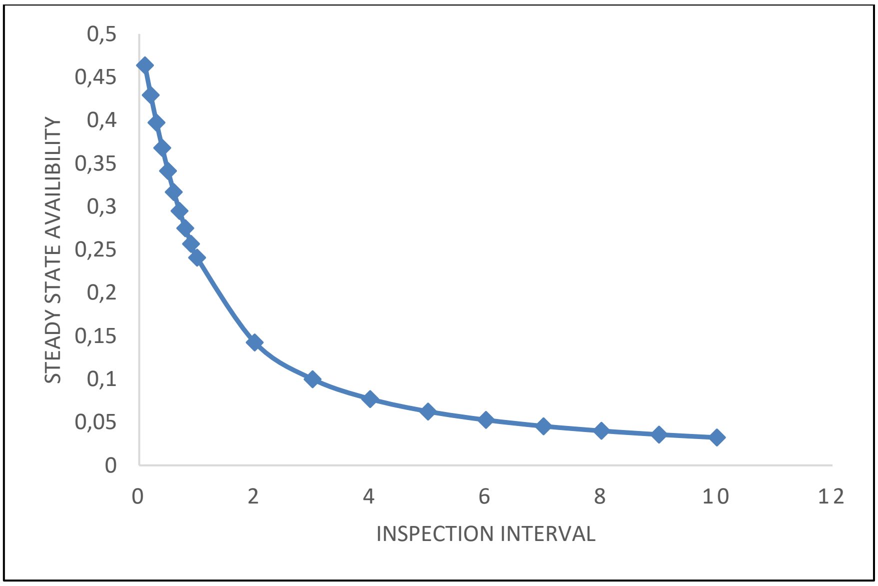

Fig. 4: *Steady-state Availability versus H.*

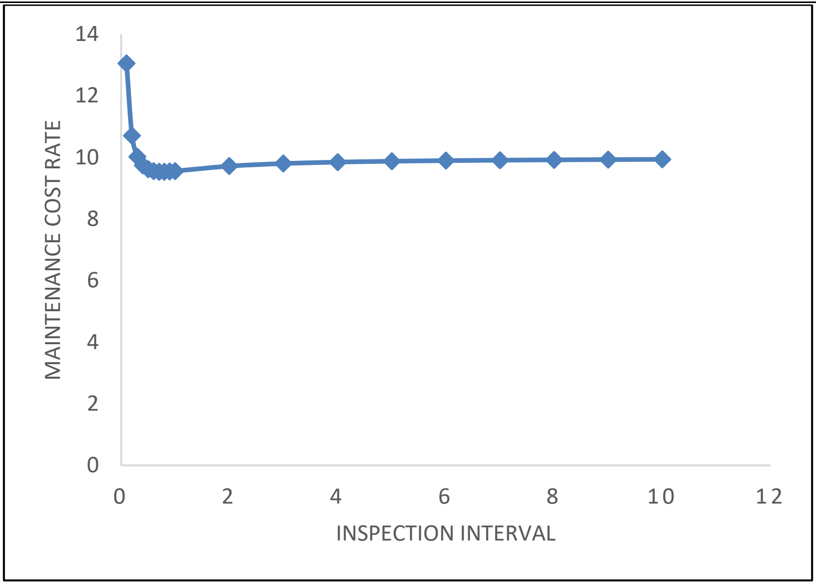

Fig. 5: *NPPS's maintenance cost rate.*

V. Discussion

The current study developed an imperfect preventive maintenance model for a periodically inspected system. The considered system is assumed to be inspected periodically for the detection of any possible failure. Imperfect preventive maintenance actions are modelled on the basis of a failure rate. The strictly increasing failure rate ensures and allows to represent the occurring failures in the system and the imperfect effect of the preventive maintenance. Furthermore, a case study on NPPS is then proposed and the results are illustrated with times of failure following exponential distribution. Also, the times to repair have been assumed to be exponentially distributed with the periodicity of inspection as one.

Conflicts of interest: Authors hereby state that there are no conflicts of interest in this manuscript. This is soul work of authors which has not submitted in any other journal. The manuscript follows ethically guidelines for the journal.

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Two-Dimensional SRGM with Delay in Debugging by Considering the Uncertainty Factor and Predictive Analysis

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Abstract

Software is a set or group of programs and instructions, which is designed to perform a well-defined function. A software fault/error can be the cause of major productivity or economic loss. The need to determine the reliability of software led to development of the software reliability growth models (SRGMs). In research literature, exist various SRGMs which have been developed without considering fault removal time and only a few models have incorporated operating environment uncertainty. However, Most of the SRGMs are characterized under the condition that the software reliability enhancement process depends only on the testing time, which is called one-dimensional SRGM. In this article, two-dimensional SRGMs have been proposed using delay in debugging, with uncertainty in the operating environment. To represent the combined effect of used resources and testing time, the Cobb – Douglas production function has been used, which converts the one-dimensional model to two-dimensional. For better understanding the results predictive analysis is performed and to validate the proposed model results are compared with existing SRGMs using five statistical comparison criteria.

Keywords: Software, SRGMs, Predictive analysis, Reliability, Cobb-Douglas production function.

I. Introduction

Software lifecycle management through development, usage, operation and maintenance of systems, requires a systematic approach. Planning, design, coding, testing and analysis are the five phases of any software lifecycle. Software testing is the critical phase to reliably measure the functionality of the software. Any error in the software may cause considerable economic and operational damage. For successful functioning of any software, the important factor is to minimize the possibility of software failure with high reliability. "Reliability is the probability that the system will execute its indeed function under definite limit" [1]. To develop reliable software, many SRGMs have been studied but most of them are insufficient for today's complex software. The present SRGMs can be specified in two types, one-dimensional and two-dimensional models. One-dimensional SRGMs consider only testing time, whereas two-dimensional SRGMs depend on the effect of testing time as well as used resources. Kapur et al. [2] proposed a two-dimensional SRGM in perfect debugging environment for multi-release software. Singh et al. [3] considered the effect of the parameters like testing-time and testing-coverage or testing-time and testing- effort, simultaneously on fault detection process with change point. Anniprincy and Sridhar [4] discussed an S-shaped model with joint effect of testing-time and testing-coverage in imperfect debugging. SRGM modeling based on a CES (constant elasticity of substitution) type time function was introduced by Minamino et al. [5]. Pachauri et al. [6] studied an SRGM in imperfect debugging environment that considers the delay in fault correction.

In traditional SRGMs, it has been studied under the common belief that when a fault occurs in the software, it can be immediately removed. These models assume that the testing and operating environment of the software are the same. The first NHPP based traditional SRGM (the NHPP exponential model) was introduced by Goel and Okumoto [7]. The inflection S-shaped SRGM, depend on the failure rate of each detectable fault was discussed in Ohba [8]. Inflection S-shaped function as a fault reduction factor was given by Pachauri et al. [9].

A few models were also developed using time lag in the fault removal process. Yamada et al. [10] considered the time delay effect between the fault removal and fault detection process. An SRGM with fault dependency and various time lags, to predict the reliability, was introduced by Pradhan et al. [11]. Huang and Lin [12] proposed that in the fault removal process the correction time cannot be neglected.

In some recent publications, authors have included uncertainty of environment in the SRGMs [13 -19]. Song et al. [14] used inflection factor of error detection with vagueness in operating environment. Pham [15] introduced a new SRGM with Vtub-shaped fault detection rate. Song et al. [17] studied an SRGM with an ideal software release time and have done sensitivity analysis. Pachauri et al. [20] proposed new SRGM under the fuzzy paradigm with optimum release time and calculated total software cost.

Predictive analysis can provide advance warning of any failure, which may prove invaluable for system operators. The predictive analysis depends on historical data to make predictions about any possible future failures and errors. Recently, Song et al. [14] have introduced predictive analysis method in their SRGM.

From the literature, it is found that the factors like environmental factor, delay in debugging, effect of testing-time and used-resources, etc. are not included in an SRGM simultaneously. Therefore, motivated by Kapur et al. [2], Pachauri et al. [6] and Song et al. [14], a perfect debugging SRGM by considering uncertainty of operating environment with delay has been proposed. The predictive analysis also has been done for a better understanding of prediction. The proposed study may be useful for any software development company to check the reliability of the software based on the testing data before launching it in the market and may be used to estimate the optimum release time of software.

In this paper, two-dimensional perfect debugging SRGMs have been proposed by considering delay in fault correction and uncertainty of environment. Calculation of Mean value function (MVF) is given in Section 2 with mathematical derivation. To validate the models, the used comparison criteria, numerical results are in Section 3 and predictive analysis is shown in section 4. Finally, the conclusion is presented in Section 5.

II. Proposed model

The following are the common assumptions for these models [6, 18]:

- The software failures phenomena follow NHPP.
- Fault detection rate and residual system faults are proportional.
- The operating environment uncertainty is the multiplication of the unit failure detection rate $b(t)$ with a random variable η.
- The software system may fail during execution, due to remaining faults in the system.

- The compound effect of the used resources and testing effort is represented using Cobb-Douglas production function.

Based on these assumptions, the rate of change in MVF with uncertainty factor is given as [15]:

$$\frac{d\,m(t)}{d(t)} = \eta[b(t)][N - m(t)], \tag{1}$$

where η is a random variable (r.v.) that represents the uncertainty of operating environments with the probability density function g, $b(t)$ is the fault-finding rate function and N is the total number of faults initially [15]. The solution of the above differential equation with the initial condition $m(0) = 0$ is given as:

$$m(t) = \int_{\eta} N\left(1 - e^{-\eta\int_0^t b(x)dx}\right) dg(\eta), \tag{2}$$

where g has the parameters $\alpha \geq 0$, $\beta \geq 0$ and η follows the gamma distribution. After applying the r.v. η in equation (2) we get:

$$m(t) = N\left(1 - \frac{\beta}{\beta + \int_0^t b(s)ds}\right)^{\alpha}, \tag{3}$$

when detection rate function $b(t)$ is given as [14],

$$b(t) = \frac{b}{1 + ae^{-bt}}, a, b > 0, \tag{4}$$

then the cumulative number of faults at time t,

$$m(t) = N\left(1 - \frac{\beta}{\beta + \ln\left(\frac{a + e^{bt}}{1 + a}\right)}\right)^{\alpha}, \tag{5}$$

where b is the constant fault detection rate and a is the inflection factor.

After considering the testing-time and used-resources simultaneously, the modified MVF is given as [2, 6];

$$m(s,u) = N\left(1 - \frac{\beta}{\beta + \ln\left(\frac{a + e^{b(s^{\gamma}u^{1-\gamma})}}{1 + a}\right)}\right)^{\alpha}, \tag{6}$$

where s is the testing time and u is used resources. By using the delay factor which is a function of time, the new MVF $m(s,u)$ is given as;

$$m(s,u) = m(s - \varphi(s), u). \tag{7}$$

$$m(s,u) = N\left(1 - \frac{\beta}{\beta + \ln\left(\frac{a + e^{b((s-\varphi(s))^{\gamma}u^{1-\gamma})}}{1 + a}\right)}\right)^{\alpha}. \tag{8}$$

Here, two types of delay functions are considered, delayed S-shaped and inflection S-shaped curve.

Model-1

In the fault detection exercise, the delay function is modeled as an S-shaped curve. It considers the learning progress because the skills of the examiners or testers are directly proportional to time [6]. A good explanation of delayed S-shaped curve is given in [1] and mathematically it is defined as;

$$\boldsymbol{\varphi(s) = \frac{1}{b}\log(1 + bs)}, \tag{9}$$

after using the delay function as defined in equation (9), then the MVF is,

$$\boldsymbol{m(s,u) = N\left(1 - \left(\frac{\beta}{\beta + \ln\left(\frac{a + e^{b\left(s - \frac{1}{b}\log(1+bs)\right)^{\gamma} u^{1-\gamma}}}{1 + a}\right)}\right)^{\alpha}\right)}. \tag{10}$$

Model-2

When the inflection S-shaped curve is used as the delay function which is defined as,

$$\boldsymbol{\varphi(s) = \frac{1}{b}\log\left(\frac{1 + \psi}{1 + \psi e^{-bs}}\right)}, \tag{11}$$

where ψ is the inflection factor of Inflected S-shaped curve [6]. Then, the MVF with delay factor is,

$$\boldsymbol{m(s,u) = N\left(1 - \left(\frac{\beta}{\beta + \ln\left(\frac{a + e^{b\left(s - \frac{1}{b}\log\left(\frac{1+\psi}{1+\psi e^{-bs}}\right)\right)^{\gamma} u^{1-\gamma}}}{1 + a}\right)}\right)^{\alpha}\right)}. \tag{12}$$

III. Numerical results and discussion

For the performance validation of the proposed models, four historical data sets have been used, which are summarized in Table-1. The estimated parameter values have been obtained using the curve fitting tool in MATLAB. The performance has been obtained in terms of mean square error (MSE), sum of squared error (SSE), root mean squared error (RMSE), R-square (R^2), and adjusted R-square ($Adj\ R^2$).

Table 1: *Reference data sets.*

Data sets	Time (t)	Total testing hours	Faults	Description	References
DS1	21 (weeks)	7476	26	Data set of telecommunication system test	[1]
DS2	22(days)	93 CPU hours	86	The pattern of discovery of error	[23]

DS3	19 (weeks)	10,272 CPU hours	120	Tandem computers	[24]
DS4	12 (weeks)	5053 CPU hours	61	Tandem computers	[24]

For DS1, the estimated parameters values are given in Table-2 with comparison criteria (MSE, SSE, RMSE, R^2, $Adj\ R^2$). From Table-2, it can be seen that the obtained values of the proposed model 1 for all five criteria are, 0.537, 8.056, 0.7328, 0.9950 and 0.9933. Similarly, for the model 2, values for comparison criteria are 0.573, 8.019, 0.7568, 0.9950 and 0.9929, respectively. From the results of DS1, both the proposed models give-better results compared to the existing literature and in between model 1 & 2, model 1 gives better performance. For more clarity, the graphical representation of total number of faults with time is shown in figure-1.

Table 2: *Parameter Estimated values and Comparison results for DS1.*

No.	Model	Estimated value	MSE	SSE	RMSE	R^2	$Adj\ R^2$
1	GO [7]	$\hat{a} = 3923854.73$, $\hat{b} = 3.2 \times 10^{-7}$	3.8672	73.477	1.9665	0.9582	0.9535
2	Y-DS [10]	$\hat{a} = 3{,}9.82198$, $\hat{b} = 0.1104$	1.4938	28.382	1.2222	0.9838	0.9820
3	O-IS [8]	$\hat{a} = 26.6845, \hat{b} = 0.2918$, $\hat{\beta} = 21.6856$	0.6745	12.141	0.8213	0.9931	0.9919
4	K-SRGM 3 [21]	$\hat{p} = 0.1385, \hat{b} = 0.1385$, $\hat{\alpha} = 0.1012, \hat{A} = 24.989$,	1.2295	20.902	1.1088	0.9881	0.9851
5	R-M-D NHPP [22]	$\hat{a} = 40.2018, \hat{\alpha} = 0.9319$, $\hat{\beta} = 0.1402, \hat{b} = 0.1152$	2.0059	34.101	1.4163	0.9806	0.9757
6	C-TC [16]	$\hat{b} = 2.234, \hat{\beta} = 15.2504$, $\hat{a} = 0.0043$, $\hat{N} = 26.833$, $\hat{\alpha} = 9959.1698$,	1.0939	17.502	1.0459	0.9900	0.9867
7	P-Vtub [15]	$\hat{\alpha} = 1.5176, \hat{\beta} = 11.3848$, $\hat{b} = 1.2978, \hat{N} = 25.7412$ $\hat{a} = 1.0985$,	0.7178	11.485	0.8472	0.9935	0.9913
8	S-3PFD [13]	$\hat{a} = 0.038, \hat{c} = 1488.598$, $\hat{\beta} = 0.002, \hat{b} = 0.292$, $\hat{N} = 26.889$	0.7590	12.144	0.8712	0.9931	0.9908
9	SONG -P [14]	$\hat{\alpha} = 0.2176, \hat{b} = 1.0047$, $\hat{\beta} = 155.501, \hat{N} = 47.797$ $\hat{a} = 108232.819$,	0.5864	9.3824	0.7658	0.9947	0.9929
10	Model 1	$\hat{a} = 371.1$, $\hat{\alpha} = 0.8046$, $\hat{\beta} = 1.344, \hat{\gamma} = 0.4027$, $\hat{b} = 0.2355, \hat{N} = 27.59$	**0.537**	**8.056**	**0.7328**	**0.9950**	**0.9933**
11	Model 2	$\hat{a} = 8.214$, $\hat{\alpha} = 0.6583$ $\hat{\beta} = 1.47, \hat{b} = 0.2948$, $\hat{N} = 26.73, \hat{\psi} = 243.6$, $\hat{\gamma} = 0.6273$	**0.573**	**8.019**	**0.7568**	**0.9950**	**0.9929**

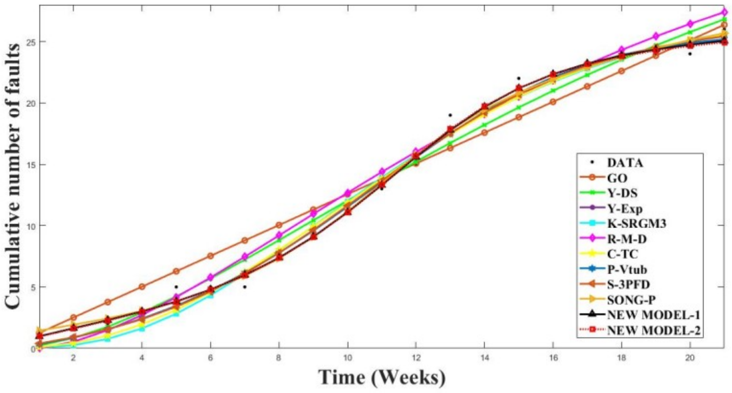

Figure 1: *Mean value function for various SRGMs for DS1.*

From Table-3, the values of the comparison criteria for DS2 are 3.3219, 53.15, 1.8226, 0.9969, 0.9959 and 3.9006, 58.51, 1.975, 0.9966, 0.9952 for model 1 & 2, respectively. These values are less than to the other existing models for the criteria MSE, SSE, RMSE and greater than for R^2, $Adj\ R^2$. Which shows the better performance for the proposed models for DS2. Again, the results of DS2 are shown in figure-2 for better understanding.

Table 3: *Parameter Estimated values and Comparison results for DS2.*

No.	Models	Estimated value	MSE	SSE	RMSE	R^2	$Adj\ R^2$
1	GO [7]	$\hat{a} = 153, \hat{b} = 0.0414$	25.190	503.818	5.019	0.9706	0.9691
2	Y-DS [10]	$\hat{a} = 94.25, \hat{b} = 0.1929$	7.645	152.883	2.765	0.9911	0.9906
3	O-IS [8]	$\hat{a} = 87.21, \hat{\beta} = 6.899,$ $\hat{b} = 0.2631$	5.909	112.317	2.431	0.9934	0.9928
4	K-SRGM 3 [21]	$\hat{A} = 10.97, \hat{p} = 1.152,$ $\hat{b} = 1.026, \hat{\alpha} = 0.890$	9.865	177.557	3.141	0.9896	0.9879
5	R-M-D NHPP [22]	$\hat{\alpha} = 1.6, \hat{b} = 0.1192$ $\hat{a} = 61.73, \hat{\beta} = 0.203,$	18.084	325.809	4.255	0.9810	0.9778
6	C-TC [16]	$\hat{a} = 0.449, \hat{\alpha} = 90.05,$ $\hat{\beta} = 1190, \hat{b} = 1.89,$ $\hat{N} = 88.34$	7.6209	129.556	2.7606	0.9924	0.9907
7	P-Vtub [15]	$\hat{a} = 15.78, \hat{b} = 1.291,$ $\hat{\beta} = 7.08, \hat{N} = 60.13,$ $\hat{\alpha} = 0.0367,$	292.20	4967.6	17.094	0.7099	0.6417
8	S-3PFD [13]	$\hat{a} = 1.117,\ \hat{c} = 136.9,$ $\hat{\beta} = 0.251, \hat{N} = 89.31$ $\hat{b} = 0.2656$	6.602	112.231	2.5694	0.9934	0.9919
9	SONG-P [14]	$\hat{a} = 58.4, \hat{\alpha} = 0.8222,$ $\hat{\beta} = 0.292, \hat{N} = 92.21$ $\hat{b} = 0.3154$	6.506	110.605	2.551	0.9935	0.9920
10	Model 1	$\hat{a} = 6.074, \hat{N} = 99.37,$ $\hat{\alpha} = 1.4E-07,$ $\hat{\beta} = 1.23, \hat{\gamma} = 0.7093,$ $\hat{b} = 0.07321$	**3.3219**	**53.15**	**1.8226**	**0.9969**	**0.9959**
11	Model 2	$\hat{a} = 4.022,\ \hat{\alpha} = 0.066,$ $\hat{\beta} = 0.919, \hat{b} = 0.066,$ $\hat{N} = 100.9 \hat{\gamma} = 0.8298$ $\hat{\psi} = 0.07275$	**3.9006**	**58.51**	**1.975**	**0.9966**	**0.9952**

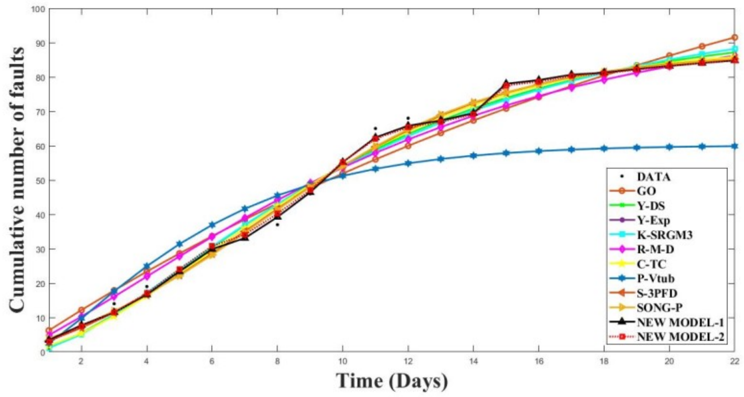

Figure 2: *Mean value function for various SRGMs for DS2.*

The results of proposed models 1, 2 for DS3 are shown in Table-4, which are 1.5055, 19.57, 1.227, 0.9992, 0.9989 and 1.5625, 18.74, 1.25, 0.9993, 0.9989, respectively. Comparison with other models in terms of MVF and time is given in figure-3. Similarly, from Table-5 the results of DS4 for both the models are 5.6644, 34.00, 2.380, 0.9932, 0.9875 and 8.8209, 44.09, 2.97, 0.9912, 0.9805, respectively. The comparative study of DS4 is shown in figure-4.

Table 4: *Parameter Estimated values and Comparison results for DS3.*

No.	Models	Estimated value	MSE	SSE	RMSE	R^2	$Adj\ R^2$
1	GO [7]	$\hat{a} = 183$, $\hat{b} = 0.0615$	26.002	442.0	5.0992	0.9824	0.9813
2	Y-DS [10]	$\hat{a} = 127.4, \hat{b} = 0.2417$	14.688	249.691	3.8325	0.9900	0.9895
3	O-IS [8]	$\hat{a} = 124.4, \hat{b} = 0.2535$ $\hat{\beta} = 3.779$	7.1268	114.0261	2.6696	0.9955	0.9895
4	K-SRGM 3 [21]	$\hat{A} = 1.858, \hat{p} = 3.791$ $\hat{b} = 2.413,$ $\hat{\alpha} = 0.9873$	18.550	278.2	4.307	0.9889	0.9867
5	R-M-D NHPP [22]	$\hat{a} = 98.13, \hat{\alpha} = 1.353,$ $\hat{b} = 0.1835,$ $\hat{\beta} = 0.215$	13.432	201.5	3.665	0.9920	0.9904
6	C-TC [16]	$\hat{\alpha} = 46.24, \hat{b} = 1.474,$ $\hat{\beta} = 26.63, \hat{N} = 125.7$ $\hat{a} = 0.0787,$	13.265	185.7	3.6422	0.9926	0.9905
7	P-Vtub [15]	$\hat{a} = 60.55, \hat{\beta} = 8.282,$ $\hat{N} = 118.7, \hat{b} = 1.157,$ $\hat{\alpha} = 0.0246$	65.594	918.3	8.099	0.9634	0.9529
8	S-3PFD [13]	$\hat{a} = 0.673, \hat{c} = 39.4,$ $\hat{\beta} = 0.326, \hat{N} = 130.6$ $\hat{b} = 0.2555,$	8.2599	115.7	2.874	0.9954	0.9941
9	SONG-P [14]	$\hat{N} = 176, \hat{a} = 3181,$ $\hat{\alpha} = 0.2208,$ $\hat{\beta} = 58.16, \hat{b} = 1.096,$	2.143	30.006	1.464	0.9988	0.9984
10	Model 1	$\hat{a} = 77.35, \hat{\beta} = 45.28,$ $\hat{\alpha} = 0.659, \hat{N} = 180.9$ $\hat{\gamma} = 0.2777,$ $\hat{b} = 0.1647,$	**1.5055**	**19.57**	**1.227**	**0.9992**	**0.9989**
11	Model 2	$\hat{a} = 35.52$, $\hat{\alpha} = 0.533,$ $\hat{\beta} = 16.66, \hat{N} = 141,$ $\hat{b} = 0.2278,$ $\hat{\gamma} = 0.3356,$	**1.5625**	**18.74**	**1.25**	**0.9993**	**0.9989**

		$\hat{\psi} = 72.77$					

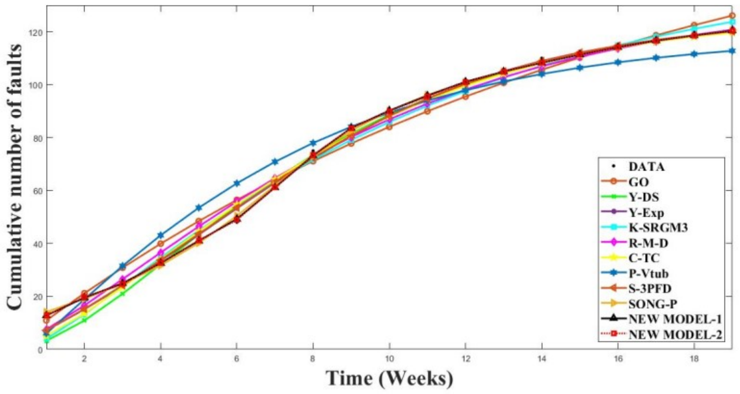

Figure 3: *Mean value function for various SRGMs for DS3.*

Table 5: *Parameter Estimated values and Comparison results for DS4.*

No.	Models	Estimated value	MSE	SSE	RMSE	R^2	$Adj\ R^2$
1	GO [7]	$\hat{a} = 244.3, \hat{b} = 0.0265$	20.894	209	4.571	0.9581	0.9539
2	Y-DS [10]	$\hat{b} = 0.2741, \hat{a} = 76.25,$	9.897	98.94	3.146	0.9802	0.9782
3	O-IS [8]	$\hat{a} = 64.4, \hat{\beta} = 11.36,$ $\hat{b} = 0.483$	6.345	57.09	2.519	0.9885	0.9860
4	K-SRGM 3 [21]	$\hat{\alpha} = 0.966, \hat{p} = 1.942,$ $\hat{A} = 3.057, \hat{b} = 2.097,$	15.296	122.4	3.911	0.9755	0.9663
5	R-M-D NHPP [22]	$\hat{a} = 59.05, \hat{b} = 0.1996,$ $\hat{\alpha} = 1.353, \hat{\beta} = 0.2526,$	20.512	145.1	4.259	0.9709	0.9600
6	C-TC [16]	$\hat{\beta} = 94.38, \hat{\alpha} = 1070,$ $\hat{b} = 1.921, \hat{a} = 0.0439$ $\hat{N} = 64.69$	10.74	75.16	3.2768	0.9849	0.9763
7	P-Vtub [15]	$\hat{\beta} = 852.4, \hat{\alpha} = 59.87,$ $\hat{b} = 0.782, \hat{a} = 1.923,$ $\hat{N} = 61.07$	6.6616	46.63	2.581	0.9906	0.9853
8	S-3PFD [13]	$\hat{a} = 29.31,\ \hat{\beta} = 2.857,$ $\hat{c} = 228.1, \hat{b} = 0.4653,$ $\hat{N} = 64.48$	9.4741	66.32	3.078	0.9867	0.9791
9	SONG-P [14]	$\hat{a} = 2262,\ \hat{\alpha} = 0.5059,$ $\hat{\beta} = 0.645, \hat{b} = 0.7314,$ $\hat{N} = 62.56$	6.287	44.01	2.507	0.9912	0.9861
10	Model 1	$\hat{a} = 48.34,\ \hat{\alpha} = 0.5042,$ $\hat{\beta} = 0.7441,\ \hat{\gamma} = 0.4634$ $\hat{b} = 0.0629, N\hat{} = 66.57$	**5.6644**	**34.00**	**2.380**	**0.9932**	**0.9875**
11	Model 2	$\hat{a} = 92.17,\ \hat{\alpha} = 0.735$ $\hat{\beta} = 0.125, \hat{b} = 0.0940$ $\hat{N} = 66.83, \hat{\gamma} = 0.8239,$ $\hat{\psi} = 0.1244$	8.8209	44.09	2.97	0.9912	0.9805

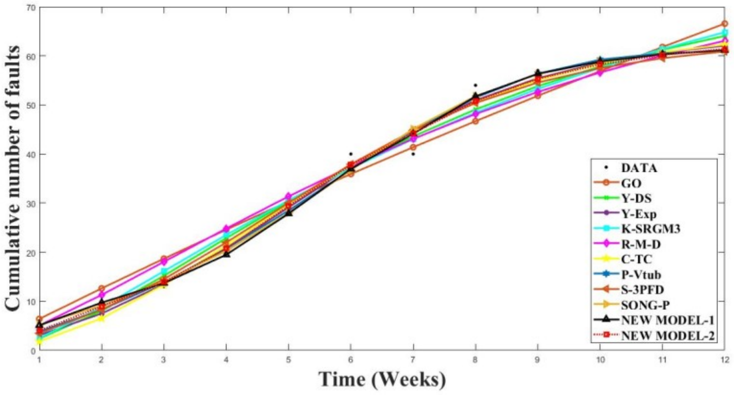

Figure 4: *Mean value function for various SRGMs for DS4.*

The relative error with time is also calculated for all the data sets, which confirms the ability to provide better accuracy. The comparison of the proposed models with other models in terms of relative error are shown in figures 5-8. Calculated values of MSE, SSE, and RMSE for proposed models are less than the existing models for all four data sets. The values of R-square and Adjusted R-square for proposed models are greater than the others. Based on these results, it can be said that these proposed models give improved results and have considerably better goodness of fit. Also, model-1 gives better results compared to model-2.

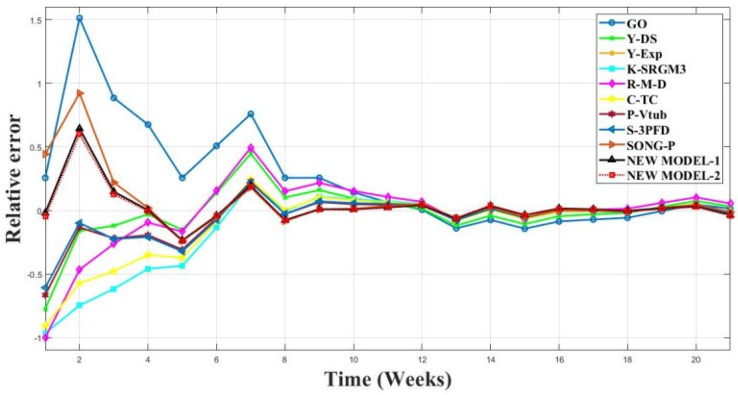

Figure 5: *Relative error curve for various SRGMs for DS1.*

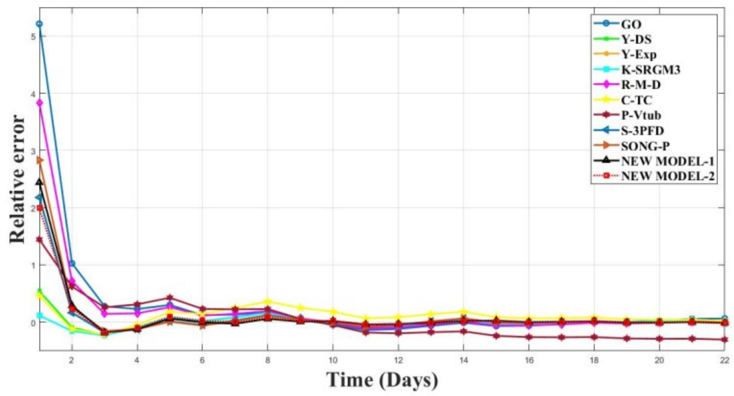

Figure 6: *Relative error curve for various SRGMs for DS2.*

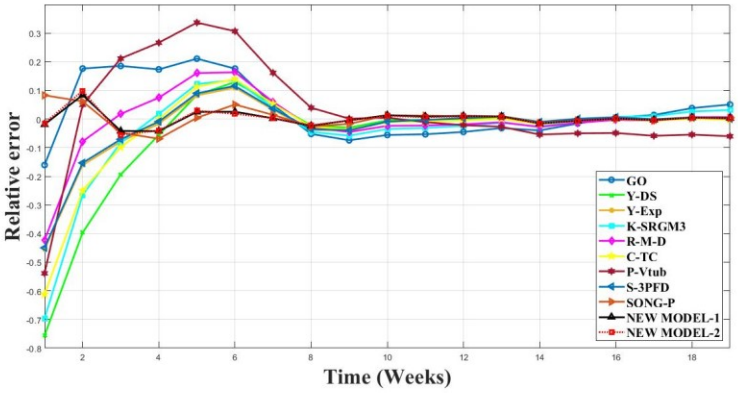

Figure 7: *Relative error curve for various SRGMs for DS3.*

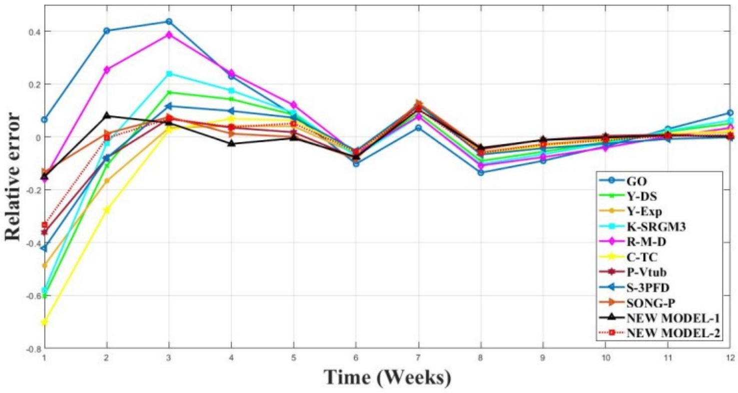

Figure 8: *Relative error curve for various SRGMs for DS4.*

IV. Predictive Analysis

In this paper, DS1 is used for the comparison of predicted values and how these differ for each model. 75% of the data set is used to estimate the parameters and the remaining 25% of the data set is used to predict the model performance. The sum of squared error of predicted value is denoted by PreSSE and it is calculated for the last 25% of the data.

The result of predictive analysis based on the six criteria (MSE, SSE, RMSE, R^2, $Adj\ R^2$ and PreSSE) is given in Table-6. From the Table-6, the values of the six criteria are 0.64, 6.417, 0.8010, 0.9924, 0.9873, 1.8684 and 0.72, 6.488, 0.8491, 0.9923, 0.9856, 1.8811 for both the models, respectively. Graphical representation of estimated MVF and predicted MVF is shown in figure-9. Based on the obtained results as shown in Table-6 and figure-9, it can be said that the proposed models give better prediction compared to the other existing models, because the SSE and PreSSE are minimum for both the proposed models.

Table 6: *Parameter Estimated values, criteria and predictions from DS1*

No	Model	Estimated value	MSE	SSE	RMSE	R^2	$Adj\ R^2$	PreSSE
1	GO [7]	$\hat{a} = 2439.196,$ $\hat{b} = 0.00052$	4.85	8.012	2.2041	0.9199	0.9076	5.756
2	Y-DS [10]	$\hat{a} = 81.516, \hat{b} = 0.0657$	0.85	11.84	0.9195	0.9861	0.9839	128.71
3	O-IS [8]	$\hat{a} = 32.236\hat{b} = 0.2378,$ $\hat{\beta} = 16.9353$	0.67	8.690	0.8176	0.9898	0.9872	36.131
4	K-SRGM 3 [21]	$\hat{A} = 43.931, \hat{p} = 0.0265$ $\hat{b} = 1.3062, \hat{\alpha} = 2.2423$	1.02	12.24	1.0099	0.9856	0.9804	264.331

5	R-M-D NHPP [22]	$\hat{a} = 153.5748, \hat{\alpha} = 1.056, \hat{\beta} = 0.046, \hat{b} = 0.0327$	0.91	10.91	0.9535	0.9872	0.9825	188.6321
6	C-TC [16]	$\hat{a} = 0.0288, \hat{\alpha} = 193.585, \hat{\beta} = 167.613, \hat{b} = 1.6825, \hat{N} = 86.0872$	1.02	11.24	1.0107	0.9868	0.9802	160.1096
7	P-Vtub [15]	$\hat{a} = 1.2816, \hat{\alpha} = 1.1473, \hat{\beta} = 0.1982, \hat{b} = 0.9844, \hat{N} = 31.4297$	0.78	8.613	0.8849	0.9899	0.9848	32.5458
8	S-3PFD [13]	$\hat{a} = 0.1496, \hat{c} = 62.4407, \hat{\beta} = 0.1982, \hat{b} = 0.2372, \hat{N} = 36.9478$	0.82	8.989	0.9040	0.9894	0.9841	45.9267
9	SONG -P [14]	$\hat{a} = 10453.17$ $\hat{\alpha} = 0.4174, \hat{\beta} = 0.1175, \hat{b} = 0.53348, \hat{N} = 24.292$	0.59	6.506	0.7691	0.9923	0.9885	2.6780
10	Model 1	$\hat{a} = 26.45, \hat{\alpha} = 0.4766, \hat{\beta} = 0.4243, \hat{b} = 0.03818, \hat{N} = 26.45, \hat{\gamma} = 0.481,$	0.64	**6.417**	0.8010	**0.9924**	0.9873	**1.8684**
11	Model 2	$\hat{a} = 11.34, \hat{\alpha} = 0.7453, \hat{\beta} = 0.403, \hat{b} = 0.2607, \hat{N} = 26.18, \hat{\psi} = 86.91, \hat{\gamma} = 0.7413,$	0.72	**6.488**	0.8491	**0.9923**	0.9856	**1.8811**

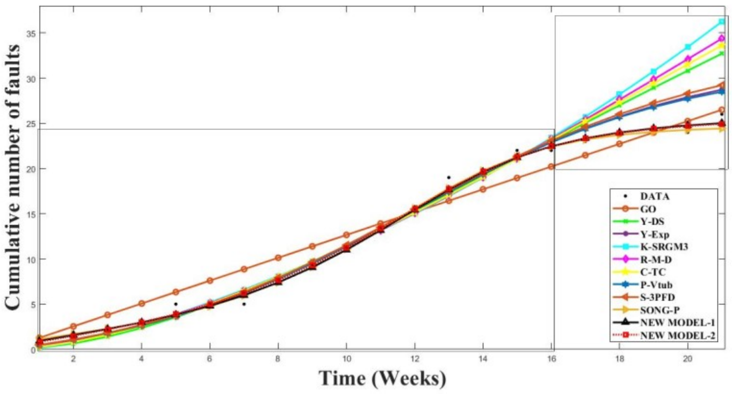

Figure 9: *Prediction of MVF for various models for DS1.*

V. Conclusion

Since-software development and testing process takes place in a controlled environment which is far away from operating environment. Therefore, uncertainty in the operating environment has been considered. To reflect the joint effect of testing time and used resources Cobb-Douglas production function has been used. Fault (if any) correction time is also affected the software costs as well as launching time. Delay in debugging has been included in two different ways, delayed S-shaped and Inflection S-shaped. In this article, two SRGMs have been proposed and these have been validated on the four real data sets. For validation, a curve fitting tool in MATLAB has been used for estimation and five statistical comparison criteria (MSE, SSE, RMSE R^2, and $Adj\ R^2$) have used to compare the results with existing literature. After validation, predictive analysis has been done to check the predictability of the proposed models using SSE and PreSSE. All the results with relative error have been figure out for better understating. On the basis of the results, it can be concluded that proposed models give significantly better performance compared to the others. In between these two models, it is found that results are improved when delayed S-shaped function has been used as delay function. In future, proposed work may include improvement of the proposed models with soft-computing techniques or introduction of additional reliability factors, i.e., change point, testing coverage, etc.

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Ranking of Software Reliability Growth Models: A Entropy-ELECTRE Hybrid Approach

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Abstract

Software reliability is estimated using software reliability growth models. In the last few decades, numerous software reliability growth models (SRGMs) have been established. Some models are developed with the consideration of perfect debugging, imperfect debugging, testing coverage, testing effort, and fault reduction factor. Generally, SRGMs are dependent on dataset and thus it is a challenging task to select SRGM appropriately based on the need of software user. As the inappropriate selection of SRGM can lead to inaccurate results and consequently delay in software release. To address this issue, we have combined the two multi-criteria decision-making (MCDM) approaches namely Entropy and Elimination et Choice Translating Reality (ELECTRE) methods. The proposed approach identifies the criterion importance as to select suitable SRGM, comparison of criteria is important. The outcomes are based on the aggregate value of dominant matrix that is being used for SRGMs ranking. The working of proposed Entropy-ELECTRE method is demonstrated on a real time data set on which ten SRGMs are compared against six evaluation criteria. The findings play an important role in determining the SRGM's appropriateness for decision-maker.

Keywords: SRGM, MCDM, Entropy, ELECTRE, software reliability

I. Introduction

Computer systems have become an integral part of our life. Software systems are now involved in almost every aspect of human life, due to this their importance and demands are increasing. To build such computer systems, we will need increasingly complex and large-scale software systems. The software applications have marked their presence from critical applications of mission like military, defense to safety application including medical process [1]. A software failure of such systems may result in a financial loss, loss of human life, or the collapse of a critical operation. Also, it's important to determine if a software system will meet consumer expectations without failing before releasing it. Furthermore, with the expansion of the existing market, the complexity and size of software has continued to grow. Since, humans have created software systems, a large number of software flaws must be incorporated into the software product during the process of development. Thus, application of relevant technologies is important in the development of a highly reliable software system. Hence, software reliability is considered as an essential quality factor. “Software reliability refers to the software's failure-free operations in a given time period and specific environment”. It measures the failure free services rendered by software to its authorized consumers. In the software industries, prediction and estimation of software reliability enables to meet complexities of software development. It is becoming more challenging for software managers to efficiently develop highly reliable software systems. In order to obtain fault-free software, there is a requirement of process to track fault content and reliability. Hence, a mathematical relationship termed as a software reliability growth model (SRGM) describing the process of finding and removing errors to increase software reliability is introduced.

A software system is tested during the software testing phase to find and rectify the remaining software defects (or errors) and hence software reliability improves. Such a failure detection process is represented by a software reliability growth model (SRGM). SRGMs are considered as a mathematical model that statistically examine software reliability and hence provide a measure of the software product's quality. SRGM establishes mathematical relations between time of testing and the failure occurrence rate at the testing time to quantify software reliability. These mathematical relations are made using statistical formulas, stochastic processes and probability. During the testing phase, SRGM make use of failure data to forecast the reliability of software over the span of its operational life. SRGM consider failure data as input to generate the prediction for reliability as an output in the form of mathematical functions. Thus, SRGM is referred as a parametric model with the parameters based on availability of software failure data. For a given data set, different SRGMs must be selected for comparison and then identification of best suited model must be done. Also comparison criteria for selecting best SRGM should be chosen wisely. The outcome for selecting best SRGM may be numerous models or there may be none at all. If there is no outcome, a model should be created based on the technique and surroundings. Further, the models have to undergo process again if the outcomes of model selection results into two or more. Sometimes, it is possible that the comparison outcome for two models may be similar to each other. Thus, the outcome of such comparison may not be accurate. Therefore, selection of best SRGM is considered as a multi- criteria decision making (MCDM) problem.

In operation research and management science, multi-criteria decision-making (MCDM) is the most prominent areas of multidisciplinary research. According to Belton and Stewart [2], MCDM provides a method for making justifiable, understandable, and rational decisions. As selecting best software reliability growth model among the given SRGMs is considered as MCDM problem so there is necessity to develop a system to resolve the MCDM problem effectively. Since SRGMs are data-dependent, therefore there is always a need to rank the SRGM. Traditional approaches such AHP, VIKOR, COPRAS, PROMETHEE, DEMATEL, MOORA, and many more have certain complexities and limitations. We tried to resolve the situation where software engineers have to choose a model for the software testing among the availability of various models. The objective of this study is to find responses to the questionnaire that have been collected based on the data gathered from research. Table 1 depicts the research questions based on software reliability growth models.

Table 1: *Research Questions*

S.No	Research Questions
1	Identification of SRGM for software engineers.
2	Selection of alternative SRGM for comparison.
3	Determination of ranking criteria.
4	Calculation of weights using Entropy method.
5	Ranking of SRGMs based on ELECTRE method.

In this paper, we have integrated two MCDM approaches namely Entropy and ELECTRE to find the best software reliability growth model for a dataset. We have considered ten well-known Non-homogeneous Poisson process model (NHPP) based software reliability growth models. These ten models were examined based on six evaluation criteria. We have used Entropy approach to find the weights of evaluation criteria and have used these weights in ELECTRE approach for the ranking of SRGMs. The primary contributions of the study are as follows:

- Proposed Entropy-ELECTRE as a combined approach.
- Application of Entropy approach to calculate weights of evaluation criteria.
- ELECTRE approach is used to rank alternative SRGMs.
- A real time data set is used to illustrate the proposed approach.

Further, remaining sections of the paper are organized as follows: Section II describes the related research on software reliability growth models (SRGMs) and multi criteria decision making (MCDM). Section III explains the methodology of combined Entropy-ELECTRE approach. Section IV specify the selection of ranking criteria and software reliability growth models for comparison. Section V illustrates proposed approach numerically on a real time data set. Lastly, section VI concludes the paper with future work.

II. Literature Review

Numerous software reliability growth models (SRGMs) has been developed for estimating the reliability of a software system. Chang et al. [3] presented a novel testing coverage software reliability model that takes into account operating environment uncertainty. Zhang et al. [4] developed a Fault removal model that incorporates efficiency of fault removal into software reliability model. Goel and Okumoto [5] developed a model that described failure detection as an NHPP and assumed that the hazard rate is proportional to the number of defects remaining in the software. Kapur et al. [6] proposed software reliability models in the presence of error generation and imperfect debugging. Dhavakumar and Gopalan [7] proposed chaotic grey wolf optimization algorithm (CGWO) as a new technique to quantify attributes of software reliability growth models. CGWO is a heuristic system that depicts execution by achieving complicated parameter optimization and solving application design challenges. Gao [8] proposed simulation approach to model Fault detection process (FDP), Fault correction process (FCP), and Fault introduction process (FIP) together and considered debuggers featuring contribution differently to FDP, FCP and FIP. Li et al. [9] proposed testability growth models on the basis of NHPP that takes into account the testability growth effort while simultaneously rectifying delay and imperfect correction.

Kaur et al. [10] proposed a mathematical model for firms that provide patching services with the assumption that sometimes corrective steps may implement infected patches. Erto et al. [11] proposed new generalized inflection S-shaped software reliability growth model. It is unique, very flexible finite failure Poisson process that covers the commonly used Goel-Okumoto model, inflection S-shaped model, and Goel generalized Nonhomogeneous Poisson process as special cases. Li and Pham [12] proposed a generalized model based on a non-homogeneous Poisson process (NHPP) that covers the uncertainty of the operating environment and imperfect debugging and its impact on fault detection rate in the evaluation of software. Zeephongsekul et al. [13] introduced a variation of EM algorithm, the expectation conditional maximization (ECM) algorithm and provided a viable option to estimate the parameters of nonhomogeneous Poisson (NHPP) software reliability growth models (SRGM). Kumar et al. [14] proposed a model to allocate resources in an effective way to reduce costs with fault correction process (FCP) and fault detection process (FDP) in a dynamic environment. Vizarreta et al. [15] focuses on several applications of SRGM framework that are critical for the adequate management based on SDN networks. Raghuvanshi et al. [16] proposed the time-variant fault detection Software Reliability Model that analyses numerous well-known algorithms on several performance measures and contains different software properties for model development.

Lee et al. [17] proposed a novel SRGM considering software failures that are interdependent. Authors used numerous evaluation criteria to assess the proposed model's goodness-of-fit with the previous results of nonhomogeneous Poisson process SRGMs on real-world datasets. Zhu and Pham [18] designed a martingale-based generalized multiple-environmental-factors software reliability growth model with related unpredictability. On the basis of randomness authors included a stochastic software fault detection procedure in the model. Kumar and Ram [19] highlighted the current advances and applications of artificial intelligence, data mining, and many other approaches in the predictive modelling and analytics in software reliability engineering. Kumar and Sahni [20] described the estimation of testing efforts in a dynamic environment with the assumption that debugging costs associated with each release follow a learning curve.

Garg and Ram [21] explained how to deal with uncertainty in reliability optimization using maintenance scheduling, soft computing, uncertainty, and fuzzy optimization scheduling strategies. Cai et al. [22] proposed a mechanism to minimize the disparities between adjacent trace files and incorporated certain unique mutation/crossover strategies into the genetic algorithm (GA). Kumar and Sahni [23] used FCP and FDP in a dynamic environment to assign testing resources in a way to reduce costs throughout the testing process. Kumar et al. [24] developed a reliability growth model based on software patching to enable software systems more cost-effective and reliable, minimizing software release time and testing cost. Kumar et al. [25] developed a model considering two stage process of fault detection and removal incorporating the effects of resources and testing time.

Over the last few decades, Multi-Criteria Decision Analysis has been used widely. Its importance in several application fields has grown dramatically, particularly new approaches have developed and existing ones have improved. Amirghodsi et al. [26] introduced Decision Making Trial and Evaluation Laboratory (DEMATEL) and ELECTRE decision-making approach on grey numbers from both quantitative and qualitative methods to address the technology provider selection problem systematically. Anser et al. [27] used analytical hierarchy process (AHP) and Fuzzy-VIKOR methods used to resolve the problem of selection of the optimal site for the installation of solar projects. Ho et al. [28] provides review work for supplier evaluation and selection on multi criteria decision making.

Sevkli [29] proposed a new approach known as fuzzy technique for ELimination Et Choix Traduisant la REalite´ (ELECTRE) for supplier selection problem by considering it multi criteria decision problem. Aruldoss et al. [30] addressed the problem in fuzzy multi criteria decision making techniques. Lin et al. [31] proposed an approach for evaluation of eutrophication based on Monte Carlo simulation (MCS) and technique for order preference by similarity to an ideal solution (TOPSIS). Rani et al. [32] proposed a new divergence measure based on fuzzy TOPSIS for evaluating and selecting renewable energy sources in multi-criteria decision-making challenges. Torlak et al. [33] used VIseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) - structural equation modeling (SEM) to develop service provider benchmarks and to analyze a multi methodology approach in the internet sector.

Mohammed [34] applied the techniques and concept of multi-criteria decision-making in a fuzzy environment to project prioritizing and selection in portfolio management. Yazdani et al. [35] introduced a combined compromise decision-making algorithm with the use of several aggregation strategies. Deveci et al. [36] proposed a technique to prioritize the benefits of different methods of real-time traffic management using fuzzy multi-criteria decision making (MCDM). Kumar et al. [37] presented a novel hybrid entropy weight based multi-criteria decision-making (MCDM) method and Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) approach to select suitable SRGM and used it for identify and rank SRGMs in the most efficient manner. Ikram at al. [38] aimed to find a way on the development of an integrated management system (IMS) using AHP-Fuzzy VIKOR approach. Arabsheybani et al. [39] applied a fuzzy multi-objective optimization model based on the ratio analysis (MOORA) to analyze the overall performance of supplier's. Kumari and Mishra [40] extended traditional complex proportional assessment (COPRAS) approach to resolve the multi criteria decision making (MCDM) problem of green supplier selection with intuitionistic fuzzy sets (IFSs).

III. The Proposed Approach

The proposed methodology is developed using the MCDM techniques. This methodology is based on the combination of Entropy method and ELECTRE method by considering ten models and six evaluation criteria. The techniques of Entropy and ELECTRE are appropriate to determine weight of evaluation criteria, rank the SRGMs, and to choose the best SRGM in the decision matrix. The weights of each evaluation criterion is obtained by using Entropy method and alternative SRGMs are ranked by using ELECTRE method to find the best SRGM. A hierarchical model is used to explain the process to determine the best SRGM. The objective is to choose the best SRGM among a given set of SRGMs. The model is divided into stages to make the process easier. The initial stage of the model includes the identification of criteria for evaluation. The second stage includes identification of alternative SRGMs followed by the calculation of SRGMs parameter using SPSS 20. In the next stage weights of each evaluation criterion are calculated by using the methodology of Entropy followed by the application of ELECTRE method to find best SRGM and hence SRGMs are ranked. The process of hierarchical model is explained by the flowchart shown in Figure 1. The methodology of Entropy and ELECTRE are discussed below.

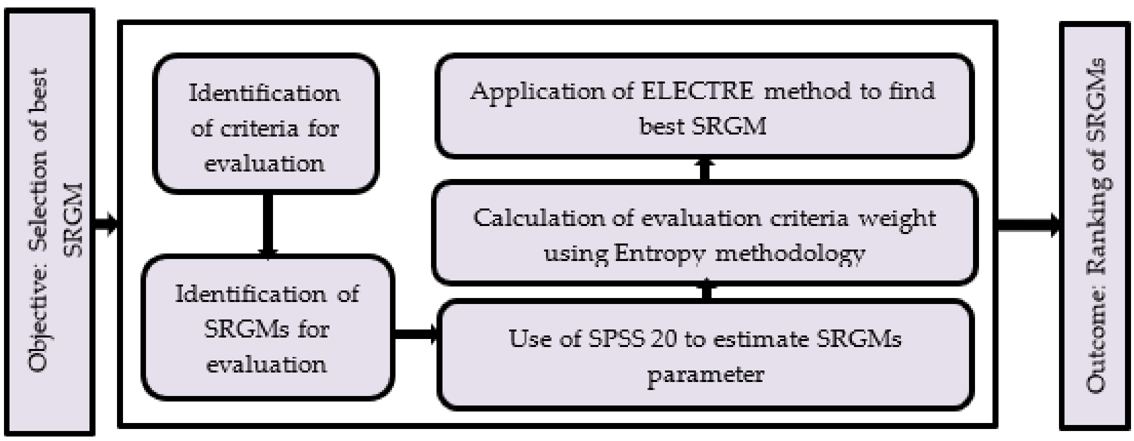

Figure 1: *Hierarchical model for best SRGM*

Shannon proposed the concept of entropy in 1948. "The Entropy method is a generic form of Monte Carlo simulation which is applied in complicated estimation and optimization problems for minimizing the error". It's a decision-making tool for determining the selection criteria weights in MCDM application. The Entropy approach should be employed for quantitative data volume measurement as well as for computing proportionate weight information [37]. Moreover, in information theory, entropy may be used to compute the predicted value of information in a given message. In this study, entropy is used to find the weight of each criterion and the steps involved in the process of Entropy method are explained in Figure 2.

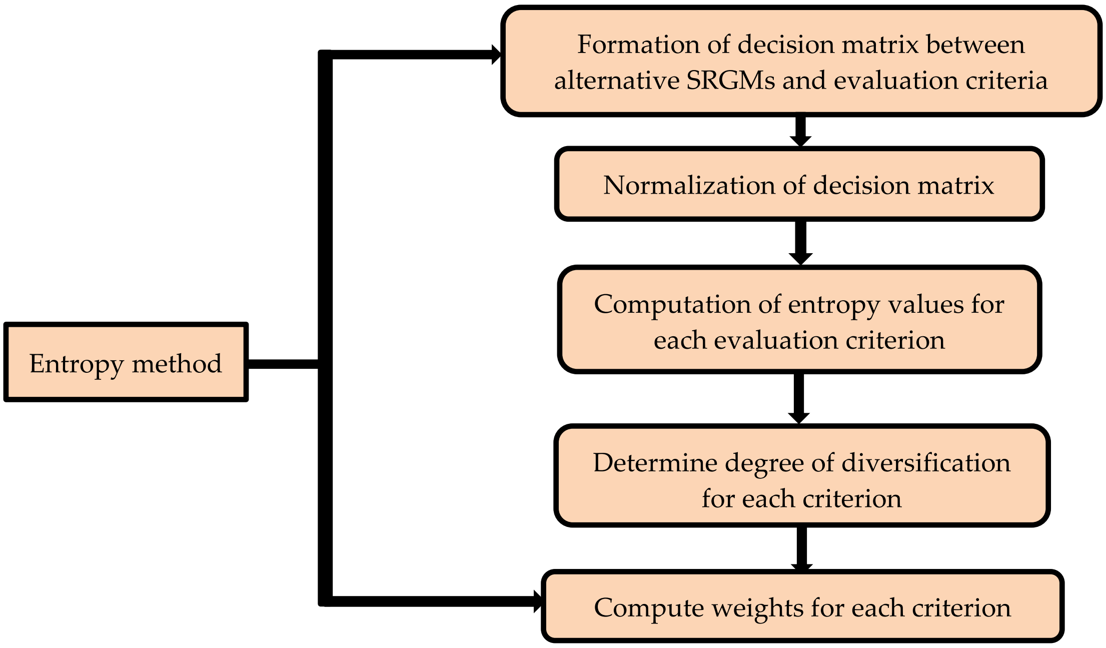

Figure 2: *Flowchart of Entropy method*

Let us assume SRGMs consisting of 1,2,3,...,p SRGMs and 1,2,3,...,q selection criterion for each alternative SRGM, where z_{pq} denotes the value of estimated parameter for p^{th} SRGM based on the q^{th} evaluation criterion and matrix A describes the collection of p^{th} SRGM and q^{th} evaluation criterion.

$$A = \begin{array}{c} \\ T_1 \\ T_2 \\ \cdots \\ \cdots \\ T_p \end{array} \begin{array}{c} \begin{array}{ccccc} D_1 & D_2 & \cdots & \cdots & D_q \end{array} \\ \begin{bmatrix} z_{11} & z_{12} & \cdots & \cdots & z_{1q} \\ z_{21} & z_{22} & \cdots & \cdots & z_{2q} \\ \cdots & \cdots & \cdots & \cdot\cdot & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ z_{p1} & z_{p2} & \cdots & \cdots & z_{pq} \end{bmatrix} \end{array}$$

where " $T_1, T_2, \ldots, T_p$ " signifies the alternative SRGM and " $D_1, D_3, \ldots, D_q$ " signifies the evaluation criteria.

The normalized decision matrix (S_{mn}) of evaluation criteria for each alternative SRGM using entropy method is represented by Equation (1) given below

$$S_{mn} = \frac{z_{mn}}{\sum_{m=1}^{p} z_{mn}}, m = 1,2,\ldots,p; n = 1,2,\ldots,q \quad (1)$$

The normalized decision matrix (S_{mn}) of evaluation criteria for each alternative SRGM given in Equation (1) is used to determine the value of entropy (e_n). The value of entropy of each evaluation criterion is calculated by using Equation (2) given below

$$e_n = \frac{-\sum_{m=1}^{p} S_{mn} \ln S_{mn}}{\ln p}; m = 1,2,\ldots,p; n = 1,2,\ldots,q \tag{2}$$

The value of entropy (e_n) described in Equation (2) is used to find the value of degree of diversification (d_n). The degree of diversification (d_n) of each evaluation criterion is obtained by using Equation (3) given below

$$d_n = 1 - e_n; n = 1,2,\ldots,q \tag{3}$$

The weights (w_n) of each evaluation criterion is measured by using value of degree of diversification (d_n) represented by Equation (3). The weights (w_n) of each evaluation criterion is calculated by using Equation (4) given below

$$w_n = \frac{d_n}{\sum_{n=1}^{q} d_n}; n = 1,2,\ldots,q \tag{4}$$

ELECTRE (Elimination et Choice Translating Reality) is a technique used to find solutions of problem for the situations involving in Multi Criteria Decision Making. The methodology of ELECTRE is based on the study of ranking relations. It analyses the relations of ranking among alternatives using indexes of concordance and discordance. The best alternative that a decision-maker chooses over the other alternative is measured using concordance and discordance indexes and eliminate the alternatives that are not suitable so the best solution or alternative can be obtained. The steps involved in the process of ELECTRE method is explained in the Figure 3 using flowchart and calculation for ELECTRE method can be obtained by equations given below involved in the process of ELECTRE method [41].

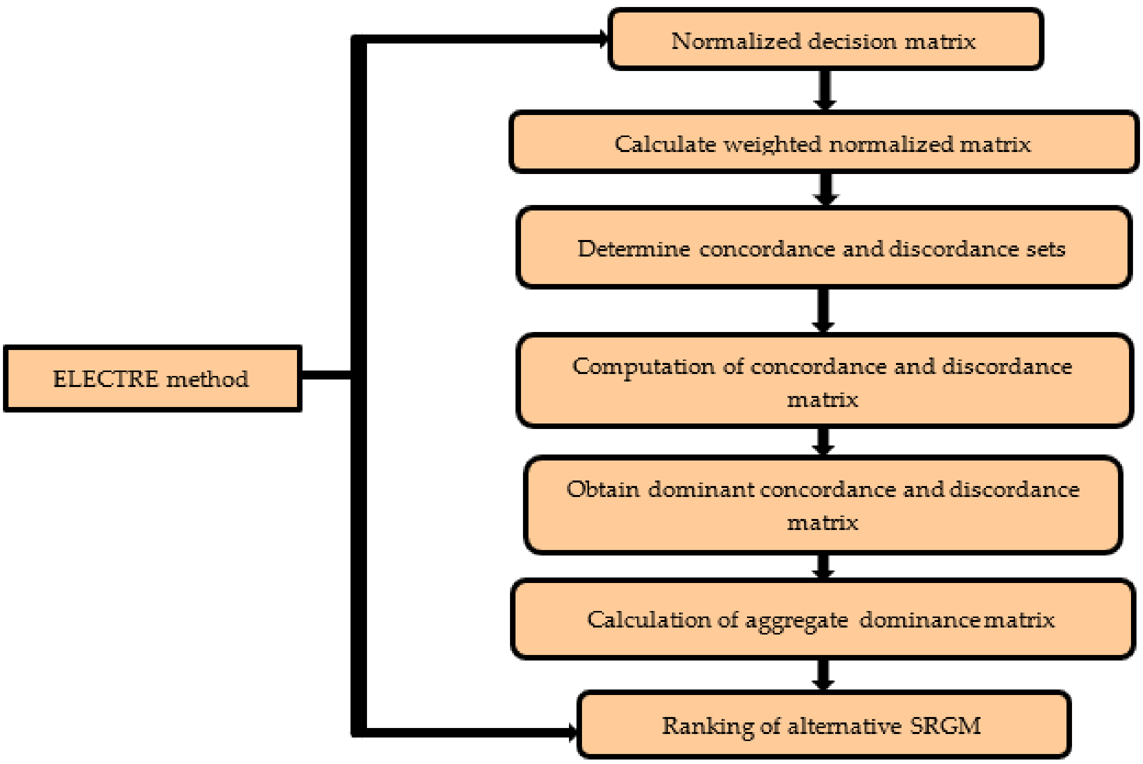

Figure 3: *Flowchart of ELECTRE method*

Equation (5) gives the normalized value (r_{mn}) of decision matrix based on ELECTRE method. Results of the normalized decision matrix of evaluation criteria for each alternative SRGM using Equation (5) will be a matrix R, which is described by equation (6).

$$r_{mn} = \frac{z_{mn}}{\sqrt{\sum_{m=1}^{p} z_{mn}^2}}; m = 1,2,\ldots,p; n = 1,2,\ldots,q \tag{5}$$

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1q} \\ r_{21} & r_{22} & \cdots & r_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ r_{p1} & r_{p2} & \cdots & r_{pq} \end{bmatrix} \tag{6}$$

The normalized decision matrix R in Equation (6) is used to find the weighted normalized matrix X of evaluation criteria for each alternative SRGM. The weights (w_n) of each criterion are predetermined by using Equation (4) and is used in Equation (7) to find the weighted normalized matrix X as shown below.

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1q} \\ x_{21} & x_{22} & \cdots & x_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ x_{p1} & x_{p2} & \cdots & x_{pq} \end{bmatrix} = \begin{bmatrix} w_1 r_{11} & w_2 r_{12} & \cdots & w_q r_{1q} \\ w_2 r_{21} & w_2 r_{22} & \cdots & w_q r_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ w_1 r_{p1} & w_2 r_{p2} & \cdots & w_q r_{pq} \end{bmatrix} \tag{7}$$

Concordance and discordance sets will be calculated by comparing the data of matrix X with every pair and their results are obtained as shown below. Concordance sets can be obtained by using Equation (8) and discordance sets can be obtained by using Equation (9).

$$\{c_{ab}\} = \{n \mid x_{an} \geq x_{bn}\} forn = 1,2,..q; a,b = 1,2,\ldots,panda \neq b \tag{8}$$

$$\{d_{ab}\} = \{n \mid x_{an} < x_{bn}\} forn = 1,2,\ldots,q; a,b = 1,2,\ldots,panda \neq b \tag{9}$$

Matrix $C = [C_{ab}]_{p \times p}$ for concordance consisting of alternative SRGM corresponding to each SRGM is calculated by adding weight values of the elements of concordance sets by using Equation (8) as shown in Equation (10). Discordance matrix $D = [D_{ab}]_{p \times p}$ consisting of alternative SRGM corresponding to each SRGM is calculated by using Equation (9) as shown below in Equation (11).

$$C_{ab} = \sum_{n \in c_{ab}} w_n; n = 1,2,\ldots,q \tag{10}$$

$$D_{ab} = \frac{\max[|x_{an} - x_{bn}|] n \in d_{ab}}{\max[|x_{an} - x_{bn}|] \forall n} \tag{11}$$

To calculate the dominant concordance matrix G consisting of alternative SRGM corresponding to each SRGM, there is need to find threshold value $\overline{C}$ by using Equation (12). Now, threshold value $\overline{C}$ is used to find dominant concordance matrix $G = [g_{ab}]_{p \times p}$ as shown below in Equation (13).

$$\overline{C} = \frac{\sum_{a=1}^{p}\sum_{b=1}^{p} c_{ab}}{p(p-1)} \tag{12}$$

$$g_{ab} = \begin{Bmatrix} 1, c_{ab} \geq \overline{C} \\ 0, c_{ab} < \overline{C} \end{Bmatrix} \tag{13}$$

To calculate the dominant discordance matrix H consisting of alternative SRGM corresponding to each SRGM, there is need to find threshold value $\overline{D}$ by using Equation (14). Now, threshold value $\overline{D}$ is used to find dominant discordance matrix $H = [h_{ab}]_{p\times p}$ as shown below in Equation (15).

$$\overline{D} = \frac{\sum_{a=1}^{p}\sum_{b=1}^{p} d_{ab}}{p(p-1)} \tag{14}$$

$$h_{ab} = \begin{Bmatrix} 1, h_{ab} \geq \overline{D} \\ 0, h_{ab} < \overline{D} \end{Bmatrix} \tag{15}$$

The value of aggregate dominance matrix $F = \left[f_{ab}\right]_{p\times p}$ consisting of alternative SRGM corresponding to each SRGM is obtained by the multiplication of matrices G and H as shown below in Equation (16).

$$f_{ab} = g_{ab} \times h_{ab} \tag{16}$$

The ranking of SRGMs is based on ascending or descending order of the sum of rows of matrix F.

IV. Selection of ranking criteria and SRGM

There are a variety of SRGMs available at present. Thus, it is required to analyze and verify the reliability of SRGM. This section describes the evaluation of ranking criteria and selection of alternative SRGM. The section I explains the evaluation of ranking criteria and section II explains the selection of alternative SRGM.

I. Evaluation of ranking criteria for SRGM

There is no model that satisfies all conditions among the SRGM. In contrast, different models anticipate very different outcomes. Sometimes a model gives good result for a given data set but the same model does not work well for other data set. Therefore, evaluation of model should be done based on specific data set and hence there is need to select criteria for evaluation. We have used the following criteria for evaluation.

- Mean Square Error (MSE) is defined as the distance between estimated and actual data and can be calculated by using the equation given below [42]

$$MSE = \frac{1}{n-N}\sum_{i=1}^{n}(y_i - \hat{m}(t_i))^2$$

n → number of observations
y_i →total number of faults detected upto to time t_i in terms of the testing data.
$m(t_i)$→estimated value of cumulative fault number up to t_i based on the mean value function, i=1,2,…,n.
N →number of parameters in the model

- R^2 is the second criteria for evaluation of SRGM and is defined as correlation index of the regression curve equation [43]. It is calculated by the formula given below

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{m}(t_i))^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}$$
$$where \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

- Adjusted R^2 is used as a third criteria for evaluation of SRGM and is obtained by the equation shown below [43]

$$Adjusted R^2 = 1 - \frac{(1-R)(n-1)}{n-P-1}$$

where R→value of R^2
P→number of predictors in the fitted model

- The predictive-ratio risk (PRR) calculates the distance between an estimated data and actual data and is measured by the following equation [44]

$$PRR = \sum_{i=1}^{n}\left(\frac{\hat{m}(t_i) - y_i}{\hat{m}(t_i)}\right)^2$$

- The fifth criterion used for evaluation of SRGM is Predictive power (PP) [44]. It measures the distance between the model's estimation and the actual data. It can be measured by the equation given below.

$$PP = \sum_{i=1}^{n}\left(\frac{\hat{m}(t_i) - y_i}{y_i}\right)^2$$

- The Akaike information criterion (AIC) [45] defines a model's capacity to maximise the likelihood function that is directly proportional to the degrees of freedom and is calculated as

$$AIC = -2LogL + 2N$$

II. Selection of SRGM

In order to make experimental analysis of proposed technique, ten NHPP SRGM are selected to determine the accuracy of proposed Entropy-ELECTRE technique. The selected ten SRGM with their mean value function are described in Table 2.

V. Numerical Example

The objective of this example is to check the performance of the proposed integrated Entropy-

ELECTRE method so that the systematic ranking of alternative SRGM could be made with the association of relevant ranking criteria for evaluation. A software failure data set is used to check the numerical applicability of proposed method. In this study, we used a real time data set to calculate the parameters of model for ten SRGMs. The data set is taken from [46]. Table 3 represents the failure data set wherein 14 weeks 38 faults were observed. The parameter values are estimated by using SPSS 20.0 for ten NHPP based SRGM. The estimated parameter values of SRGMs for data set is presented in Table 4 and estimated values of ranking criteria for each alternative SRGM is shown in Table 5.

Table 2: *Selected SRGMs and their Mean value functions*

No	Model Name	Model Type	Mean Value Function
1	**Li-Pham model [1]**	S-Shaped	$m(t)=a(1+\alpha' t)-a\left(\frac{\beta}{\beta+pt^r}\right)^{\alpha}-a\alpha'\left(\frac{\beta}{\beta+pt^r}\right)^{\alpha}\left[t+\sum_{n=1}^{\infty}\left(\frac{p}{\beta+pt^r}\right)^{n}\frac{t^{nr+1}}{n(nr+1)}\frac{1}{beta(n,\alpha)}\right]$
2	**Chang et al. [3]**	S-Shaped	$m(t)=N\left\{1-\left(\frac{\beta}{\beta+(at)^b}\right)^{\alpha}\right\}$
3	**Fault removal model [4]**	S-Shaped	$m(t)=\frac{a}{p-\beta}\left\{1-\left(\frac{(1+\alpha)e^{-bt}}{1+\alpha e^{-bt}}\right)^{\frac{c}{b}(p-\beta)}\right\}$
4	**Goel and Okumoto model [5]**	Concave	$m(t)=a(1-e^{-bt})$
5	**Kapur et al. model [6]**	S-Shaped	$m(t)=\frac{A}{1-\alpha}\left[1-\left(\left(1+bt+\frac{b^2t^2}{2}\right)e^{-bt}\right)^{p(1-\alpha)}\right]$
6	**HD/GO model [47]**	Concave	$m(t)=\log[(e^{a}-c)/(e^{ae^{-bt}}-c)]$
7	**Roy et al. model [48]**	Concave	$m(t)=a\alpha(1-e^{-bt})-\frac{ab}{b-\beta}(e^{-\beta t}-e^{-bt})$
8	**Teng-Pham model [49]**	S-Shaped	$m(t)=\frac{a}{p-q}\left\{1-\left(\frac{\beta}{\beta+(p-q)\ln\left(\frac{c+e^{bt}}{c+1}\right)}\right)^{\alpha}\right\}$
9	**Yamada exponential model [50]**	Concave	$m(t)=a(1-e^{-\gamma\alpha(1-e^{-\beta t})})$
10	**Yamada Rayleigh model [50]**	S-Shaped	$m(t)=a(1-e^{-\gamma\alpha(1-e^{-\beta t^2/2})})$

Table 3: *Failure Data Set*

Week No.	Faults	Cumulative detected faults
1	2	2
2	11	13
3	2	15
4	4	19
5	3	22
6	1	23
7	1	24
8	2	26
9	4	30
10	0	30
11	4	34
12	1	35
13	3	38
14	0	38

Table 4: *Estimated Parameter values of SRGMs*

No	Model Name	Estimated Parameter
1	Li-Pham model [1]	$\alpha' = 0.1904, \hat{\beta} = 9.451, \hat{a} = 10.67,$ $\hat{\alpha} = 59, \hat{p} = 0.04222, \hat{r} = 3$
2	Chang et al model [3]	$\hat{N} = 584, \hat{a} = 0.3318, \hat{\alpha} = 0.03498, \hat{b} = 1.11, \hat{\beta} = 0.9627$
3	Fault removal model [4]	$\hat{\beta} = 0.3616, \hat{a} = 25.24, \hat{\alpha} = 2.701e-06$ $\hat{b} = 7.944e-06, \hat{c} = 0.2162, \hat{p} = 0.9085$
4	Goel and Okumoto model [5]	$\hat{a} = 46.14, \hat{b} = 0.1182$
5	Kapur et al. model [6]	$\hat{A} = 24.25, \hat{\alpha} = 0.3188, \hat{b} = 1,$ $\hat{p} = 0.6313$
6	HD/GO model [47]	$\hat{a} = 46.14, \hat{b} = 0.1182, \hat{c} = 1$
7	Roy et al. model [48]	$\hat{a} = 473.9, \hat{\alpha} = 1.038, \hat{b} = 0.3889, \hat{\beta} = 0.003857$
8	Teng-Pham model [49]	$\hat{a} = 78.53, \hat{\alpha} = 0.06219, \hat{b} = 0.3843, \hat{\beta} = 0.291,$ $\hat{c} = 3.186e-08, \hat{p} = 0.853, \hat{q} = 0.6711$
9	Yamada exponential model [50]	$\hat{a} = 68.09, \hat{\beta} = 0.05826, \hat{\gamma} = 0.8222, \hat{\alpha} = 1.741$
10	Yamada Rayleigh model [50]	$\hat{a} = 38.09, \hat{\beta} = 0.03646, \hat{\gamma} = 1.656, \hat{\alpha} = 1.4$

Table 5: *Estimated values of ranking criteria for each alternative SRGM*

S.No	SRGM	MSE	R^2	Adjusted R^2	PRR	PP	AIC
1	Li -Pham model [1]	1.46	0.9916	0.9864	0.1071	0.1933	62.7092
2	Chang et al. [3]	4.0544	0.9738	0.9622	0.4961	2.9906	355.24
3	Fault removal model [4]	5.4512	0.9687	0.9491	0.528	2.5785	70.8309
4	G-O model [5]	3.6342	0.9687	0.9661	0.5283	2.5742	62.8309
5	Kapur et al. model [6]	14.52	0.8958	0.8768	2.4295	0.6419	76.6596
6	HD/GO model [47]	3.9645	0.9687	0.963	0.5283	2.5742	64.831
7	Roy et al. model [48]	3.287	0.9764	0.9693	0.5115	4.2796	66.7588
8	Teng-Pham model [49]	5.2514	0.9736	0.951	0.5125	3.5656	72.6973
9	Yamada exponential model [50]	4.116	0.9704	0.9616	0.5188	2.8112	66.7085
10	Yamada Rayleigh model [50]	16.1	0.8844	0.8498	1.9072	0.5447	82.0585

To determine the best SRGM among alternative SRGM, ranking of ten SRGM is done based on the proposed method as discussed in section III. The proposed methodology is applied on ten SRGM and the results are shown in following Tables. Initially we will use Entropy method to find weights of each criterion. Equation (2) is used to compute the values of normalized entropy (e_n) for each criterion. The result of the calculated entropy is shown in Table 6. The weights (w_n) of each criterion is calculated by using Equation (4) and the results obtained is shown in Table 7.

Table 6: *Normalized* (e_n) *entropy for each criterion*

Criteria(D_n, n=1 to 6)	Normalized entropy (e_n, n=1 to 6)
D_1	0.89474
D_2	0.99972
D_3	0.99956677
D_4	0.86562
D_5	0.91207
D_6	0.89199

Table 7: *Weights* (w_n) *of each criterion*

Criteria(D_n, n=1 to 6)	Weights (w_n, n=1 to 6)
w_1	0.24126
w_2	0.00065
w_3	0.000992966
w_4	0.308
w_5	0.20154
w_6	0.24756

Table 8: *Normalized decision (R) matrix of ranking criteria for each alternative SRGM using ELECTRE method*

S.No	SRGM	MSE	R^2	Adjusted R^2	PRR	PP	AIC
1	**Chang et al. [3]**	0.669650496	0.313079277	0.309992524	0.072804014	1.08043627	305.980859
2	**Fault removal model [4]**	1.210540268	0.309808538	0.301609114	0.082467858	0.80318718	12.164564
3	**G-O model [5]**	0.538037637	0.309808538	0.312510547	0.082561598	0.80051056	9.57188754
4	**HD/GO model [47]**	0.640282825	0.309808538	0.310508211	0.082561598	0.80051056	10.1909922
5	**Kapur et al. model [6]**	8.588715974	0.264933513	0.257407737	1.746025885	0.04977569	14.2489914
6	**Li -Pham model [1]**	0.086836182	0.324629377	0.325781668	0.003393093	0.00451384	9.53484301
7	**Roy et al. model [48]**	0.440143626	0.314753323	0.314584225	0.077394152	2.2125284	10.8060773
8	**Teng-Pham model [49]**	1.123427907	0.312950689	0.302817904	0.077697064	1.53584624	12.8140847
9	**Yamada exponential model [50]**	0.690153573	0.310896876	0.30960604	0.079619016	0.95469794	10.7897996
10	**Yamada Rayleigh model [50]**	10.55958281	0.258233303	0.241798704	1.075992755	0.03584241	16.326691

Table 9: *Weighted Normalized Matrix (X) of ranking criteria for each alternative SRGM*

S.No	w_n	**0.241256715**	**0.000648498**	**0.000992966**	**0.30799854**	**0.20154373**	**0.24755955**
	SRGM	MSE	R^2	Adjusted R^2	PRR	PP	AIC
1	**Chang et al. [3]**	0.161557679	0.000203031	0.000307812	0.02242353	0.21775516	75.7484832
2	**Fault removal model [4]**	0.292050968	0.00020091	0.000299488	0.02539998	0.16187734	3.01145396
3	**G-O model [5]**	0.129805193	0.00020091	0.000310312	0.025428852	0.16133789	2.36961215
4	**HD/GO model [47]**	0.154472531	0.00020091	0.000308324	0.025428852	0.16133789	2.52287741
5	**Kapur et al. model [6]**	2.072085401	0.000171809	0.000255597	0.537773424	0.01003198	3.52747387
6	**Li -Pham model [1]**	0.020949812	0.000210522	0.00032349	0.001045068	0.00090974	2.36044143
7	**Roy et al. model [48]**	0.106187605	0.000204117	0.000312371	0.023837286	0.44592123	2.67514761
8	**Teng-Pham model [49]**	0.271034526	0.000202948	0.000300688	0.023930582	0.30954018	3.172249
9	**Yamada exponential model [50]**	0.166504184	0.000201616	0.000307428	0.024522541	0.19241339	2.6711179
10	**Yamada Rayleigh model [50]**	2.547570259	0.000167464	0.000240098	0.331404198	0.00722381	4.04182825

Now ELECTRE method is used to rank alternate SRGM so that best SRGM can be determined. Normalized decision matrix is obtained by using Equation (5) and the obtained result is shown in Table 8. Equation (7) is used to compute the weighted normalized matrix. The results for weighted normalized matrix is shown in Table 9. Concordance and discordance matrices consisting of alternative SRGM corresponding to each SRGM are calculated by using Equation (10) and Equation (11). The results of the concordance and discordance matrices are shown in Table 10 and Table 11. Equation (13) and Equation (15) are used to find the values of dominant concordance and discordance matrices. The obtained results for dominant concordance and discordance matrices consisting alternative SRGM corresponding to each SRGM are shown in Table 12 and Table 13.

Table 10: *Concordance matrix (C) consisting alternative SRGM corresponding to each SRGM*

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
C_1	0.0000	0.7578	0.6910	0.6910	0.4507	0.9984	0.4888	0.2492	0.4507	0.4507
C_2	0.5493	0.0000	0.6910	0.6910	0.2032	0.9984	0.7968	0.5493	0.7968	0.2032
C_3	0.3090	0.3096	0.0000	0.5112	0.2032	0.9984	0.5493	0.3090	0.3090	0.2032
C_4	0.3090	0.3096	0.9990	0.0000	0.2032	0.9984	0.5493	0.3090	0.3090	0.2032
C_5	0.5493	0.5493	0.7968	0.7968	0.0000	0.9984	0.7968	0.7968	0.7968	0.5112
C_6	0.0016	0.0016	0.0016	0.0016	0.0016	0.0000	0.0016	0.0016	0.0016	0.0016
C_7	0.5112	0.2032	0.4507	0.4507	0.2032	0.9984	0.0000	0.2032	0.4507	0.2032
C_8	0.7508	0.4507	0.6910	0.6910	0.2032	0.9984	0.7968	0.0000	0.6910	0.2032
C_9	0.5493	0.2032	0.6910	0.6910	0.2032	0.9984	0.5493	0.3090	0.0000	0.2032
C_{10}	0.5493	0.7968	0.7968	0.7968	0.4888	0.9984	0.7968	0.7968	0.7968	0.0000

Table 11: *Discordance matrix (D) consisting alternative SRGM corresponding to each SRGM*

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}
D_1	0.0000	0.0018	0.0000	0.0000	0.0265	0.0000	0.0031	0.0015	0.0001	0.0333
D_2	0.0018	0.0000	0.0000	0.0000	1.0000	0.0000	0.0031	1.0000	0.0897	1.0000
D_3	0.0000	1.0000	0.0000	1.0000	1.0000	0.0001	0.3055	1.0000	1.0000	1.0000
D_4	1.0000	1.0000	0.0000	0.0000	1.0000	0.0001	1.0000	1.0000	1.0000	1.0000
D_5	1.0000	0.2899	0.0779	0.0789	0.0000	0.0000	0.2217	0.1663	0.0957	1.0000
D_6	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	1.0000	1.0000	1.0000	1.0000
D_7	1.0000	1.0000	0.0773	0.1697	1.0000	0.0000	0.0000	1.0000	0.2379	1.0000
D_8	1.0000	0.1307	0.0019	0.0023	1.0000	0.0000	0.2744	0.0000	0.0012	1.0000
D_9	1.0000	1.0000	0.0030	0.0061	1.0000	0.0001	1.0000	1.0000	0.0000	1.0000
D_{10}	1.0000	0.0686	0.0637	0.0644	0.4012	0.0000	0.1797	0.1328	0.0778	0.0000

Table 12: *Dominant concordance matrix (G) consisting alternative SRGM corresponding to each SRGM*

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}
G_1	0	1	1	1	0	1	0	0	0	0
G_2	1	0	1	1	0	1	1	1	1	0
G_3	0	0	0	1	0	1	1	0	0	0
G_4	0	0	1	0	0	1	1	0	0	0
G_5	1	1	1	1	0	1	1	1	1	1
G_6	0	0	0	0	0	0	0	0	0	0
G_7	1	0	0	0	0	1	0	0	0	0
G_8	1	0	1	1	0	1	1	0	1	0
G_9	1	0	1	1	0	1	1	0	0	0
G_{10}	1	1	1	1	0	1	1	1	1	0

Table 13: *Dominant discordance matrix (H) consisting alternative SRGM corresponding to each SRGM*

	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}
H_1	1	1	1	1	1	1	1	1	1	1
H_2	1	1	1	1	0	1	1	0	1	0
H_3	1	0	1	0	0	1	1	0	0	0
H_4	0	0	1	1	0	1	0	0	0	0
H_5	0	1	1	1	1	1	1	1	1	0
H_6	0	0	0	0	0	1	0	0	0	0
H_7	0	0	1	1	0	1	1	0	1	0
H_8	0	1	1	1	0	1	1	1	1	0
H_9	0	0	1	1	0	1	0	0	1	0
H_{10}	0	1	1	1	1	1	1	1	1	1

Table 14: *Aggregate dominance matrix (F) of alternative SRGM*

	SRGM											Sum	Rank
1	Chang et al. [3]	0	1	1	1	0	1	0	0	0	0	4	5
2	Fault removal model [4]	1	0	1	1	0	1	1	0	1	0	6	3
3	G-O model [5]	0	0	0	0	0	1	1	0	0	0	2	7
4	HD/GO model [47]	0	0	1	0	0	1	0	0	0	0	2	7
5	Kapur et al. model [6]	0	1	1	1	0	1	1	1	1	0	7	1
6	Li -Pham model [1]	0	0	0	0	0	0	0	0	0	0	0	10
7	Roy et al. model [48]	0	0	0	0	0	1	0	0	0	0	1	9
8	Teng-Pham model [49]	0	0	1	1	0	1	1	0	1	0	5	4
9	Yamada exponential model [50]	0	0	1	1	0	1	0	0	0	0	3	6
10	Yamada Rayleigh model [50]	0	1	1	1	0	1	1	1	1	0	7	1

The final step is to determine the value of aggregate dominance matrix based on the Equation (16). The obtained result is shown in Table 14. The SRGM that has maximum number of 1 in its row will be ranked one. Based on the result Kapur et al. [6] and Yamada Rayleigh model [50] is ranked one as it has maximum number of 1 in a row. The graphical representation of rank of models is shown in Figure 4.

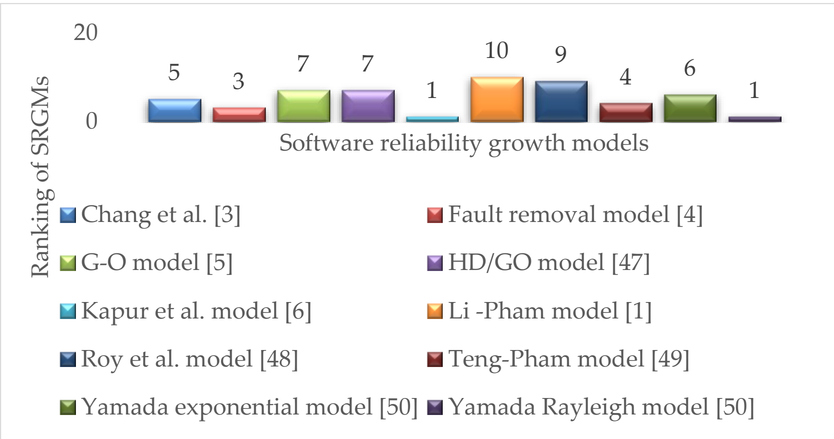

Figure 4: *Graphical representation of ranking of models*

VI. Conclusion

The objective of this study was the ranking of various software reliability growth models (SRGMs) to select suitable SRGM. In this paper, we have combined the two well-known MCDMs approach to rank and select the suitable SRGM. The renowned Shannon entropy approach is combined with Elimination et Choice Translating Reality (ELECTRE) approach to find the appropriate result. The Shannon entropy approach is applied to determine the weights of each criteria and ELECTRE is applied to rank the SRGMs so as to find suitable SRGM. The application of proposed approach shows that Kapur at al. [6] and Yamada Rayleigh model [50] are the most suitable software reliability growth models. The results obtained shows that the proposed method is suitable to rank SRGMs considering multiple criteria to determine suitable SRGM. The present study can be extended by comparing large number of criteria for comparison as it will give more accurate results. Further, proposed approach can be compared with the existing multi criteria decision making approaches.

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A Two State Time-Dependent Bulk Queue Model with Intermittently Available Server

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Abstract

This paper studies the time-dependent first-in-first-out (FIFO) queuing model with a single intermittently available server and variable-sized bulk arrivals and bulk departures. The time between arrivals, servicing time, and server availability time follows an exponential distribution. The difference-differential equations are used for developing system equations, and the Laplace transforms (L.T.), and inverse transforms are used for the solution. Explicit two-state transient recursive probabilities are obtained for an exact number ofbulk arrivals and bulk departures, and in the end, few particular casesare derived.

Keywords: Bulk, Intermittently available server, two-state queuing model, Markovian

I. Introduction

The development of (i) time-dependent queueing models and (ii) continuous-time Markov chains theory and applications is feasible due to Markovian queues. Many researchers contributed in this direction, like Asmussen [1], Gross and Harris [2], Kleinrock [3], Medhi [4], Chung [5], Freedman [6], Syski [7], etc. The time-dependent analysis ofqueuing model is comparatively tedious than the corresponding steady-state analysis. Due to this, there are limited explicit expressions available in the literature, even for simple models. Also, with the above, studying queueing systems with bulk arrivals and departures make it more tedious.

The classical literature deals with queuing models, where arrivals and departures are rendered in bulk. However, there are many real-life queuing situations in which arrivals and departures are rendered in bulk. Bulk queues have been used for modeling in various cases, such as communication systems, production/manufacturing systems, centralized parallel processing computer systems, restaurants, etc., apart from theoretical structures. Erlang's M/Ek/1 [8] solution was found to be the initiation of bulk arrival queues. Gaver [9] gave the explicitsolution to the batch arrival queue. Explicit consideration of bulk arrival queues seems to have begun several years after Bailey [10] on bulk service. Later, Premchand [11] generated time-dependent probabilities of a transient queueing system, where departures occur in variable-sized batches.

Chen et al. [12] developed a Markovian bulk queues model with general state-dependent control. Ayyapann et al. [13] analyzed a single server bulk queuing model with the repairable server, heterogeneous service, multiple vacations and standby server. Nithya [14] developed a simulation model to study queues in the production system.

The assumption of the server's instantaneously availability seems plausible only in the case when the server is automatic. But, there are real-life situations where server is not available instantaneously, i.e., interruptions in taking a new unit into service just after completion of in-hand service. This type of interruption, because the server is not instantaneously available, is unpredictable, and for an improved solution, aprobability distribution may be associated. Agarwal [15] developed a model where random interruptions in service occur after completing the in-hand service. Later, Sharda and Garg [16] worked on an intermittently available server. The server can rest or work on another important task whenever the queue length ≥ 0, but before an interruption in service, the server is bound to finish the in-hand service.

Sharda [17] presented the queuing model's solution by joining the concepts of variable-sized bulk queues and intermittently available servers. Pegden and Rosenshine [18] did the pioneering work in 1982 on the two-state queuing model concept. Also, Indra and Vijay [19, 20] obtained the transient probabilities of a model with intermittently available server and variable size bulk (i) departures, and (ii) arrivals.

Chenetal. [21] discussed a bulk queue model with quasi-stationary distributions and decay properties. Banerjee etal. [22] analyzed a bulk queueing system in which service capacity is variable and service is batch size-dependent. Niranjan etal. [23] studied the state-dependent bulk arrival retrial Bernoulli feedback queues with multiple vacations, and threshold. Shanthi etal. [24] presented a computational approach for a working vacation bulk service transient queuing model.
We study the time-dependent, first-come-first-served, intermittently available single server bulk queuing model with variable-sized batch arrivals and departures in the present work.The queuing model can be applied to the data switching systems. In data switching systems, processors have Poisson streams of primary processes requiring attention. An example of such a task would be the routing of packets to an appropriate outgoing line. The processor may also be required to execute small maintenance routines whenever it is necessary. Here, the processor's primary aim is the routing of packets. Still, when maintenance is needed, the processor, after completing the packet's service, goes for maintenance by keeping the queue packets. The maintenance time is corresponding to the server's intermittently availability time.

II. The Model

The following are the assumptions of the model:

- Arrivals follows Poisson distribution and occur in variable size batches with parameter 'λ' and 'a_i' is the probability of i arrivals, where $\sum_{i=1}^{k} a_i = 1$ k represents the maximum batch capacity.
- Service times follow an exponential distribution with the parameter 'μ'. Before each service, the batch size isdetermined, either equal to the total units waiting or equal to the service channel capacity, whichever is less. 'd_j' represents the probability that server can serve j units, where $\sum_{j}^{k} d_j = 1$ k represents the maximum capacity.
- Server's availability time follows exponential distribution and 'v' is the parameter.
- Queue discipline is FIFO.

- Stochastic processes involved are: (i) units/ customer's arrivals, (ii) service times, and (iii) server availability time is statistically independent.

III. Model Definitions and Notations

'$P_{i,j,F}(t)$' represents the probability of 'i' occurrences and 'j' services by any time t and server is available, i.e., no unit is in waiting; $i \geq j \geq 0$

'$P_{i,j,B}(t)$' represents the probability of 'i' occurrences and 'j' services by any time t and server is busy, i.e., units are in waiting; $i>j>0$

'$P_{i,j}(t)$' represents the probability of 'i' occurrences and 'j' services by any given time t; $i,j \geq 0$.

The L.T. of $P_{i,j}(t)$ is:

$$\bar{P}_{i,j}(s) = \int_0^\infty e^{-st} P_{i,j}(t)\, dt \tag{1}$$

$\sum\limits_{\sum_{t=1}^{u} r_t = n} 1$, The sum is for all permutations of total n objects taken u (=1, 2…n) at a time, such that $\sum_{t=1}^{u} r_t = n$, $r_t > 0$.

$$\delta_{i,j} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}, \qquad M_{\alpha,\beta} = \left(\sum_{\sum_{t=1}^{u} r_t = \alpha} 1 \prod_{t=1}^{u} a_{r_t} \left(\frac{\lambda}{s + \lambda + \beta} \right)^{u} \right)$$

IV. The solution to the Problem

Initially,

$$\left. \begin{aligned} P_{0,0,B}(0) &= 0 \\ P_{0,0,F}(0) &= 1 \end{aligned} \right\} \tag{2}$$

The system governing difference-differential equations are:

$$\frac{d}{dt} P_{i,i,F}(t) = -\lambda P_{i,i,F}(t) + \mu \sum_{\ell=1}^{i} \sum_{m=\ell}^{k} d_m P_{i,i-\ell,B}(t);\ k > i \geq 0 \tag{3}$$

$$\frac{d}{dt} P_{i,i,F}(t) = -\lambda P_{i,i,F}(t) + \mu \sum_{\ell=1}^{k} \sum_{m=\ell}^{k} d_m P_{i,i-\ell,B}(t);\ i \geq k \tag{4}$$

$$\frac{d}{dt} P_{i,j,F}(t) = -(\lambda + v) P_{i,j,F}(t) + \lambda \sum_{\ell=1}^{i-j} a_\ell P_{i-\ell,j,F}(t) + \mu \sum_{\ell=1}^{j} d_\ell P_{i,j-\ell,B}(t)(1-\delta_{j,0});\ \ i > j \geq 0 \tag{5}$$

$$\frac{d}{dt} P_{i,j,B}(t) = -(\lambda + \mu) P_{i,j,B}(t) + \lambda \sum_{\ell=1}^{i-j-1} a_\ell P_{i-\ell,j,B}(t)(1 - \delta_{i,j+1}) + vP_{i,j,F}(t);\ i > j \geq 0 \tag{6}$$

Taking Laplace transforms of (3) to (6) along with (2) and solving recursively,

$$\overline{P}_{i,0,F}(s)=\left(\frac{1}{s+\lambda}\right)\left(\sum_{\sum_{t=1}^{u} r_t=i}{}^{1}\prod_{t=1}^{u} a_{r_t}\left(\frac{\lambda}{s+\lambda+v}\right)^{u}\right)^{(1-\delta_{i,0})};\ i\geq 0 \tag{7}$$

$$\overline{P}_{i,0,B}(s)=\left(\frac{v}{(s+\lambda)(s+\lambda+\mu)}\right)\sum_{\ell=1}^{i}\left\{\left(\sum_{\sum_{t=1}^{u} r_t=i-\ell}{}^{1}\prod_{t=1}^{u} a_{r_t}\left(\frac{\lambda}{s+\lambda+\mu}\right)^{u}\right)^{(1-\delta_{i,\ell})}\left(\sum_{\sum_{t=1}^{u} r_t=\ell}{}^{1}\prod_{t=1}^{u} a_{r_t}\left(\frac{\lambda}{s+\lambda+v}\right)^{u}\right)\right\};\ i>0 \tag{8}$$

$$\overline{P}_{i,j,F}(s)=\sum_{b_j=j}^{i}\left(\sum_{\sum_{t=1}^{u} r_t=i-b_j}{}^{1}\prod_{t=1}^{u} a_{r_t}\left(\frac{\lambda}{s+\lambda+v}\right)^{u}\right)^{(1-\delta_{i,b_j})}\sum_{g=1}^{j}\left(\frac{\mu\left(d_g\right)^{(1-\delta_{g,1})}}{s+\lambda}\right)^{\delta_{b_j,j}}\left(\frac{\mu d_g}{s+\lambda+v}\right)^{(1-\delta_{b_j,j})}$$

$$\overline{P}_{b_j,j-g,B}(s);\ i\geq j\geq 1 \tag{9}$$

$$\overline{P}_{i,j,B}(s)=\left(\frac{v}{s+\lambda+\mu}\right)\sum_{\ell_j=j+1}^{i}\left(\sum_{\sum_{t=1}^{u} r_t=i-\ell_j}{}^{1}\prod_{t=1}^{u} a_{r_t}\left(\frac{\lambda}{s+\lambda+\mu}\right)^{u}\right)^{(1-\delta_{i,\ell_j})}\sum_{b_j=j}^{\ell_j}\left(\sum_{\sum_{t=1}^{u} r_t=\ell_j-b_j}{}^{1}\prod_{t=1}^{u} a_{r_t}\left(\frac{\lambda}{s+\lambda+v}\right)^{u}\right)^{(1-\delta_{\ell_j,b_j})}$$

$$\sum_{g=1}^{j}\left(\frac{\mu\left(d_g\right)^{(1-\delta_{g,1})}}{s+\lambda}\right)^{\delta_{b_j,j}}\left(\frac{\mu d_g}{s+\lambda+v}\right)^{(1-\delta_{b_j,j})}\overline{P}_{b_j,j-g,B}(s);\ i>j>0 \tag{10}$$

The inverse transforms of equations (7) to (10) are

$$P_{0,0,F}(t)=e^{-\lambda t} \tag{11}$$

$$P_{i,0,F}(t)=\left\{\sum_{\sum_{t=1}^{u} r_t=i}{}^{1}\prod_{t=1}^{u} a_{r_t}\left\{\left(\lambda^{u}\right)\left\{\frac{e^{-\lambda t}}{v^{u}}+\frac{e^{-(\lambda+v)t}}{(-v)}\left(\sum_{h=1}^{u}(-1)^{h+1}\left(\frac{t^{u-h}}{(u-h)!(-v)^{h-1}}\right)\right)\right\}\right\}\right\};i>0 \tag{12}$$

$$P_{i,0,B}(t)=\sum_{\ell=1}^{i}\left[\left[\left\{\sum_{\sum_{t=1}^{u} r_t=i-\ell}{}^{1}\prod_{t=1}^{u} a_{r_t}\left\{\left(\lambda^{u}\right)\left(\frac{e^{-(\lambda+\mu)t}t^{u-1}}{(u-1)!}\right)\right\}\right\}^{(1-\delta_{i,\ell})}\right]*\left[\left\{\sum_{\sum_{t=1}^{u} r_t=\ell}{}^{1}\prod_{t=1}^{u} a_{r_t}\left\{\left(\lambda^{u}v\right)\left\{\frac{e^{-\lambda t}}{\mu v^{u}}+\frac{e^{-(\lambda+\mu)t}}{(-\mu)(v-\mu)^{u}}\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.+\frac{e^{-(\lambda+v)t}}{(-v)(\mu-v)}\left(\sum_{p=1}^{u}\sum_{\ell=0}^{p-1}(-1)^{p+1}\left(\frac{t^{u-p}}{(u-p)!(\mu-v)^{\ell}(-v)^{p-1-\ell}}\right)\right)\right\}\right\}\right\}\right]\right];i>0 \tag{13}$$

$$P_{i,i,F}(t)=\sum_{\ell=1}^{i}\left[\mu\left(e^{-\lambda t}\right)\left(d_{\ell}\right)^{(1-\delta_{\ell,1})}\right]*P_{i,i-\ell,B}(t);\ i\geq 1 \tag{14}$$

$$P_{i,j,F}(t) = \sum_{b_j=j}^{i}\sum_{g=1}^{j}\Bigg[\Bigg[\Bigg[\Bigg\{\sum_{\sum_{t=1}^{u} r_t = i-b_j}^{1}\prod_{t=1}^{u} a_{r_t}\Bigg\{\left(\lambda^u \mu (d_g)^{(1-\delta_{g,1})}\right)\Bigg\{\frac{e^{-\lambda t}}{v^u}+\frac{e^{-(\lambda+v)t}}{(-v)}\left(\sum_{h=1}^{u}(-1)^{h+1}\left(\frac{t^{u-h}}{(u-h)!(-\mu)^{h-1}}\right)\right)\Bigg\}\Bigg\}\Bigg\}^{\delta_{b_j,j}}$$

$$\Bigg\{\sum_{\sum_{t=1}^{u} r_t = i-b_j}^{1}\prod_{t=1}^{u} a_{r_t}\Bigg\{\left(\lambda^u \mu d_g\right)\left(\frac{e^{-(\lambda+v)t}\,t^u}{u!}\right)\Bigg\}\Bigg\}^{(1-\delta_{b_j,j})}\Bigg]^{(1-\delta_{i,b_j})}\Bigg[\left(\mu d_g\right)e^{-(\lambda+v)t}\Bigg]^{\delta_{i,b_j}}\Bigg] * P_{b_j,j-g,B}(t)$$

$$; i > j > 0 \qquad (15)$$

$$P_{i,j,B}(t) = \sum_{\ell_j=j+1}^{i}\Bigg[\Bigg\{\sum_{\sum_{t=1}^{u} r_t = i-\ell_j}^{1}\prod_{t=1}^{u} a_{r_t}\Bigg\{\left(\lambda^u v\right)\left(\frac{e^{-(\lambda+\mu)t}\,t^u}{u!}\right)\Bigg\}\Bigg\}^{(1-\delta_{i,\ell_j})}\Bigg\{v\,e^{-(\lambda+\mu)\,t}\Bigg\}^{\delta_{i,\ell_j}}\Bigg] * \Bigg[\sum_{b_j=j}^{\ell_j}\sum_{g=1}^{j}$$

$$\Bigg[\Bigg[\Bigg[\Bigg\{\sum_{\sum_{t=1}^{u} r_t = \ell_j-b_j}^{1}\prod_{t=1}^{u} a_{r_t}\Bigg\{\left(\lambda^u \mu (d_g)^{(1-\delta_{g,1})}\right)\Bigg\{\frac{e^{-\lambda t}}{v^u}+\frac{e^{-(\lambda+v)t}}{(-v)}\left(\sum_{h=1}^{u}(-1)^{h+1}\left(\frac{t^{u-h}}{(u-h)!\,(-v)^{h-1}}\right)\right)\Bigg\}\Bigg\}\Bigg\}^{\delta_{b_j,j}}$$

$$\Bigg\{\sum_{\sum_{t=1}^{u} r_t = \ell_j-b_j}^{1}\prod_{t=1}^{u} a_{r_t}\Bigg\{\left(\lambda^u \mu d_g\right)\left(\frac{e^{-(\lambda+v)t}\,t^u}{u!}\right)\Bigg\}\Bigg\}^{(1-\delta_{b_j,j})}\Bigg]^{(1-\delta_{\ell_j,b_j})}\Bigg[\left(\mu d_g\right)e^{-(\lambda+v)t}\Bigg]^{\delta_{\ell_j,b_j}}\Bigg]^{(1-\delta_{\ell_j,j+1})}$$

$$\Bigg[\mu\,e^{-\lambda t}\Bigg]^{\delta_{\ell_j,j+1}}\Bigg] * P_{b_j,j-g,B}(t)\Bigg]; \; i > j > 0 \qquad (16)$$

From equations (7) - (10), following is observed

$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\overline{P}_{i,j}(s) = \frac{1}{s}$, therefore $\sum_{i=0}^{\infty}\sum_{j=0}^{i}P_{i,j}(t) = 1$ isverified.

The L.T. $\overline{P}_{i,.}(s)$ of $P_{i,.}(t)$, i.e., occurrences of i units:

$$\overline{P}_{i,.}(s) = \sum_{j=0}^{i}\overline{P}_{i,j}(s) = \left(\frac{1}{s+\lambda}\right)\left\{M_{i,0}\right\}^{(1-\delta_{i,0})}, \quad i \geq 0$$

1. Replacing $a_k = 1$, for k=1; $a_k = 0$, for k>1, we get

$$\overline{P}_{i,.}(s) = \sum_{j=0}^{i} P_{i,j}(s) = \left\{\frac{\lambda^i}{(s+\lambda)^{i+1}}\right\}, \quad i \geq 0 \qquad (17)$$

and hence

$$P_{i,.}(t) = \frac{(\lambda t)^i}{i!}e^{-\lambda t}, \quad i \geq 0 \qquad (18)$$

Equ (18) shows arrivals are distributed according to Poission distribution.

2. The L.T. of average of occurrences by any time ‘t’ is

$$\sum_{i=0}^{\infty} i\,\bar{P}_{i,.}(s) = \left\{\frac{\lambda}{s^2}\right\} \tag{19}$$

and the inverse is

$$\sum_{i=0}^{\infty} iP_{i,.}(t) = \lambda t \tag{20}$$

Particular Cases:

I. Case 1

I. Case 1-(a): When the units are served singly, then by replacing a_k as 1 for k=1; a_k as 0 for k>1in the equations (7) to (10), we have

$$\bar{P}_{i,0,F}(s) = \left(\frac{\lambda^i}{(s+\lambda)(s+\lambda+v)^i}\right);\ i \geq 0 \tag{21}$$

$$\bar{P}_{i,0,B}(s) = \left(\frac{\lambda^{i}\, v}{(s+\lambda)\ (s+\lambda+\mu)\ (s+\lambda+v)}\right)\sum_{\ell=0}^{i-1}\left(\frac{1}{(s+\lambda+\mu)^{i-1-\ell}\ (s+\lambda+v)^{\ell}}\right);i>0 \tag{22}$$

$$\bar{P}_{i,j,F}(s) = \sum_{b_j=j}^{i}\left(\frac{\lambda}{s+\lambda+v}\right)^{i-b_j}\sum_{g=1}^{j}\left\{\frac{\mu\left((d_g)^{(1-\delta_{g,1})}\right)^{\delta_{b_j,j}}\left(d_g\right)^{(1-\delta_{b_j,j})}}{(s+\lambda)^{\delta_{b_j,j}}\ (s+\lambda+v)^{(1-\delta_{b_j,j})}}\right\}\bar{P}_{b_j,j-g,B}(s);\ i\geq j>0 \tag{23}$$

$$\bar{P}_{i,j,B}(s) = \sum_{\ell_j=j+1}^{i}\sum_{b_j=j}^{\ell_j}\left(\frac{\lambda^{i-b_j}v}{(s+\lambda+\mu)^{i+1-\ell_j}(s+\lambda+v)^{\ell_j-b_j}}\right)\sum_{g=1}^{j}\left\{\frac{\mu\left((d_g)^{(1-\delta_{g,1})}\right)^{\delta_{b_j,j}}\left(d_g\right)^{(1-\delta_{b_j,j})}}{(s+\lambda)^{\delta_{b_j,j}}(s+\lambda+v)^{(1-\delta_{b_j,j})}}\right\}.$$

$$\bar{P}_{b_j,j-g,B}(s);\ i>j>0 \tag{24}$$

Equations (21) to (24) coincide with equations (20) to (24) of Indra and Vijay [19]

II. Case 1-(b): Along with case 1-(a), when departures are also singly then by substituting d_k as 1 for k=1; d_k as 0 for k>1 in equations (21) to (24), we have

$$\bar{P}_{i,0,F}(s) = \left(\frac{\lambda^i}{(s+\lambda)(s+\lambda+v)^i}\right);\ i\geq 0 \tag{25}$$

$$\bar{P}_{i,0,B}(s) = \left(\frac{\lambda^i\ v}{(s+\lambda)\ (s+\lambda+\mu)\ (s+\lambda+v)}\right)\sum_{\ell=0}^{i-1}\left(\frac{1}{(s+\lambda+\mu)^{i-1-\ell}\ (s+\lambda+v)^{\ell}}\right);\ i>0 \tag{26}$$

$$\bar{P}_{i,j,F}(s) = \sum_{b_j=j}^{i}\left[\left(\frac{\lambda}{s+\lambda+v}\right)^{i-b_j}\left(\frac{\mu}{(s+\lambda)^{\delta_{b_j,j}}\ (s+\lambda+v)^{(1-\delta_{b_j,j})}}\right)\right]\bar{P}_{b_j,j-1,B}(s);\ i\geq j\geq 1 \tag{27}$$

$$\bar{P}_{i,j,B}(s) = \sum_{\ell_j=j+1}^{i}\sum_{b_j=j}^{\ell_j}\left\{\left(\frac{\lambda^{i-b_j}v}{(s+\lambda+\mu)^{i+1-\ell_j}(s+\lambda+v)^{\ell_j-b_j}}\right)\left(\frac{\mu}{(s+\lambda)^{\delta_{b_j,j}}(s+\lambda+v)^{(1-\delta_{b_j,j})}}\right)\right\}\bar{P}_{b_j,j-1,B}(s)$$

$$;i>j>0 \qquad 28)$$

Equations (25) to (28) coincide with equations (6) to (10) of Sharda and Garg [16].

III. Case 1-(c): Along with case 1-(b), when the server is instantaneously available, then by

letting
$v \to \infty$ in (25) to (28), we have

$$\bar{P}_{i,0}(s) = \left[\bar{P}_{i,0,F}(s) + \bar{P}_{i,0,B}(s)\right] = \left(\frac{\lambda^i}{(s+\lambda)(s+\lambda+\mu)^i}\right);\ i \ge 0 \tag{29}$$

$$\bar{P}_{i,i}(s) = \bar{P}_{i,i,F}(s) = \left(\frac{\mu}{s+\lambda}\right)\bar{P}_{i,i-1}(s);\ \ i \ge 0 \tag{30}$$

$$\bar{P}_{i,j}(s) = \left[\bar{P}_{i,j,F}(s) + \bar{P}_{i,j,B}(s)\right] = \sum_{\ell_j=j}^{i}\left[\left(\frac{\lambda}{s+\lambda+\mu}\right)^{i-\ell_j}\left(\frac{\mu}{(s+\lambda)^{\delta_{\ell_j,j}}(s+\lambda+\mu)^{(1-\delta_{\ell_j,j})}}\right)\right]\bar{P}_{\ell_j,j-1}(s)\ ;\ i > j > 0 \tag{31}$$

Equations (29) to (31) reduce to equation (5) of Pegden and Rosenshine [18].

II. Case 2

When units depart singly, then by substituting d_k as 1 for k=1; d_k as 0 for k>1 in equations (7) to (10), we have

$$\bar{P}_{i,0,F}(s) = \left(\frac{1}{s+\lambda}\right)\left(\sum_{\sum_{t=1}^{u} r_t = i}{}^{1}\prod_{t=1}^{u} a_{r_t}\left(\frac{\lambda}{s+\lambda+v}\right)^{u}\right)^{(1-\delta_{i,0})}\ ;\ i \ge 0 \tag{32}$$

$$\bar{P}_{i,0,B}(s) = \left(\frac{v}{(s+\lambda)(s+\lambda+\mu)}\right)\sum_{\ell=1}^{i}\left\{\left(\sum_{\sum_{t=1}^{u} r_t = i-\ell}{}^{1}\prod_{t=1}^{u} a_{r_t}\left(\frac{\lambda}{s+\lambda+\mu}\right)^{u}\right)^{(1-\delta_{i,\ell})}\left(\sum_{\sum_{t=1}^{u} r_t = \ell}{}^{1}\prod_{t=1}^{u} a_{r_t}\left(\frac{\lambda}{s+\lambda+v}\right)^{u}\right)\right\};\ i > 0 \tag{33}$$

$$\bar{P}_{i,j,F}(s) = \sum_{b_j=j}^{i}\left[\left(\sum_{\sum_{t=1}^{u} r_t = i-b_j}{}^{1}\prod_{t=1}^{u} a_{r_t}\left(\frac{\lambda}{s+\lambda+v}\right)^{u}\right)^{(1-\delta_{i-b_j})}\left(\frac{\mu}{(s+\lambda)^{\delta_{b_j,j}}(s+\lambda+v)^{(1-\delta_{b_j,j})}}\right)\right]\bar{P}_{b_j,j-1,B};\ i \ge j > 0 \tag{34}$$

$$\bar{P}_{i,j,B}(s) = \left(\frac{v}{s+\lambda+\mu}\right)\sum_{\ell_j=j+1}^{i}\left(\sum_{\sum_{t=1}^{u} r_t = i-\ell_j}{}^{1}\prod_{t=1}^{u} a_{r_t}\left(\frac{\lambda}{s+\lambda+\mu}\right)^{u}\right)^{(1-\delta_{i-\ell_j})}\sum_{b_j=j}^{\ell_j}\left[\left(\sum_{\sum_{t=1}^{u} r_t = \ell_j-b_j}{}^{1}\prod_{t=1}^{u} a_{r_t}\left(\frac{\lambda}{s+\lambda+v}\right)^{u}\right)^{(1-\delta_{\ell_j-b_j})}\right.$$

$$\left.\left(\frac{\mu}{(s+\lambda)^{\delta_{b_j,j}}(s+\lambda+v)^{(1-\delta_{b_j,j})}}\right)\right]\bar{P}_{b_j,j-1,B}\ ;\ i > j > 0 \tag{35}$$

Equations (32) to (35) coincide with equations (6) to (9) of Indra and Vijay [20].

V. Results and Discussions

In this paper,transient probabilities are obtained using difference-differential equations. Explicit recursive probabilities of an exact number of bulk arrivals and departures are also obtained for this two-state bulk queuing model with an intermittently available server using Laplace transformsand inverse transforms. Particular cases are also derived, which shows the similarity with the already existing theoretical models. This theoretical model can be used at different businesses to utilize server time effectively.

VI. Future Scope

Future research couldbe extended by finding all the queueing system performance measures with numerical investigations using MATLAB in this direction. Also,the concept of multiple vacations with intermittently available servers can also be introduced.

VII. Conclusions

This theoretical paper has shown that the Poisson arrivals and exponential services are in variable-sized batches, with server availability as intermittent. Service times, inter-arrival times, intermittently available times are exponentially distributed. Difference-differential equations govern developed queuing systems. Laplace transforms and inverse transforms are used to get a feasible solution. Finally, recursive expressions of transient probabilities of an exact number of bulk arrivals and departures are obtained. To verify the system correctness, all the probabilitiesadded, which shows the sum equals one. The probabilities of thesemodels can be used in data switching systems, where processors have Poisson streams of primary processes requiring attention. An example of such a task would be the routing of packets to an appropriate outgoing line. The processor may also be required to execute small maintenance routines whenever it is necessary. Here, the processor's primary aim is the routing of packets. Still, when maintenance is needed, the processor, after completing the packet's service, goes for maintenance by keeping the queue packets. The maintenance time is corresponding to the server's intermittently availability time.

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Flexible Production Inventory Model with Time Dependent Holding Cost and Reliability Process

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Abstract

The deterministic inventory model for decayed items concerns with reliability process and flexible production rate. The rate of decayed items is assumed to be a two parameters Weibull distribution. In this study, complete backlogging shortages are acceptable and holding cost is taken as linear. Further the influence of inflation has been developed and the model is established with numerically and sensitivity analysis which has been given the impact of parameters and also interpreted through graphically. The system is designed for the producer to validate the model to optimum level.

Keywords: Deterministic demand, Time dependent holding cost, Reliability process, Inflation, Shortages, Volume flexibility

I. Introduction

In manufacturing systems, the constant production rate has been explicitly assumed by numerous researchers. Such an assumption holds only if the demand of a product is known with certainty. However, with constant changes in the markets and recent trends, the demand of a product fluctuates in the long run, which may lead to shortages or high storing costs, depending upon the rise or fall. Thus, volume flexibility is an adequate tool to deal with this situation precisely.

In practicality, objects are not always absolute and their perfection depends on the reliability of manufacturing process, those are employed by the manufacturer. For making the system reliable, manufacturer has to follow several steps for the machinery as well as for the objects also. This level of optimization is only possible when the machinery of this system utilizes their energy within the best extent. For this labour force should be efficient, machinery should be extensive, no imperfect items should be there and if there is any imperfectness amongst items then, those items should be repudiated.

Schweitzer and Seidmann, (1991) were the first one to developed the model with flexibility in the machine production rate and the optimization of processing rate. Khouza (1997) extended the manufacturing model with the inconstant rate of production. Sana and Chaudhuri (2003) considered the EPQ model with flexibility for fixed deteriorating items with stock dependent demand. Bag et al. (2008) presented a mathematical structure with uncertain demand and also used the term of flexibility and reliability. In (2010) a vendor and buyer model for decaying items and volume flexible with stochastic lead time developed by Singhal et al. Singhal and Singh (2012) given the probabilistic inventory system for selling price demand with the use of volume flexible approach. Al Masud et al. (2014) worked on the reliability parameters of a production inventory model to optimize. Singhal and Singh (2014) refined a remanufacturing model with exponential demand, volume agility and probabilistic decaying items. Singhal et al. (2016) suggested a flexible production structure with time dependent demand using stochastic rate of backlogging. An analyze of the factor of reliability and time-based demand rate on inventory management developed by Mahapatra et al. (2017). Vishnoi et al. (2018) deals a vendor buyer model with expiration date, variable holding cost in an inflationary condition. Sarkar et al. (2019) proposed a mathematical multi-item model with the system reliability where the holding cost is time dependent and used conception of sustainability. Recently, manufacturing model for a deteriorating item formulated by Shaikh et al. in (2020) where the demand is price dependent and also allowing for inflation and reliability.

Numerous researchers considered that while production take place, reliability of the product can be increased by a fixed set-up cost and if producer wants to increase the reliability of the product so he has to raise the cost and set up cost can be balanced by improvement of flexibility. Many cases can be also found where holding cost vary as it depends on time and the production system. So, considering mentioned parameters we developed a mathematical deterministic model for deteriorating items where the holding cost is time dependent and inflation is also being considered.

Two parameters Weibull distribution is considered in a situation objects deteriorate over time at varying rate. In this study demand rate considered as certain and shortages are permissible with the assumption that the defective items will trade with price cut and fresh units are escalated over the unit production cost. As it's a profit, so it will be maximized. Sensitivity of important parameters is interpreted through examples.

II. Assumptions and Notations

These assumptions and notations have been assumed:

Assumptions

1. The rate of production is assumed to be variable.
2. Weibull distribution deterioration is taken in this study.
3. Holding cost is a taken as linear.
4. Unreliable items are sold at a minimum cost.
5. The permissible of shortages with complete backlogging.
6. The effect of inflation is used.
7. Demand is less than the total cost of fresh units i.e. RK>D.

Notations

I(t) Level of Inventory

K Production rate per unit time

$C_h + \gamma t$ Time dependent carrying cost per unit time, $\gamma > 0$

C_0 Set up cost/cycle

C_d Deterioration cost/unit time

C_S Shortage cost/unit time

S_1 Fresh units selling price, $S_1 = m\eta_0(K)$, m >1

S_2 Defective units selling price, $S_2 = m_1\eta_0(K)$, $0 < m_1 \leq 1$

R Rate of reliability

$Y(C_0, R) = aC_0^{-b}R^c$, Total cost of interest and depreciation and it's inversely related to set up cost and directly related to reliability with power function

r inflation rate

$\eta_0(K)$ Cost of unit production and $\eta_0(K) = N + \frac{G}{K} + HK$, where N, H and G are material cost, tool or die cost and energy and labor cost respectively.

t Product life (time to deterioration), t > 0

$\alpha\beta t^{\beta-1}$ Weibull deterioration rate, where $\alpha > 0$ and $\beta > 0$ are scale and shape parameters respectively.

III. Mathematical Formulation

This model deals with the flexibility of production under reliability process where the demand is deterministic. Initially, production starts at t=0 with zero inventory and it is further considered that production rate is always greater than the demand rate 'D' at t = T1. When inventory is 'S' i.e. maximum, production come to an end. Between the period (T1,T2), the inventory level reduces due to both demand and deterioration and at t = T2, inventory level reaches zero. Shortages starts at t = T2 and it reaches a maximum shortage level at t = T3. At T3, production began again to fulfill the shortages and the stock level end up to zero at t = T. The level of inventory with time by the following equations:

$$I'(t) + \alpha\beta t^{\beta-1} I(t) = RK - D \qquad 0 \le t \le T_1 \tag{1}$$

$$I'(t) + \alpha\beta t^{\beta-1} I(t) = -D \qquad T_1 \le t \le T_2 \tag{2}$$

$$I'(t) = -D \qquad T_2 \le t \le T_3 \tag{3}$$

$$I'(t) = RK - D \qquad T_3 \le t \le T \tag{4}$$

With these conditions $I(0) = 0$, $I(T_1) = S$, $I(T_2) = 0$, $I(T) = 0$,

Solutions from Eq. (1) to Eq. (4) are as follows Eq. (5) to Eq. (8) respectively,

$$I(t) = (RK - D)(t + \frac{\alpha t^{\beta+1}}{\beta+1})e^{-\alpha t^{\beta}} \qquad 0 \le t \le T_1 \tag{5}$$

$$I(t) = D[(T_2 - t) + \frac{\alpha}{\beta+1}(T_2^{\beta+1} - t^{\beta+1})]e^{-\alpha t^{\beta}} \qquad T_1 \le t \le T_2 \tag{6}$$

$$I(t) = D(T_2 - t) \qquad T_2 \le t \le T_3 \tag{7}$$

$$I(t) = (RK - D)(t - T) \qquad T_3 \le t \le T \tag{8}$$

At, t = T_1, $I(T_1) = S$, from equation (6), one can get

$$S = D[(T_2 - T_1) + \frac{\alpha}{\beta+1}(T_2^{\beta+1} - T_1^{\beta+1})]e^{-\alpha T_1^{\beta}} \tag{9}$$

From equation (7) and (8), one can get

$$T_3 = \frac{(RK - D)T}{RK} + \frac{DT_2}{RK} \tag{10}$$

The present worth cost of carrying inventory during [0, T_1] and [T_1, T_2] is given by:

$$HC = [\int_0^{T_1} (C_h + \gamma t)e^{-rt} I(t)dt + \int_{T_1}^{T_2} (C_h + \gamma t)e^{-rt} I(t)dt]$$

$$= (RK - D)[C_h(\frac{T_1^2}{2} - \frac{rT_1^3}{3} - \frac{\alpha r T_1^{\beta+3}}{(\beta+1)(\beta+3)} - \frac{\alpha\beta T_1^{\beta+2}}{(\beta+1)(\beta+2)}) + \gamma(\frac{T_1^3}{3} - \frac{rT_1^4}{4} - \frac{\alpha r T_1^{\beta+4}}{(\beta+1)(\beta+4)}$$

$$-\frac{\alpha\beta T_1^{\beta+3}}{(\beta+1)(\beta+3)})]+D[C_h(\frac{(T_2-T_1)^2}{2}-\frac{\alpha T_1T_2^{\beta+1}}{(\beta+1)}+\frac{\alpha\beta T_2^{\beta+2}}{(\beta+1)(\beta+2)}-\frac{\alpha\beta T_1^{\beta+2}}{(\beta+1)(\beta+2)}-\frac{rT_2^3}{3}$$

$$+\frac{rT_2T_1^2}{2}-\frac{rT_1^3}{3}-\frac{\alpha rT_2^{\beta+3}}{2(\beta+3)}+\frac{\alpha rT_1^2T_2^{\beta+1}}{2(\beta+1)}-\frac{\alpha rT_1^{\beta+3}}{(\beta+1)(\beta+3)}+\frac{\alpha T_2T_1^{\beta+1}}{(\beta+1)})+\gamma(\frac{T_2^3}{6}-\frac{T_2T_1^2}{2}+\frac{T_1^3}{6}$$

$$+\frac{\alpha\beta T_2^{\beta+3}}{2(\beta+2)(\beta+3)}-\frac{\alpha T_1^2T_2^{\beta+1}}{2(\beta+1)}-\frac{rT_2^4}{12}+\frac{rT_2T_1^3}{3}-\frac{rT_1^4}{4}-\frac{\alpha rT_2^{\beta+4}}{3(\beta+4)}+\frac{\alpha rT_1^3T_2^{\beta+1}}{3(\beta+1)}$$

$$-\frac{\alpha rT_1^{\beta+4}}{(\beta+1)(\beta+4)}+\frac{\alpha T_1^{\beta+1}T_2}{(\beta+2)})] \tag{11}$$

The present worth of the cost of deterioration during [0,T1] and [T1,T2] is given by:

$$DC=C_d[\int_0^{T_1}\alpha\beta t^{\beta-1}e^{-rt}I(t)dt+\int_{T_1}^{T_2}\alpha\beta t^{\beta-1}e^{-rt}I(t)dt]$$

$$=C_d[\alpha\beta(RK-D)\{\frac{T_1^{\beta+1}}{(\beta+1)}+\frac{\alpha T_1^{2\beta+1}}{(\beta+1)(2\beta+1)}-\frac{(\alpha+r)T_1^{\beta+2}}{(\beta+2)}-\frac{\alpha rT_1^{2(\beta+1)}}{2(\beta+1)^2}\}$$

$$+\alpha\beta D\{\frac{T_2^{\beta+1}}{\beta(\beta+1)}-\frac{T_2T_1^{\beta}}{\beta}+\frac{T_1^{\beta+1}}{(\beta+1)}+\frac{\alpha T_2^{2\beta+1}}{\beta(\beta+1)}-\frac{\alpha T_1^{\beta}T_2^{\beta+1}}{\beta(\beta+1)}+\frac{\alpha\beta T_2^{2\beta+1}}{(\beta+1)(2\beta+1)}$$

$$-\frac{\alpha\beta T_1^{2\beta+1}}{(\beta+1)(2\beta+1)}-\frac{rT_2^{\beta+2}}{(\beta+1)(\beta+2)}+\frac{rT_2T_1^{\beta+1}}{(\beta+1)}-\frac{rT_1^{\beta+2}}{(\beta+2)}-\frac{\alpha rT_2^{2(\beta+1)}}{2(\beta+1)^2}+\frac{\alpha rT_1^{\beta+1}T_2^{\beta+1}}{(\beta+1)^2}$$

$$-\frac{\alpha rT_1^{2(\beta+1)}}{2(\beta+1)^2}-\frac{\alpha T_2^{2\beta+1}}{2\beta}+\frac{\alpha T_1^{2\beta}T_2}{2\beta}\}] \tag{12}$$

The present worth of the shortage cost during [T2, T3] and [T3, T] is given by:

$$SC=C_S[\int_{T_2}^{T_3}(-I(t))e^{-rt}dt+\int_{T_3}^{T}(-I(t))e^{-rt}dt]$$

$$=C_S[D(\frac{e^{-rt_2}}{r^2}-\frac{e^{-rT}}{r^2}+\frac{T_2e^{-rt_3}}{r}-\frac{Te^{-rt_3}}{r})+RK(\frac{e^{-rT}}{r^2}-\frac{e^{-rt_3}}{r^2}-\frac{T_3e^{-rt_3}}{r}+\frac{Te^{-rt_3}}{r})] \tag{13}$$

The present worth cost of production during [0,T1] and [T3,T] is given by:

$$PC=(N+\frac{G}{K}+HK)[\int_0^{T_1}Ke^{-rt}dt+\int_{T_3}^{T}Ke^{-rt}dt]$$

$$=(NK+G+HK^2)(\frac{1}{r}-\frac{e^{-rT_1}}{r}+\frac{e^{-rT_3}}{r}-\frac{e^{-rT}}{r}) \tag{14}$$

The present worth of the selling price of the fresh units during [0,T2] and [T3,T] is given by:

$$SP=(N+\frac{G}{K}+HK)[\int_0^{T_1}RKme^{-rt}dt+\int_{T_1}^{T_2}RKme^{-rt}dt+\int_{T_3}^{T}RKme^{-rt}dt]$$

$$=(NK+G+HK^2)Rm[\frac{1}{r}-\frac{e^{-rT_2}}{r}+\frac{e^{-rT_3}}{r}-\frac{e^{-rT}}{r}] \tag{15}$$

The present worth of the selling price of the defective units during [0,T2] and [T3,T] is given by:

$$SPD=(N+\frac{G}{K}+HK)[\int_0^{T_1}(1-R)m_1Ke^{-rt}dt+\int_{T_1}^{T_2}(1-R)m_1Ke^{-rt}dt+\int_{T_3}^{T}(1-R)m_1Ke^{-rt}dt]$$

$$=(NK+G+HK^2)(1-R)m_1[\frac{1}{r}-\frac{e^{-rT_2}}{r}+\frac{e^{-rT_3}}{r}-\frac{e^{-rT}}{r}] \tag{16}$$

The present worth of the set up cost during [0,T] is given by:

$$SUP = [\int_0^T C_0 e^{-rt} dt]$$

$$= \frac{C_0}{r}[1 - e^{-rT}] \tag{17}$$

The interest cost and depreciation cost is given by:

$$Y(C_0, R) = aC_0^{-b} R^{-c} \tag{18}$$

The total profit of the system is given by:

Total profit = Selling price of fresh units+ Selling price of defective units-Holding cost-
Deterioration cost-Shortage cost- Production cost- Set up cost-
Interest cost and depreciation cost

TP=SP+SPD-HC-DC-SC-PC-SUP-Y(C_0,R) (19)

Where SP, SPD, HC, DC, SC, PC, SUP and Y(C_0,R) are given by the equations (15), (16), (11), (12), (13), (14), (17) and (18) respectively.

IV. Solution Procedure

Total profit becomes a function of an independent variables, T_1, T_2 and T. This is the objective function. Hence, for optimal solution

$$\frac{\partial TP}{\partial T_1} = 0, \frac{\partial TP}{\partial T_2} = 0 \text{ and } \frac{\partial TP}{\partial T} = 0 \tag{20}$$

Provided, the values of T_1, T_2 and T satisfy the following conditions

$$\frac{\partial^2 TP}{\partial {T_1}^2} < 0,\ \frac{\partial^2 TP}{\partial {T_2}^2} < 0 \text{ and } \frac{\partial^2 TP}{\partial T^2} < 0 \tag{21}$$

To maximize the objective function for optimal solution the equation (19) is differentiate with respect to independent variables. Equation (19) gives an estimate of profit function. The equations (19) and (20) are nonlinear. These equations are solved by the software MATHEMATICA 11.3. For maximizing the profit, numerical illustration developed the optimal solutions of the model.

V. Numerical Illustrations

Adopting the parameters from the previous studies in proper units, which are as follows:
D = 50, α = 0.05, β = 2, r = 0.06, K= 80, C_h = 3, γ = 0.02, C_S = 5, a = 800, b = 0.50,
c = 0.75, R= 0.7, N =100, G = 200, H = 0.04, C_0 = 150, m = 2, m_1 = 0.6, C_d = 0.03

The optimum solutions is obtained T_1=11.6023, T_2=18.8667, T_3=25.3725, T=40.1264, Profit=204232.35

VI. Sensitivity Analysis

The analysis of sensitivity is performed by changing some parameters as Demand 'D', Deterioration parameters ('α' and 'β'), Inflation rate 'r' and Reliability parameter 'R' with the percentages of -50, -25, 25, 50 and get the variation with time and profit.

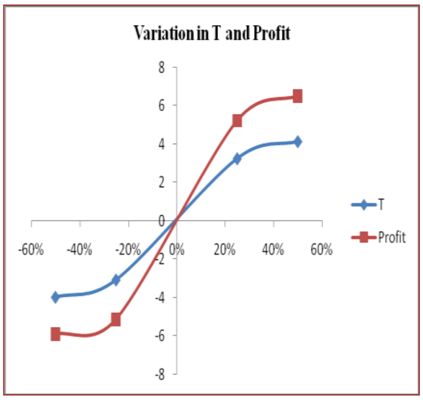
Table 1: % Change *of Optimal Solution w.r.t. time and profit*

Parameter	% Change	T_1	T_2	T_3	T	Profit
D	-50%	-8.25	-5.75	-2.08	-3.96	-5.90
	-25%	-6.61	-4.29	-1.65	-3.06	-5.12
	+25%	+5.74	+4.72	+2.19	+3.26	+5.24
	+50%	+7.82	+6.51	+2.84	+4.15	+6.53
α	-50%	-2.49	-3.87	0	-4.62	+5.33
	-25%	-1.25	-3.45	0	-3.56	+4.18
	+25%	+1.68	+3.65	0	+3.55	-4.27
	+50%	+2.46	+3.98	0	+4.34	-5.32
β	-50%	-4.26	-3.63	0	-3.06	+4.74
	-25%	-3.27	-2.84	0	-1.35	+3.90
	+25%	+3.46	+2.90	0	+1.27	-3.37
	+50%	+5.72	+3.28	0	+3.86	-4.82
r	-50%	-5.22	-3.89	0	-4.70	+7.40
	-25%	-4.72	-2.78	0	-2.63	+5.20
	+25%	+4.78	+3.24	0	+2.63	-5.24
	+50%	+5.63	+4.04	0	+4.70	-7.39
R	-50%	+6.91	+5.74	+5.02	+4.44	-6.34
	-25%	+5.37	+4.36	+4.71	+3.78	-4.69
	+25%	-5.35	-4.21	-4.73	-3.12	+4.84
	+50%	-6.92	-5.68	-5.02	-4.45	+6.26

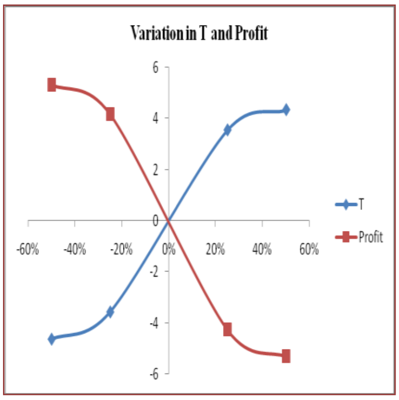
VII. Observations

The major conclusions are drawn from the numerical study. From the analysis it has been observed that:

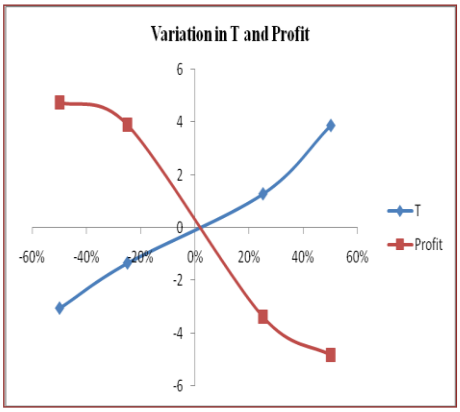
- T_1, T_2 and the profit are moderately sensitive and T_3 and T are slightly sensitive to change the parameter of demand 'D'.
- When the deterioration rates α and β increasing, then T_1, T_2, T and the profit are quite sensitive to change the scale and shape parameters of deterioration ('α'and 'β').
- It has been observed that when the net discount rate of inflation 'r' is increasing, the optimal profit is decreasing. T_1, T_2, T and the profit are reasonably sensitive to change the parameter of inflation 'r'.
- T_1, T_2, T_3, T and the profit are somewhat sensitive to change the parameter of reliability 'R'.
- With the increment of reliability factor, the optimal time T_1, T_2, T_3 and T are decreases and the profit is increases. With the decrement of reliability factor, the optimal time T_1, T_2, T_3 and T are increases and the profit is decreases.



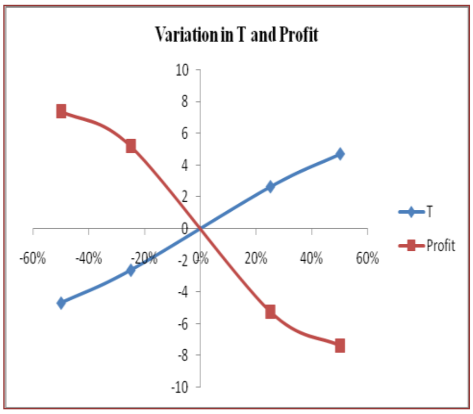
Graph 1: *Representation of the T and Profit w.r.t. 'D'*



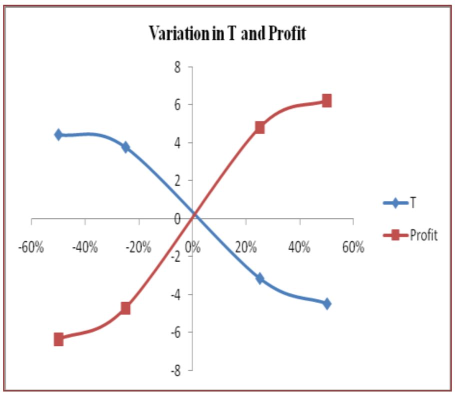
Graph 2: *Representation of the T and Profit w.r.t. 'α'*



Graph 3: *Representation of the T and Profit w.r.t. 'β'*



Graph 4: *Representation of the T and Profit w.r.t. 'r'*



Graph 5: *Representation of the T and Profit w.r.t. 'R'*

VIII. Conclusion

Flexibility and reliability for time dependent decaying items under inflation has been discussed in this deterministic inventory model. The requirement of a model with a greater level of flexibility has been highlighted. As a change has been seen throughout the globe in customer's demands across, a change has been observed in every minute and as a result the market forces like inflation change. A model which professes suppleness is sure to have a greater impact and acceptability by the organization.

The objective of this mathematical interpretation can be used to find the time dependent holding cost with the concept of volume flexibility and reliability. If a producer do not produce reliable product then, his organization will be out of contention. By dint of reliable products, producer will be able to face the upcoming challenge to survive and compete with running competitive market. Reliable of the product meant to maintain the standard quality of the object. Reliability and worth of the product have the caliber to stay in the market for long time. For instance, now-a-days (in pandemic) people have a firm believer in Swadeshi or Ayurvedic products as they have no side effects.

Sensitivity analysis is used as a mathematical technique to elucidate the results of the models. With the help of numerical illustrations, the feasibility of all the parameters has been shown. The results are found to be quite suitable and stable. This interpretation can be hold forth with fuzzy and partial backlogging.

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An EOQ Policy for Decaying Items with Constant Rate of Demand and Decreased Willingness of Buying over the Life Cycle of Item

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Abstract

In present study an EOQ model is proposed for decaying items with constant demand in which the process of demand is related to the willingness to receive the good quality items by giving the power to the costumers of checking the quality of items before reaching the effective purchasing. The cost of disposing is also considered for unsold units alongside the regular costs for storage and acquirement. This study presents a mathematical analysis to acquire the EOQ policy under certain conditions. To minimize the total expected inventory cost, a linear decrement is assumed in purchasing during the cycle of product life. Then the optimal parameters are found for the model with the help of a numerical example. In the last a numerical sensitivity analysis is presented to prove that the traditional EOQ model for decaying items is approximated by the proposed study during the sufficiently large life cycle.

Key words: Inventory, decaying items, demand, willingness, life cycle, costs.

I. Introduction and Literature review

In real situations it can be observed that many products like beverages, medicines, unpreserved drinks and foods etc. have a short life cycle. Due to the quick deterioration of products, there is a decrement in the willingness of customers to purchase such kind of products and so a big issue is there to manage all related management costs. This makes a situation of elaborating an inventory policy considering all related limitations that come from this situation. For case, it must be discussed here that the product quality over time, or the effect of deterioration which results the interest of customers to purchase the products. The deterioration of products results waste material which has to be getting rid of and it requires some cost which has an important role in market business or production industry. Thus, concept of deterioration of products is always required in the study of inventory control theory.

During the last few decades numerous researchers have elaborated inventory policies (models) for decaying products such as food items, blood banks, volatile liquids, medicines, volatile liquids, dairy products, electronic equipments etc..In [1] Ghare and Schrader were the first promoters for elaborating an inventory model with exponential decay. After it Covert and Philip [2] extended study [1] by taking constant rate of deterioration. In [3] an inventory model is considered for decaying products with time related demand without shortages. In [4] the study [3] is extended by allowing the shortages. Later, in [5] the demand pattern is generalized to any log-concave function. Further in studies [6] and [7] the demand function is generalized to include any continuous function (non-negative) which changes with time. Lately, in [8] an important survey is presented on the latest patterns in the study of decaying items. The conventional EOQ scheme [9] recommends that the rate of demand is always constant over the time which is responsible for inventory depletion. In [10] a detailed discussion is presented on depletion in inventory by taking an exponential decrement of demand. In [11] the decaying products are classified in two distinct parts, one with fixed decrement and second with finite life services. Authors such as in [12] and [13] studied an updated review on decaying products. In study [14] an EOQ policy is elaborated by taking cycle time related demand with holding and ordering costs. In articles [15], [16], [17]and [18] the EOQ models are developed with inventory reduction. Such models are identified by the direct modeling of inventory as a derivational relation and including only the holding and ordering costs. In study [19] a safety stock placement is presented. In study [20] a pricing factor is discussed in two separate studies and a policy developed for a system of decisions to optimize it. Instead of product deterioration level in many cases, customers want to buy any other product and the interest to buy will depend on the quality of product only. Besides to decay, price has a big impact on demand of products. Practically, a decrement in price of a product results to significant demand and sales volume. Thus, the price scheme is a basic factor that customers and sellers use to optimize gain and therefore studies with price-related demands have a great importance in the literature of inventory theory. In article [21] a deteriorating inventory policy is elaborated with price-related demand. In study [22] a pricing and replenishment scheme is studied for decaying items with a price related rate of demand that decreased with time. In article [23] the dynamic pricing and problem of lot size for decaying items with partial backlogging are presented.

Some other articles like [24] and [25] represent that there are various ways to control inventories in warehouses related to the business firms. In article [26] a single-item policy is elaborated with stock-level related demand. The researchers also present a price depletion model later the decay of the items which enhance more number of the customers. In article [27], the researchers studied a more realistic pricing model in a supply chain system. In study [28] researchers presented a perishable model that manages to a Weibull distribution. It is more realistic and practical work that contains costs due to the shortage and deterioration of the items. In [29] a pricing model is proposed for products with short life and variable demand. In study [30] a pricing model is proposed for products with shot life in a closed-loop system of supply with arbitrary orders.

In this study the charge of dumping of deteriorated or unsold items is an important factor. In some studies, the researchers presented a pricing model related to the evaluation of dynamic quality and check the effects of timings of occurrence and concession. This work presents two different ideas for an EOQ schemes: first one, the units or amounts that persist later every cycle bring out a disposing cost and ,other one, the request for a spoilable accords to an event, that is distinct to the sales process. The demand is always related to the interest of a client to get an item, whereas the sale correlates to the real effectual buying willingness of the client. Such situations occur only for decaying products; it is known that during the lifetime cycle of an item the willingness of buying will be decreased and once the customer observed it, there is a lack in interest to buy.

In this study, it is assumed that the readiness to buy a product reduces linearly across the lifetime of a decaying item. The demand is assumed as a constant.. The study is modeled for the cost of disposal for deteriorated items. Thus an EOQ model for decaying items is presented where the buying capacity from the customer side declines slowly during the life phase of product. Further the simulation results are there which present a high level correctness in forecasted results by the preferred mathematical equations.

The next part of this work is arranged as follows: The required assumptions and notations are presented in Section-**2**.The description of model and its solution method given in Section-**3**. The numerical example and its solution presented in in Section-4. The managerial implications of the study are presented in section-5. In the last, the conclusion and future scope of the study are presented in Section 6.

II. Used Assumptions and Notations

Assumptions

Following are the required and used assumptions throughout the whole study.

- The readiness to buy from the customer side declines linearly to the deterioration of an item and becoming zero at the end.
- Shortages of products are not permitted.
- Rate of demand is constant.
- Deteriorated or unsold products are disposed of at the end of their expiry.
- Instantaneous replacement.
- However this study is modeled to show the nature for a single product, but the sale and demand of an item is unconstrained to that of any other item so that the presented equations hold.

Notations

Following are the required and used assumptions throughout the whole study.

d : Per unit time demand.
Q : Quantity of order.
A : Quantity of demand (per year).
n :Annual number of orders.
Z : Life cycle of product.
T : Time between two consecutive buying orders.
K(j) : Damage or deterioration at the end of cycle j.
$I_1(t, j)$: Level of inventory for cycle j at time t.
$I_2(j)$: Level of average inventory for cycle j.
X(t, j) : Selling for cycle j at time t .
Y(t, j) : Collected selling for cycle t at time t.
P(t, i) : Probability of selling of demand for cycle j at time t.
H : Per unit holding cost (annual).
O : Cost of ordering.
D : Cost of disposing the product (per unit).

III. Model Description and Solution method

With the help of above notations and assumptions it is obvious that for the phase $j \leq n, t \leq T$,the probability of selling of one unit (at time t of phase j) is:

$$P(t,j) = 1-t/Z \tag{1}$$

If d is the willingness to buy an item at time t of phase j then to determine the counting of sold units of products at that time

Let $x_1, x_2, x_3, \ldots\ldots, x_d$ be the random variables which denote the sales for all the d units. If these sales are independent then selling at time t for the cycle j is given by

$$X(t, j) = x_1 + x_2 + x_3, \ldots\ldots, + x_d$$

The Expected value of X(t, j) is given by

$$E(X(t, j)) = E(x_1 + x_2 + x_3, \ldots\ldots, + x_d)$$
$$= E(x_1)+E(x_2)+E(x_3), \ldots\ldots, +E(x_d)$$
$$= (1-t/Z) + (1-t/Z)+ (1-t/Z)+\ldots\ldots+ (1-t/Z) = d(1-t/Z) \text{ i.e}$$
$$E(X(t, j)) = d(1-t/Z) \qquad .(2)$$

This expected value of X(t, j) is required to find the total cost equation detailed later .

To find the per cycle expected average inventory

For a cycle j at time t , the sales are given by X(t, j). If the process is continuous over time then the collective selling Y(t, j) cannot exceed the integral of X(t, j):

$$Y(t, j) = \int_0^t X(t, j)dt \qquad (3)$$

In real situations, the level of inventory is the gap between the sold and ordered units:

$$I_1(t, j) = Q - Y(t, j) = Q - \int_0^t X(t, j)dt \qquad (4)$$

The level of average inventory for cycle j:

$$I_2(j) = \frac{\int_0^T I_1(t, j)dt}{T} = \frac{\int_0^T [Q - Y(t, j)]dt}{T}$$
$$= \frac{\int_0^T Qdt - \int_0^T Y(t, j)dt}{T} = \frac{\int_0^T Qdt - \int_0^T \left[\int_0^t X(t, j)dt\right]dt}{T} \qquad (5)$$

The expected value of $I_2(j)$ is given as

$$E(I_2(j)) = E\left(\frac{\int_0^T Qdt - \int_0^T \left[\int_0^t X(t, j)dt\right]dt}{T}\right) = \frac{E\left(\int_0^T Qdt\right) - E\left(\int_0^T \left[\int_0^t X(t, j)dt\right]\right)dt}{T}$$
$$= \frac{\int_0^T Qdt - \int_0^T \left[\int_0^t E(X(t, j))dt\right]dt}{T} = \frac{\int_0^T Qdt - \int_0^T \left[\int_0^t d(1 - t/Z)dt\right]dt}{T}$$
$$= \frac{\int_0^T Qdt - d\int_0^T (t - t^2/2Z)dt}{T} = \frac{QT - d\left(\frac{T^2}{2} - \frac{T^3}{6Z}\right)}{T} \qquad (6)$$
$$= Q - d\left(\frac{1}{2} - \frac{T}{6Z}\right) = Q - Q\left(\frac{1}{2} - \frac{Q}{6dZ}\right) = Q\left(\frac{1}{2} + \frac{Q}{6dZ}\right), \forall T < Z$$

If the order quantity, $Q \geq dZ$, the solution of equation (6) can be evaluated up to Z, provided that in the extra range of time T−Z the inventory will be zero.

$$E(I_2(j)) = \frac{QZ - d\left(\frac{Z^2}{2} - \frac{Z^3}{6Z}\right)}{T} = \frac{QZ}{T} - \frac{dZ^2}{3T} = dZ - \frac{d^2Z^2}{3Q}, \forall T \geq Z \qquad (7)$$

This result shows that every cycle has the nature that the average. If the time of cycle is smaller than the life of product then average inventory will be $Q\left(\frac{1}{2}+\frac{Q}{6dZ}\right)$. Therefore, there is a classic average inventory Q/2 plus a factor $Q^2/6dZ$. The second factor is less than Q/6 in magnitude and approaches to zero as Z grows.

To find per cycle expected number of damaged or spoiled units

The number of damaged units will be difference between ordered and sold quantities after the ending of cycle T i.e

$$K(j) = Q - Y(T,j) = Q - \int_0^T X(t,j)dt \tag{8}$$

The expected value of K(j) is given as

$$\begin{aligned} E(K(j)) &= E\left(Q - \int_0^T X(t,j)dt\right) = E(Q) - E\left(\int_0^T X(t,j)dt\right) \\ &= Q - \int_0^T E(X(t,j))dt = Q - \int_0^T d(1-t/Z)dt \end{aligned} \tag{9}$$

There are cases to solve the integral in equation (9)

Case-I: If Q <dZ then integral in (9) will be until T i.e

$$\begin{aligned} E(K(j)) &= Q - d(T - T^2/2Z) = Q - dT(1 - T/2Z) \\ &= Q - Q(1 - Q/2dZ) = Q^2/2dZ, \forall T < Z \end{aligned} \tag{10}$$

Case-II: If $Q \geq dZ$ then integral in (9) will be until Z i.e

$$E(K(j)) = Q - d(Z - Z^2/2Z) = Q - dZ/2, \forall T \geq Z \tag{11}$$

If the result in equation (10) is analyzed then it can be noticed that Q/2 is greater than $Q^2/2dZ$ (Q <dZ) and approaches to zero as the life of product rises .

To find the EOQ that depletes the total annual expected total cost

The total inventory cost involved in this model is given as the sum of the stock holding cost, total cost of ordering the materials, and the total cost of damage or spoilage per unit:

$$TIC = O\frac{A}{Q} + HI_2 + D\sum_{i=1}^{n} K(i) \tag{12}$$

The expected total inventory cost is given as

$$\begin{aligned} E(TIC) &= E(O\frac{A}{Q}) + E(HI_2) + E(D\sum_{i=1}^{n} K(i)) \\ &= O\frac{A}{Q} + HI_2 + D\sum_{i=1}^{n} E(K(i)) \end{aligned} \tag{13}$$

If Q <dZ then from equations (6), (10) and (13):

$$\begin{aligned} E(TIC) &= O\frac{A}{Q} + HQ\left(\frac{1}{2}+\frac{Q}{6dZ}\right) + D\sum_{i=1}^{n}\frac{Q^2}{2dZ} = O\frac{A}{Q} + HQ\left(\frac{1}{2}+\frac{Q}{6dZ}\right) + D\frac{Q^2}{2dZ}n \\ &= O\frac{A}{Q} + HQ\left(\frac{1}{2}+\frac{Q}{6dZ}\right) + D\frac{Q^2}{2dZ}\frac{A}{Q} = O\frac{A}{Q} + HQ\left(\frac{1}{2}+\frac{Q}{6dZ}\right) + D\frac{QA}{2dZ} \end{aligned} \tag{14}$$

Differentiating (14) with respect Q and putting equal to zero

$$\frac{dE(TIC)}{dQ} = -O\frac{A}{Q^2} + \frac{H}{2} + \frac{HQ}{3dZ} + \frac{DA}{2dZ} = 0$$

$$\Rightarrow 2HQ^3 + 3dZHQ^2 + 3DAQ^2 - 6dZOA = 0 \Rightarrow 2HQ^3 + 3(AD + dHZ)Q^2 - 6dZOA = 0$$

$$\Rightarrow Q^3 + \frac{3}{2}.\frac{(AD + dZH)}{H}Q^2 - \frac{3dZOA}{H} = 0 \tag{15}$$

If $Q \geq dZ$ then from equations (7), (11) and (13):

$$E(TIC) = O\frac{A}{Q} + H\left(dZ - \frac{2d^2Z^2}{3Q}\right) + D\sum_{i=1}^{n}\left(Q - \frac{dZ}{2}\right) = O\frac{A}{Q} + H\left(dZ - \frac{2d^2Z^2}{3Q}\right) + \frac{D(Q - dZ/2)A}{Q}$$

After solving

$$E(TIC) = HdZ + DA + \frac{(6AO - 4Hd^2Z^2 - 3DAdZ)}{6Q} \tag{16}$$

If $6AO - 4Hd^2Z^2 - 3DAdZ$ = W then equation (16) will be decreasing if W > 0. Since *Q* may have a biggest value/ year as *A*, it will be the best ordering value for the developed conditions. If W < 0 then equation (16) will be increasing and Q will have the minimum value dZ as an optimal value. Although equations (14) and (16) have the similar value in Q = dZ that proposes, the probability of a quantity Q <dZ may persist that the cost decreases in equation (14). Definitely it is unclear that decaying items get single order only per year. so the total optimal cost will be assumed from the solution of equation (15). It is convenient to determine Q and if the base equation is considered for T < Z ,the stock holding cost approaches to be one of the mentioned costs in the model(if the complete time of life cycle approaches to be very big). The cost related with disposed units approaches to be decreased when the life of item approaches to grow.

Solution Method

Consider equation (15) and put

$$g(Q) = Q^3 + \frac{3}{2}.\frac{(AD + dZH)}{H}Q^2 - \frac{3dZOA}{H}$$

$$\Rightarrow g'(Q) = 3Q^2 + \frac{3Q(AD + dZH)}{H}, \text{which is positive for every value of } Q \geq 0.$$

Thus g(Q) is an increasing function in the intervals $]-\infty, -(AD + dZH)/H]$ and $[0, \infty[$ and it is decreasing in $]-(AD + dZH)/H, 0[$ .Definitely , Q=0 is a comparative lowest , known that *g*(0) < 0. This statement assurances the occurrence of one real and positive root of *g*(*Q*).

From equation (14), it can be observed that total expected annual cost increases with the increment in D or H. Although this cost decreases with the increment in d and Z. Generally the results of total expected annual cost are careful to the parameter Z which is involved in two components of cost. The limit of cost function as Z tends to infinity, given as below:

$$\lim_{Z\to\infty} E(TIC) = \lim_{Z\to\infty}\left[O\frac{A}{Q} + HQ\left(\frac{1}{2} + \frac{Q}{6dZ}\right) + D\frac{QA}{2dZ}\right] = O\frac{A}{Q} + \frac{HQ}{2}$$

This result shows that the model tends to the normal EOQ for the products with large life cycle. To find the positive root of cubic equation g(Q) = 0, the Cardano approximation method is used. For the cubic equation $ax^3 + bx^2 + cx + d = 0$ this method is given as below:

$$x_1 = L_1 + L_2 - \frac{b}{3a},\ x_2 = -\frac{L_1 + L_2}{2} - \frac{b}{3a} + \frac{i\sqrt{3}}{2},\ x_3 = -\frac{L_1 + L_2}{2} - \frac{b}{3a} - \frac{i\sqrt{3}}{2}$$

$$L_1 = \sqrt[3]{R + \sqrt{S^3 + R^2}},\ L_2 = \sqrt[3]{R - \sqrt{S^3 + R^2}} \text{ and in turn}$$

$$R = \frac{3ac - b^2}{9a^2},\ S = \frac{9abc - 27a^2 d - 2b^3}{54a^3}$$

If Q_1, Q_2 and Q_3 be the roots of g(Q) = 0, then these roots can be obtained by using the above formula.

IV. Numerical Example and Solution

Consider the following values of the various parameters:
A=25,000 units, D=$520/unit, H=$110/unit, Z=30 days,O=$100.000/Order. By considering 360 functioning days in a year and applying the method of Cardano approximation the order quantity Q is obtained:

*Q** ≈ 812.36 *E(TIC*(*Q**)) ≈ $4'974.164.6.

To study the probabilistic nature of the interest to buy, a case of the study is simulated by software with the related parameters. The calculation is done with 20 random cases. In every case, the optimal expected cost is determined and the gap percentage is calculated with the help of optimal cost determined by the simulation. For every simulated instance, The optimal order quantity for each simulated instance is calculated theoretically as per the solution of equation (15) by the method of Cardano approximation.

The results of the study are shown in Table-1. The maximum gap is observed as 3.42%, and it is less than 1% usually. This shows that there is an accuracy of high level in the derived equations even under the random situations. The values of *Q** are estimated to the closest integer.

Table: 1

A	O	D	H	Z	Q*	Predicted optimal cost	Simulated optimal cost	Percentage gap
1,000,000	200,500	150	20	12	2782	$12.2304	$12.2182	.09
21,000	45,000	1200	450	25	302	$ 5.4412	$5.4642	.36
62,000	350,000	22,000	2550	65	522	$66.4202	$66.5638	.22
500,000	150,000	250	65	40	9510	$15.7846	$15.7583	.17
1250	5.200,000	100,00	35,00	100	180	$69.8632	$69.9896	.18
500	32,000	55,000	22,000	52	10	$ 3.5142	$ 3.5250	.30
2100	32,000	1000	550	17	72	$ 1.7195	$ 1.7305	.62
2600	250	5	2	25	121	$ 8.7284	$ 8.8152	.98
24,500	5500	45	14	70	1072	$ 230.1542	$234.4342	1.82
85,600	10,000	2100	360	50	342	$ 5.2726	$ 5.2961	.44
100	250	25	12	22	6	$ 5.8326	$ 5.6326	3.42
12,500	450	30	5	12	97	$ 103.0564	$ 102.4120	.62
500	100	5	1	30	45	$ 2.5732	$ 2.5611	.47
7600	150	3	3	4	88	$ 21.2346	$ 20.9710	.77
36,000	240	7	5	6	192	$ 82.8815	$ 83.3654	.59
9600	1000	100	10	48	165	$ 125.0456	$ 122.7650	1.82
300	2500	90	32	75	424	$ 22.8758	$ 22.7086	.73
66,000	140	3	1	12	54	$ 38.6456	$ 38.8786	.60

33,000	700	45	27	65	402	$ 105.1172	$ 105.3960	.264
25,000	11,000	250	12	100	782	$ 633.7036	$ 638.5234	.75

V. Managerial Implications

This study offers a more suitable and realistic structure for inventory modeling of decaying products, which creates domain-relevant results achieving the acceptability of research to the industry activities. In particular, the over time deterioration effect for such products may relent in scrap which has to be get rid of, thus a cost of disposal is required and bad effect on functioning is there. Normally, it is decided by the managers that what to do with the products that could decay. Hence, firms are counseled to recognize and determine the suitable levels of inventory for decaying, which may suggest their operations for supply chain to settle and thereby ignore partly consequences. Earlier study represents many ideas in which the inventory for decaying products is analyzed by managers. In addition, few of the studies propose the models with gradual deterioration of products. Although, here it is argued that the managers now have to focus to the interest of customers to buy, because of the deteriorated quality of the products. The executives should include the welfares coming from a more appropriate inventory model for decaying products, while they should be in position to do effort to good recognition the nature of a client whose interest to buy a product could deplete with the deteriorating quality of the products.

VI. Conclusion and Future Scope

In this study, an EOQ model is elaborated for decaying products with constant demand while the probability of buying from the side of customers depletes linearly over the life phase of product. The study`s aim is to find the quantity of ordering that optimizes the expected total annual cost. With the help of considered assumptions an equation of third order is constructed to determine the value of Q which is used to lessen the entire cost and this cost decreases with the increment in the life product. An unique solution is proved to this equation and this equation has a positive root which is determined by the method of Cardano approximation. In this study the resulting cost is very tactful to the product life phase.

In situations where (i) time of replenishment is small relatively, (ii) the units are to be disposed in the end of every cycle and (iii) the loss of the product quality is linear throughout its life phase, then this study could be applied well.

The results are validated through the simulation by taking the random behavior in the interest of customers to buy. The simulated cases presented a perfect exactness in the projection of the generated equations, specified that the forecasted cost is deflected upwards by 3.42 % as per the simulated optimal cost.

Further, the model can be elongated for future research by including new alternatives of this study, such as probabilistic demand, non-linear natures for the quality loss in decaying products, penalization due to unsatisfied demand and other things that which are helpful to the related studies.

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A Two-State Retrial Queueing Model with Feedback and Balking

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Abstract

Present paper discusses a two-state retrial queueing model with feedback and balking. If a customer on arrival finds the server free it is served immediately. Else either it joins the retrial orbit as a secondary customer or balks from the system due to impatience. Primary and secondary arrivals both follow Poisson process. If the customer feels unsatisfied after service, it may join the orbit as a feedback customer. Service times follow Exponential distribution. The transient state probabilities for exact number of arrivals and departures when the server is busy or idle are obtained by solving difference-differential equations. Numerical solution is obtained and presented graphically.

Keywords: Arrivals, Departures, Queueing, Retrial, Feedback, Balking.

I. Introduction

Apart from classical queueing systems there exists a new class of queueing systems that is referred to as retrial queueing systems. In recent years, a lot of work has been done in this direction. Here if a customer on arrival finds the server free, it is served immediately. Else it joins the orbit (virtual queue) and retries for service from the orbit after a random amount of time (as shown in Figure 1). Some of the real life phenomena where these systems are successfully used are telecommunication systems, computer network systems, telephone switching systems. For detailed overview and main results [1], [2], [3] and [4] could be referred.

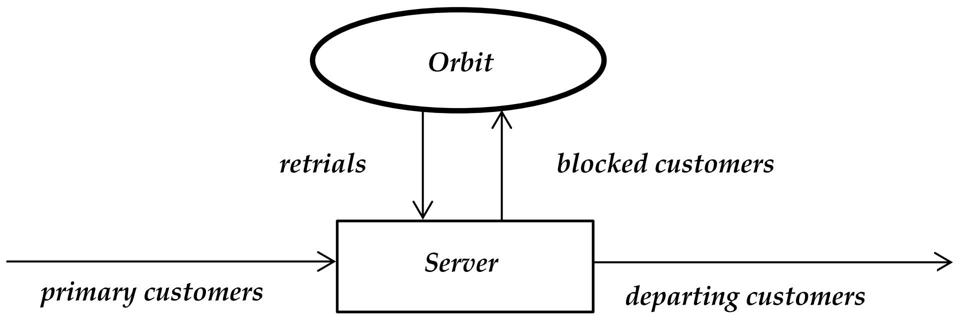

Figure 1: *Basic Structure of a Retrial Queueing System*

If on encountering a busy server, a customer leaves the system forever instead of joining the orbit (due to impatience), it is known as balking. Impatience can be commonly observed in many queueing systems dealing with which could lead to profits. [5] analyzed `Retrial queueing system with balking, optional service and vacation' where the steady state distributions of server state and number of jobs in the orbit are obtained.

If a customer feels unsatisfied after service, it may join the orbit as a feedback customer in order to obtain a satisfied service. This feature of feedback has also been widely discussed in retrial queueing theory. `A single server feedback retrial queue with collisions' was analyzed by [6]. [7] worked on `Modified vacation policy for M/G/1 retrial queue with balking and feedback' where some important measures were obtained. [8] published `Performance evaluation of two Markovian retrial queueing model with balking and feedback' in which the joint distribution of server state and retrial queue was derived.

[9] worked on `Some new results for the M/M/1 queue' where solution is obtained for the probability that exactly `i' number of arrivals, `$j$' number of services occur over a time interval $t$. In standard queueing models, total number of units in the system is considered whereas in this approach the exact number of arrivals and departures are considered. `A Single Server Retrial Queue with Impatient Customers' was studied by [10] considering the number of arrivals and departures from the orbit. [11] worked on `A two-state multiserver retrial queueing model with balking'. [12] analyzed `A Two-State Retrial Queueing Model with Feedback having Two Identical Parallel Servers' where the transient state probabilities were obtained.

The novelty of the work in the present paper is that here the solution of two-state model considering balking on the basis of immediate need and providing feedback facility to unsatisfied customers is obtained.

The present paper is categorized into various sections as under:

Section II gives the model description along with the difference-differential equations governing the system. The transient state probabilities are evaluated in section III. In section IV various performance measures are obtained. Numerical and graphical solutions are illustrated in section V. In section VI, the busy period probabilities are presented numerically and graphically. Finally, the paper is concluded in section VII which is followed by the references at the end.

II. Model Description

We consider a two-state retrial queueing model with feedback and balking. The fresh customers follow a Poisson process. On encounter with a busy server, the customer may join the orbit in order to retry for service else it balks from the system due to impatience. Service times follow Exponential distribution. The secondary customers repeatedly request for service from the orbit following a Poisson process. Also, an unsatisfied customer may join the orbit as a feedback customer in order to receive a satisfied service.

- Primary arrivals follow Poisson process with parameter λ.
- On encountering a busy server, arriving customer either joins the retrial orbit with probability β or leaves the system without joining i.e., balk from the system with parameter 1-β.
- Secondary arrivals follow Poisson process with parameter θ.
- Service times follow Exponential distribution with parameter μ.
- After receiving service, the customer joins the orbit with probability γ (when unsatisfied) and departs from the system with probability 1-γ.

The input flow of primary calls, intervals between repetitions, service times are statistically independent.

Laplace Transformation of $\bar{f}(s)$ of $f(t)$ is given by:

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt; \quad Re(s) > 0$$

The Laplace inverse of

$\frac{Q(p)}{P(p)} = \sum_{k=1}^{n} \sum_{l=1}^{m_k} \frac{t^{m_k - l} e^{a_k t}}{(m_k - l)!(l-1)!} \times \frac{d^{l-1}}{dp^{l-1}} \left(\frac{Q(p)}{P(p)}\right) (p - a_k)^{m_k}$ ∀ p= a_k, $a_i \neq a_k$ for i ≠ k

where, P(p)= $(p - a_1)^{m_1} (p - a_2)^{m_2} \ldots\ldots\ldots (p - a_n)^{m_n}$

Q(p) is a polynomial of degree < m_1+m_2+m_3 +…………m_n-1.

The Laplace inverse of $\bar{N}^{a,b,c}_{n_1,n_2,n_3}(s) = \frac{1}{(s+a)^{n_1}(s+b)^{n_2}(s+c)^{n_3}}$ is

$$N^{a,b,c}_{n_1,n_2,n_3}(t) = \sum_{l=1}^{n_3} \sum_{m=1}^{l} \frac{e^{-at}\, t^{n_3-l} (-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{g_1=0}^{l-m-1}(n_1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_2+g_2)\right)}{(n_3-l)!(m-1)!\,(b-a)^{n_2+m-1}(c-a)^{n_1+l-m}}$$
$$+ \sum_{l=1}^{n_2} \sum_{m=1}^{l} \frac{e^{-bt}\, t^{n_2-l} (-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{g_1=0}^{l-m-1}(n_1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_3+g_2)\right)}{(n_2-l)!(m-1)!\,(a-b)^{n_3+m-1}(c-b)^{n_1+l-m}}$$
$$+ \sum_{l=1}^{n_1} \sum_{m=1}^{l} \frac{e^{-ct}\, t^{n_1-l} (-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{g_1=0}^{l-m-1}(n_2+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_3+g_2)\right)}{(n_1-l)!(m-1)!\,(a-c)^{n_3+m-1}(b-c)^{n_2+l-m}}$$

If L^{-1}\{f(s)\} = F(t) and L^{-1}\{g(s)\} = G(t), then

L^{-1}\{f(s) g(s)\} = $\int_0^t F(u)G(t-u)du$ = F * G,

F * G is called the convolution of F and G.

Two-Dimensional State Model

Definitions:

$P_{i,j,0}(t)$=Probability that there are exactly *i* number of arrivals, *j* number of departures from the system by time *t* and server is idle.

$P_{i,j,1}(t)$=Probability that there are exactly *i* number of arrivals, *j* number of departures from the system by time *t* and server is busy.

$P_{i,j}(t)$ =Probability that there are exactly *i* number of arrivals, *j* number of departures from the system by time *t*.

$$P_{i,j}(t) = P_{i,j,0}(t) + P_{i,j,1}(t)\ \forall i,j;\ \ i \geq j$$

$$P_{i,j,0}(t) = 0; i < j \quad P_{i,j,1}(t) = 0; i \leq j$$

Initially

$$P_{0,0,0}(0) = 1;\ P_{i,j,0}(0) = 0\ i \geq j;\ i,j \neq 0$$
$$P_{i,j,1}(0) = 0;\ \forall\, i,j$$

The Difference-Differential Equations Governing the System are:

$$\frac{d}{dt}P_{i,j,0}(t) = -(\lambda + (i-j)\theta)P_{i,j,0}(t) + \mu(1-\gamma)P_{i,j-1,1}(t) + \mu\gamma P_{i,j,1}(t); \quad i \geq j \geq 0 \qquad (1)$$

$$\frac{d}{dt}P_{1,0,1}(t) = -(\lambda\beta + \mu)P_{1,0,1}(t) + \lambda P_{0,0,0}(t) + \theta P_{1,0,0}(t); \qquad (2)$$

$$\frac{d}{dt}P_{i,j,1}(t) = -(\lambda\beta + \mu)P_{i,j,1}(t) + \lambda P_{i-1,j,0}(t) + \lambda\beta\left(1 - \delta_{i-1,j}\right)P_{i-1,j,1}(t) + (i-j)\theta P_{i,j,0}(t); \quad i > 1, i > j \geq 0 \qquad (3)$$

where

$$\delta_{i-1,j} = \begin{cases} 1; & i-1 = j \\ 0; & otherwise \end{cases}$$

Using Laplace Transform $\bar{f}(s)$ of $f(t)$ given by:

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt; \qquad Re(s) > 0$$

and using initial condition in equations (1) to (3), we have:

$$(s + \lambda)\bar{P}_{0,0,0}(s) = \bar{P}(0)$$
$$(s + \lambda + (i-j)\theta)\bar{P}_{i,j,0}(s) = \mu(1-\gamma)\bar{P}_{i,j-1,1}(s) + \mu\gamma\bar{P}_{i,j,1}(s); \qquad i \geq j \geq 0 \qquad (4)$$

$$(s+\lambda\beta+\mu)\bar{P}_{1,0,1}(s)=\lambda\bar{P}_{0,0,0}(s)+\theta\bar{P}_{1,0,0}(s) \quad (5)$$

$$(s+\lambda\beta+\mu)\bar{P}_{i,j,1}(s)=\lambda\bar{P}_{i-1,j,0}(s)+\lambda\beta\bar{P}_{i-1,j,1}(s)+(i-j)\theta\bar{P}_{i,j,0}(s);\ \ i>1, i>j\geq 0 \quad (6)$$

III. Solution of the Problem

Solving equations (4) to (6) recursively, we have

$$\bar{P}_{0,0,0}(s)=\frac{1}{s+\lambda} \quad (7)$$

$$\bar{P}_{1,0,1}(s)=\frac{\lambda}{s+\lambda\beta+\mu}\left(\frac{1}{s+\lambda}\right)+\frac{\theta}{s+\lambda\beta+\mu}\bar{P}_{1,0,0}(s) \quad (8)$$

$$\bar{P}_{1,1,0}(s)=\frac{\mu(1-\gamma)}{s+\lambda}\left[\frac{\lambda}{s+\lambda\beta+\mu}\bar{P}_{0,0,0}(s)+\frac{\theta}{s+\lambda\beta+\mu}\bar{P}_{1,0,0}(s)\right] \quad (9)$$

$$\bar{P}_{i,0,0}(s)=\frac{\mu\gamma}{s+\lambda+i\theta}\bar{P}_{i,0,1}(s); \qquad i\geq 1 \quad (10)$$

$$\bar{P}_{i,1,0}(s)=\frac{\mu(1-\gamma)}{s+\lambda+(i-1)\theta}\bar{P}_{i,0,1}(s)+\frac{\mu\gamma}{s+\lambda+(i-1)\theta}\bar{P}_{i,1,1}(s); \qquad i\geq 2 \quad (11)$$

$$\bar{P}_{i,i-1,0}(s)=\frac{\mu(1-\gamma)}{s+\lambda+\theta}\bar{P}_{i,i-2,1}(s)+\frac{\mu\gamma}{s+\lambda+\theta}\bar{P}_{i,i-1,1}(s); \qquad i\geq 3 \quad (12)$$

$$\bar{P}_{i,0,1}(s)=\frac{\lambda}{s+\lambda\beta+\mu}\bar{P}_{i-1,0,0}(s)+\frac{\lambda\beta}{s+\lambda\beta+\mu}\bar{P}_{i-1,0,1}(s)+\frac{i\theta}{s+\lambda\beta+\mu}\bar{P}_{i,0,0}(s); i\geq 2 \quad (13)$$

$$\bar{P}_{i,i-1,1}(s)=\frac{\lambda}{s+\lambda\beta+\mu}\bar{P}_{i-1,i-1,0}(s)+\frac{\theta}{s+\lambda\beta+\mu}\bar{P}_{i,i-1,0}(s); \qquad i\geq 2 \quad (14)$$

$$\bar{P}_{i,i,0}(s)=\frac{\mu(1-\gamma)}{s+\lambda}\left[\frac{\lambda}{s+\lambda\beta+\mu}\bar{P}_{i-1,i-1,0}(s)+\frac{\theta}{s+\lambda\beta+\mu}\bar{P}_{i,i-1,0}(s)\right]; \qquad i\geq 2 \quad (15)$$

$$\bar{P}_{i,1,1}(s)=\frac{\lambda}{s+\lambda\beta+\mu}\bar{P}_{i-1,1,0}(s)+\frac{\lambda\beta}{s+\lambda\beta+\mu}\bar{P}_{i-1,1,1}(s)+\frac{(i-1)\theta}{s+\lambda\beta+\mu}\bar{P}_{i,1,0}(s);\ i\geq 3 \quad (16)$$

$$\bar{P}_{i,j,1}(s)=\sum_{k=1}^{i-j}\left(\frac{1}{s+\lambda\beta+\mu}\right)^{i-j-k}\lambda^{\psi_k'}\,(\lambda\beta)^{(i-j-k-1)\psi_k'}\,\eta_k'(s)\,\bar{P}_{j+k,j,0}(s)$$
$$+\left(\frac{\lambda\beta}{s+\lambda\beta+\mu}\right)^{i-j-1}\bar{P}_{j+1,j,1}(s); \qquad i\geq j+2, j\geq 1 \quad (17)$$

where

$$\eta_k'(s)=\begin{cases}1; & k=1\\ 1+\dfrac{k\theta\beta}{s+\lambda\beta+\mu}; & k=2\ to\ i-j-1\\ \dfrac{k\theta}{s+\lambda\beta+\mu}; & k=i-j\end{cases}$$

$$\psi_k'=\begin{cases}1; & k=1\ to\ i-j\\ 0; & k=i-j+1\end{cases}$$

$$\bar{P}_{i,j,0}(s)=\frac{\mu(1-\gamma)}{s+\lambda+(i-j)\theta}\left\{\left[\sum_{k=1}^{i-j+1}\left(\frac{1}{s+\lambda\beta+\mu}\right)^{i-j-k+1}\lambda^{\psi'_k}(\lambda\beta)^{(i-j-k)\psi'_k}\eta'_k(s)\,\bar{P}_{j+k-1,j-1,0}(s)\right]+\left(\frac{\lambda\beta}{s+\lambda\beta+\mu}\right)^{i-j}\bar{P}_{j,j-1,1}(s)\right\}+\frac{\mu\gamma}{s+\lambda+(i-j)\theta}\left[\sum_{k=1}^{i-j+1}\left(\frac{1}{s+\lambda\beta+\mu}\right)^{i-j-k+1}\lambda^{\psi'_k}(\lambda\beta)^{(i-j-k)\psi'_k}\phi'_k(s)\,\bar{P}_{j+k-1,j,0}(s)\right];\quad i>j>1 \quad (18)$$

where

$$\eta'_k(s)=\begin{cases}1; & k=1\\ 1+\dfrac{k\theta\beta}{s+\lambda\beta+\mu}; & k=2\ to\ i-j\\ \dfrac{k\theta}{s+\lambda\beta+\mu}; & k=i-j+1\end{cases}$$

$$\phi'_k(s)=\begin{cases}1; & k=1\\ 1+\dfrac{(k-1)\theta\beta}{s+\lambda\beta+\mu}; & k=2\ to\ i-j\\ \dfrac{(k-1)\theta}{s+\lambda\beta+\mu}; & k=i-j+1\end{cases}$$

$$\psi'_k=\begin{cases}1; & k=1\ to\ i-j\\ 0; & k=i-j+1\end{cases}$$

Taking Inverse Laplace of equations (7) to (18), we get the time dependent probabilities as:

$$P_{0,0,0}(t)=e^{-\lambda t} \quad (19)$$

$$P_{1,0,1}(t)=\lambda e^{-\lambda t}\left\{\frac{1}{\frac{\mu}{\beta}}-\frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\frac{\mu}{\beta}}\right\}+\theta e^{-(\lambda\beta+\mu)t}*P_{1,0,0}(t) \quad (20)$$

$$P_{1,1,0}(t)=\mu(1-\gamma)\lambda e^{-\lambda t}\left\{\frac{1}{\frac{\mu}{\beta}}-\frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\frac{\mu}{\beta}}\right\}*P_{0,0,0}(t)+\mu(1-\gamma)\theta e^{-\lambda t}\left\{\frac{1}{\frac{\mu}{\beta}}-\frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\frac{\mu}{\beta}}\right\}*P_{1,0,0}(t) \quad (21)$$

$$P_{i,0,0}(t)=\mu\gamma e^{-(\lambda+i\theta)t}*P_{i,0,1}(t);\quad i\geq 1 \quad (22)$$

$$P_{i,1,0}(t)=\mu(1-\gamma)e^{-(\lambda+(i-1)\theta)t}*P_{i,0,1}(t)+\mu\gamma e^{-(\lambda+(i-1)\theta)t}*P_{i,1,1}(t);\quad i\geq 2 \quad (23)$$

$$P_{i,i-1,0}(t)=\mu(1-\gamma)e^{-(\lambda+\theta)t}*P_{i,i-2,1}(t)+\mu\gamma e^{-(\lambda+\theta)t}*P_{i,i-1,1}(t);\quad i\geq 3 \quad (24)$$

$$P_{i,0,1}(t)=\lambda e^{-(\lambda\beta+\mu)t}*P_{i-1,0,0}(t)+\lambda\beta e^{-(\lambda\beta+\mu)t}*P_{i-1,0,1}(t)+i\theta e^{-(\lambda\beta+\mu)t}*P_{i,0,0}(t);\quad i\geq 2 \quad (25)$$

$$P_{i,i-1,1}(t)=\lambda e^{-(\lambda\beta+\mu)t}*P_{i-1,i-1,0}(t)+\theta e^{-(\lambda\beta+\mu)t}*P_{i,i-1,0}(t);\quad i\geq 2 \quad (26)$$

$$P_{i,i,0}(t)=\mu(1-\gamma)\lambda e^{-\lambda t}\left\{\frac{1}{\frac{\mu}{\beta}}-\frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\frac{\mu}{\beta}}\right\}*P_{i-1,i-1,0}(t)+\mu(1-\gamma)\theta e^{-\lambda t}\left\{\frac{1}{\frac{\mu}{\beta}}-\frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\frac{\mu}{\beta}}\right\}*P_{i.i-1,0}(t);\quad i\geq 2 \quad (27)$$

$$P_{i,1,1}(t) = \lambda e^{-(\lambda\beta+\mu)t} * P_{i-1,1,0}(t) + \lambda\beta e^{-(\lambda\beta+\mu)t} * P_{i-1,1,1}(t) + (i-1)\theta e^{-(\lambda\beta+\mu)t} * P_{i,1,0}(t); \qquad i \geq 3 \quad (28)$$

$$\begin{aligned} P_{i,j,1}(t) = {} & \lambda^{i-j-1}\beta^{i-j-2}\frac{t^{i-j-2}}{(i-j-2)!}e^{-\left(\lambda+\frac{\mu}{\beta}\right)t} * P_{j+1,j,0}(t) \\ & + \sum_{k=2}^{i-j-1}\lambda^{i-j-k}\beta^{i-j-k-1}\frac{t^{i-j-k-1}}{(i-j-k-1)!}e^{-\left(\lambda+\frac{\mu}{\beta}\right)t} * P_{j+k,j,0}(t) \\ & + \sum_{k=2}^{i-j-1}(k\theta)(\lambda\beta)^{i-j-k}\frac{t^{i-j-k}}{(i-j-k)!}e^{-\left(\lambda+\frac{\mu}{\beta}\right)t} * P_{j+k,j,0}(t) + (i-j)\theta e^{-\left(\lambda+\frac{\mu}{\beta}\right)t} * P_{i,j,0}(t) \\ & + (\lambda\beta)^{i-j-1}\frac{t^{i-j-2}}{(i-j-2)!}e^{-\left(\lambda+\frac{\mu}{\beta}\right)t} * P_{j+1,j,1}(t); \; i \geq j+2, j \geq 1 \quad (29) \end{aligned}$$

$$\begin{aligned} P_{i,j,0}(t) = {} & \mu(1-\gamma)\lambda^{i-j}\beta^{i-j-1}e^{-(\lambda+(i-j)\theta)t}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j}} - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-r}}\right\} * P_{j,j-1,0}(t) \\ & + \mu(1-\gamma)\lambda e^{-(\lambda+(i-j)\theta)t}\sum_{k=2}^{i-j}(\lambda\beta)^{i-j-k}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k+1}} - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-k}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k-r+1}}\right\} \\ & * P_{j+k-1,j-1,0}(t) \\ & + \mu(1-\gamma)e^{-(\lambda+(i-j)\theta)t}\sum_{k=2}^{i-j}(k\theta)(\lambda\beta)^{i-j-k+1}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k+2}} - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-k+1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k-r+2}}\right\} \\ & * P_{j+k-1,j-1,0}(t) + \mu(1-\gamma)(i-j+1)e^{-(\lambda+(i-j)\theta)t}\left\{\frac{1}{\frac{\mu}{\beta}} - \frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\frac{\mu}{\beta}}\right\} * P_{i,j-1,0}(t) \\ & + \mu(1-\gamma)(\lambda\beta)^{i-j}e^{-(\lambda+(i-j)\theta)t}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j}} - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-r}}\right\} * P_{j,j-1,1}(t) \\ & + \mu\gamma\lambda^{i-j}\beta^{i-j-1}e^{-(\lambda+(i-j)\theta)t}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j}} - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-r}}\right\} * P_{j,j,0}(t) \\ & + \mu\gamma\lambda e^{-(\lambda+(i-j)\theta)t}\sum_{k=2}^{i-j}(\lambda\beta)^{i-j-k}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k+1}} - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-k}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k-r+1}}\right\} \\ & * P_{j+k-1,j,0}(t) + \mu\gamma e^{-(\lambda+(i-j)\theta)t}\sum_{k=2}^{i-j}(k-1)\theta(\lambda\beta)^{i-j-k+1}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k+2}}\right. \\ & \left. - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-k+1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k-r+2}}\right\} * P_{j+k-1,j,0}(t) \\ & + \mu\gamma(i-j)\theta e^{-(\lambda+(i-j)\theta)t}\left\{\frac{1}{\frac{\mu}{\beta}} - \frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\frac{\mu}{\beta}}\right\} * P_{i,j,0}(t); \qquad i > j > 1 \quad (30) \end{aligned}$$

IV. Some Performance Measures

- The Laplace transform $\bar{P}_{i.}(s)$ is given by:

$$\bar{P}_{i.}(s) = \sum_{j=0}^{i} \bar{P}_{i,j}(s) = \frac{\lambda^i}{(s+\lambda)^{i+1}}; \quad i > 0$$

and its Laplace Inverse is:

$$P_{i.}(t) = \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

which verifies the basic assumption that primary arrivals follow Poisson process.

- The probability that exactly *j* customers depart from the system by time *t* is given by:

$$\bar{P}_{.j}(t) = \sum_{i=j}^{\infty} P_{i,j}(t)$$

- Summing equations (7)-(18) over *i* and *j* we get:

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\{\bar{P}_{i,j,0}(s) + \bar{P}_{i,j,1}(s)\} = \frac{1}{s}$$

and hence

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\{P_{i,j,0}(t) + P_{i,j,1}(t)\} = 1$$

which is a verification of our results.

- Define $Q_{n,m}(t)$ = Probability that there are exactly *n* customers in the orbit when *m (m*=0, 1*)* i.e., either the server is idle or busy at time *t*.

For idle server we represent it by probability $Q_{n,0}(t)$

$$Q_{n,0}(t) = \sum_{j=0}^{\infty} P_{j+n,j,0}(t)$$

The number of customers in the orbit, in this case are calculated with the following formula:
n = (number of arrivals − number of departures)
When the server is busy, it is represented by probability $Q_{n,1}(t)$

$$Q_{n,1}(t) = \sum_{j=0}^{\infty} P_{j+n+1,j,1}(t)$$

The number of customers in the orbit in this case is calculated by the following formula:
n = (number of arrivals − number of departures − 1)
Using above definitions in (1)-(3) and letting *γ=0*, the equations we get under statistical equilibrium are:

$$(\lambda + n\theta)Q_{n,0} = \mu Q_{n.1}; \qquad n \geq 0 \qquad (31)$$

$$(\lambda\beta + \mu)Q_{n,1} = \lambda Q_{n,0} + \lambda\beta Q_{n-1,1} + (n+1)\theta Q_{n+1,0}; \qquad n \geq 2 \qquad (32)$$

which coincides with the result (3.68) of [13].

V. Numerical Solution and Graphical Representation

Using MATLAB programming for the case *ρ=0.7*, *η=0.5*, *γ=0.5* and *1-β=0.4* the numerical solutions are generated. Some of which are given in Table 1 to Table 5. Observing the below tables for various time instants it is observed that the sum of probabilities approaches to 1.

Table 1: *At t=1*

$P_{0,0,0}$	$P_{1,0,0}$	$P_{1,1,0}$	$P_{4,4,0}$	$P_{7,1,0}$	$P_{1,0,1}$	$P_{2,0,1}$	$P_{2,1,1}$	$P_{6,5,1}$	Sum
0.4966	0.0623	0.0736	0.2735	0.0492	0.0165	0	0	0	0.9717

Table 2: *At t=5*

$P_{0,0,0}$	$P_{1,0,0}$	$P_{1,1,0}$	$P_{2,0,0}$	$P_{2,1,0}$	$P_{2,2,0}$	$P_{3,0,0}$	$P_{3,1,0}$	$P_{3,2,0}$	$P_{3,3,0}$
0.0302	0.0379	0.0792	0.0238	0.0699	0.0552	0.0101	0.0344	0.0409	0.0171

$P_{4,2,0}$	$P_{1,0,1}$	$P_{2,0,1}$	$P_{2,1,1}$	$P_{3,0,1}$	$P_{3,1,1}$	$P_{3,2,1}$	$P_{4,0,1}$	$P_{4,1,1}$	$P_{4,2,1}$
0.0172	0.0551	0.069	0.0812	0.0436	0.0795	0.039	0.0185	0.00415	0.0328

$P_{4,3,1}$	$P_{5,1,1}$	$P_{5,2,1}$	Sum
0.0092	0.0151	0.015	0.9154

Table 3: *At t=15*

$P_{2,2,0}$	$P_{3,3,0}$	$P_{4,2,0}$	$P_{4,3,0}$	$P_{4,4,0}$	$P_{5,3,0}$	$P_{5,4,0}$	$P_{5,5,0}$	$P_{6,4,0}$	$P_{6,5,0}$
0.0028	0.0076	0.007	0.0143	0.0122	0.0157	0.0206	0.0129	0.0128	0.0204

$P_{6,6,0}$	$P_{7,0,0}$	$P_{7,4,0}$	$P_{7,5,0}$	$P_{7,6,0}$	$P_{8,3,0}$	$P_{8,4,0}$	$P_{8,5,0}$	$P_{8,6,0}$	$P_{8,7,0}$
0.0199	0.0095	0.015	0.018	0.0137	0.0118	0.0244	0.0366	0.04	0.0302

$P_{8,8,0}$	$P_{4,2,1}$	$P_{4,3,1}$	$P_{5,2,1}$	$P_{5,3,1}$	$P_{5,4,1}$	$P_{6,2,1}$	$P_{6,3,1}$	$P_{6,4,1}$	$P_{6,5,1}$
0.0128	0.0142	0.0123	0.0196	0.0272	0.0164	0.0193	0.0333	0.0326	0.0147

$P_{7,2,1}$	$P_{7,3,1}$	$P_{7,4,1}$	$P_{7,5,1}$	$P_{7,6,1}$	$P_{8,2,1}$	$P_{8,3,1}$	$P_{8,4,1}$	$P_{8,5,1}$	$P_{8,6,1}$
0.0149	0.0293	0.0362	0.0268	0.0095	0.0187	0.0416	0.0619	0.0624	0.0404

$P_{8,7,1}$	Sum
0.0134	0.9029

Table 4: *At t=25*

$P_{2,1,0}$	$P_{3,2,0}$	$P_{5,5,0}$	$P_{7,6,0}$	$P_{8,4,0}$	$P_{8,5,0}$	$P_{8,6,0}$	$P_{8,7,0}$	$P_{8,8,0}$	$P_{6,4,1}$
0	0.0001	0.0015	0.0065	0.0142	0.0406	0.0939	0.1804	0.2744	0.0033

$P_{7,4,1}$	$P_{8,3,1}$	$P_{8,5,1}$	$P_{8,6,1}$	$P_{8,7,1}$	Sum
0.0059	0.0141	0.0744	0.104	0.0893	0.9026

Table 5: *At t=35*

$P_{6,0,0}$	$P_{8,4,0}$	$P_{8,6,0}$	$P_{8,7,0}$	$P_{8,8,0}$	$P_{4,2,1}$	$P_{5,4,1}$	$P_{6,4,1}$	$P_{8,4,1}$	$P_{8,5,1}$
0	0.0014	0.029	0.1263	0.7119	0	0	0.0001	0.0041	0.0133

$P_{8,6,1}$	$P_{8,7,1}$	Sum
0.0349	0.0689	0.9899

Various probabilities are graphically presented against time t in Figures 2 to 5. Here, traffic intensity from primary calls is $\rho = \frac{\lambda}{\mu}$ and traffic intensity from secondary calls is $\eta = \frac{\theta}{\mu}$.

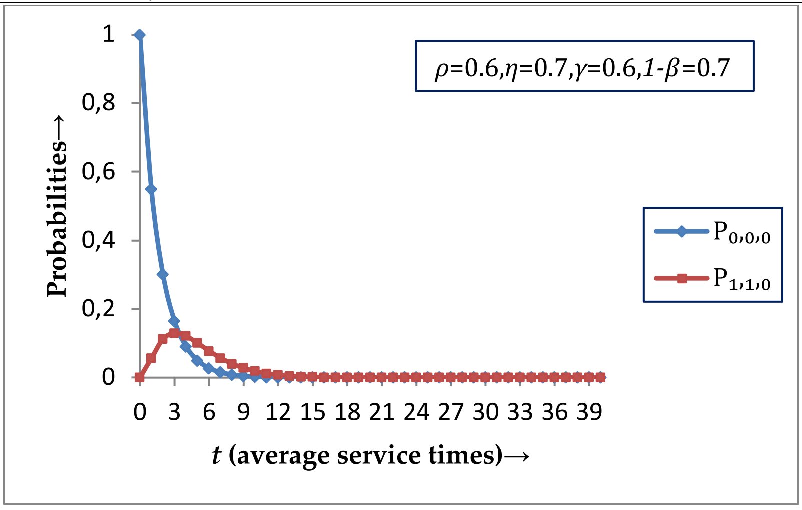

Figure 2: *Probabilities $P_{0,0,0}$ and $P_{1,1,0}$ against t (average service times)*

The probabilities $P_{0,0,0}$ and $P_{1,1,0}$ are compared in Figure 2 by plotting against time t for the case ρ=0.6, η=0.7, γ=0.6 and $1-\beta$=0.7. It can be seen from the plot that the probability $P_{0,0,0}$ with initial value 1 at t=0 decreases rapidly whereas probability $P_{1,1,0}$ initiates with value 0 at t=0 increases in the beginning and then decreases gradually with time.

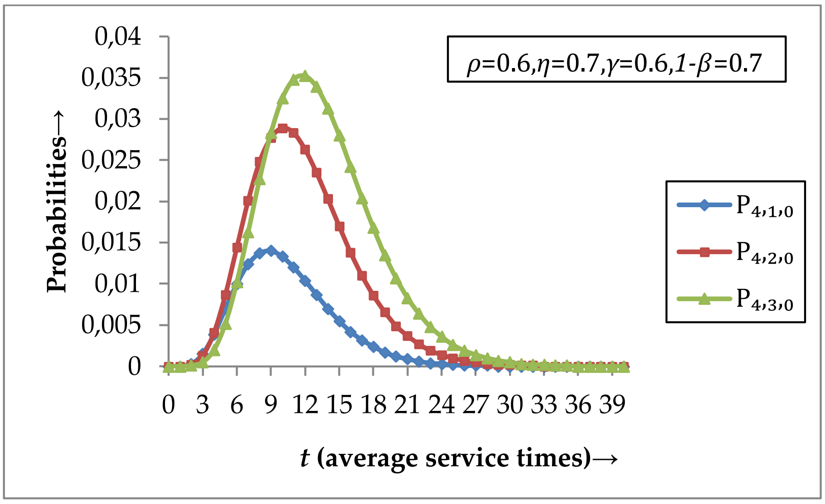

Figure 3: *Probabilities $P_{4,1,0}$, $P_{4,2,0}$ and $P_{4,3,0}$ against t (average service times)*

In Figure 3, the probabilities $P_{4,1,0}$, $P_{4,2,0}$ and $P_{4,3,0}$ are plotted against time t for the case where ρ=0.6, η=0.7, γ=0.6 and $1-\beta$=0.7. It can be observed from the plot that the probabilities increase initially and then decrease gradually. In general, the probabilities are higher for larger number of departures.

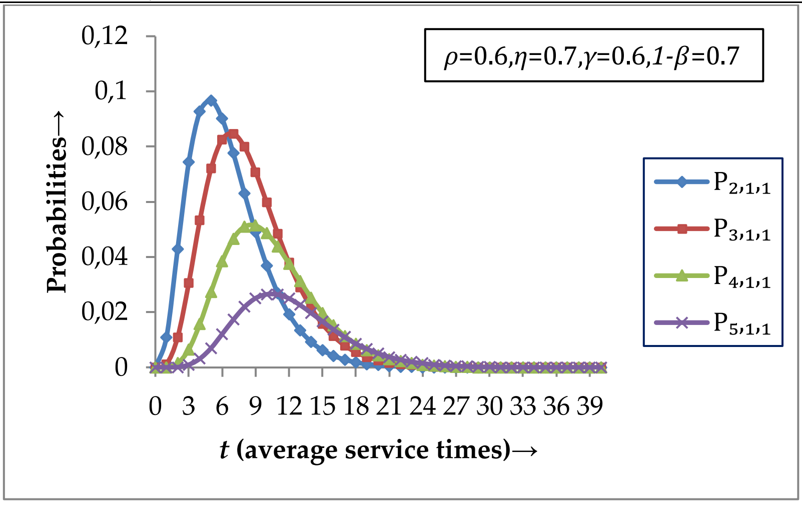

Figure 4: *Probabilities $P_{2,1,1}$, $P_{3,1,1}$, $P_{4,1,1}$ and $P_{5,1,1}$ against t (average service times)*

Figure 4 shows the comparison between the probabilities $P_{2,1,1}$, $P_{3,1,1}$, $P_{4,1,1}$ and $P_{5,1,1}$ when plotted against time t. It is interpreted from the graph that these probabilities increase initially from the value 0 at t=0. Highest achieved values of various probabilities are higher for lower i (number of arrivals). After reaching their respective peaks, probabilities start decreasing and the trend gets reversed i.e., now the probabilities take higher values for larger i (number of arrivals).

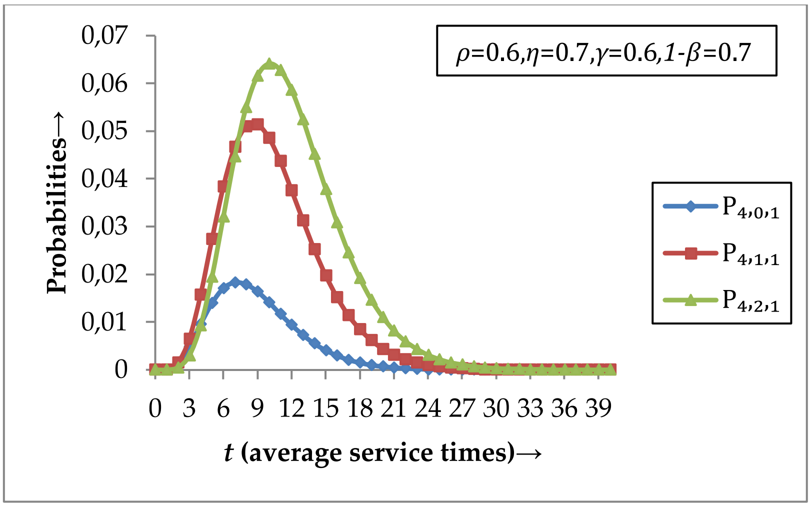

Figure 5: *Probabilities $P_{4,0,1}$, $P_{4,1,1}$ and $P_{4,2,1}$ against t (average service times)*

The probabilities $P_{4,0,1}$, $P_{4,1,1}$ and $P_{4,2,1}$ are plotted against time t in Figure 5. Beginning with value 0 at t=0, all the probabilities increase rapidly to their highest values and then decrease gradually. Also, it is observed that $P_{4,2,1}$ is higher than $P_{4,1,1}$ which is in turn greater than $P_{4,0,1}$ i.e., probabilities are higher for larger j (number of departures).

VI. Busy Period Probabilities

The probability that server is busy is given by

$$P(Server\ is\ busy) = \sum_{i>j\geq 0} P_{i,j,1}(t) \tag{33}$$

The probability that system is busy is given by

$$P(System\ is\ busy) = \sum_{i>j\geq 0} (P_{i,j,0}(t) + P_{i,j,1}(t)) \qquad (34)$$

Numerical and Graphical Representation of Busy Period Probabilities:

Following the work of [14] and using MATLAB programming, the numerical results are obtained. Here the probabilities for system busy as well as for server busy are obtained which are presented in the Table 6 below for various values of ρ keeping values of η, γ and 1-β constant.

Table 6: *Probabilities of System busy and Server busy to study the effect of ρ*

t	Probability(System Busy)			Probability(Server Busy)		
	ρ=0.3	ρ=0.6	ρ=0.9	ρ=0.3	ρ=0.6	ρ=0.9
0	0	0	0	0	0	0
1	0.2231	0.3937	0.5245	0.1772	0.3152	0.4232
2	0.3565	0.5774	0.7166	0.2508	0.4159	0.5281
3	0.4455	0.6798	0.8081	0.2946	0.4676	0.5764
4	0.5085	0.7438	0.8599	0.3255	0.5019	0.608
5	0.555	0.7872	0.8927	0.3491	0.5275	0.6317
6	0.5905	0.8184	0.915	0.3678	0.5478	0.6506
7	0.6183	0.8416	0.9308	0.383	0.5645	0.6661
8	0.6405	0.8596	0.9424	0.3957	0.5784	0.6786
9	0.6586	0.8738	0.9512	0.4063	0.5903	0.6886
10	0.6735	0.8853	0.958	0.4153	0.6003	0.6959

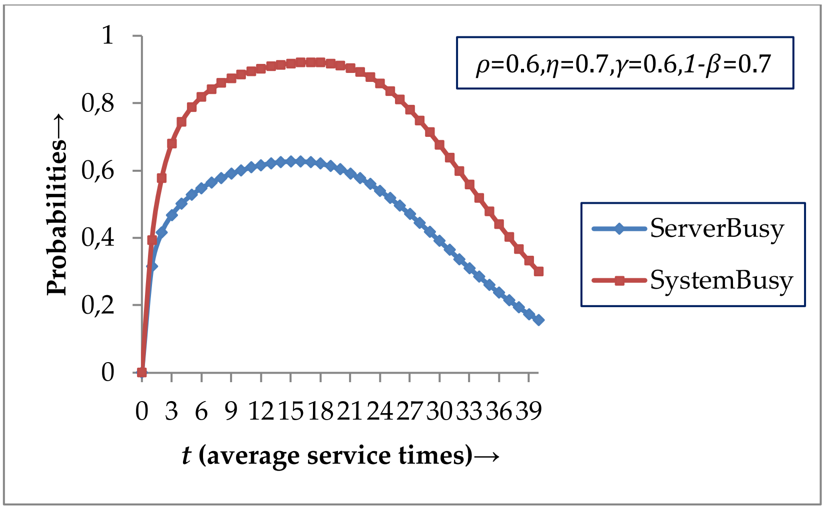

Figure 6: *Probabilities of System busy and Server busy against t (average service times)*

The probabilities for System busy and Server busy are compared in Figure 6 for the case ρ=0.6, η=0.7, γ=0.6 and 1-β=0.7. It is clearly visible that the probabilities for System busy remained higher than that of Server busy throughout, as expected. General trend shows that probabilities start increasing in the beginning, achieve some highest values and then start decreasing.

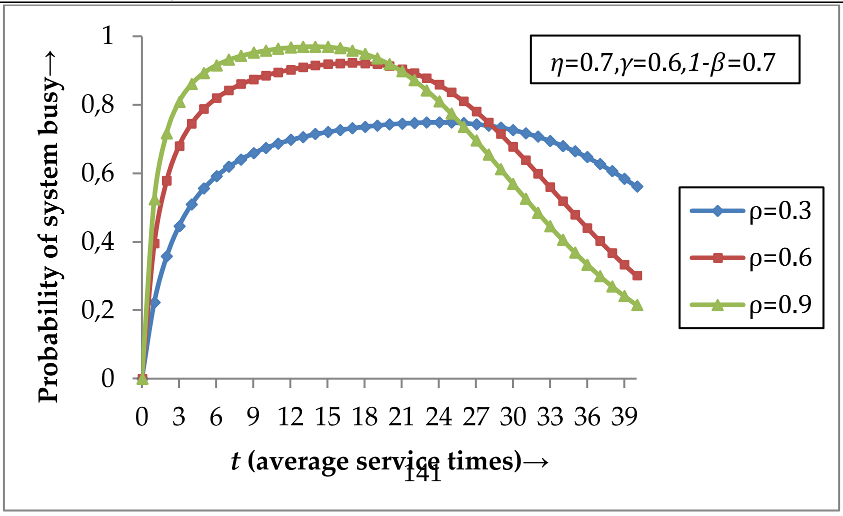

Figure 7: *Effect of ρ on System busy against t (average service times)*

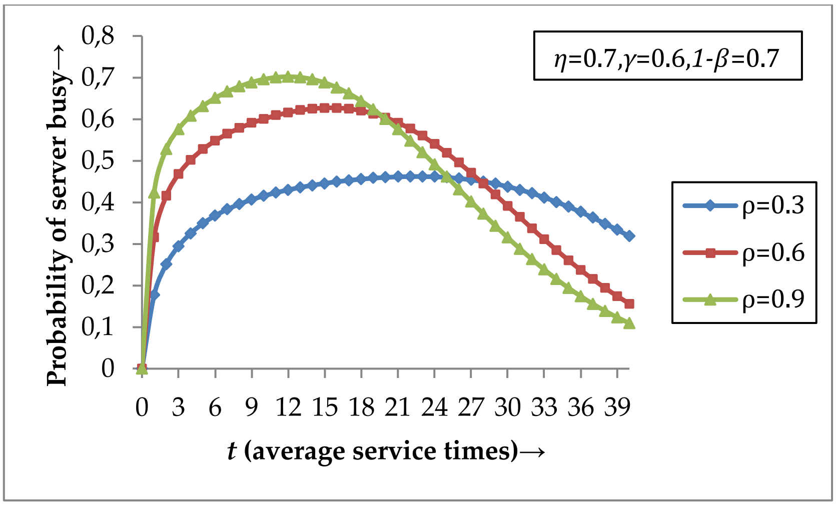

Figure 8: *Effect of ρ on Server busy against t (average service times)*

The effect of changing primary customers traffic intensity i.e., $\rho = \left(\frac{\lambda}{\mu}\right)$ on probability of system busy and probability of server busy is studied through Figure 7 and Figure 8 respectively. In both the graphs the trend followed is similar. The probabilities increases in the beginning and are higher for larger values of ϱ but the trend gets reversed for higher values of *t*.

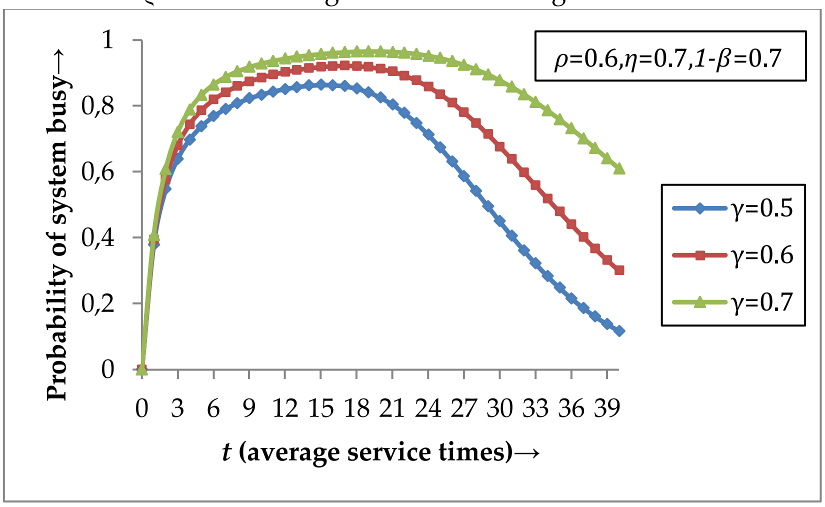

Figure 9: *Effect of γ on System busy against t (average service times)*

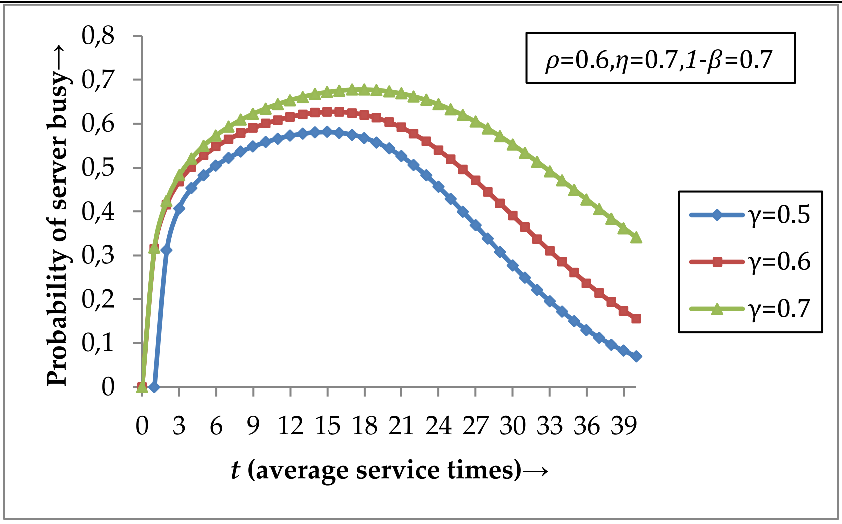

Figure 10: *Effect of γ on Server busy against t (average service times)*

The effect of change of γ (feedback factor) on probability of system busy and server busy is observed through Figures 9 and 10 respectively. It is interpreted from the plots that initially both the probabilities increase rapidly from the value 0 at t=0 and then decrease gradually for higher t. In both the cases, the probabilities are higher for larger value of γ.

VII. Conclusion

In this paper we analyzed a two-state retrial queueing system with feedback and balking. As we know balking is one aspect of impatient customers. Managing impatience leads to profit in business. The time dependent probabilities for exact number of arrivals and exact number of departures from the system are derived. Here due to dealing with two-state probabilities, results are more quantified and informative. Some performance measures are obtained in order to verify results. Numerical results are generated using MATLAB programming. Also, the graphical illustrations are provided in order to understand the effect of change of various parameters. The present model can serve as a base for the future research to model various practical situations applying the concept of balking and feedback where more than one homogeneous or heterogeneous servers would be required.

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Inventory Policy for Deteriorating Items with Two-Warehouse and Effect of Carbon Emission

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Abstract

In today's era, humans have shifted from the stage of fulfilling their needs to the stage of freedom where they indiscriminately use all the resources. This has increased the concern for protecting our nature and developing sustainable practices. So, an EOQ inventory model is formulated for decaying items with capacity constraint policy where demand is stock-dependent. Shortages are allowed with condition of partial backlogging. The impact of carbon emission is taken on holding cost and deterioration cost. Due to different conditions of storage, the deterioration rate is also different in different warehouses. The main focus of this study is to optimize total cost with carbon emission and total cycle time. An algorithm for the inventory model to find the best output is formulated here. Numerical example with sensitivity is also discussed to show the impact of carbon emission.

Keywords: Deterioration, Carbon-Emission, Two-Warehouse, EOQ Model, Shortages.

I. Introduction

Carbon emission is one among many greenhouse gas emissions that takes place after human activities or other processes. It refers to the release of carbon dioxide gas in the atmosphere from vehicles, industries, etc. Carbon dioxide gas traps the incoming solar radiations which results in global warming. When the heat does not escape, it leads to inflation in the earth's temperature. This increase in the temperature in not beneficial for all the species living on earth. It leads to change in climate of regions affecting agriculture, monsoon patterns, leads to melting of icebergs, glaciers etc. It also increases respiratory diseases in humans. It is very important to reduce carbon emissions. It can be reduced by taking measures to reduce the production of CO_2 by using filters, using public transportation or carpooling, adopting the ideas of composting for bio-degradable wastes and cycling or walking over small distances is a very healthy option to decrease carbon emission.

Due to environmental conditions, the quality of an item changes that is referred as deterioration phenomena. Deterioration occurs because of change in temperature, transportation and also the storage standard play an important role. Some of items deteriorate with time and lose their quality. Some items have specific lifetime because of some specific things which they are made of. After the completion of the time period, the item turns to waste and with nearing of time its quality decreases.

Warehousing is necessary for continuous, seasonal demand. The limited space and high rental costs in large market places such as malls, supermarkets etc., has made it difficult to have large showrooms there. The management decides to buy large number of items either when there is an impressive discount in cost for bulk order or when there are some issues in often acquirement and also if the item demand is inevitably high. The available space then falls small for the recently purchased items, to solve the problem of storage a rental warehouse is hired. The RW (Rented Warehouse) could be located in close proximity to the owned warehouse or at some distance with enough space for the items. The holding cost in RW increases than the one in OW (Owned Warehouse). To decrease the total holding cost demands are first fulfilled from RW. The demand rate in supermarkets is influenced by volume of stock-level. Over the past many years, a lot of interest has been shown to the situations in which the demand rate depends on the on-hand stock level. It is nearly not possible to stock high quantities in store because of increased holding costs and during high demands this issue leads to storage in the inventory systems. Thus, we assume inventory system for decaying goods with stock-based demand rate which has different rate of decaying in various warehouses, where shortages are allowed.

II. Literature Review

Many researchers have done work on two-warehouse inventory model. Hartley [1] firstly formulated the two-warehouse inventory structure. He considers the rented warehouse causing the higher total carrying cost rather than owned warehouse. Sharma [2] recommended two-warehouse system for decaying items. Mandal & Phujdar [3] offered model for two-warehouse with stock-based demand for decaying items. Pakkala and Achary [4] established model for two-warehouse with stock-based demand for decaying items under the condition of shortages. Shah [5] offered a survey of literature on structure of inventory. Zhou and Yang [6] preferred model for two-warehouse facilities with stock-based demand for decaying items. Alfares [7] initiated work on model for two-warehouse for decaying items with variable carrying cost and the rate of demand is stock-based. Singh et al. [8] suggested a system with two-warehouse under the condition of partially backlogged for decline goods. Yang et al. [9] explored a system for two-warehouse with the impact of inflation and shortages. The rate of demand is stock-based. Singh et al. [10] considered system for two-warehouse with the impact of trade credit and shortages. Here, the rate of demand is based on presently stock. Bhunia et al. [11] proposed structure for two-warehouse with shortages and the rate of demand is constant and known. Rastogi and Rathore [12] projected system for two-warehouse with the impact of preservation technology and shortages. Here, the rate of demand to be stock-based.

Sarkar et al. [13] developed policy related to inventory which gave an effect on environment with the condition of partial backlogging. An offer of multi-trade-credit period is given to the dealer to increase the business. Panda et al. [14] proposed policy related to inventory with the effect on trade credit under the condition of partial backlogging. An offer of multi-trade-credit period is given to the dealer to increase the trade. Demand is the combination of advertisement, cost and stock. Harit et al. [15] developed policy related to inventory with the effect on trade credit and inflation. Chauhan and Yadav [16] discussed the policy of inventory for decline goods with capacity constraint using genetic algorithm and stock -dependent demand. Yadav et al. [17] initiated model with stock-based and ramp type demand function for decline items, with reverse money and shows the impact of carbon emission. Gautam et al. [18] offered a sustainable production policy under the effect of volume agility and the technology of preservation with price-reliant demand. Sarkar et al. [19] explored the policy of inventory for deteriorating items with carbon emission. Mishra et al. [20] formulated model with controllable carbon emission for deteriorating items.

Sepehri [21] initiated model with for deteriorating Item with maximum lifetime and carbon emissions under trade credit policy. Al Arjani [22] offered a sustainable online-to-offline model for a supply chain management where lead time is controllable and function of demand is a variable.

The next part of this work is arranged as follows: Introduction with literature review is presented in Section 1, Section 2 includes assumptions & notations used in the paper, Section 3 deals with mathematical formulation of model, the process of solution with numerical example is presented in Section 4 and Section 5 contains sensitivity analysis and in the last, the conclusion and future scope of the study are presented in section 6.

III. Assumptions and Notations

3.1 Assumptions:

Following are the required and used assumptions throughout the whole study.

(i) The rate of demand D(t) is a function of immediate inventory level and taken as: $D(t) = \begin{Bmatrix} p + q(t), 0 \leq t \leq t_1 \\ p, t_1 \leq t \leq T \end{Bmatrix}$ where $p, q > 0$.Shortages are allowed and unfulfilled demand is partially backlogged. The fraction of backordered is $\frac{1}{1+\lambda(T-t)}$ where λ is a positive constant.

(ii) Time limit is infinite.

(iii) Size of OW is limited and RW has unlimited size.

(iv) Firstly, consumed the RW goods and then consumed the OW goods.

(v) Lead time is zero.

(vi) For the best result, we consider maximum decaying amount for times in OW, $bI_1 < D$.

(vii) Carbon emission is assumed for the best result of keeping the decaying goods and warehousing the goods.

3.2 Notations:

$Q_r(t)$: At time t level of positive stock in RW

$Q_o(t)$: At time t level of positive stock in OW

C_o: ordering cost

C_d: Cost of deterioration

I_1: Capacity of OW

I_2: Capacity of RW

C_{ho}: Holding cost in OW

C_{hr}: Holding cost in RW

C_{re}: Carbon emission cost for holding goods in RW

C_{oe}: Carbon emission cost for holding goods in OW

C_s: Shortage cost

C_l: Lost sale cost

a: deterioration cost in RW, $0 \leq a < 1$

b: deterioration cost in OW, $0 \leq b < 1$

C_{ae}: Carbon emission cost form deteriorating goods in RW

C_{be}: Carbon emission cost form deteriorating goods in OW

t_1: Time of zero stock keeping unit in RW

t_2 : Time of zero stock keeping unit in OW

TC: Total average cost

p: Initial rate of demand
q: Positive rate of demand

IV. Formulation of Mathematical Model

In formulation of mathematical model assume the intervals [0, t_1], [t_1, t_2] and [t_2, T]. The level of RW decline due to decay and demand in [0, t_1] and reaches to zero at $t=t_1$. After this demand of customer is fulfilled by the stock available in OW in [t_1, t_2]. After this shortage takes place that are partially backlogged. The level of OW decline due to decay and demand and reaches to zero at $t=t_2$ as shown in Figure1.

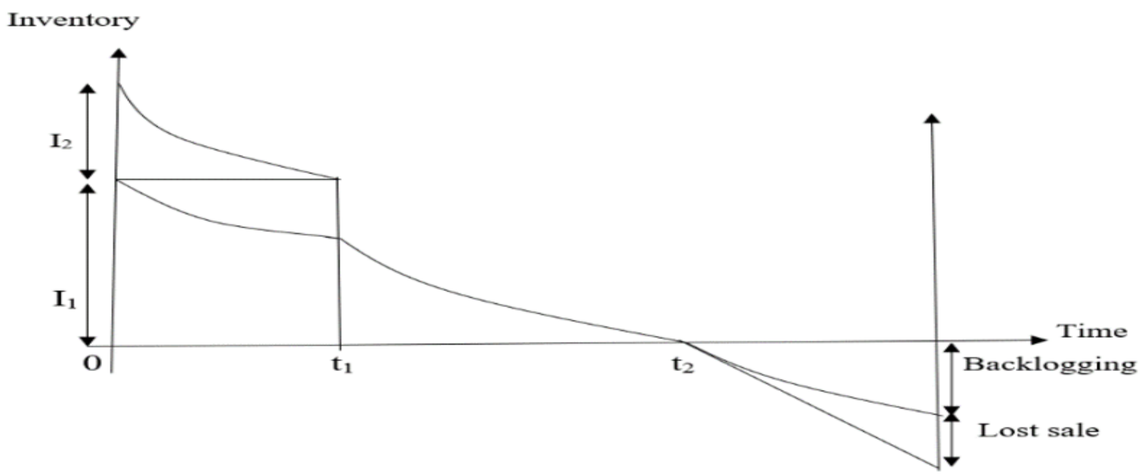

Figure 1: *Two-warehouse Inventory System*

The level of stock between $0 \le t \le T$ is represented by the following differential equations:

$$\frac{dQ_r(t)}{dt} = -(p + qQ_r(t)) - aQ_r(T), \ 0 \le t \le t_1 \ \text{ With } Q_r(t_1) = 0 \tag{1}$$

$$\frac{dQ_o(t)}{dt} = -bQ_o(t), \ 0 \le t \le t_1 \ \text{ With } Q_o(0) = I_1 \tag{2}$$

$$\frac{dQ_o(t)}{dt} = -(p + qQ_o(t)) - bQ_o(t), \ t_1 \le t \le t_2 \ \text{ With } Q_o(t_2) = 0 \tag{3}$$

Using equations (1), (2) and (3),

$$Q_r(t) = \frac{p}{(q+a)}[e^{(q+a)(t_1-t)} - 1], \ 0 \le t \le t_1 \tag{4}$$

$$Q_o(t), = I_1 e^{-bt}, 0 \le t \le t_1 \tag{5}$$

$$Q_o(t) = \frac{p}{(q+b)}[e^{(q+b)(t_2-t)} - 1], t_1 \le t \le t_2 \tag{6}$$

At $t = t_1$, by equations (5) and (6),

$$I_1 e^{-bt_1} = \frac{p}{(q+b)}[e^{(q+b)(t_2-t_1)} - 1] \tag{7}$$

Now $t_2 = t_1 + \frac{1}{(q+b)}\log(1+\frac{(q+b)I_1 e^{-bt_1}}{p})$ (8)

Here t_2 is a function of t_1. In $[t_2, T]$ shortages starts and level of stock based on demand and some of the demand is lost. So,

$$\frac{dQ_o(t)}{dt} = -\frac{p}{1+\lambda(T-t)}, \ t_2 \le t \le T \quad With\ Q_o(t_2) = 0 \tag{9}$$

$$Q_o(t) = -\frac{p}{\lambda}\left[\{\log(1+\lambda(T-t_2)\} - \{\log(1+\lambda(T-t)\}\right] \tag{10}$$

Now we proceed to find the various cost associated with stock as follows:
Ordering cost (OC)= C_o
Holding cost (HC)is the combination of holding cost and carbon emission cost. So, (from equation (4))

$$\text{HC}_{\text{RW}} = (C_{hr} + C_{re})\int_0^{t_1} Q_r(t)dt$$

$$HC_{RW} = \frac{(C_{hr} + C_{re})p}{(q+a)^2}[e^{(q+a)t_1} - (q+a)t_1 - 1] \tag{11}$$

$$HC_{OW} = (C_{ho} + C_{oe})\left(\int_0^{t_1} Q_o(t)dt + \int_{t_1}^{t_2} Q_o(t)dt\right)$$

$$HC_{OW} = (C_{ho} + C_{oe})\left[\frac{I_1}{b}(1-e^{-bt_1}) + \frac{p}{(q+b)^2}\left(e^{(q+b)(t_2-t_1)} - (q+b)(t_2-t_1) - 1\right)\right] \tag{12}$$

Shortage cost (SC)= $C_s \int_{t_2}^{T} -Q_0(t)dt$

$$SC = \frac{pC_s}{\lambda^2}\left(\lambda(T-t_2) - \{\log(1+\lambda(T-t_2)\}\right) \tag{13}$$

Lost Sale Cost (LSC) = $C_l p\int_{t_2}^{T}\left(1-\frac{1}{1+\lambda(T-t)}\right)dt$

$$LSC = \frac{pC_l}{\lambda}\left(\lambda(T-t_2) - \{\log(1+\lambda(T-t_2)\}\right) \tag{14}$$

Cost of deterioration in RW: $D_R = Q_r(0) - \int_0^{t_1} D(t)dt$

$$D_R = \frac{pa}{(q+a)^2}[e^{(q+a)t_1} - (q+a)t_1 - 1]$$

Cost of deterioration in OW: $D_O = Q_o(0) - \int_{t_1}^{t_2} D(t)dt$

$$D_O = I_1 - p(t_2 - t_1) - \frac{pb}{(q+b)^2}[e^{(q+b)(t_2-t_1)} - (q+b)(t_2 - t_1) - 1]$$

Deterioration cost (DC) is the combination of deterioration cost and carbon emission cost. So,

$$D_c = (C_d + C_{ae})D_R + (C_d + C_{be})D_O$$

$$D_c = (C_d + C_{ae})\{D_R + D_O\}$$

$$D_C = \begin{bmatrix} (C_d + C_{ae})\left(\frac{pa}{(q+a)^2}[e^{(q+a)t_1} - (q+a)t_1 - 1]\right) + \\ (C_d + C_{be})\begin{pmatrix} I_1 - p(t_2 - t_1) - \frac{pb}{(q+b)^2}[e^{(q+b)(t_2-t_1)} - \\ (q+b)(t_2 - t_1) - 1] \end{pmatrix} \end{bmatrix} \quad (15)$$

Total average cost = [OC+ HC+ SC+ LSC+DC]

$$TC(t_1, T) = \frac{1}{T}\begin{Bmatrix} C_o + \frac{(C_{hr} + C_{re})p}{(q+a)^2}[e^{(q+a)t_1} - (q+a)t_1 - 1] + \\ (C_{ho} + C_{oe})\left[\frac{I_1}{b}(1 - e^{-bt_1}) + \frac{p}{(q+b)^2}\begin{pmatrix} e^{(q+b)(t_2-t_1)} \\ -(q+b)(t_2 - t_1) \\ -1 \end{pmatrix}\right] + \\ \frac{p(C_s + \lambda C_l)}{\lambda^2}\left(\lambda(T - t_2) - \{\log(1 + \lambda(T - t_2)\}\right) + \\ (C_d + C_{ae})\left(\frac{pa}{(q+a)^2}[e^{(q+a)t_1} - (q+a)t_1 - 1]\right) + \\ (C_d + C_{be})\begin{pmatrix} I_1 - p(t_2 - t_1) - \\ \frac{pb}{(q+b)^2}[e^{(q+b)(t_2-t_1)} - (q+b)(t_2 - t_1) - 1] \end{pmatrix} \end{Bmatrix} \quad (16)$$

To optimize the total cost function, the necessary conditions are $\frac{\delta TC(t_1, T)}{\delta t_1} = 0 \ \& \ \frac{\delta TC(t_1, T)}{\delta T} = 0$

and sufficient are
$$\frac{\partial^2 TC(t_1, T)}{\partial t_1^2} > 0 \ \& \ \frac{\partial^2 TC(t_1, T)}{\partial T^2} > 0 \ and$$
$$\left(\frac{\partial^2 TC(t_1, T)}{\partial t_1^2}\right)\left(\frac{\partial^2 TC(t_1, T)}{\partial T^2}\right) - \left(\frac{\partial^2 TC(t_1, T)}{\partial t_1 T}\right) > 0$$

V. Numerical Example

Algorithm:

Step 1.

(i) Begin with $t_1(1) = t_2$.

(ii) Put the value of t_1 in to equation $\frac{\delta TC(t_1,T)}{\delta t_1} = 0$ and obtain $T_{(1)}$.

(iii) By using $T_{(1)}$, obtain $T_{1(2)}$ from equation $\frac{\delta TC(t_1,T)}{\delta T} = 0$.

(iv) Reoccur (ii) and (iii) until changeless t_1 & T comes.

Step 2: - (i) If $t_1 \le t_2$, then use step (3).

(ii) If $t_2 < t_1$, then set $t_1 = t_2$, find T from equation $\frac{\delta TC(t_1,T)}{\delta T} = 0$ and then use step (3).

Step 3: - Calculate Total Cost.

Numerical Example: To find the optimal result of total cost, suppose the following data has been taken in Table 1:

Table 1: *Value of Different Parameter*

C_o	C_{hr}	C_{re}	C_{ho}	C_{oe}	C_s	C_l
500	4	0.03	6	0.02	20	10
b	I_1	C_d	p	q	C_{ae}	C_{be}
0.06	150	4	1000	15	0.04	0.02
C_d	t_1	a	λ	b		
4	0.1250	0.05	0.2	0.06		

From this value, we find t_2 = 0.9801, T= 95.622, TC= 113603.5551

VI. Sensitivity Analysis

We explore that the acuteness for the optimal solution of the system by using the distinct values of the variables. The following results are observed.

Table 2: *Impact of Demand and Deterioration on Policy of Inventory*

Parameter	Change in cost in %	t_2	T	Total Cost
p	-20	1.018	161.69	96278.24
	-10	0.997	122.49	103592.72
	10	0.966	78.31	123753.14
	20	0.954	66.29	133216.94
C_d	-20	0.838	14.86	188657.78
	-10	0.911	28.46	135562.90
	10	1.057	173.13	117136.17
	20	1.131	884.44	109339.43

From Table-2, We noticed that when demand function rise, T & t_2 also rise & Total cost fall with concerning demand function. As demand function falls, T & t_2 also changed and Total cost with concerning demand function. While demand function & t_2 reduce then total cost also rises. We see that the effect of deterioration cost on t_2, T& total cost. As effect of deterioration cost the reduce, Total cost increase while t_2& T fall and as the impact of deterioration cost increase, Total cost fall while t_2& T rise.

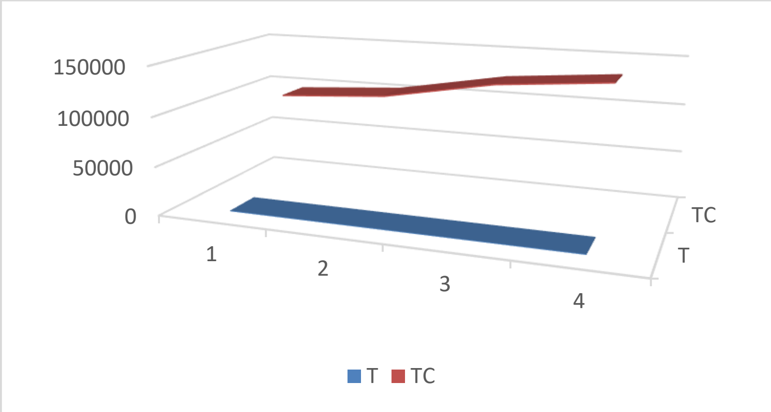

Figure 2: *Impact of demand function on Total cost*

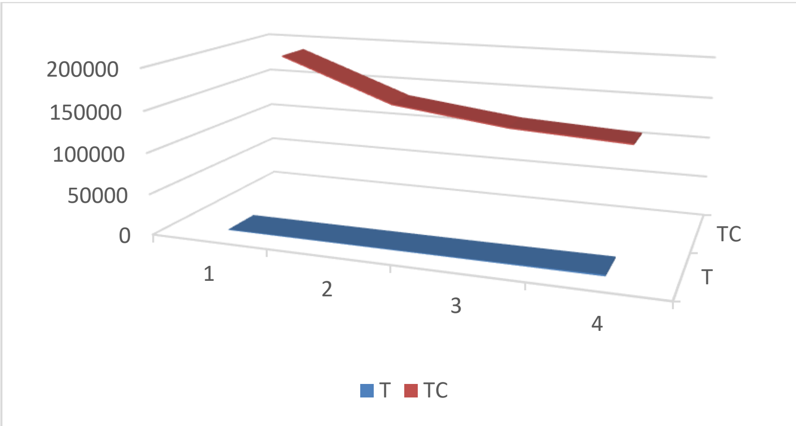

Figure 3: *Impact of Deterioration cost on Total Cost*

VII. Conclusion

In the present study, a two-warehouse inventory model for deteriorating items is formulated with the effect of carbon emissions where demand is a function of stock available in warehouses which also permits shortages. There are different conditions of storage in different warehouses. Therefore, the rate of deterioration is also different for each warehouse. The impact of sustainability is taken with holding cost and deterioration cost. To demonstrate the work a numerical example with analysis of sensitivity regarding various variables is taken and it is observed that total average cost is favorably accessible for the different the value of demand function, deterioration cost and cycle time. It is favorably receptive for the deteriorating parameter. t_2 is favorably receptive for the different value of deteriorating parameter and highly receptive for the cycle time. The future extension for this model is different trade credit period, effect of inflation and time-dependent ordering cost etc.

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An Order Level EOQ Inventory Model for Perishable Products with Price Reliant Demand, Time Reliant Holding Cost and Shortages

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Abstract

In present climate of competition, each prime company wants to increase the planning of pricing in order to get more profit; controlling the cost of inventories is one of the finest policies of it. The present research study elaborates an order level inventory policy for perishable items where the rate of decay varies with time directly. The holding and ordering costs are taken the functions of time. The policies of optimality and decision rules are developed to optimize the total inventory cost,. The shortages are totally backlogged and allowed. A numerical example is given to explain and verify the study. In order to examine the effect of major parameters on decision-making, a sensitivity analysis is performed. Eventually, conclusions are presented along with some managerial perceptions.

Keywords: Perishable products, demand, deterioration, holding cost, shortages.

I. Introduction

In the present era mathematical modelling is an urgent need in different areas of research and development. The study of inventory control theory is not possible without mathematical ideas and modelling. To minimize the inventory related total cost, it is important for the management to agree how much and when to produce or order and it is much important when there is deterioration or decay in the inventory. The meaning of deterioration or decay is the damage or loss of the real value of the inventory, which results a big loss and this loss cannot be ignored. Many products such as flowers, medicines, vegetables, cosmetic items, electronic equipments facing the problem of deterioration and lost their values in the period of normal storage. In this study of inventory controls, the optimal inventory policies are determined for such type of perishable items and the total loss is to be minimized. In the study of inventory theory demand is the main factor, demand may be dependent and independent. If the demand of an inventory of a product depends to another product then it is dependent otherwise independent. In many inventory systems the rate of demand is supposed as constant, but in case of present solid goods it not feasible to consider constant demand always. The different rate of demand may be considered such as stock dependent, cost dependent, time dependent, selling price dependent etc. In our study an order level inventory policy for decaying products is developed with selling price dependent rate of demand as there is a big role of selling price demand in present business environment.

II. Literature Review

In [25], authors presented a research article on economic lot size inventory policy with selling price related demand by considering freight and quantity discounts. In [28], authors proposed a research article for ameliorating items with selling price related rate of demand. In [29], authors studied an inventory policy with time and list price related demand. In maximum models, the cost of holding is well-known and considered as constant but it is not always constant; many studies are there where cost of holding is taken as variable. In the study of EOQ models, many researchers such as in [1] and [2]. The researchers in [11] and [22] used various functions for holding cost. In article [13], researchers developed some refined policies for linear type of demands. In articles [19] and [21] EOQ policies for perishable products, presented with time- related demand.

In the present study an order level inventory policy for perishable products is presented for a system of sole warehouse. Rate of demand is considered as a function of list price (selling). Here the shortages are completely backlogged and allowed. The purpose of the current study is to present an inventory policy for deteriorating items with selling price related demand and time related holding cost with shortages. There are two types of selling price related demand one is linear and other is exponential. If there are uniform changes in the rate of demand of any item per unit selling price, then it is linear selling price dependent demand. But in general it is not noticed in the present era market. Other is exponentially selling price dependent demand which is also unrealistic as the rate of exponential change is really on top level and rarely such high exponential rate of change exists in the market. Therefore researchers have an alternate approach of more realistic selling price dependent demand. A brief literature review working with selling price dependent demands is given as follow.

In the study of inventory models, researchers take interest mainly on two factors first one is the rate of deterioration and the second one is variation in rate of demand with time In [3], authors presented as cheme of approximate solution in all cases of deterministic and time related pattern of demand. In [7], authors presented an inventory model for a demand of linear trend without shortages but the calculation of the solution was very complicate. Researchers in [12], extended the model [7] and presented an analytical study for positive linear pattern demand. In [9] , the authors also worked on the model of [7] and determined an exact formula for EOQ model with demand which increases linearly.

In [8],an easy solution is provided to adjust the EOQ for both the cases of linear demand pattern, either it is increasing or decreasing. But till now the consideration of shortages and deterioration were not included properly. In [10], authors studied an inventory procedure for decaying items where the demand was varying with time. Later, Sachan extended the same and covered the option of backlogging. In [17], a heuristic inventory model is developed to determine economic order quantities with constant rate of inventory deterioration and time proportional demand over time. In study [14], an inventory model is presented for replenishment policies for the items with deterministic demand nature, positive linear trend and shortages. In [15], the same work extended by Murdesh war to minimize the total inventory cost and given an analytic study to find the decision policy for the selection of times and with finite time-horizon. The same contribution was given by many researchers such as [16], [18], [20] and [23].In [4], a model is presented by using two parameters Weibull and relaxed the assumption of constant rate of deterioration. Further in [5], the author extended the same model by using Weibull distribution of three-parameters. Later in [6], authors also used a Weibull distribution of two parameters for decay and presented an inventory scheme with fixed rate of replenishment. Recently many researchers are following these investigations and working on these. In [39],a model is developed to control the best capital within a working environment. In articles [24] and [27], models are presented for perishable items with time-related demand and shortages. In article [26], the structural properties of an inventory system is elaborated with various demands. In article [30], an extension of inventory policies with discrete holding costs. In articles [31], [32], [33] and [34}, researchers discussed various inventory methods for decaying items with price and stock- level related demands. In articles [35], [36], [37] and [38], researchers discussed various inventory schemes for decaying items with different demand rates and shortages. In articles [40] and [41], researchers studied inventory and rating strategy for defective products with sales returns exploration inaccuracies and incomplete reorders under inflationary environment. In article [42], optimal manufacturing transportation schemes are discussed for vendor and producer in a conditional closed-loop SPM for commutable delivery wrapping by usingmetaheuristics policy.

III. Useful Assumptions and Notations

Assumptions

To elaborate the present mathematical study, the following assumptions and limitation are used
1. A single warehouse is considered.
2. Shortages are taken backlogged and allowed.
3. The period of schedule is constant without supplying lead time.
4. The primary level of stock is raised to order level at the starting of every period.
5. When deterioration of the units has been received into the inventory then it is considered.
6. The rate of demand depends on selling price and given in the form of $M(x) = ux^{-v}, u, v > 0$,where x denotes the selling price.
7. At any time $t > 0$, the deterioration rate heed the Weibull distribution with two-parameters, given as $\phi = \mu\delta t^{(\delta-1)}$,where $\mu(0 < \mu < 1)$ and $\delta(\delta > 0)$ which is a shape parameter.

Notations

To elaborate the present mathematical study, the following notations and limitation are used through the study.

M(x)	: Demand rate function.
ϕ	: Rate of deterioration.
I(t)	: The level of inventory at time t.
T	: Time period of a schedule
S_m	: Level of maximum Shortage
S_i	: Level of primary stock at the starting of each inventory.
P_1	: Cost of shortage
P_2	: Per unit cost of shortage
H_c	: Holding Cost
H_i	: Per unit holding cost
D_c	: Per unit cost of deterioration
M_ϕ	: Total-deteriorating units

IV. Mathematical formulation and Analysis

There are two mainly factors, responsible for decreasing the inventory level. The first one is demand and the second one is deterioration. At time t = 0, the initial stock is S_i, the level of stock approaches to 0 at $t = t_1$. Next the shortages arise and reach to $t\,S_m$ level at t = T. In the interval (0, T) the generated differential equations for the inventory level I(t) at time t, are given as below

$$\frac{dI(t)}{dt} + \mu\delta t^{(\delta-1)} I(t) = -ux^{-v}, \qquad 0 \le t \le t_1 \tag{1}$$

$$\frac{dI(t)}{dt} = -ux^{-v}, \qquad t_1 \le t \le T \tag{2}$$

Using the boundary conditions $I(0) = S_i, I(t_1) = 0$ and $I(T){=}S_m$, to solve the equations (1) and (2), We have

$$I(t) = \left[S_i - \mu x^{-\delta} \left(t + \frac{\mu}{\delta+1} t^{\delta+1} \right) \right] e^{-\mu t^{\delta}} \tag{3}$$

$$I(t) = u(t_1 - t)x^{-v} \tag{4}$$

Substitute $t = t_1$ in equation (4), we have

$$S_i = ux^{-v} \left(t_1 + \frac{\mu}{\delta+1} t^{\delta+1} \right) \tag{5}$$

Now, substitute t =T in equation (5), we have

$$S_m = u(T - t_1)x^{-v} \tag{6}$$

Now we find the total deteriorating units in the interval (0,T)

$$M_{\phi} = \int_0^{t_1} \phi I(t)dt \;= S_i \left(\mu t_1^{\delta} - \frac{\mu^2}{2} t_1^{\delta} \right) - ux^{-v} \left[\frac{\mu\delta}{(\delta+1)} t_1^{\delta+1} - \frac{\mu^2\delta}{(\delta+1)(2\delta+1)} t_1^{2\delta+1} - \frac{\mu^3\delta}{(\delta+1)(3\delta+1)} t_1^{3\delta+1} \right] \tag{7}$$

Now the deteriorating cost DC= $D_c M_{\phi}$

$$= D_c S_i \left(\mu t_1^{\delta} - \frac{\mu^2}{2} t_1^{\delta} \right) - ux^{-v} D_c \left[\frac{\mu\delta}{(\delta+1)} t_1^{\delta+1} - \frac{\mu^2\delta}{(\delta+1)(2\delta+1)} t_1^{2\delta+1} - \frac{\mu^3\delta}{(\delta+1)(3\delta+1)} t_1^{3\delta+1} \right] \quad .(8)$$

During the period (0,T), the holding cost is given as

$$H_c = \int_0^{t_1} I(t)dt \;= H_i S_i \left(t_1 - \frac{\mu}{(\delta+1)} t_1^{\delta+1} \right) - uH_i x^{-v} D_c \left[\frac{t_1}{2} - \frac{\mu\delta}{(\delta+1)(\delta+1)} t_1^{\delta+2} - \frac{\mu^2}{(\delta+1)(2\delta+2)} t_1^{2\delta+2} \right] \tag{9}$$

Now the cost of shortages is given as

$$P_1 = P_2 \left(-\int_{t_1}^{T} I(t)dt \right) = uP_2 x^{-v} \left(\frac{T^2 + t_1^{\,2}}{2} - Tt_1 \right) \tag{10}$$

Now the total inventory cost is given as

$$TIC = DC + H_c + P_1$$

$$
\begin{aligned}
&=D_c S_i\left(\mu t_1^{\delta}-\frac{\mu^2}{2} t_1^{\delta}\right) \\
&-ux^{-v} D_c\left[\frac{\mu\delta}{(\delta+1)} t_1^{\delta+1}-\frac{\mu^2\delta}{(\delta+1)(2\delta+1)} t_1^{2\delta+1}-\frac{\mu^3\delta}{(\delta+1)(3\delta+1)} t_1^{3\delta+1}\right] \\
&+H_i S_i\left(t_1-\frac{\mu}{(\delta+1)} t_1^{\delta+1}\right)-uH_i x^{-v} D_c\left[\frac{t_1}{2}-\frac{\mu\delta}{(\delta+1)(\delta+1)} t_1^{\delta+2}-\frac{\mu^2}{(\delta+1)(2\delta+2)} t_1^{2\delta+2}\right] \\
&+uP_2 x^{-v}\left(\frac{T^2+t_1^2}{2}-Tt_1\right)
\end{aligned} \tag{11}
$$

Equation (11) gives the total inventory cost associated with model.

V. Numerical Verification and Sensitivity Analysis

In this section, the present study is illustrated by using the following numerical example
Consider an order level inventory scheme for decaying products with the following parameter values
u=10, v =1, μ =0.0052, δ =0.41, x =6.00, H_i =5.00, P_2 =4.00, D_c =2.00 Substitute these parameters values, we get, S_i =29.9879 = 30units (units approximately) and the total inventory cost TIC=2967.97

Table: 1*Sensitivity analysis of the Model*

Change in parameter Values		S_i	t_1	TIC
H_i	2	44.86	26.60	1783.65
	3	38.47	22.88	2291.52
	4	33.64	19.99	2671.21
	5	29.90	17.82	2968.38
	6	26.87	15.92	3205.12
P_2	1	11.16	6.76	1112.54
	2	19.20	11.41	1907.22
	3	25.21	14.87	2503.21
	4	29.82	17.77	2967.25
	5	33.45	20.00	3204.25
D_c	0.5	29.87	17.72	2967.91
	1	29.87	17.72	2968.05
	1.5	29.87	17.72	2968.12
	2	29.87	17.72	2968.34
	2.5	29.87	17.72	2968.46
T	25	18.65	11.10	1159.12
	30	22.40	13.39	1669.42
	35	26.15	15.51	2272.55
	40	29.86	17.70	2968.25
	45	33.59	19.92	3757.56
X	3	59.57	17.74	5936.72
	4	44.83	17.74	4453.44
	5	35.86	17.74	3562.10
	6	29.66	17.74	2968.32
	7	25.62	17.74	2544.32

The following graphs are showing the impact of different parameters on total inventory cost.

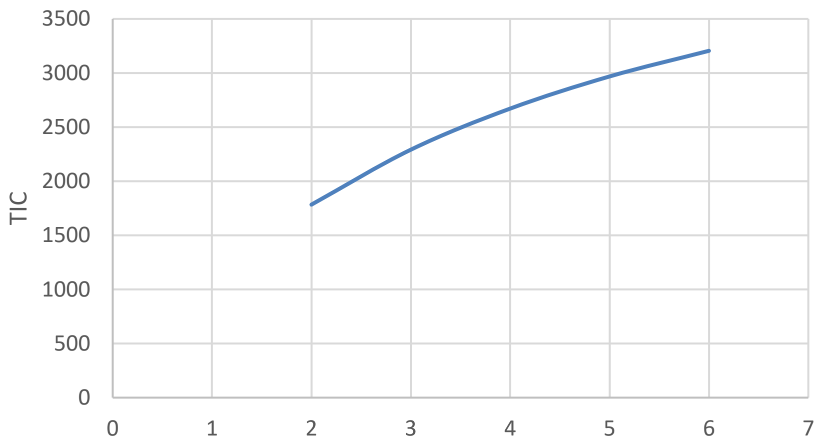

Figure-01:Change in parameter Hi

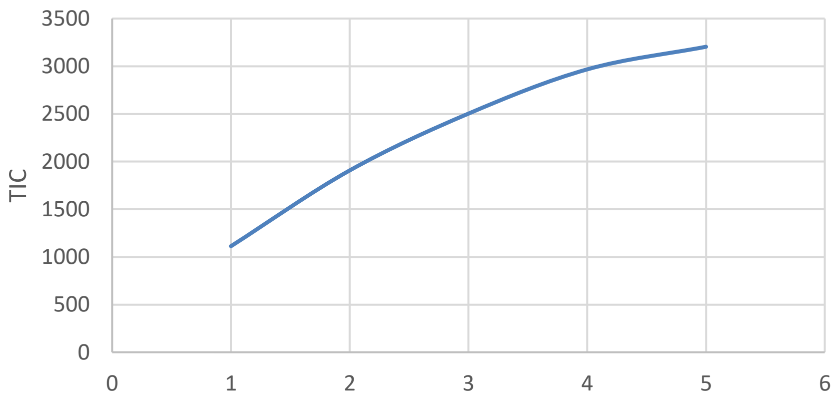

Figure-2: Change in parameter P2

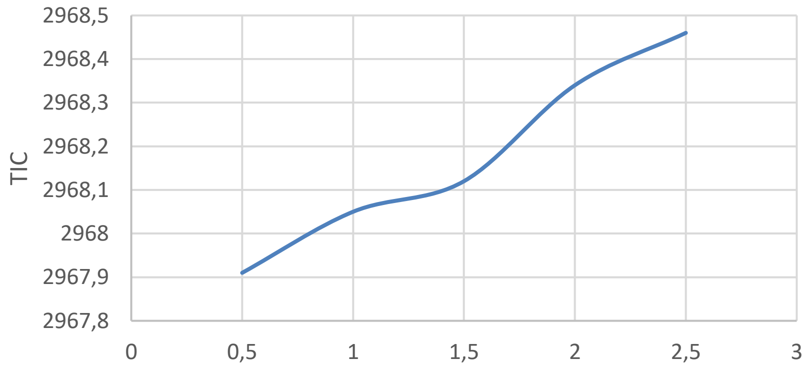

Figure-03: Change in parameter Dc

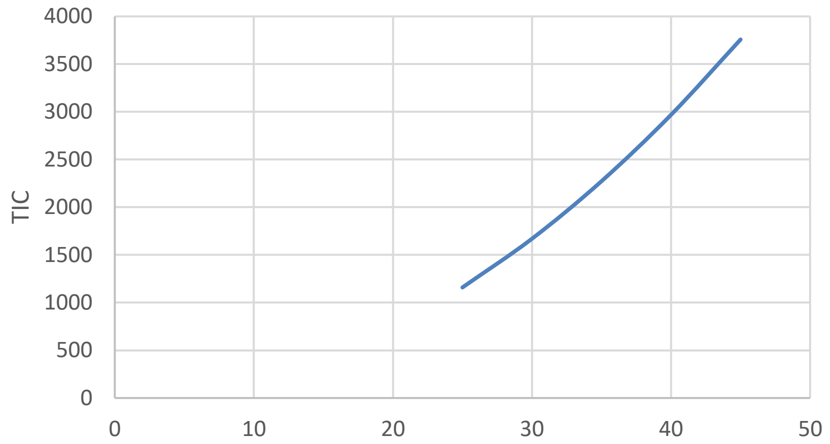

Figure-04: Change in parameter T

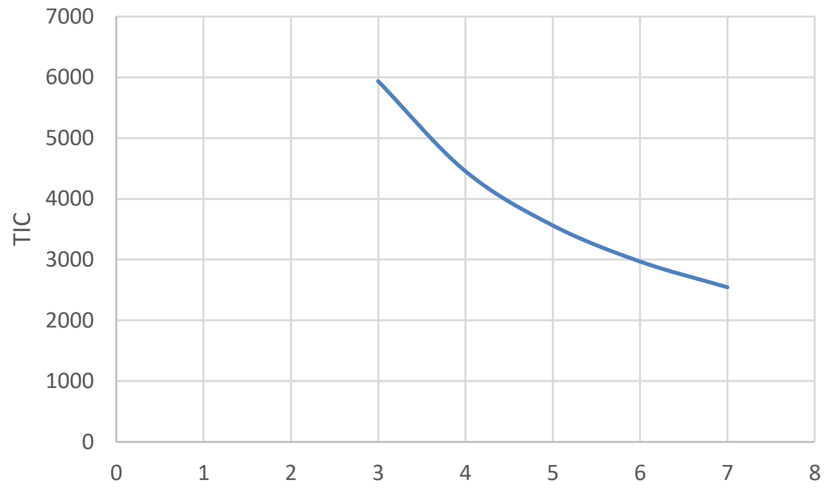

Figure-5: Change in parameter X

Observations

The observations of the above table are as follows.

1. The TIC i.e cost of total inventory increases as the shortage cost (P_2) per unit increases.
2. The (TIC) increases with the holding cost (H_i) (per unit).
3. The (TIC) increases with the time period (T).
4. The (TIC) decreases as the price (x) changes.
5. After increases in deteriorating cost there is a little change in TIC.

V. Conclusion and Future Scope

This study presents an order level inventory EOQ policy for deteriorating items. The rate of demand which is deterministic is taken as a function of price (selling). Here the shortages are considered fully backlogged and allowed in this current study. Also the cost of holding is taken as the function of time. In many conditions the stock out cannot be ignored as many stocked items give a high value of profit as comparison to its backorder cost. Decay is a natural process in any system of inventories and it is always necessary to consider it, so it also considered in the present study. The study is verified by a numerical example. Sensitivity is performed to observe the behaviour of the decision variables.

This study can be extended by inculcating other constraints and different preservation technologies. It can be extended for inventory models with two warehouses system in which PT can be used in any of the warehouses. Further, it can be extended for order quantity of production and to study the models for perishable items by taking different demand rates and costs.

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Machine Learning-Based Abnormality Detection Approach for Vacuum Pump Assembly Line

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Abstract

The fundamental basis of Industry 4.0 is to make the manufacturing sector more productive and autonomous. In the manufacturing sector, practitioners always long for product quality improvement, reducing reworking costs, enhancing first pass yield in production or assembly line, in this regard anomaly detection, is becoming popular and is widely used. With the integration of anomaly detection models and Artificial Intelligence-based condition monitoring systems, industries have attained promising results in achieving these goals. However, it is a highly complex task to extract meaningful information from the large amount of data generated by the manufacturing systems. Hence, in this paper, effective machine-learning-based anomaly detection and prediction model has been proposed. A two-phase model is presented in this study and is validated for abnormality detection in the assembly line of vacuum pumps. In the first phase, a random forest algorithm is used to predict the pump vacuum. Based on the actual and predicted values, the error is computed. Then, EWMA (Exponentially Weighted Moving Average) chart is employed to detect the anomalies. In the second phase of the proposed model, based on the EWMA chart and calculated error, anomaly prediction is done. For better prediction results, statistical features are extracted from the error values and used as input for the second phase. To validate the proposed approach, other machine learning models SVR, Decision Tree, Logistic Regression, KNN and SVC have been compared. A statistical method EWMA chart is also integrated with random forest.

Keywords: Abnormality detection, Abnormality Prediction, Machine learning, process control, Assembly line, mechanical vacuum pump.

I. Introduction

New technologies have been adopted by the industrial industry as quality measurement tools, resulting in a data-rich environment that lays the groundwork for the use of machine learning (ML) techniques for data collected from various sources to reduce production costs and improve product quality [1]. Assembly machines are considered one of the vital components in the manufacturing sector and are frequently employed in production lines; each machine is responsible for guaranteeing the end product's quality [2].

Anomaly detection technique is used in many areas like healthcare, computer security, credit card image processing, and muchother application [2], [4].
The following are types of anomalies:
a. Point anomalies: If one piece of data differs significantly from the rest, it is considered anomalous. Detecting credit card fraud based on "amount spent" is a business use case [5].
b. Contextual anomalies: When data behave contextually different from thewhole dataset is called acontextual anomaly, for example, if water consumption in summer is more than the rest of the year. Contextual anomaly is also known as aperiodic anomaly. Collective anomalies: A group of data samples working together to find abnormalities is known as collective anomalies.
c. Business use case: Someone tries to copy data from a remote workstation to a local host without permission, which would be identified as a possible cyber assault [5].Reducing the reworking cost and enhancing the first pass yield, thefalse alarm can cause waste of capital, time, and productivity, so amachine-learning-based anomaly detection algorithm would work best to improve accuracy and prevent failure. The objectives addressed in this study are depicted in Table 1.

Table 1: *Objectives*

Model	Purpose	Input features	Output
Vacuum performance prediction	Vacuum prediction and construction of EWMA chart based on theerror term	X1-x14	Y:minimum vacuum created by the pump
Abnormality prediction	Prediction of anomaly by taking EWEWMA chart as input		Y: 0 means abnormal 1 means normal

II. Literature Review

Anomaly detection has been a center of many studies and research articles throughout the years, as per Xu et al. in [2], it is a major field. As a result, various methods and techniques for identifying anomalies and improving the effectiveness of current strategies for detecting anomalies have been provided in the existing literature. This section will examine the most cutting-edge approaches in the area. Numerous approaches were presented in [2], with the primary objective of determining the relative efficacy of techniques for extremely large unlabeled benchmark datasets. The difficulty comes when we have to find or draw the boundary between normal and abnormal data. X.liu et al. developed an approach that employed an ensemble learning algorithm to find abnormalities in the exceedingly difficult dataset. Anomalies are discovered using the Angle-based methods k-Nearest Neighbors Outlier Detection which iscombined to form the input to the local outlier factor (LOF). The authors used Rank power, precision, and area beneath roc curve characteristics to quantify their model's performance in order to discover various models would deliver the best results. Fast ABOD (-angle-based outlier detection) and KNN had the best area under the receiver operating characteristic curve (AUROC) performance in the dataset, recognizing up to 75% of real positive anomalies. The authors of [7] also employed a large-scale, high-dimensional data collection method. They also ran across the same problems as those detailed in [8].Researched to have a better understanding of the many types of classification algorithms that might be used for deep intrusion detection. Finally, they proposed combining a one-class SVM (OC-SVM) with a deep neural network (OC-NN) model to create a hybrid model [9].Local outlier factor and isolation forest model also perform well on -large-scale data, according to [10] and [11]. Additionally, they utilized Precision, Recall, and F1-score as performance metrics in addition to F1-score, except for Galante, who included an extra approach, One-Class Support Vector Machines (OCSVM) as well as AUROC as a statistic. Additionally, this group of test participants ran into the same problem while working with high-dimensional space, which was rectified using PCA. We obtained a 64 percent F1 score by extracting new characteristics that better properly characterize the data.

The authors of [6] compared 10 distinct benchmark datasets using several unsupervised methods. They computed seven sets of outlier scores using -one-class SVM, KNN, CBLOF, and HBOS. Additionally, they had to consider how they determined the boundary between anomalous and non-anomalous data values. The authors conducted tests to evaluate these approaches' performance on a variety of datasets of varied sizes and in high-dimensional space. Additionally, they employed AUROC as a performance metric, along with Precision and Rank Power. To do this, it is observed that a dataset with more no of variables AUROC is better compared to low no. of variables in thecase of KNN and LOF. They discovered abnormalities in the network traffic dataset when they utilized cluster-based anomaly detection algorithms such as NKICAD, K-means, CBLOF, and LDCOF [7]. The authors used an unclassified dataset but used statistical analytic approaches to produce labels for their algorithms. To begin, the authors validated their methodology by researching a diverse group of benchmark datasets. They then applied this approach to their dataset of network traffic. On average, these cluster-based methods can identify 87 percent of network traffic abnormalities. In [8], OCSVM, LOF, and Random Forests were used to detect abnormalities. Thus, the author in [8] stated that their dataset is free of anomalies that would render it non-specific in high-dimensional data, resulting in an AUROC of 70%. That is, they employed semi-supervised anomaly detection to train their models but did so contrary to the findings in [9], [6].Additionally, the authors in [10] employed similar approaches, but with the addition of the following metrics: F-score, Recall, and Precision. Simultaneously, the method was reported to have a high rate of detection and a low incidence of false positives. Unsupervised approaches include cluster analysis and classification inside a single class, as well as autoencoders. Additionally, unsupervised learning techniques such as the convolutional autoencoder (which has been used to identify faulty components in the manufacturing industry) and the generative adversarial network which has been adopted in the manufacturing sector for identifying defective components. The major approaches for detecting time series anomalies are recurrent autoencoder (RAE) [11] and one-class SVM [12].When abnormal conditions arise, these models are taught to recognise the process's true state, and any deviation from that state is considered abnormal. This product has applications in finance [13], security [14], and healthcare [15]. When it comes to anomaly prediction, the problem is far more challenging, and little effort has been made to address it. There is a suggestion to anticipate energy price anomalies using the pattern sequence forecasting (PSF) technique. A wavelet-based multi-feature classification model and extreme learning machine technique were developed for predicting the possibility of a stock market anomaly happening. Later in the petroleum industry, a nonlinear adaptive ELM (also known as an adaptive sparse dynamic window filter) and a wavelet transform-based method was employed. Compared to other models, the experimental findings indicated that the forecast of theanomaly was improved [16]. The majority of the evaluated studies assessed just the outcomes of previous research on run-to-run identification of aberrant patterns in SMT using standard Shewhart control charts, this shows that little scientific research has been conducted on prognosis techniques. The proposed research employs advanced machine learning techniques to develop a framework for the prediction of aberrant assembly line trends. This approach contributes to increased overall assembly line productivity while also lowering the cost of fault repair.

III. FRAMEWORK AND MODELLING

I. Framework

For this problem, a two-phase model is proposed as shown in Figure 1. A mechanical vacuum pump assembly line dataset is used in this study. For confidentiality purposes, the name of the industry is not revealed here. In this work, a -two-phase model is proposed, first, thevacuum created by thepump is predicted and the error term between actual

And the predicted vacuum is calculated. Then Na EWMA chart of prediction error is constructed. This EMWA chart is used as input for the second phase of the model which is the abnormality prediction phase.

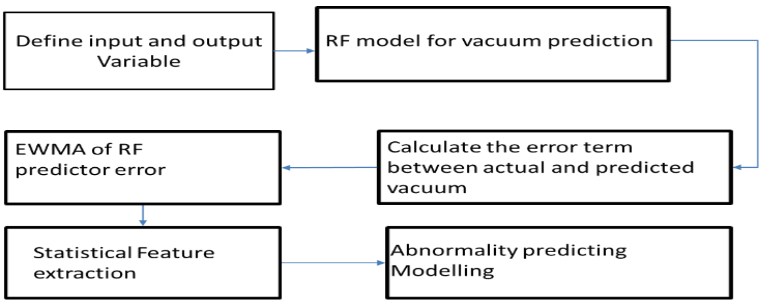

Figure 1: *Framework of proposed methodology*

The adopted dataset has a total of six workstations. For the development of the vacuum prediction model, all the variables from workstation 1 to workstation 5 has been taken as input for our model, and theminimum vacuum collected in workstation 6 is taken as the output for our model. The random forest-based algorithm is used as amodel for this study and thefurther error term is calculated between the predicted and actual vacuum, thereafter based on the error term, andanEWMA chart is constructed. This EWMA chart is hereby used as the input for abnormality prediction modeling. To produce the best results, the frame size is set to nine observations in this work. To do so, the predicted error observations are divided into chunks of nineobservations and the statistical feature are extracted. Normal and Abnormal data shows different features. Many statistical features are extracted from each set of observations and these extracted features are used as the input for abnormality prediction modeling as proposed in the framework. The list of statistical features is given in Table 2.

Table 2: *Statistical features*

Features	Calculation formula

Mean	$X_{mean} = \frac{1}{n}\sum_{i=1}^{n} x_i$
Variance	$\sigma^2 = \sum \frac{(x-\mu)^2}{N}$
Skewness	$X_{skew} = \frac{1}{n(X_{std})^3}\sum_{i=1}^{n}(x_i - X_{mean})^3$
Kurtosis	$X_{kurt} = \frac{1}{n(X_{std})^4}\sum_{i=1}^{n}(x_i - X_{mean})^4$
RMS	$RMSE = \sqrt{\sum_{i=1}^{n}\frac{(\hat{x}_i - x_i)^2}{n}}$
Maximum	$X_{max} = max(x_i); (i = 1,2,3 \dots \dots, n)$
Minimum	$X_{min} = min(x_i); (i = 1,2,3 \dots \dots, n)$
Coefficient of Variation	$CV = \frac{X_{std}}{X_{mean}} \times 100\%$

II Random forest model:

Ensemble learning methods are used for supervised learning using Random Forest Regression. When using this ensemble learning, several machine learning models are combined to yield a more accurate prediction than a single model. Even though the trees don't interact with each other, we can realize that they line up in parallel. The Random Forests construct multiple decision trees and the mean of all of the classes is used as the prediction of all of the trees. The steps of the Random Forest algorithm are discussed briefly as follows:

1. Pick randomly from the training set the k data points you would like to use.
2. Use this dataset to build a decision tree.
3. You must decide on the number of trees you want to build, and then you must repeat the first two steps.
4. Make four of your N-tree trees predict the value of y for the new data point and average their predictions.

Table 3: *Pseudo Random Forest Algorithm*

Algorithm 1: Random Forest for Regression and classification

1. For a = 1 to A, the following is true:

(a) From the training data, create a bootstrap sample Z of size N.

(b) Fit the bootstrapped data to a random-forest tree T a by iteratively repeating the following procedures for each terminating node of the tree until the minimal node size n min is attained.

i. At random, choose m variables from the predictor variables.

ii. Choose the optimal variable/split-point from them.

iii. Dividing the node into two child nodes

2. Produce the tree ensemble $\{T_a\}_1^a$

To create a forecast about a new point x, use the following syntax:

Regression: $\hat{f}_{rf}^A(x) = \frac{1}{A}\sum_{a=1}^{A} T_a(x)$

Classification: Let $\hat{C}_a(x)$ be the class prediction of the a_{th} random-forest tree.

Then $\hat{C}_{rf}^A(x) = majorityvote\{\hat{C}_a(x)\}_1^A$

II. RF-Based EWMA Chart Construction Phase

EWMA control based on RF regression in the first phase Chart was developed. The independent's simultaneous impacts minimum vacuum variables generated by the pump are considered. The process is shown in the figure. 4. To begin, first define the input and output variable with the help of thedesign of theexperiment (DOE).Then a nonlinear regression model is precisely suited. X through Y are derived from the set of remarks (X_i, Y_i), and the mathematical model defines the real link between X and Y.

$$Y = X\beta + \delta \tag{1}$$

Where Y is the minimum vacuum generated by the pump and X is a vector representation of a set of k control variables is a vector with k + 1 parameters that reflects the actual value. Δ represents the error term in a relationship between variables that aredistributed regularly and independently regularly and independently with a nearly uniform distribution. The standard deviation is σ and the mean is 0. Nevertheless, observing the entire population and determining the truth is tough. As a result, the term is calculated based on a set of assumptions. By decreasing the total, we may obtain a set of data that is representative. Equation 2 represents the computation of β.

$$min_\beta \sum_{i=1}^{n} \delta_i^2 \tag{2}$$

δ = Y -Xβ by definition, as a result of equation (1).

The equation is differentiated with respect to and the resulting is set to zero to minimize the expression in (2).

We can discover the value by solving that, and we can use equation (3) to forecast the value of Y for minimal vacuum by solving that.

$$\hat{Y} = f(X, \hat{\beta}) \tag{3}$$

Here Y and X are the predictors, respectively. After computing Y and utilizing the residual of a subsequent matched observation occurring at time t, the formulas error = $\hat{Y}(t) - Y(t)$ are used to compute X (t) and Y (t). Errors are charted using the EWMA control chart of error terms. If the sample statistic plotted deviates from the bounds, the process has gotten out of hand. Even if all samples are controlled in a variety of real-world circumstances, considerable variance is still a possibility. The patterns on the chart, in general, do not follow a random pattern.

This is quite advantageous, as it provides critical diagnostic information. When you observe abnormality patterns, it indicates that the process is not proceeding as planned and that it must be restored to regular operating circumstances.

IV. Control chart showing the projected error term using EWMA

Regression control charts are designed to track dependent variables. This is typically done while the operation is being performed. Numerous more process variables might have a significant influence on the output concurrently. The EWMA control chart was developed to increase control precision. Shewhart control charts constrained them by failing to account for minor and moderate changes. Since they employ information from the most recent observation, the estimate's accuracy is less likely to change over time. When you notice a slight shift, the EWMA chart records the change. Most current observations, as well as the most recent data evidence from the past, has greater weight while explaining the process. FirstEWMA regression control chart must be produced. The first step of process function$\hat{f}$, is estimated and then used to forecast$\hat{Y}$, whenever you wish Next, design an EWMA control the error term vector, seen here as a chart, utilizes exponential smoothing. The equation should look like this. When the stage is done, the function f has moved, indicating that the process it previously represented no longer exists, and the error term has been shifted to minimize errors.

$$Z_i = \lambda R_i + (1 - \lambda) Z_{I-1} \qquad (4)$$

Where R_i is the ith observation residual time I = 1, 2...., n) andλ is the exponential smoothing constant that we determine; a bigger value of implies that the most recent observation has a stronger influence.

The EWMA chart is created under the premise that the error term follows a normal distribution, although EWMA is insensitive to this assumption.

V. Abnormality Prediction Phase using EWMA chart of error as input

In the second step, a KNN classifier is used to anticipate abnormalities. The input is the produced by EWMA chart. The duration of observation is set at nine in this study to achieve the best results. Normal and abnormal data have distinct characteristics. Numerous statistical characteristics are collected from each set of observations, and these recovered features are utilized as input for the framework's suggested abnormality prediction modeling.

KNN Classifier Algorithm: KNN assumes that items identical to the training examples exist nearby. What I want to say is that similar things are nearby. A part of what we need as children are used to finding the distance between points on a graph with KNN. There are several methods of measuring distance, and it may be useful to use a different one depending on the problem we are trying to solve. Although the straight-line distance (also known as the Euclidean distance) is prominent and readily understood, it is not always the best choice.

Table 4: *KNN Classifier Algorithm*

Algorithm 2: KNN Pseudocode
1. Fill the database with data. 2. When you have chosen the number of neighbors, initialize K to that number. For each piece of data, there is an example in the data. 3. To find out how far apart two data points are, you must use a formula that contains the distance between them. 4. In an ordered collection, add the distance and the index of the example.

5. Use the shorter distances as the start and progressively use the longest distances as the finish to get an ordered list of distance and index values (in ascending order).
6. Choose the first -K-sorted items, see which of the selected K items have their labels.
7. Return the mean of the K-labeled variables if regression is required.
8. If you wish to return the mode of the K labels, classify the items.

The process of selecting a proper value for K: When we run the KNN algorithm several times with different values of K, we pick the K that provides the best results with the fewest mistakes while also allowing the system to perform properly with new data that it has never seen before.

IV. Experimentation and Results

I. Data Description: The data obtained from six workstations consists of two types of variables. The first one is assembly parameters like speed, torque, angle at which different components of the mechanical pump are mounted and thesecond type of variable is thefunctional variable which is a test variable used to check the performance of mechanical vacuum pump like thepneumatic test, vacuum test, leakage test pressure. Figure 2 is the typical flow chart that is followed in this work.

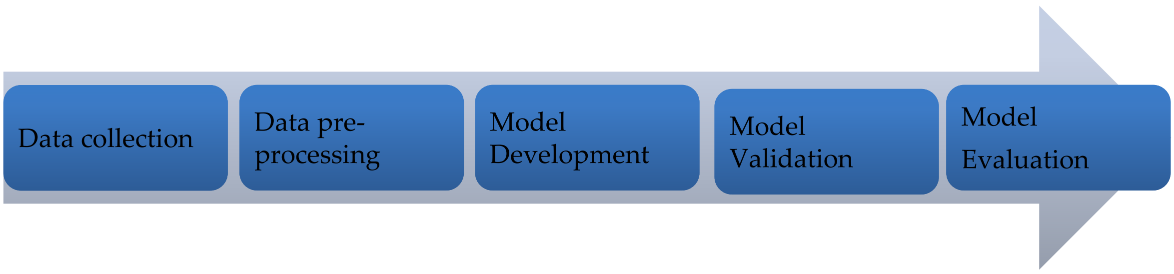

Figure 2: *Flow Chart*

II.Data Pre-Processing: Dataset preparation is required for machine learning algorithms to utilize the dataset. Duplicating datasets, redistributing data, and organizing the data, high values, distinctive characteristics, measurements. Feature scaling and duplicate removal processes are also performed in thedata preprocessing step. The dataset used in this study consists of 40 columns and 25000 rows. Features of some of the columns are described in the Table 5.

III.Vacuum performance prediction model:A machine learning model is executed on the given data and compared the performance of other models to determine which one was the best. The algorithms that were chosen are Random forest, SVR, and Decision Tree. To test the consistency, multiple runs are performed for each algorithm. According to the authors of [18], anomaly detection models are prone to failure. Due to its uncontrolled nature, it suffers from volatility. Additionally, Unsupervised learning, as discussed in Section II examination of comparable works. The accuracy of several models is seen in Table 6. Random forest with hyper parameter optimization is the underlying model with 70.48 percent accuracy. SVR is the -second-best model for vacuum prediction in this dataset.

Table 5: *Description of Dataset*

Feature	Type	Description
Last Station	Numeric	This tells about the last station of pump
Entry Time	Time	Time of starting assembly
Exit Time	Time	Time of completing assembly
Cycle Time	Numeric	Time is taken to complete assembly and testing of

		one pump
Final outcome Ok	Binary	It tells the pump working properly
Final outcome Not Ok	Binary	It tells the pump is not working properly
Piece Reworked	Numeric	How many pieces reworked
Workstation	Numeric	At which workstation a pump is assembled
Couple	Numeric	Torque at which screw is inserted
Angle	Numeric	The angle at Which Screw is inserted
Pressure	Numeric	To check leakage
Flow rate	Numeric	Flow rate at which air is leaking from the pump

Table 6*: Prediction accuracy of different models*

Sr.no	Model	Accuracy	Standard Deviation
1	Random Forest	70.48%	3.40%
2	Decision Tree	41.23%	6.56%
3	SVR	68.86%	3.75%
4	RF-Regressor (n=14)	68.78%	3.76%
5	PCA+Regressor(n=5)	68.66%	3.48%
6	PCA+Regressor(n=7)	66.24%	3.27%

IV.RF-based EWMA chart: This complicated link between the minimum vacuum and the specified independent variables is captured by the RF predictor. Fivefold -cross-validation is performed to test the model's effectiveness. Several RF prediction models can have a higher probability of correct identification (also known as index measures or indicators) performance-related measures, like mean-absolute error (MAE). In mathematical terms, they are equal to (5), (6), and (7), respectively. R^2Indicates how well the variation in the dependent variable (minimum vacuum) is explained by the factors specified. To say that the model fits exactly with the data is to say that the model has a R^2 of 1. MAE is used to calculate the prediction error (i.e. the absolute difference between the real and predicted values)The RMSE limits a larger disagreement, whereas the MAE averages the expected deviations. Y values have the same RMSE and MAE units. Both ranges are 0 to infinity. On the whole, small RMSE and MAE values indicate that the model is fitting the data well. The red point in the chart below is showing the detection of and an anomaly.

$$\mathrm{R}^2 = \frac{\sum_{i=1}^{n}(\hat{Y}_i - Y^2)}{\sum_{i=1}^{n}(Y_i - Y^2)} \tag{5}$$

$$\mathrm{RSME} = \sqrt{\frac{1}{n}\sum_{i-1}^{n}(\hat{Y}_i - Y_i^2)} \tag{6}$$

$$\mathrm{MAE} = \frac{1}{n}\sum_{i=1}^{n}\left|\hat{Y}_i - Y_i\right| \tag{7}$$

Where Y_i and $\hat{Y}_i$ signify the observed and expected minimum vacuum a, and n signifies the total number of pumps Figure 3 showing the EWMA chart of errors.

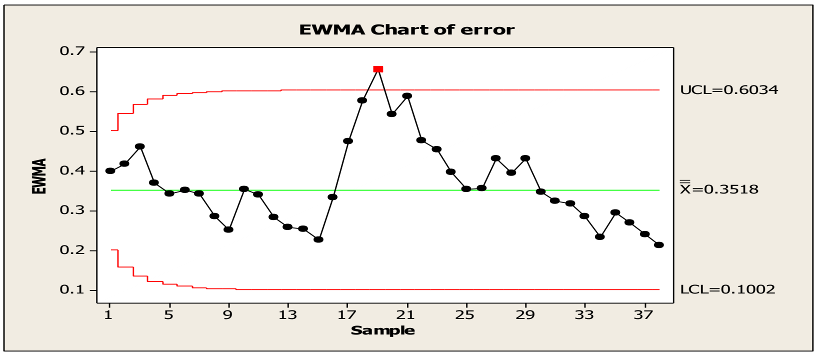

Figure 3: *EWMA Chart*

V.Anomaly detection Model: Using the information from first model , we constructed a machine learning model and evaluated the performance of many models to choose the best one. The selected algorithms are Logistic Regression, Random forest, SVC, and KNN. The accuracy of these models is shown in Table 7.

Table 7: *Accuracy of different model*

Model	Accuracy
Logistic Regression	0.97
KNN	0.98
SVC	0.96
Random forest	0.98

Model Validation: Single main validation strategy: percentage split is used because the datasets had a significant number of samples. This approach evaluates models using a fraction of the available data. Divide the dataset in half to create a training set of 80% of the data and a test set of 20% of the data

Model Evaluation: Given that the ultimate goal is assurance of quality and to provide only correct results, the ideal model should be capable of distinguishing aberrant data points from normal data points. The model must be capable of discriminating between right and incorrect positive occurrences. To evaluate our model we used two performance indicators confusion matrix and F1-score. For a binary classification model, one can use several performance indicators from the confusion matrix. A Confusion matrix is a N x N matrix that is used to evaluate the performance of a classification model, where N is the number of target classes. In the matrix, the machine learning model's prediction values are compared to the actual target values. By assessing the model's overall performance and pointing out any type of mistake, we get a holistic picture of how our classification model is doing.

From Table 8, by comparing the performance of different models it can be stated that the proposed model works effectively for anomaly detection and prediction of vacuum pump assembly line. Random forest showing best result across all performance metrics.

Table 8: *comparison of different prediction accuracy*

Model	Accuracy (%)	Recall (%)	Specificity (%)	Precision (%)	F1-score (%)	Training Time
Logistic Regression	93.40	88.0	87.4	97	92.28	3.1
KNN	93.40	87.9	86.4	98	92.67	4.3
SVC	92.04	85.4	84.3	96	90.39	3.5
Random Forest	94.85	91.2	90.2	98	94.47	4.2

V. Conclusion and Future Scope

In manufacturing sectors, it is quite challenging to attain product quality improvement, reducing reworking costs, enhancing first pass yield in production or assembly line. Anomaly detection approaches have shown promising results in overcoming the challenges to achieve the goals. In this paper, RF-EWMA based anomaly detection and prediction model has been proposed. A two-phase model is presented in this study for abnormality detection in the assembly line of vacuum pumps. Results show that the number of rejected pumps can be reduced by implementing this framework. The r the random forest model is showing thebest accuracy for vacuum performance prediction. KNN and random forest are showing thebest accuracy in classification models. It is to be noted that the initial cost for implementation is higher, however, in long run, it will truly create value.

This work can be further developed in numerous different ways in the future. Feature selection strategies can improve the model's efficiency. Another way, developing a classification model that can differentiate aberrant patterns, such as trend shift and cyclic pattern, from each other, which helps with diagnosing system faults and enhancing fault diagnostic decision-making. Additional modeling may be performed to enhance the proposed framework's flexibility by suggesting machine modifications via process parameter control to restore the process to normal condition when the model detects an abnormal state. Finally, a mix of Adaboost-based machine learning algorithms may be used to enhance the model. As a result, multivariate SPC approaches open up new avenues of investigation.

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Declaration of Conflicting Interests:
The authors declare that there is no conflict of interest.

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Availability-Cost Optimization of Butter Oil Processing System by Using Nature Inspired Optimization Algorithms

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Abstract

The challenge of upgrading the complex industrial systems is basically to cope up with the ever-increasing demands of the real world. For the maximum reliability of complex industrial systems, decisions of management depend on experience. This is because the pattern of the chance of success is not easy to predict due to limited and rough available information. Thus, the task of the researchers lies here to increase the operational time of the individual components of a system for maintaining higher system reliability to increase productivity and profit of an organization. In this paper, an optimum choice of the mean time between failure (MTBF), mean time to repair (MTTR), and associated costs in a suitable design unit has been showcased to bring as much efficiency as possible. The motive is to minimize the cost satisfying the availability constraints of the system by using a few recent nature-inspired optimization techniques named Grey Wolf Optimization (GWO) technique and Cuckoo Search Algorithm (CSA). The computational parameters produced to improve the efficiency of the designed system with the application of GWO and CSA techniques, which not only achieve the target of minimum cost but also stand out much competitively in terms of performance. The results obtained by these two algorithms for butter oil processing system are compared and this comparative study shows that the GWO is superior to CSA for this availability-cost optimization problem of butter oil processing system.

Keywords: **A**vailability, Reliability, Cost function, Metaheuristics, Grey Wolf Optimizer, Cuckoo Search Algorithm.

I. Introduction

It is not possible for any system to be perfectly reliable even if the researchers and the stakeholders work to the best of their efforts. So, the increasing complexity of present-day equipment has brought into focus two other aspects known as maintainability and availability. Maintenance plays a very crucial role as a preventive and corrective measure so as to achieve continuous and longer availability. Maintainability means the probability that the system will resume operation in a given prescribed time after the repairing is completed as per the specified condition. Availability is associated with the concept of maintainability. Availability refers to the probability that the system is operating within a given time. It means the proportion of time for which the system is available for use that is excluding the downtime (when it is under maintenance). Though availability is not an indicator of the number of failures but depends on both failure and repair rates and it integrates both reliability and maintainability. The input costs and availability are very important in any operation and are the deciding factors for increasing the reliability of any complex system. There are three types of availability depending upon the time elements. (a) Inherent availability (b) Achieved availability (c) Operational availability. To understand the different types of availability it is important to understand the concepts of MTBF and MTTR. MTBF is the mean time between the breakdowns or failures during which the system is unavailable and undergoes repairs. MTBM is the mean of the time periods between the maintenance which could be either scheduled (preventive) maintenance or corrective maintenance due to failure. MTTR is the average mean time calculated as the total repair time during a given period divided by the number of malfunctions during the same interval. For any system down time is the total time for which it is down for corrective or preventive maintenance. MTBF does not include the preventive maintenance. The Up time is the time for which system is under active operation. Now the three types of availability are explained as follows:

I. Inherent availability

Inherent availability is the availability in the presence of defined conditions in an ideal promoting environment without considering the preventive maintenance at any given time. It is expressed as

$$A_i = \frac{MTBF}{MTBF+MTTR} \tag{1}$$

Where, $MTBF = \frac{1}{\lambda}$ and $MTTR = \frac{1}{\mu}$

II. Achieved availability

Achieved availability refers to the chance that a system shall operate satisfactorily taking into account the preventive down time also. It is expressed as

$$A_c = \frac{MTBM}{MTBM+M} \tag{2}$$

Where MTBM is the mean time between the maintenance, which could be either scheduled (preventive) or corrective maintenance due to failure and M is the mean active--maintenance downtime resulting from both preventive and corrective maintenance.

III. Operational availability

Operational availability is the availability when the system operates under actual supply environment at any given time considering the administrative or supply downtime. It is expressed as

$$A_o = \frac{MTBM}{MTBM+MDT} \tag{3}$$

where MDT is the mean actual down time.

For achieving the goal of maximum reliability of any complex system matching the global standards and also making the estimated profit it is imperative for the management to specify the availability and cost related to each individual component reliability. Most recently for the minimization of the total costs of the system, various researchers have suggested the availability allocation models. The set availability of the system, which is already achieved after optimization as determined by some other technique, behaves as a constraint. The availability models can be classified as (a) formulation of a suitable model of system availability and (b) allocation of availability to each individual component depending upon the system requirements. The major focus of the paper is on required minimum performance of each component which can be done through failure avoidance of each component or redundancy allocation for it along with the cost minimization factor. Several researchers have devoted their study to the reliability optimization problems. Verma and Chari [43] emphasized the influence of common cause shock failures and individual failures individually as well as both together on the determination of availability of a repairable system and also developed related formulae. Ramirez and Bernal [35] used Evolutionary Algorithm for reliability and cost optimization for distribution networks expansion. Stochastic analysis of a Reheating-furnace system subject to preventive maintenance and repair was proposed by Upreti [42] using Markov model and exponential distribution. Garg and Sharma [9] studied reliability, availability and maintainability and did the analysis of these in synthesis unit in fertilizer plant. Different multi-objective and Single-objective constrained and unconstrained problems have been successfully solved to give competitive results using GWO. Fouad et al. [8] found additional number of neighboring nodal points using GWO technique. Mosavi et al. [27] applied three data sets including Iris, Lenses and Sonar to train the multi-layer perception neural networks, using GWO. Gupta and Saxena [10] applied GWO for finding parameters for the successful automatic power dispatch in two interconnected areas. Whereas, Jaya Bharati et al. [11] used crossover and mutation with GWO to solve economic power transmission problem. Zhang et al. [47] used GWO technique for minimizing the fuel cost and avoiding the threat areas in the (unmanned) ACV problem. Manikandan et al. [22] did the gene selection on the of micro array data using binary and mutated GWO approaches. Kamboj et al. [13] proposed GWO for the non-convex economic load dispatch problem. Multi–Objective GWO was proposed by Mirjalili et al. [25] in which an archive defining the global optimum solution is introduced into the original GWO for retrival of the Pareto Optimal solution. Kumar A [14] proposed GA and fuzzy logic for reliability of industrial systems. Kumar et al. [16] used GWO for complex system reliability optimization due to its highly efficient results to optimize reliability and cost of life support system in a space capsule and complex bridge system. Also, Kumar et al. [15] proposed the use of GWO for the comparison and analysis of availability and cost of the engineering systems in series configuration. Kumar et al. [17] continued further and proposed the use of GWO for the safety system of a nuclear power plant to optimize the reliability cost of the residual heat removal system. Negi et al. [28] presented a review and applications of the various forms and hybrids of GWO. Uniyal et al. [41] presented an overview of the reliability applications of few Nature inspired optimization techniques Various forms of GWO have been proposed to solve complex systems reliability optimization problems with very competitive results. In the case of WSNs. Li et al. [20] proposed Modified Discrete GWO (MDGWO) for multi-level image thresholding in which the optimized function Kapur's entropy was used along with the discrete nature of the threshold values. Mirjalili et al. [20] presented Multi-objective GWO (MOGWO) using Pareto-optimal solutions for solving global engineering problems. Other varied forms include Chaotic GWO [23] and Refraction Learning GWO [44]. No free lunch theorem [45] says that no single meta-heuristic can solve all complex problems of optimization. Pant et al. [31] proposed the method of solution for nonlinear system of equations using metaheuristics. Also, Pant et al. [30] presented an advanced approach of Particle Swarm optimization for reliability optimization. In addition to this they [29] also proposed a State of Art review of the flower pollination algorithm development. Pant et al. [32] also applied multi-objective particle swarm optimization (MOPSO) technique for solving reliability optimization problem. Pant et al. [33] presented modified PSO algorithm for nonlinear optimization problems. Li and Haimes [19] proposed decomposition method for the reliability optimization of large complex systems. Developed by Kennedy and Eberhart, [7] PSO has been used to solve many real-world engineering problems to get much competitive results. With further development Coelho [6] solved reliability-redundancy optimization problem using an efficient PSO approach for mixed integer programming problem. Kumar et al. [18] solved the reliability optimization problems of complex systems using CSA. Baskan [2] proposed CSA with L'evy Flights to determine optimal link capacity expansions in road networks. Buaklee and Hongesombut [5] proposed the CSA for solving optimal DG allocation in a smart distribution grid.

Hybridized Optimization Algorithms are those metaheuristics which use the characteristics of each of the involved algorithms in the best possible way in order to give much competitive results in terms of convergence rates, stability, efficiency and quality results than the individual algorithm alone. Some of these are GWO-ACO [1], GWO-GA [38], and GWO-ANN [40].

For optimal convergence rate and highly competitive results as compared to the existing methods leading to global optimum solution, nature inspired algorithm called the Metaheuristics can play a major role. Broadly, they are classified as population oriented (PSO, ACO, GWO, GA.) or trajectory oriented (SA).

Section II deals with the illustration of the different stages of the butter oil processing system. Section III explains GWO and CSA used for the minimization of expenditure in a butter oil processing system. The mathematical model devised for the optimization problem is presented in section IV. In section V the outcomes obtained by the GWO algorithm are discussed along with the investigation of the statistics and sensitivity analysis done thereby. Section VI proposes the conclusions and further scope of the research.

II. Demonstration of the industrial system considered

A butter oil processing plant is discussed below to demonstrate the suggested approach of GWO technique. It is assumed to be a repairable industrial system of a kind based in Northern India.

Description of six sub-units of butter oil processing and manufacturing industrial plant is presented below [36].

I. Separator (Sub-unit I):

Separator uses the law of centrifugal force to separate cream from the milk. To separate the cream (which contains fats) from the milk, chilled milk is introduced into the separator from the refrigerators. This removes 40-50% of fats from the milk and the skimmed milk which remains in the silos is used for making milk powder. Sub-part I is composed of three components in series which are motor, bearings and high-speed gearbox.

II. Pasteurizer (Sub-unit 2):

In this sub-unit pasteurization of cream is done. In this process cream is heated to at least 71°C which may go to 80-82°C in actual practice as long as the process of pasteurization is completed. It involves destruction of unwanted organisms and pathogenic organisms. The enzymes present become inactivated and the volatile substances are also removed. The substances which tan the contents also get removed in the heating process. Then on one side pasteurized milk goes out of this sub-unit through the outlets and on the other side storage of the pasteurized cream takes place in the double-coated tank for the next processing step. The flow of the milk gradually gets obstructed as some residue particles of milk stick around the outlet and form sludge with the passage of time leading to blockage in the outlet causing the sub-unit to fail. The sub-unit 2 has a series of motor and bearings.

III. Butter preparation without break (sub-unit 3):

The storage tank pours the butter into the butter preparation machine where butter is made continuously. Butter granules are formed due to continuous churning process in the machine which produces butter milk also. Then raw milk silos pump back the buttermilk produced during churning process. The butter granules formed are put to further processing with purpose of getting a homogeneous mass of butter. With the help of trolleys the homogeneous butter is shifted to melting vats. There is a series of gearbox, motor and bearings in the butter making machine.

IV. Melting vats (sub-unit 4):

This unit is a double coated tank for carrying out process of melting of butter. Heating butter to 107°C very gently evaporates water from the melting butter. After melting, it is important to keep the melted butter undisturbed for at least half an hour. This sub-unit is composed of mono block pumps, motors and bearings in series.

V. Butter-oil cleanser (sub-unit 5):

From the melting vats butter-oil is shifted to settling tanks to let the butter-oil settle for few hours. The butter-oil residue formed in the settling period is then removed and the residue free butter-oil is stored in the storage tanks. For storing butter-oil suitably, it is allowed to cool to 28-30°C. In this sub-unit a motor and gear box are connected in series.

VI. Packaging (sub-unit 6):

With the help of a pouch-filling machine, packets of processed butter are made in this sub-unit. The machine automatically fills, flows the packets and seals them. There is a printed circuit board and a pneumatic cylinder connected in series in this sub-unit [36].
All these sub-units are connected in series.

III. Nature Inspired Optimization techniques

I. Grey Wolf Optimizer:

I. The impulse that led to GWO

The two important phenomena that led to the development of the GWO algorithm are the social intelligence and hierarchical attitude among the wolves, which can be collectively defined as their social intelligence to carry out an efficient hunting mechanism. In the entire hunting process the four predominant types of wolves taking part can be categorized as alpha, beta, delta and omega in the of their leading capacity. These become the four candidates for initial solution. The alpha being the strongest leads the entire hunting process and the others follow to mechanism successful. This very effective mechanism has been simulated to develop an algorithm to find global optimum solution to many real-world engineering problems. The wolves of different capacities become the four candidates for solutions, which are improved in the iterations that follow, become the four candidates for initial solution.

II. Mathematical Model formulation of the GWO Algorithm

The detailed model:

- Tracking (approaching).
- Encompassing.
- Attacking.

The equations constructed to carry out the simulation are as follows.

$$D = \left|C.X_p(t) - X(t)\right| \quad (4)$$

$$X(t+1) = X(t) - A.D \quad (5)$$

Note that, in the equations, use of vectors help the use of the model to the required number of dimensions. Here, $X(t+1)$ expresses the location the wolf reaches in time $(t+1)$. $X(t)$ is the present location of the wolf, A is a coefficient matrix and D defines the location of the prey Xp. Here, A and C are represented as follows:

$$A = 2a.r_1 - a \quad (6)$$

$$C = 2.r_2 \quad (7)$$

where, r_1 and r_2 are random vectors in the interval [0,1]. The components of the vector a are linearly decreased from 2 to 0 over the course of iterations. The value of A ranges from -2 to 2 as there are random variables in the expression. The premises that alpha, beta and delta are the three best solutions in GWO is taken considering that they have good idea of the position due their strength in the entire population. So, the other wolf should try to update their position as follows:
where, r_1 and r_2 are random vectors in the interval [0,1]. The components of the vector a are linearly decreased from 2 to 0 over the course of iterations. The value of A ranges from -2 to 2 as there are random variables in the expression. The premises that the alpha, the beta and the delta are three best solutions in GWO is taken considering that they have good idea of the position due their strength in the entire population. So, the other wolf should try to modify their position as follows:

$$X(t+1) = \frac{1}{3}X_1 + \frac{1}{3}X_2 + \frac{1}{3}X_3 \quad (8)$$

where, $X_1, X_2,$ and X_3 are evaluated with the equations:

$$X_1 = X_\alpha(t) - A_1.D_\alpha$$

$$X_2 = X_\beta(t) - A_2.D_\beta$$

$$X_3 = X_\delta(t) - A_3 . D_\delta \quad (9)$$

Here, D_α, D_β, D_δ are calculated as follows:

$$D_\alpha = |C_1 . X_\alpha - X|$$
$$D_\beta = |C_2 . X_\beta - X|$$
$$D_\delta = |C_3 . X_\delta - X| \quad (10)$$

Initialize the grey wolf population $X_i \; (i = 1, 2, \ldots\ldots, n)$
Initialize a, A and C
Calculate the fitness of each search agent
$\vec{X}_\alpha$ = the best search agent
$\vec{X}_\beta$ = the second best search agent
$\vec{X}_\delta$ = the third best search agent

while (t < Max number of iterations)
for each search agent
Update the position of current search agent by equation (8)
end for
Update a, A, and C
Calculate the fitness of all search agents
Update $\vec{X}_\alpha$ $\vec{X}_\beta$ $\vec{X}_\delta$
t = t+1
end while
return $\vec{X}_\alpha$

Fig. 1 Pseudo code of the GWO algorithm

Pseudo code of the GWO algorithm is given in Figure 1 [24].

III. Proper survey (exploration) and effective utilization (exploitation) in the hunting mechanism:

Surveying enough before attacking is very important to make the process successful. The decisions of the surveying wolves lead to the effective positioning of the following wolves. To simulate this, the values of the parameters a and A have to be chosen in their ranges to so to get the best value of A. It has been established that IAI > 1. As the process of exploration or surveying and approaching reaches its peak then the attacking decisions depend on the parameter A and it should be and IAI < 1. Now here it is important that unless there is appropriate approaching of the prey, the attacking process won't be that effective. So, choosing the parameters within the range, according to the constraints is very important firstly to properly survey and explore the search space enough before utilizing and exploiting so as to avoid any local convergence of the solution. Thus, achieving global solution is the objective behind the required amount of investigation of the search space and utilizing the results of the investigation to get the optimum solution via proper exploitation as shown in fig. 2. GWO gives an efficiently converged result as compared to existing optimization methods like PSO, ACO, GA, cuckoo search, and few more.

II. Cuckoo Search Algorithm (CSA):

I. Cuckoo's breeding strategy

CSA [46] has its roots in the hostile and vigorous strategy of reproducing its young-ones in some species of fascinating bird cuckoo which can make beautiful sounds. The cuckoos belong to the

Cuculidae family of birds. Some of them are brood parasites which search for a nest of the host birds of different species probably to lay and hide their eggs. The host bird either tries to engage in direct conflict with invading cuckoo and tries to throw away the eggs of the invading cuckoos or leave its own nest and builds a new nest altogether. To increase their reproductivity some species of cuckoos like Tapera, mimic even some characteristics like color, pattern of the eggs and call of the chicks of the host species which really help in reducing abandoning of their eggs. Specific timing of egg laying in the host nest by cuckoos so that they can be hatched earlier than the host eggs is also a strategic pattern of cuckoos to throw the host eggs out of the nest. Cuckoos have developed basically three types of parasitic nature: nest takeover, cooperative parasitism and intraspecific parasitism. To increase the share of food for the cuckoo's chick in the host nest the cuckoo throws the host eggs out of the nest.

II. Idea of Levy Flights

Animals in nature, look for the food in an effective manner which is often much random and quasi-random way. Every next move is dependent on the present position. The shift to the new location and the direction chosen are probabilistic in nature which can hence be mathematically modelled. Levy flights [4, 34] characteristics have been observed in many animals and insects. Be it the landscape exploring by the fruit flies Drosophila melanogaster or the human behaviour such as the hunter gatherer Ju/'Hoansi [4] or the pattern of light all show the characteristics of Levy flights. The outstanding performances [37] shown by the application of such behaviour to the optimization problems for global optimal search have been tested successfully.

III. Cuckoo Search Model

Before presenting the actual model, the premises which lead to the model can be as follows.

- Every cuckoo in particular ensures laying one egg in one time in a nest chosen randomly;
- The highly potent eggs (solutions) of the ideal nests have the capability of being transferred to the next generations;
- The probability of revealing the stranger egg is from 0 and 1 which is approximately equal to the fraction of the number of nests being renewed and built. The probability obtained which can lead to removal of the stranger egg or building of a new nest by the host bird. Also, every cuckoo has only a fixed number of nests for laying their eggs.

The fitness of a solution is important and for a maximization problem it has a fixed ratio to the objective function.

A new solution x (t + 1) for say kth cuckoo can be generated by applying the L'evy flight feature as follows [38]

$$x\,(t{+}1)\,i = x\,(t)\,i + \alpha \oplus L'evy(\lambda) \tag{11}$$

where, $\alpha > 0$ is the size of the step and the suitable problems can be based on the same scale and it can be $\alpha = O\,(1)$. The product $\oplus$ represents the multiplication at each entry. The L'evy flight represent the random steps in the random walk whereas for the large steps L'evy distribution is applicable as follows:

$$L'evy \sim u = t - \lambda, \quad (1 < \lambda \leq 3) \tag{12}$$

This produces infinite mean and variance which explain the steps taken by the cuckoo in succession and is based on power-law step length distribution with a heavy tail. Since the probability of a cuckoo egg getting identified by the host bird is very less it is more important that the fitness function should be a function of the difference in solutions. Thus, random walk and random steps process chosen is very suitable. Pseudo code of CSA algorithm is given in Figure 2 [30].

begin

Objective function $f(x),\ x=(x_1,x_2,\ldots\ldots\ldots,x_d)^T$;

Initial a population of n host nests $x_i\ (i=1,2,\ldots\ldots\ldots,n)$;
while (t <MaxGeneration) or (stop criterion);
Get a cuckoo (say i) randomly by L´evy flights;
Evaluate its quality/fitness F_i;
Choose a nest among n (say j) randomly;
if ($F_i > F_j$),
Replace j by the new solution;
end
Abandon a fraction (p_a) of worse nests
[and build new ones at new locations via L´evy flights];
Keep the best solutions (or nests with quality solutions);
Rank the solutions and find the current best;
end while
Postprocess results and visualisation;

end

Fig. 2 Pseudo Code of Cuckoo Search Algorithm

IV. Formulation of the Mathematical model of the proposed problem

It is not possible to predict the behavior of a system perfectly even from the past records so, it is important to analyze the available parameters in an appropriate manner and some assumptions can be helpful in formulation of availability model the series-parallel system and use the GWO algorithm for cost optimization. Before formulating the mathematical model of the problem following important premises are notable.

- The components or sub system are not dependent on each other and so the failing and repairing of one the component is independent of the other and do not interfere with each other.
- The components do not fail simultaneously.
- The failure ($\boldsymbol{\lambda_i}$) repair rate (μ_i) are constants such that $\boldsymbol{\lambda_i} < \mu_i$.
- The repair and maintenance start in the event of failure of a component immediately with separate maintenance system available for each component.

The proposed optimization model requires expression for cost minimization along with the constraint that the system availability should be greater than the minimum availability criteria.

I. Availability and total cost

The constituent components of the proposed industrial system are as arranged and put in the reliability block diagram (RBD). The system consists of the series-parallel configuration for which the availability expressions with the basic parameters are as follows:

I. Series system.

$$Av_s = Av_1.Av_2.\ldots.Av_n \sim 1-\left(\frac{\lambda_1}{\mu_1}+\frac{\lambda_2}{\mu_2}+\ldots\ldots+\frac{\lambda_n}{\mu_n}\right) \tag{13}$$

where, $\lambda_s \sim \lambda_1+\lambda_2+\cdots+\lambda_n$ and $\mu_s \sim \frac{\lambda_1+\lambda_2+\cdots+\lambda_n}{\frac{\lambda_1}{\mu_1}+\frac{\lambda_2}{\mu_2}+\cdots\frac{\lambda_n}{\mu_n}}$

II. Parallel system

$$Av_s \sim 1-\frac{\lambda_1.\lambda_2\ldots\lambda_n}{\mu_1.\mu_2\ldots\mu_n} \tag{14}$$

where, $\lambda_s \sim \frac{\lambda_1.\lambda_2.....\lambda_n(\mu_1+\mu_2+\cdots\mu_n)}{\mu_1.\mu_2....\mu_n}$ and $\mu_s \sim \mu_1 + \mu_2 + ... \mu_n$

Here, Av_s and Av_i denote the availability of the system and i^{th} component, λ_i and μ_i denote the failure and repair rate respectively for the i^{th} component of the system and system failure and repair rate are denoted by
λ_s and μ_s.

The expressions for availability, failure rate and repair rate are from [3]. Thus, from the definitions and expressions for availability, following expression for availability (Av_s) for the proposed system can be presented approximately as follows:

$$Av_s = f(MTBF_1, MTBF_2, ..., MTBF_n, MTTR_1, MTTR_2 .... MTTR_n) \quad (15)$$

Failure rate of a system depends on MTBF. The higher value of the MTBF of any component causes decrease in the failure rate of the component. This generally leads to an increase in the cost sharply [21] and at the same time also the reliability of the system is increased as a whole. The relation between MTBF and manufacturing cost [39] can be expressed as follows:

$$CMTBF_i = \alpha_i.(MTBF_i)^{\beta_i} + \gamma_i \quad (16)$$

where, the manufacturing cost and $MTBF$ of the ith component are denoted by $CMTBF_i$ and $MTBF_i$ respectively, α_i, β_i and γ_i are constants which represent the physical properties of the i^{th} component and value of $\beta_i > 1$.

The output of a system depends on failure rate and reduces the efficiency of the system as a whole. Timely repairing of the failed component can help not to affect the efficiency and output of the system to some extent. Maintenance and repair of the failed component as soon as possible can be carried out with help of experts and repairing by standard equipment. $MTBF_i$ and repairing cost of the individual components ($CMTTR_i$) are linearly related to each other and mathematically can be represented as follows [12]:

$$CMTTR_i = a_i - b_i.(MTTR_i) \quad (17)$$

where, a_i and b_i are constants related to the i^{th} component of the system. From Equations (12) and (13), total cost can be expressed as:

$$T_c = \sum_{i=1}^{n}(\alpha_i.(MTBF_i)^{\beta_i} + \gamma_i) \ \sum_{i=1}^{n}(a_i - b_i.MTTR_i) \quad (18)$$

II. Optimization model for the cost minimization of butter oil plant:

Using equations, (1) and (4) optimization model of the problem is framed as follows:

Minimize T_c

Subject to $Av_s \geq A_{mini}$

$LbMTBF_i \leq MTBF_i \leq UbMTBF_i$

$LbMTTR_i \leq MTTR_i \leq LbMTTR_i$

$i = 1, 2, ...6 \ \ All\ variables \geq 0$

$Av_s = 1 - [5\frac{MTTR_1}{MTBF_1} + \frac{MTTR_2}{MTBF_2} + \frac{MTTR_3}{MTBF_3} + \frac{MTTR_4}{MTBF_4}]$

$$T_c = \sum_{i=1}^{6}(\alpha_i.(MTBF_i)^{\beta_i} + \gamma_i) + \sum_{i=1}^{6}(a_i - b_i.MTTR_i) \quad (19)$$

$A_{mini} = 0.96$

where, lower and upper bounds of MTBF and MTTR for i^{th} component are denoted by $LbMTBF_i$, $UbMTBF_i$, $LbMTTR_i$, $UbMTTR_i$ out of the total 6 components of the given plant. GWO algorithm solves the formulated optimization problem quite efficiently. The values of α, β and γ are respectively taken as 0.92, 1.94 and 1250. The respective values of a and b are taken [14], [18], [150] and [50]. The range of lower and upper bounds of mean time between failure (MTBF) and mean time to repair (MTTR) for various components are 4000 hours to 4200 hours and 2 hours to 6 hours respectively.

V. Results and Analysis

GWO has an edge over other nature inspired optimization algorithms as in it the search agent and fitness function are not directly correlated. In GWO various search agents modify their position in accordance with the positions taken by the wolf alpha, beta, and delta. With this feature, GWO finds application to solve problem of any type of constraints with its mechanism remaining the same. This model for minimization of the expenditure in the butter oil processing plant system, uses the simplest method of constraints handling like penalty functions. For this cost minimization problem of butter oil processing plant, 100 grey wolves have been fixed and we run GWO algorithm with iterations around 200. On the other hand, in cuckoos search algorithm, number of nests have been fixed at 30 with the chance of finding the alien eggs/solutions is kept at 0.35. Total number of iterations have been set as 1000. After that, the GWO algorithm and Cuckoo search algorithm has been run in the MATLAB and table 1 shows the results, which are better the earlier in some respects definitely.

The search history of GWO algorithm is tabulated in the following manner for the same problem. The minimum system cost 5.61615071665e+07 obtained by GWO is similar to that obtained by CSA but there exists a difference in the function evaluation as shown in Fig. 3. GWO takes only 20000 function evaluations on the other hand CSA takes 60000 FE for the same cost. Both GWO and CSA are kept at system availability as shown in table 1.

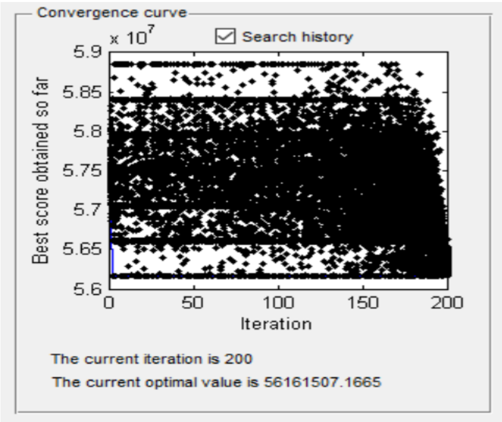

Fig. 3 Search history of GWO for butter oil processing plant

Table 1. Comparison results for butter oil processing plant

	Grey Wolf Optimizer (GWO)		Cuckoos Search Model (CSA)	
Components	**Mean Time Between Failure (MTBF in hours)**	**Mean Time to Repair (MTTR in hours)**	**Mean Time Between Failure (MTBF in hours)**	**Mean Time to Repair (MTTR in hours)**
Motors	4025	5	4025	5
Bearings	4100	3	4100	3
Gear Box	4075	5.5	4075	5.5
Pumps	4150	3.5	4150	3.5
Circuit Box	4070	3	4070	3
Cylinder	4115	3.5	4115	3.5
System Cost	5.61615071665e+07		5.616150716646019e+07	
System Availability	0.978716807		0.978716807	
Number of Iterations	200		1000	
FE	20000		60000	

VI. Conclusion and further scope:

For a series-parallel system, exact methods of reliability optimization are not enough to get effective results. This is because it may lead to an unnecessary rise in the costs of the whole system. Since the aim of any industrial unit is profit generation along with the satisfaction of the other constraints of weight, volume, maintenance policies, maximum performance in terms of reliability and availability so, nature inspired optimization algorithms like GWO and CSA work quite well under all these conditions to get better results as this butter-oil processing plant system show. These optimization techniques work to calculate the optimum values of MTBF and MTTR so well that they consider the constraints to gain maximum out of the series-parallel system even with limitations of its structure. The efficient results of the GWO and CSA algorithms to the present problems help the decision makers to derive the properties of the components to be chosen in future to get the best results. Together with this, comparatively GWO show high performance over CSA algorithms with regard to total number of functions evaluated and hence can save time of decision makers (DM). Hence, the DM can further decide about the policies of the design and repair based on GWO to improve the performance to meet the other constraints if any.

Declaration of Conflicting Interests:

The Authors have no conflict of interests.

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An Economical Order Quantity Inventory Model for Time-Dependent Deterioration Rate with Price Dependent Demand under Permitted Delay

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Abstract

In the current study, we develop an inventory model for the deteriorating items under the variable holding cost. The decline (or deterioration) rate depends on time; also, the demand rate depends on the price of the item. The shortage cost is linear in nature. The interest rate is a component of selling prices and shortages. Supplier offers a trade-credit period to the customer during which there is no interest charged, but upon the prescribed time limit expiry, the supplier will charge some interest. This study validates with a numerical example and explains the sensitivity analysis, and the optimal solution not only exists but also feasible.

Keywords: Inventory, permitted delay, deterioration and time varying holding cost.

I. Introduction

Generally, an inventory is typical of idle resources in organizations for upcoming use. Manufacturing organizations naturally have lists of raw materials, apparatuses, tools, equipment, semi-finished items, finished items, etc. In service organizations such as banks, financial institutions, hospitals, etc., the inventory consists of various items to be used in the multiple types of forms (for various banking operations), brochures, and pamphlets (for details of different banking policies, schemes, and instruments), etc. Banks also have inventories of currency notes and coins. Hospitals have medical equipment stocks such as a syringe, thermometer, drip, various one-time instruments used by medical professionals, etc. other accessories such as bandages, cotton, spirit, etc., to multiple medicines. Thus, no organization works without inventory.

Due to the effects of deterioration, this assumption is not always applicable to certain commonly used physical commodities such as wheat, rice, or some other sort of grains, vegetables, organic products, and so on. A few parts of these products have been damaged, decayed, gasifier, or influenced by various elements. These degraded parts are not so much lost to the stock management office. The decrease in items is one of the severe variables in any stock and creation framework. The acknowledgment of this factor provoked modellers to consider the corruption factor as one of the displaying angles. The weakening stock issue has been widely concentrated by various analysts now and then. Covert and Philip [4] proposed a model of things under a consistent rate of weakening; Deb and Chaudhuri [5] finished the absolute most recent work here, Aggarwal et al. [2] broadened the Goyal model when the circumstance deteriorated. Goyal and Giri [9] projected a typical model affected by item weakening. Roy [17] proposed an inventory model for deteriorating items with price dependent demand and time-varying holding cost. Kumar et al. [14] built up a stock model that decayed after some time; Kingsman [11] developed the effect of payment rules on ordering and stockholding in purchasing. Aarya and Kumar [1] proposed a stock model with various restrictions at a constant decaying rate.

An EOQ model accepts that the retailer must compensate the provider following getting the merchandise in practical situations. Providers may regularly permit retailers to direct advance financing to expand requests or diminish stock. This implies the vendor will support the purchaser with a credit period to settle the sum owed, and during this period, the sum owed won't acquire any intrigue. Goyal [8] built up a stock model affected by professional credit; additionally, Goh [7] presented an EOQ model with general demand and holding cost functions; Teng et al. [18] established an optimum pricing and assembling strategy under permitted delay, Mondal et al. [16] demonstrated a model of upgrading objects under demand rate depends on price, Kumar et al. [13] built up a model for the diverse interest rates under exchange credit and in [12] a stock model of the incremental holding cost with the admissible instalment delay proposed, Weiss [19] presented an economic order quantity model with non-linear holding cost. Giri and Chaudhuri [6[presented a heuristic model for deteriorating items with shortages and time-varying demand and costs, and Yu [20] proposed an inventory policy for products with price and time-dependent demands.

In the field of inventory management, many authors use various other types of needs and factors. Some researchers believe that the demand rate is static, linear, price-related, and inventory-related. The actual target demand may be related to time, inventory, and price. Haley [10] offered Inventory policy and trade credit financing, Chapman et al. [3] developed Credit policy and inventory control, Kumar et al. [15] presented a model on preservation technology with trade credits under demand rate dependent on an advertisement, time and selling price.

In view of the above writing survey, we set up a degradation model. When the corruption rate is relative to time, and the holding cost is variable, the interest rate is an element of selling price and all-out deficiency, bringing about lack and total accumulation. On account of postponed instalment authorization, the interest rate is an element of the business cost, and we utilize a numerical example to check the model.

Assumptions and Notations:

Assumptions:

- Deterioration rate changes with time.
- Shortage is permitted and 'completely backlogged'.
- Demand function is a component of trade cost.
- Cost of ownership is linear.
- Renewal or 'replacement' rate is immediate.
- Leading- time is zero.
- Trade credit is permitted.

Notations:

1) C_1 : Shortage cost per unit time.
2) C_2 : Cost of an item per unit.
3) $\theta(t) = \theta t$: Deterioration rate.
4) $f(p) = (\alpha p^{-\beta}) > 0$ is the demand rate, where $\alpha, \beta > 0$.
5) $H(t) = (h + at)$, where $a > 0$, $h > 0$ is holding cost per unit time.
6) q: Order quantity per cycle.
7) p : Marketing charge per unit item.
8) M is the trade credit period.
9) A is the ordering cost.

10) T is the time period.
11) In period $0 < t < T_1$ the inventory is positive.
12) In time (T_1) the stock is drained because of the crumbling and request of the thing. At time (T_1) the stock goes to zero and shortage starts.
13) I_e: Interest received for each unit time.
14) I_p : Interest paid for each unit time with $I_p > I_e$.

II. Model Formulation and Solution

Modeling, and Solution of Proposed Model

$$\frac{dI(t)}{dt} + \theta(t)\, I(t) = -f(p), \qquad 0 \le t \le T_1 \tag{1}$$

$$\frac{dI(t)}{dt} = -f(p), \qquad T_1 \le t \le T \tag{2}$$

with initial condition $I(T_1) = 0$

Solution of equations (1) and (2) are given as follows:

$$I(t) = (\alpha p^{-\beta}) \left[(T_1 - t) + \theta\left(\frac{T_1^3}{6} - \frac{T^3}{3} - \frac{T_1 t^2}{2}\right)\right.$$

$$\left. + \theta^2\left(\frac{T_1^5}{40} - \frac{t^5}{15} - \frac{t^2 t_1^3}{12} + \frac{t_1 t^4}{8}\right)\right], \quad 0 \le t \le T_1 \tag{3}$$

and $I(t) = -(\alpha p^{-\beta})(t - T_1) = (\alpha p^{-\beta})(T_1 - t), \ T_1 \le t \le T$ (4)

Holding cost

$$HC = \int_0^{T_1} H(t)\ I(t)\ dt$$

$$= (\alpha p^{-\beta})\ h\ .\left[\frac{T_1^2}{2} + \frac{\theta T_1^4}{12} + \frac{\theta^2 T_1^6}{90}\right] + a(\alpha p^{-\beta}) \left[\frac{T_1^3}{6} + \frac{\theta T_1^5}{40} + \frac{\theta^2 T_1^7}{336}\right] \tag{5}$$

Shortage cost

$$SC = -C_1 \int_{T_1}^{T} [-(t - T_1) f(p)]\ dt$$

$$= C_1 \frac{(\alpha p^{-\beta})}{2} (T - T_1)^2 \tag{6}$$

Stock loss due to deterioration

$$D = (\alpha p^{-\beta}) \int_0^{T_1} e^{\frac{\theta t^2}{2}} dt - (\alpha p^{-\beta}) \int_0^{T_1} dt$$

$$= (\alpha p^{-\beta}) \left[\frac{\theta T_1^3}{6} + \frac{\theta^2 T_1^5}{40}\right] \tag{7}$$

Order quantity

$$q = C_2 \left[D + \int_0^T (\alpha p^{-\beta})\ dt\right]$$

$$= C_2(\alpha p^{-\beta}) \left[\frac{\theta T_1^3}{6} \ \frac{\theta^2 T_1^5}{40}\right] + (\alpha p^{-\beta})\ T \tag{8}$$

Presently, there are two prospects with respect to the delay period M of allowable deferral in installments.

Case I: $M \le T_1$,

Case II: $M > T_1$

Case I: $M \le T_1$: In this case, the interest payable for each period of unsold stock after the maturity date (M) is

$$IP_1 = C_2 I_p \int_M^{T_1} I(t)\ dt$$

$$= C_2(\alpha p^{-\beta}) I_p \left[\left(\frac{T_1^2}{2} - \frac{\theta T_1^4}{12} + \frac{\theta^2 T_1^6}{90}\right) - \left(M T_1 - \frac{M^2}{2}\right)\right.$$

$$\left. - \theta\left(\frac{M T_1^3}{6} - \frac{M^4}{12} - \frac{M^3 T_1}{6}\right) - \theta^2\left(\frac{M T_1^2}{40} - \frac{M^6}{90} - \frac{M^3 T_1^3}{36} + \frac{M^5 T_1}{40}\right)\right] \tag{9}$$

In the calculation, the interest earned (IE_1) in each period is found as

$$IE_1 = C_2 I_e \int_0^{T_1} t.f(p)\ dt$$

$$= C_2 I_e\ (\alpha p^{-\beta})\ \frac{T_1^2}{2} \tag{10}$$

Total profit function is

$$\mathrm{Z(T,\ T_1,\ p)\ =\ p.f(p) - \frac{1}{T}[OrderingCost + ShortageCost + HoldingCost + Orderquantity + IP_1 - IE_1]}$$

$$= (\alpha p^{1-\beta}) - \frac{1}{T}\left[A + \frac{C_1(\alpha p^{-\beta})}{2}(T-T_1)^2 + (\alpha p^{-\beta})h\left\{\frac{T_1^2}{2} + \frac{\theta T_1^4}{12}\right.\right.$$
$$\left.+\frac{\theta^2 T_1^6}{90}\right\} + a(\alpha p^{-\beta})\left\{\frac{T_1^3}{6} + \frac{\theta T_1^5}{40} + \frac{\theta^2 T_1^7}{336}\right\}$$
$$+C_2(\alpha p^{-\beta})\left\{T + \frac{\theta T_1^3}{6} + \frac{\theta^2 T_1^5}{40}\right\} + C_2(\alpha p^{-\beta})I_p\left\{\left(\frac{T_1^2}{2} - \frac{\theta T_1^4}{12} + \frac{\theta^2 T_1^6}{90}\right)\right.$$
$$-\left(MT_1 - \frac{M^2}{2}\right) - \theta\left(\frac{MT_1^3}{6} - \frac{M^4}{12} - \frac{M^3 T_1}{6}\right)$$
$$\left.\left.-\theta^2\left(\frac{MT_1^5}{40} - \frac{M^6}{90} - \frac{M^3 T_1^3}{36} + \frac{M^5 T_1}{40}\right)\right\} - \frac{C_2 I_e(\alpha p^{-\beta})T_1^2}{2}\right]$$

Let $T_1 = aT;\ \ 0 < a < 1$

Thus, total profit is

$$Z(T,p) = (\alpha p^{1-\beta}) - \frac{1}{T}\left[A + \frac{C_1(\alpha p^{-\beta})}{2}(1-a)^2T^2 + h(\alpha p^{-\beta})\left\{\frac{a^2T^2}{2} + \frac{\theta a^4 T^4}{12}\right.\right.$$
$$\left.+\frac{\theta^2 a^6 T^6}{90}\right\} + a(\alpha p^{-\beta})\left\{\frac{a^3T^3}{6} + \frac{\theta a^5T^5}{40} + \frac{\theta^2 a^7 T^7}{336}\right\} + C_2(\alpha p^{-\beta})\left\{T + \frac{\theta a^3 T^3}{6} + \frac{\theta^2 a^5 T^5}{40}\right\}$$
$$+C_2(\alpha p^{-\beta})I_p\left\{\left(\frac{a^2T^2}{2} - \frac{\theta a^4 T^4}{12} + \frac{\theta^2 a^6 T^6}{90}\right) - \left(MaT - \frac{M^2}{2}\right) - \theta\left(\frac{Ma^3T^3}{6} - \frac{M^4}{12} - \frac{M^3 aT}{6}\right)\right.$$
$$\left.\left.-\theta^2\left(\frac{Ma^5T^5}{40} - \frac{M^6}{90} - \frac{M^3a^3T^3}{36} + \frac{M^5 aT}{40}\right)\right\} - \frac{C_2 I_e(\alpha p^{-\beta})a^2T^2}{2}\right].$$

Presently, our goal is to optimize $Z(T,\ p)$. The essential situations for expanding the profit are $\frac{\partial Z(T,\ p)}{\partial T} = 0$ and $\frac{\partial Z(T,p)}{\partial p} = 0$

We obtain

$$\left[-\frac{A}{T^2} + \frac{C_1(\alpha p^{-\beta})(1-a)^2}{2} + h(\alpha p^{-\beta})\left\{\frac{a^2}{2} + \frac{\theta a^4 T^2}{4} + \frac{\theta^2 a^6 T^4}{18}\right\}\right.$$
$$+a(\alpha p^{-\beta})\left\{\frac{a^3 T}{3} + \frac{\theta a^5 T^3}{10} + \frac{\theta^2 a^7 T^5}{56}\right\}$$
$$+C_2(\alpha p^{-\beta})\left\{\frac{\theta a^3 T}{3} + \frac{\theta^2 a^5 T^3}{10}\right\}$$
$$+C_2(\alpha p^{-\beta})I_p\left\{\left(\frac{a^2}{2} - \frac{\theta a^4 T^2}{4} + \frac{5\theta^2 a^6 T^4}{4}\right) - \left(\frac{M^2}{2T^2}\right) - \theta\left(\frac{Ma^3T}{3} + \frac{M^4}{12T^2}\right)\right.$$
$$\left.\left.-\theta^2\left(\frac{Ma^5T^3}{10} + \frac{M^6}{90T^2} - \frac{M^3a^3T}{18}\right) - \frac{C_2 I_e(\alpha p^{-\beta})a^2}{2}\right] = 0 \tag{11}$$

and

$$\alpha(1-\beta)p^{-\beta} - \frac{1}{T}\left[-\frac{C_1(\alpha\beta p^{-\beta-1})(1-a^2)}{2}T^2 - h(\alpha\beta p^{-\beta-1})\left\{\frac{a^2T^2}{2} + \frac{\theta a^4 T^4}{12} + \frac{\theta^2 a^6 T^6}{90}\right\}\right.$$
$$-a(\alpha\beta p^{-\beta-1})\left\{\frac{a^3T^3}{6} + \frac{\theta a^5 T^5}{40} + \frac{\theta^2 a^7 T^7}{336}\right\} - C_2(\alpha\beta p^{-\beta-1})\left\{T + \frac{\theta a^3 T^3}{6} + \frac{\theta^2 a^5 T^5}{40}\right\}$$
$$-C_2 I_p(\alpha\beta p^{-\beta-1})\left\{\left(\frac{a^2T^2}{2} - \frac{\theta a^4 T^4}{12} + \frac{5\theta^2 a^6 T^6}{4}\right) - \left(MaT - \frac{M^2}{2}\right)\right.$$
$$\left.\left.-\theta\left(\frac{Ma^3T^3}{6} - \frac{M^4}{12} - \frac{aTM^3}{6}\right) - \theta^2\left(\frac{Ma^5T^5}{40} - \frac{M^6}{90} - \frac{M^3a^3T^3}{36} + \frac{M^5aT}{40}\right)\right\} + \frac{C_2 I_e(\alpha\beta p^{-\beta-1})a^2T^2}{2}\right] = 0 \tag{12}$$

Case II: $M > T_1$

For this situation, the interest unpaid for each cycle is zero, when $T_1 < M \leq T$on the grounds that the provider can be forked over the required funds at the time M, the allowable delay. In this manner, the premium received in each cycle is the premium received during the great stock time frame in addition to the premium earned from the money contributed during the timeframe $(T_1,\ M)$ after the stock is depleted at time T_1, and it is given by

$$IE_2 = C_2 I_e \int_0^{T_1} f(p).t dt + C_2 I_e (M - T_1) \int_0^{T_1} f(p)\, dt$$

$$= C_2 I_e f(p) \frac{T_1^2}{2} + C_2 I_e (M - T_1) f(p).T_1$$

$$IE_2 = C_2 I_e \left(\alpha p^{-\beta}\right) T_1 \left(M - \frac{T_1}{2}\right) \quad (13)$$

Total profit is

$$Z(T, T_1, p) = p.f(p) - \frac{1}{T}[A + \text{ShortageCost} + \text{HoldingCost} + q + IP_2 - IE_2]$$

$$= \left(\alpha p^{1-\beta}\right) - \frac{1}{T}\left[A + \frac{C_1\left(\alpha p^{-\beta}\right)}{2}(T - T_1)^2 + \left(\alpha p^{-\beta}\right) h \left\{\frac{T_1^2}{2} + \frac{\theta T_1^4}{12} + \frac{\theta^2 T_1^6}{90}\right\}\right.$$
$$\left. + a\left(\alpha p^{-\beta}\right) \left\{\frac{T_1^3}{6} + \frac{\theta T_1^5}{40} + \frac{\theta^2 T_1^7}{336}\right\} - C_2 I_e \left(\alpha p^{-\beta}\right) T_1 \left(M_1 - \frac{T_1}{2}\right)\right] \quad (14)$$

Let $T_1 = aT$; $0 < a < 1$.

Thus, the profit function is

$$Z(T, p) = \left(\alpha p^{1-\beta}\right) - \frac{1}{T}\left[A + \frac{C_1\left(\alpha p^{-\beta}\right)}{2}(1 - a)^2 T^2 + h\left(\alpha p^{-\beta}\right)\left\{\frac{a^2 T^2}{2} + \frac{\theta a^4 T^4}{12} + \frac{\theta^2 a^6 T^6}{90}\right\}\right.$$
$$\left. + a\left(\alpha p^{-\beta}\right) \left\{\frac{a^3 T^3}{6} + \frac{\theta a^5 T^5}{40} + \frac{\theta^2 a^7 T^7}{336}\right\} - C_2 I_e a T \left(\alpha p^{-\beta}\right)\left(M_1 - \frac{\alpha T}{2}\right)\right] \quad (15)$$

Presently, our goal is to optimize $Z(T, p)$. The essential conditions for expanding the profit are $\frac{\partial Z(T,p)}{\partial T} = 0$ and $\frac{\partial Z(T,p)}{\partial p} = 0$.

$$\Rightarrow \left[-\frac{A}{T^2} + \frac{C_1(\alpha p^{-\beta})(1 - a)^2}{2} + h(\alpha p^{-\beta})\left\{\frac{a^2}{2} + \frac{\theta a^4 T^2}{4} + \frac{\theta^2 a^6 T^4}{18}\right\}\right.$$
$$\left. + a(\alpha p^{-\beta})\left\{\frac{a^3 T}{3} + \frac{\theta a^5 T^3}{10} + \frac{\theta^2 a^7 T^5}{56}\right\} - C_2 a(M - aT)(\alpha p^{-\beta})\right] = 0 \quad (16)$$

And
$$\alpha(1 - \beta)p^{-\beta} - \frac{1}{T}\left[-\frac{C_1(1-a)^2(\alpha\beta p^{-\beta-1})}{2} T^2 - h\left(\alpha\beta p^{-\beta-1}\right)\left\{\frac{a^2 T^2}{2} + \frac{\theta a^4 T^4}{12} + \frac{\theta^2 a^6 T^6}{90}\right\}\right.$$
$$\left. - a\left(\alpha\beta p^{-\beta-1}\right)\left\{\frac{a^3 T^3}{6} + \frac{\theta a^5 T^5}{40} + \frac{\theta^2 a^7 T^7}{336}\right\} + C_2 I_p a T\left(\alpha\beta p^{-\beta-1}\right)\left(M - \frac{\alpha T}{2}\right)\right] = 0 \quad (17)$$

Solving equations (11) to (17), we find the T^*, and p^*. Also, optimize the function $Z^*(T, p)$ using the essential conditions for maximizing of $Z(T, p)$ are

$$\frac{\partial^2 Z(T,p)}{\partial T^2} < 0, \quad \frac{\partial^2 Z(T,p)}{\partial p^2} < 0 \;\; and \;\; \frac{\partial^2 Z(T,p)}{\partial T^2} . \frac{\partial^2 Z(T,p)}{\partial p^2} - \frac{\partial^2 Z(T,p)}{\partial T \partial p} > 0 \;\; at\; (T^*, p^*).$$

III. Illustrative Example

Table-1: *Sensitivity analysis table*

Parameter	Parameter Changing (%)	Value of Parameter	T	p	Z	Changing in Z*(%)
A	-50%	125	54	13759	413	-8%
	-25%	187.5	61	16842	426	-5%
	0%	250	69	24434	450	0%
	25%	312.5	72	24444	449	-0.2%
	50%	375	74	25061	450.77	0.17%
α	-50%	50	72	14394	205.91	-54%
	-25%	75	69	17424	319.85	-29%
	0%	100	69	24434	450	0%
	25%	125	64	20659	549.54	22.1%
	50%	150	68	35908	683.68	51.93%
β	-50%	0.425	30	182990	106120	23482%
	-25%	0.6375	60	151030	7534	1574%

	0%	0.85	69	24434	450	0%
	25%	1.0625	58.8	1546.4	56	-87.6%
	50%	1.275	62.48	580.2293	11.84	-97.37%
M	-50%	0.0325	67	20658	438.97	-2%
	-25%	0.04875	69	24434	450.5349	0%
	0%	0.065	69	24434	450	0%
	25%	0.08125	69	24434	450.5357	0.1%
	50%	0.0975	69	24434	450.5361	0.12%
H	-50%	0.25	69	24493	450.71	0%
	-25%	0.375	72	32079	469.84	4%
	0%	0.5	69	24434	450.5353	0%
	25%	0.625	67	20659	438.97	-2.6%
	50%	0.75	67	20658	438.96	-2.57%
C_2	-50%	10	70	15247	419.29	-7%
	-25%	15	67	46841	497.62	11%
	0%	20	69	24434	450	0%
	25%	25	66	24444	450.33	0.1%
	50%	30	61	16914	425.0958	-5.53%
C_1	-50%	0.65	71	28270	461.04	2%
	-25%	0.975	69	24443	450.69	0%
	0%	1.3	69	24434	450	0%
	25%	1.625	66	20685	438.87	-2.5%
	50%	1.95	69	28271	460.48	2.33%
a	-50%	0.1	56.2	2230.5	306	-32%
	-25%	0.15	67.8	6162.7	363.28	-19%
	0%	0.2	69	24434	450	0%
	25%	0.25	53	17454	426.89	-5.1%
	50%	0.3	44	176669	606.61	34.80%
θ	-50%	0.01	0.75	12227	608.86	35%
	-25%	0.015	69	1.5294	419.18	-7%
	0%	0.02	69	24434	450	0%
	25%	0.025	62	20659	4388.72	875.3%
	50%	0.03	62	28875	461.83	2.63%
I_p	-50%	0.08	70	15279	419.14	-7%
	-25%	0.12	67	16844	425.39	-5%
	0%	0.16	69	24434	450	0%
	25%	0.2	69	3.2079	469.711	4.4%
	50%	0.24	65	24434	450.33	0.07%
I_e	-50%	0.065	70	28257	460.7	2%
	-25%	0.0975	67	20659	438.96	-2%
	0%	0.13	69	24434	450	0%
	25%	0.1625	72	32103	469.9	4.4%
	50%	0.195	67	20658	439.01	-2.44%

We applied our program to a major cosmetics retailer store in a mega-city to explain the proposed model. In advertising products, including TV/Internet, sunscreens, powders, lipsticks, and baby products, these products were initially promoted, but the products' sales declined slightly.

For the validation of the model numerically, we consider the values of the parameters are as follows:

A = 250, α = 100, β = 0.85 M = 0.065, C_2 = 20, H = 0.5, C_1 = 1.3, a = 0.2, θ = 0.02, Ip = 0.16, Ie = 0.13.
Using Mathematica software, we get the optimal values are as follows:
(Z*) = 450, p* = 24434, and T* = 69.

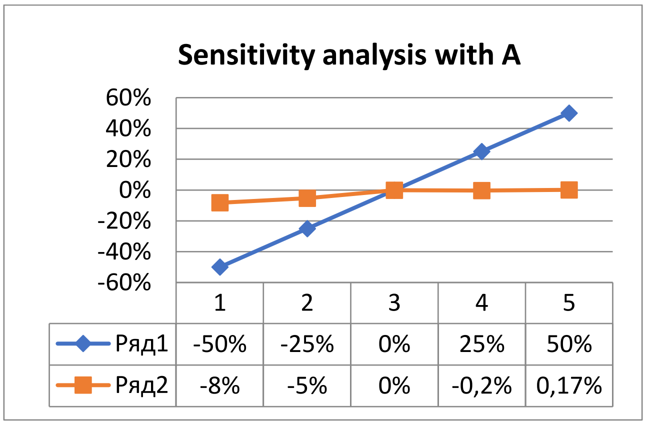
IV. Discussion

For the justification of the proposed model numerically, data for the numerical section, and using Mathematica software, we obtained the optimal values are as follows:
Profit (Z*) = 450, Price (p*) = 24434, and Time (T*) = 69.
An affectability investigation is performed to contemplate the impacts of parameter values on the optimal solution. The Sensitivity analysis table shows the consequences of the model. The following decisions are as follows:

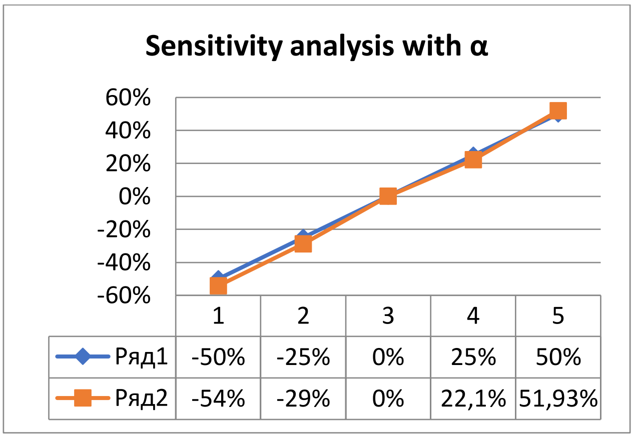
- If the parameters β, C_2, and Ie are changed at the rate of 50%, 25%, -25%, and -50%, then the profit function Z decreases.

- If the parameters A, α, M, H, C_1, a, ɵ, and Ip, are changes at the rate of 50%, 25%, -25%, and -50%, then profit function Z increases.

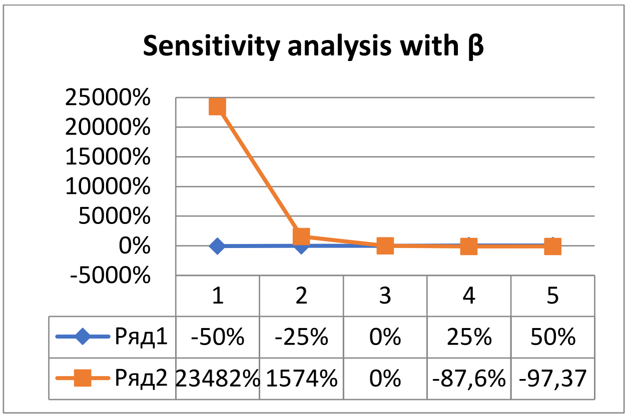
The following figures shows the affectability investigation as for parameters: *A, α, β, M, H, C_2, C_1, a, θ, I_p, I_e* :



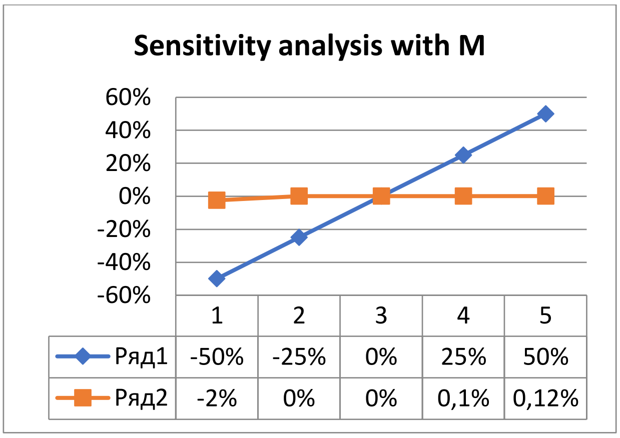
(Figure 1: *w. r. to parameter A***)**



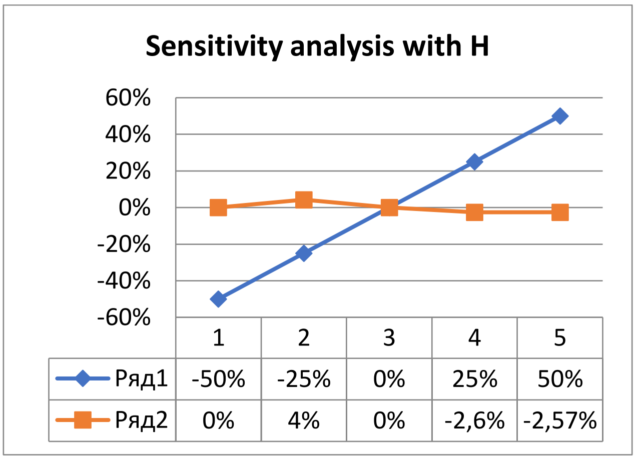
(Figure 2: *w. r. to parameter α***)**



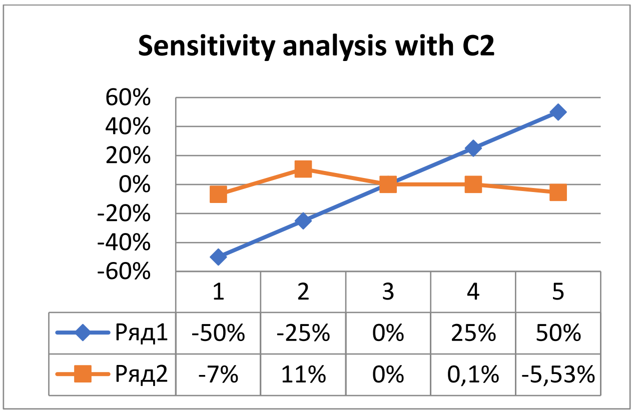
(Figure 3: *w. r. to parameter β***)**



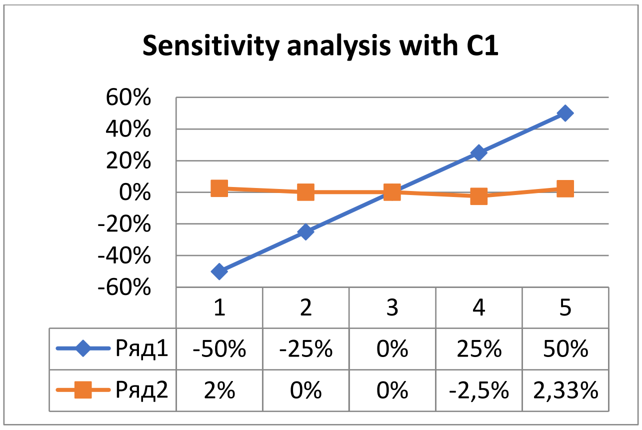
(Figure 4: *w. r. to parameter M***)**



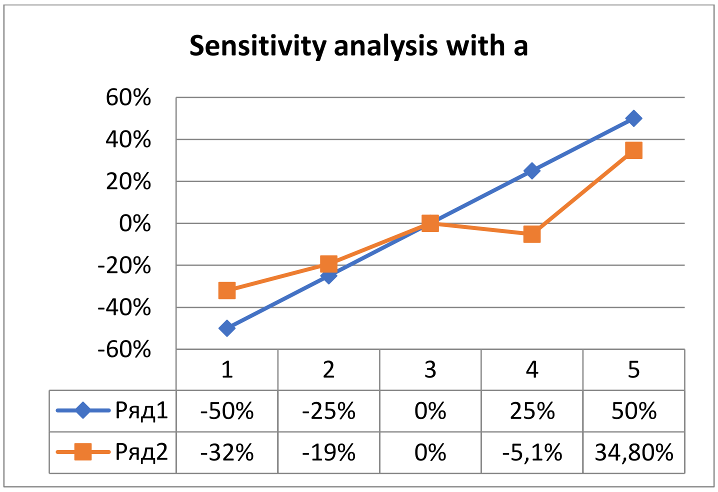
(Figure 5: *w. r. to parameter H***)**



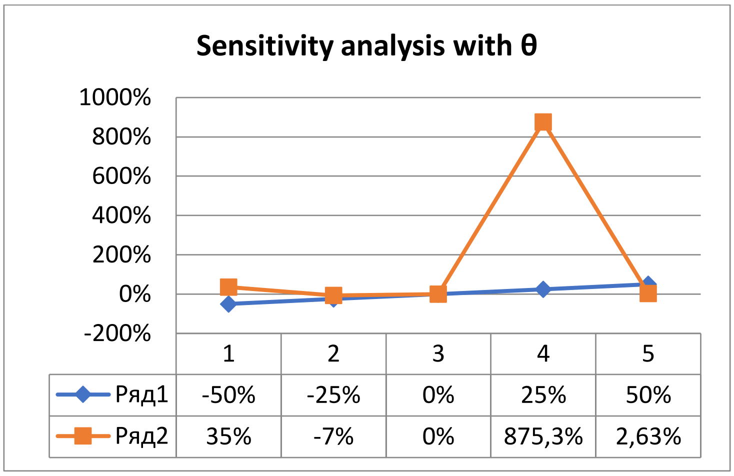
(Figure 6: *w. r. to parameter* C_2)



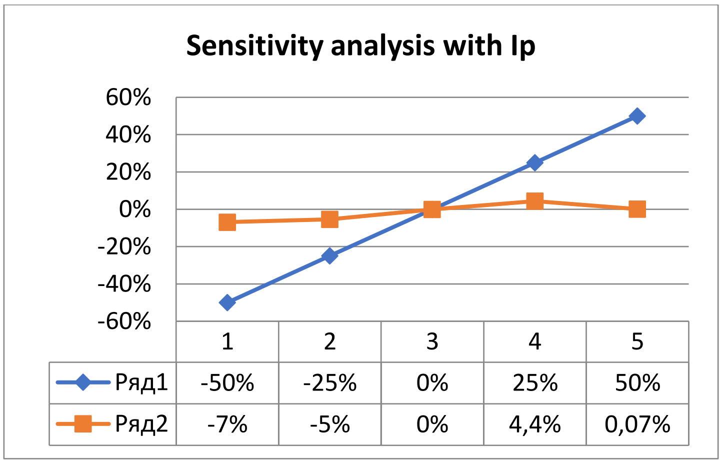
(Figure 7: *w. r. to parameter* C_1**)**



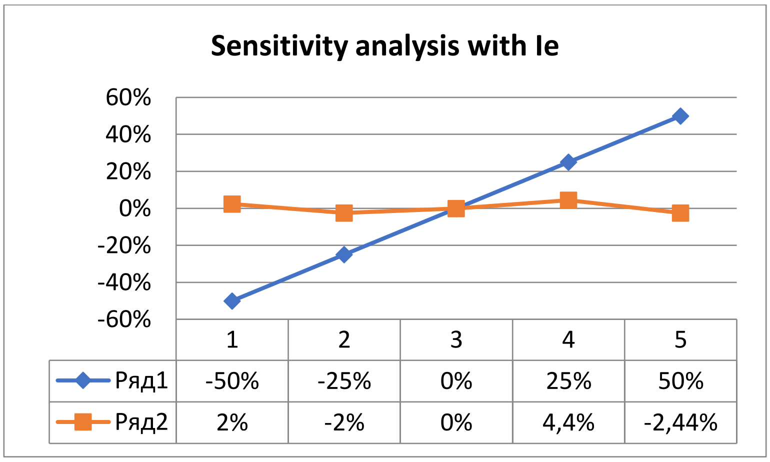
(Figure 8: *w. r. to parameter a***)**



(Figure 9: *w. r. to parameter* θ**)**



(Figure 10: *w. r. to parameter* I_p**)**



(Figure 11: *w. r. to parameter* I_e**)**

V. Conclusions

This study established an inventory model for the linear decline rate and the price-related demand rate under the holding cost. Shortages are permitted and are entirely reproduced and allowed a delayed payment period. Supplier offers a credit limit to the customer during which there is no interest charged, but the supplier will charge some interest upon the prescribed time limit expiry. However, the retailer has stored to make the installment, choosing to profit by as far as possible. Also, approve the model with the assistance of a mathematical model and study the sensitivity analysis. This model further developed with the inflation rate.

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A Production Inventory Model for Deteriorating Items with Effect of Price Discount under the Stock Dependent Demand

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Abstract

In most of the production-based model, the effect of deterioration at initial stage of the production is not assumed even if assumed than considered to be constant, in few productions-based model deteriorations is assumed time-dependent but during initial level of production its effect was not considered. For a product whose life is high this phenomenon is justifying but for those products whose life is very less this is not faring to not assume the effect of deterioration. In the present study, author have considered the effect of deterioration at the initial level of the model and for the rest cycle of the model as well. After a fixed time, a price discount is also offered for partially deteriorated items where a discount rate is offer at the original one, Shortages are not allowed, production is assumed as demand dependent, and demand is assumed as stock dependent where holding cost of the inventory is assumed as a function of time. A numerical example is also discussed.

Keywords: Stock Dependent Demand, Economic Production Quantity, Time dependent Holding Cost.

I. Introduction

Present study contains an inventory control model that decides the amount of a single commodity that satisfies the market requirement over an infinite planning horizon.

One of the important aspects of the production inventory model is deterioration. In previous research, so many researchers have been taken into an account on deterioration. The four major concerns of any manufacturing firm are production, planning, quality, and maintenance. In the present time, each business has a competition day by day for a superior quality item and provides better customer services. Because of globalization and new specialized advancement, the manufacturing infrastructure likewise changes quickly.

In this paper, the non-instantaneous deterioration model under production policy has been considered. Here the deterioration starts with the beginning time (from one day) because the proper storage condition is not good, so it will start to decline. In this circumstance to encourage the sales with more attractive offer on the decayed unit is known as the rebate rate. In the current paper, author studied and derived a production-based inventory model under the effect of stock dependent demand rate and fixed deterioration rate. Also, author have taken a linear holding cost. The production inventory system has completed in four stages in each cycle.

Author assume that the production starts with zero inventory level in each cycle. After a certain time, demands will be fulfilled then break the production. The collected inventory is then quietly empty out and after some time it reached zero due to the effect of demand and deterioration. The decaying item which loss self-original value sells at a discounted rate. When the inventory falls down to zero then one cycle is complete, the same process will be repeated in the whole process.

Assuming constant rate of deterioration [1] analyzed the effect of Economic Production Quantity model of decaying items, [2] elaborate on the production model of deteriorating items that minimize the total cost. Also [3] focused on the inventory model for the exponential demand rate. An approach based on inventory model developed by [4] and explained a multi-lot size inventory-based system with constant demand and production rates. For different demand rates for different stage of inventory [5], [6], [7], [8] and [9] studied the Economic Production Quantity model under the consideration of partial backlogged.

[10] studied a deteriorating inventory model with exponentially decreasing demand, where author have assumed a finite planning horizon in the model. [11] developed an Economic Production Quantity model with time-varying demand and partially backlogged model. [12] explored the inventory production-based inventory model by considering shortages. [13], investigated and formulated an optimal returned policy-based model by considering reverse logistics with backorders where [14] extended these models with new criteria of limited storage facility under inflation. In all inventory models, author seen that the product almost all items either software and hardware have fixed shelf life. Because of the expiration, new technology, time-consuming, more efforts, non-auto start, etc. substantial decay of inventory system cannot be neglected, it was another major feature of the real world. In this field, [15] developed an Economic Production Quantity model with constant demand and exponentially decaying items.

The stock decreases due to demand which is a function of the on-hand inventory and deterioration which is constant. It is seen that items have a lifetime which cessation when advantages become zero of the on-hand inventories. It is noticed that products have a lifetime which cessation when the benefit becomes zero. Considering the concept of permissible delay in payment, [16] introduced a perishable inventory model with a parabolic rate of demand along with partial backlogging.

For the product of low cycle [17] presented a quality consideration deteriorating inventory model. [18] considered a multivariate demand model for decaying items having shortages. [19] presented a demand dependent production inventory model with price-sensitive demand and shortages. In the last few decades researcher pay attention to the deterioration-based models, they considered the different deterioration rates for different environment conditions. Now author discuss the non – instantaneous deteriorating items in our study so many researchers contribute to this field, but author have discussed some of them.

[20] developed an inventory control model under stock dependent consummation rate. [21] also did work on Gupta’s model. [22] and considered different inventory models based on the time value of money and inflation. After that [23] developed the deterministic model where non-linear holding cost was considered for decay items with stock dependent consummation rate. [24] explored an inventory model with a stock dependent rate of demand and holding cost was time-dependent. After that [25] investigated the model in which demand and holding cost both are stock dependent.

[26] built up the echelon inventory model for decay products and assumed variable holding cost with stock dependent demand. In view of non-instantaneous deterioration, under the impact of inflation [27] analyzed and developed an inventory model with completely backlogged shortages

where price and stock dependent demand was taken. By considering trade-credit strategy and two warehouse [28] presented an Economic Production Quantity model for non-instantaneous deteriorating items. [29] analyzed and developed a two-shop based stock model for non-instantaneous and assumed variable holding cost with stock dependent demand. Cardenas Barron et al. (2020) built up an Economic Production Quantity model and examined a uniform inventory model with non- non-linear stock dependent holding cost and demand. Based on Price discount facility [30] developed an EOQ model for deteriorating items with stock-dependent demand and partial backlogging. For multi-item deteriorating [31] presented a two-echelon inventory model with price-and stock-dependent demand.

For deteriorating items [32] described discount facility and discussed a study of inventory model for deteriorating items with price and stock dependent demand under-price discount facility. In some of the study a concept of overtime production was considered in the same way [33] presented an inventory model for deteriorating items under overtime production for deteriorating items with nonlinear price and stock dependent demand.

For the product with time varying holding cost by offering some quantity discounts [34] developed and discussed partial backlogging inventory model with price and stock level dependent demand.

In most of the inventory models, it is commonly seen that the ordering cost is assumed to be fixed but it is not certifiable in the present emulative market. Under this presumption, many attempts were made by various research scholars in the direction to developed inventory control model for decaying items under the effect of exponential demand where holding cost has been time-dependent. But in this study deterioration was considered at the initial level of the production. In the current paper, considering the more practical phenomena author have discussed the stock dependent rate of demand and time-varying holding cost with rebate rate in the production model and author have assumed the impact of decay at the initial level of production.

II. Assumptions and Notations

- The demand rate for the product is assumed to be stock-based.
- The produced unit of product is always available to face the demand of the market.
- Products start deteriorates at the initial level of the model but price discount on deteriorated items is offered only when production is stopped.
- Shortages are not permitted in the model.
- There is no replacement or repair of perishable items.
- The holding cost of the inventory units is a function of time.
- $p = k.D$, where P is the production rate and k ≥ 1, where D is the demand.

The notations are as follows used in the present model

A = Setup cost

C_h = Holding cost per unit per unit time

C_p = production cost per unit

C_d = Deterioration cost per unit

p = Production rate per unit time

r = Price discount per unit cost

t_1 = The time where production was stopped

T = Duration of complete production cycle

$I_1(t)$ = Level of inventory at time t between the intervals $0 \le t \le t_1$

$I_2(t)$= Level of inventory at time t between the intervals $t_1 \le t \le T$

III. Mathematical Formulation of the Model

The stock level develops during the interval $[0, t_1]$. Thus, the stock level in a creation period is represented by the differential equation (1) where the inventory is increasing due to production and reached to the highest level at t_1. The stock level exhausts along the $[t_1, T]$. The stock level in a non-creation period is represented by the differential equation (2) also in the time slot due to combined effect of demand and deterioration the inventory level is decreasing and reached to the lowest point at T. This model is shown by the following Figure. 1

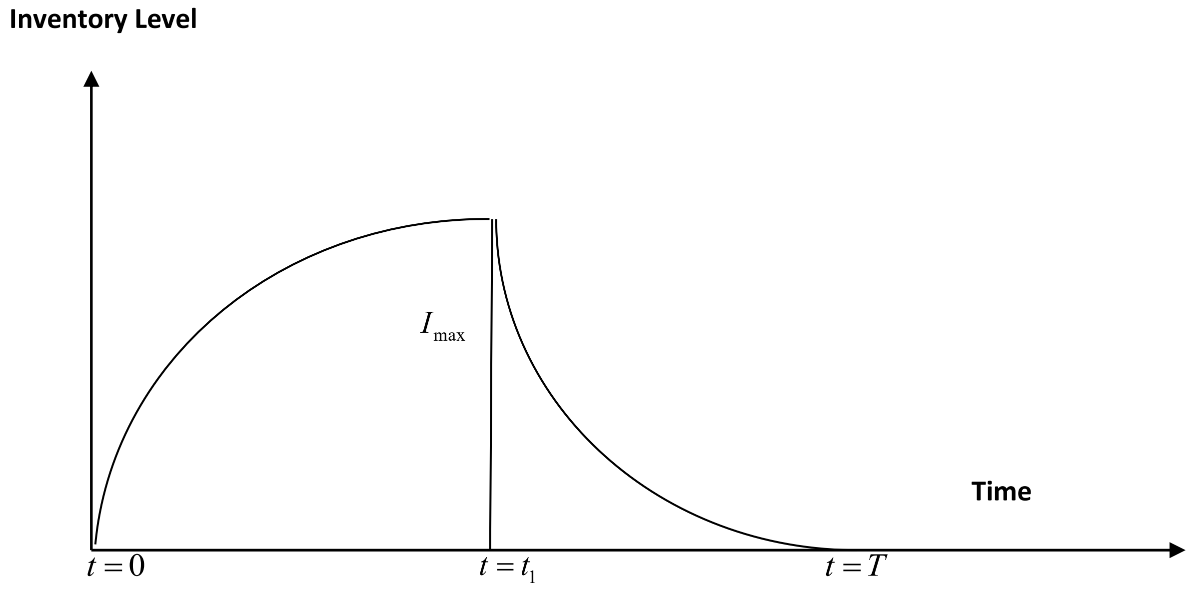

Figure 1 Graphical representation of the inventory with respect to time

$$\frac{dI_1(t)}{dt} + \theta t I_1(t) = p - \left(a + bI_1(t)\right) \qquad 0 \le t \le t_1 \qquad (1)$$

$$\frac{dI_2(t)}{dt} + \theta t I_2(t) = -\left(a + bI_2(t)\right) \qquad t_1 \le t \le T \qquad (2)$$

Author have the following boundary conditions

$I_1(0) = 0 = I_2(T)$

By using the boundary conditions,

$$I_1(t) = \frac{(a-p)}{b+\theta}\left(e^{-t(b+\theta)} - 1\right) \qquad 0 \le t \le t_1 \qquad (3)$$

$$I_2(t) = \frac{a}{b+\theta} e^{-t(b+\theta)}\left(e^{T(b+\theta)} - e^{t(b+\theta)}\right) \qquad t_1 \le t \le T \qquad (4)$$

Maximum inventory level

$$I_{Max} = I_1(t_1) = \frac{(a-p)}{b+\theta}\left(e^{-t_1(b+\theta)} - 1\right) \qquad (5)$$

Using the condition of continuity

$I_1(t_1) = I_2(t_1)$

$$\frac{(a-p)}{b+\theta}\left(e^{t_1(-b-\theta)} - 1\right) = \frac{a}{b+\theta}\left(e^{T(b+\theta)} e^{-t_1(b+\theta)} - 1\right)$$

$$\left(e^{t_1(-b-\theta)} - 1\right)(a - p) = a\left(e^{(b+\theta)(T-t_1)} - 1\right)$$

$$t_1 = \frac{a}{p} T \qquad (6)$$

Now various costs associated with the models are: -

Production cost

$$PC = C_p p t_1 \tag{7}$$

Setup cost

$$SC = A \tag{8}$$

Cost for Holding the Inventory

$$HC = C_h\left[\int_0^{t_1} I_1(t)dt + \int_{t_1}^{T} I_2(t)dt\right]$$

$$HC = \frac{a}{2(b+\theta)^3}\begin{pmatrix} -2h_2 - (b+\theta)\begin{pmatrix} 2\big(1+(T-t_1)(b+\theta)\big)h_1 \\ +\big(2T+T^2(b+\theta)-t_1{}^2(b+\theta)\big)h_2 \end{pmatrix} \\ +2e^{(T-t_1)(b+\theta)}\big(h_2+(b+\theta)(h_1+t_1h_2)\big) \end{pmatrix} - \frac{(a-p)}{2(b+\theta)^3}\begin{pmatrix} -2\big((b+\theta)h_1+h_2\big)+t_1(b+\theta)^2(2h_1+t_1h_2) \\ +2e^{-t_1(b+\theta)}\big(h_2+(b+\theta)(h_1+t_1h_2)\big) \end{pmatrix} \tag{9}$$

Deterioration cost

$$DC = C_d\left[\int_0^{t_1}\theta I_1(t)dt + \int_{t_1}^{T}\theta I_2(t)dt\right]$$

$$= \frac{C_d\theta}{(\theta+b)}\left((a-p)\left(\frac{1-e^{-t_1(b+\theta)}}{\theta+b}-t_1\right)-\frac{a}{(\theta+b)}\left((T-t_1)(\theta+b)+1-e^{(T-t_1)(b+\theta)}\right)\right) \tag{10}$$

Price Discount

$$PD = C_p r\left[\int_0^{t_1}\big(a+bI_1(t)\big)dt + \int_{t_1}^{T}\big(a+bI_2(t)\big)dt\right]$$

$$= C_p r\left[at_1+\frac{b(a-p)}{b+\theta}\left(\frac{1-e^{-t_1(b+\theta)}}{b+\theta}-t_1\right)+\frac{a}{(b+\theta)^2}\left((T-t_1)\theta^2+b\big(e^{(T-t_1)(b+\theta)}+(T-t_1)\theta-1\big)\right)\right] \tag{11}$$

The average Total Cost per unit time

$$TC(t_1,T) = \frac{1}{T}[PC+SC+HC+DC+PD]$$

$$= \frac{1}{T}\left[\begin{array}{l} C_p p t_1 + A + \frac{a}{2(b+\theta)^3}\begin{pmatrix} -2h_2-(b+\theta)\begin{pmatrix} 2\big(1+(T-t_1)(b+\theta)\big)h_1 \\ +\big(2T+T^2(b+\theta)-t_1{}^2(b+\theta)\big)h_2 \end{pmatrix} \\ +2e^{(T-t_1)(b+\theta)}\big(h_2+(b+\theta)(h_1+t_1h_2)\big) \end{pmatrix} \\ \quad -\frac{(a-p)}{2(b+\theta)^3}\begin{pmatrix} -2\big((b+\theta)h_1+h_2\big)+t_1(b+\theta)^2(2h_1+t_1h_2) \\ +2e^{-t_1(b+\theta)}\big(h_2+(b+\theta)(h_1+t_1h_2)\big) \end{pmatrix} \\ +\frac{C_d\theta}{(\theta+b)}\left((a-p)\left(\frac{1-e^{-t_1(b+\theta)}}{\theta+b}-t_1\right)-\frac{a}{(\theta+b)}\left((T-t_1)(\theta+b)+1-e^{(T-t_1)(b+\theta)}\right)\right) \\ +C_p r\left(at_1+\frac{b(a-p)}{b+\theta}\left(\frac{1-e^{-t_1(b+\theta)}}{b+\theta}-t_1\right)+\frac{a}{(b+\theta)^2}\left((T-t_1)\theta^2+b\big(e^{(T-t_1)(b+\theta)}+T\theta-t_1\theta-1\big)\right)\right) \end{array}\right] \tag{12}$$

Substituting the value of $t_1 = \frac{a}{p}T$,by (6) in (12),

Total Cost per unit time $TC = \dfrac{A}{T} + aC_p$

$$+\frac{1}{2T(b+\theta)^3}\left\{\begin{array}{l} (a-p)\begin{pmatrix} 2\big((b+\theta)h_1+h_2\big)-\frac{aT(b+\theta)^2}{p}\left(2h_1+\frac{aTh_2}{p}\right) \\ -2e^{-\frac{aT(b+\theta)}{p}}\left(h_2+(b+\theta)\left(h_1+\frac{aTh_2}{p}\right)\right) \end{pmatrix} \\ -a\begin{pmatrix} 2h_2+(b+\theta)\begin{pmatrix} 2\left(1+\left(T-\frac{aT}{p}\right)(b+\theta)\right)h_1+ \\ \left(2T+(b+\theta)\left(T^2-\frac{a^2T^2}{p^2}\right)\right)h_2 \end{pmatrix} \\ -2e^{\left(T-\frac{aT}{p}\right)(b+\theta)}\left(h_2+(b+\theta)\left(h_1+\frac{aTh_2}{p}\right)\right) \end{pmatrix} \end{array}\right\}$$

$$+\frac{C_d\theta}{T(b+\theta)}\left(\begin{array}{c}(a-p)\left(-\frac{aT}{p}+\frac{1-e^{-\frac{aT(b+\theta)}{p}}}{b+\theta}\right)\\ -\frac{a}{(b+\theta)}\left(1-e^{\left(T-\frac{aT}{p}\right)(b+\theta)}+\left(T-\frac{aT}{p}\right)(b+\theta)\right)\end{array}\right)$$

$$+\frac{C_p r}{T}\left(\begin{array}{c}\frac{a^2T}{p}+\frac{b(a-p)\left(\frac{1-e^{-\frac{aT(b+\theta)}{p}}}{b+\theta}-\frac{aT}{p}\right)}{b+\theta}\\ +\frac{a\left(\left(T-\frac{aT}{p}\right)\theta^2+b\left(e^{\left(T-\frac{aT}{p}\right)(b+\theta)}+T\theta-\frac{aT\theta}{p}-1\right)\right)}{(b+\theta)^2}\end{array}\right) \quad (13)$$

IV. Solution Procedure

Now the optimum value of T which minimize the total cost.
The values of T for which
$\frac{\partial TC(T)}{\partial T}=0$, Satisfying the condition $\frac{\partial^2 TC(T)}{\partial T^2}>0$
The optimal solution of the equation (13) is obtained by using Mathematica software. Above said process can also be seen through the following example.

V. Numerical Example

Considering $A\to 50, a\to 20, b\to 1.2, C_p\to 15, h_1\to 2, h_2\to 0.15, p\to 40, \theta\to .006, C_d\to 3, C_p\to 2, r\to .02$ in appropriate units. The optimal value of $T^*=2.20006, t_1^*=1.10003, TC^*=349.78825$

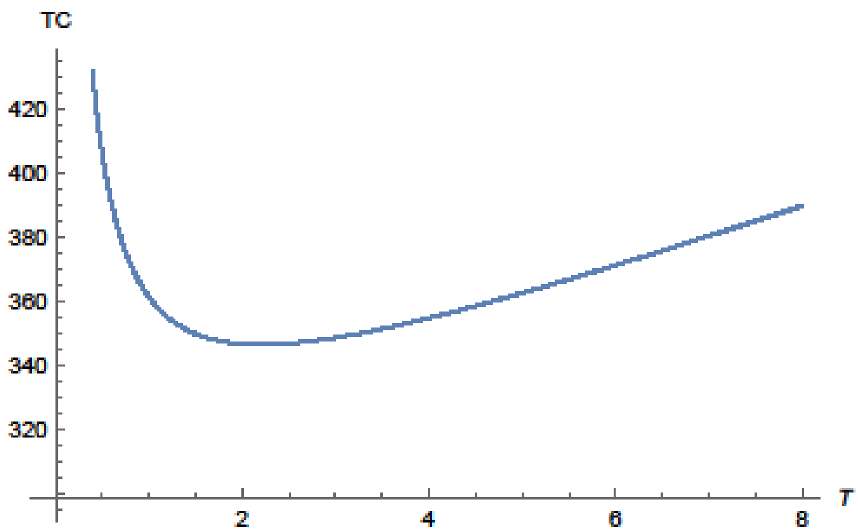

Figure 2. Concavity of the profit function.

VI. Sensitivity Analysis

Based on the values used in above example in the model author have examined the sensitivity analysis by changing some parameters one at a time and keeping the rest fixed.

Table 1 Sensitivity Analysis w.r.t. Various Parameters

Parameter	%	Changed value	t_1	T	I_{max}	TC
A	+50%	75	1.34725	2.69451	169.0449	359.918
	+25%	62.5	1.22741	2.40402	130.22971	354.026
	0	50	1.10003	2.20006	114.77403	349.788
	-25%	37.5	0.97816	1.98821	71.21354	341.498
	-50%	25	0.87699	1.68724	32.21354	333.805
a	+50	30	1.43291	1.09105	19.33721	489.125
	+25	25	1.22886	1.10554	53.97692	431.226
	0	20	1.10003	2.20006	114.77403	349.788
	-25	15	0.92214	2.31269	148.29284	276.394
	-50	10	0.78764	2.39991	189.76542	200.947
b	+50	1.8	1.09371	2.18739	90.53786	354.985
	+25	1.5	1.09685	2.19369	102.73451	352.063
	0	1.2	1.10003	2.20006	114.77403	349.788
	-25	0.9	1.10324	2.20647	133.95031	348.072
	-50	0.6	1.11009	2.22018	149.24731	346.857
P	+50	60	0.53185	1.59475	107.98073	361.688
	+25	50	0.86818	1.82913	110.76742	357.296
	0	40	1.10003	2.20006	114.77403	349.788
	-25	30	1.33422	2.62627	143.91819	335.358
	-50	20	1.67151	2.91326	201.21387	317.963

Author have calculated the sensitivity analysis based on different parameters. The outcome of the result is compared.

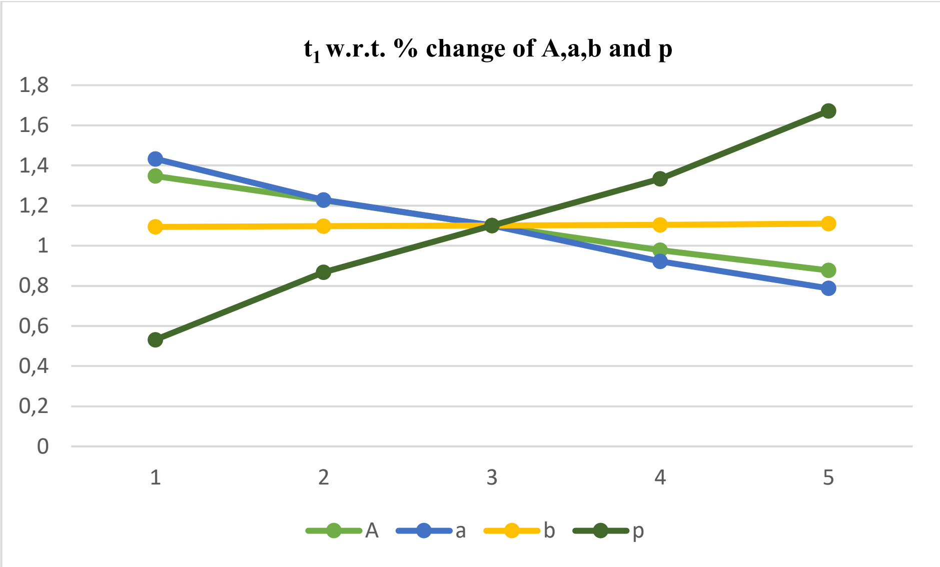

Figure 3. t_1 v/s change in parameters

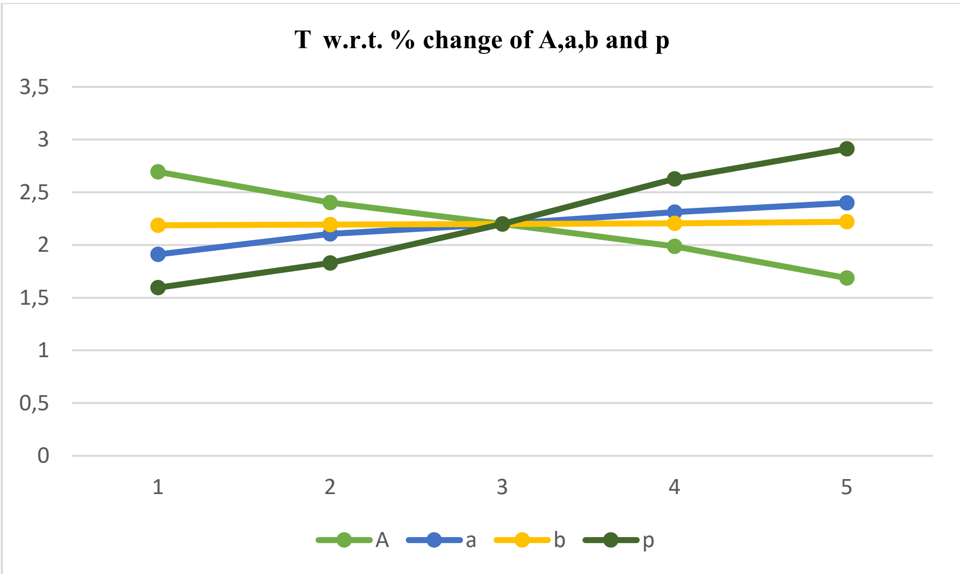

Figure 4. T v/s change in parameters

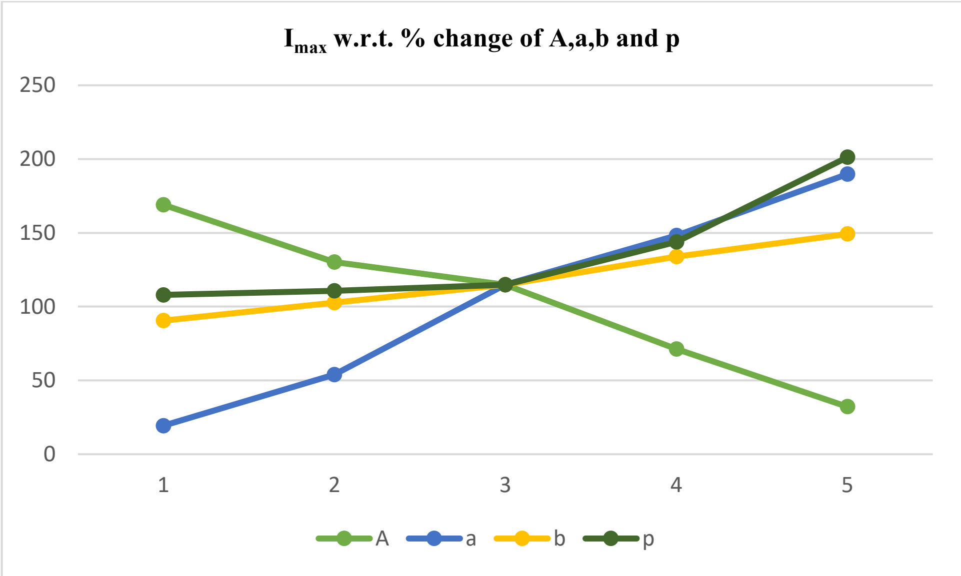

Figure 5. I_{max} v/s change in parameters

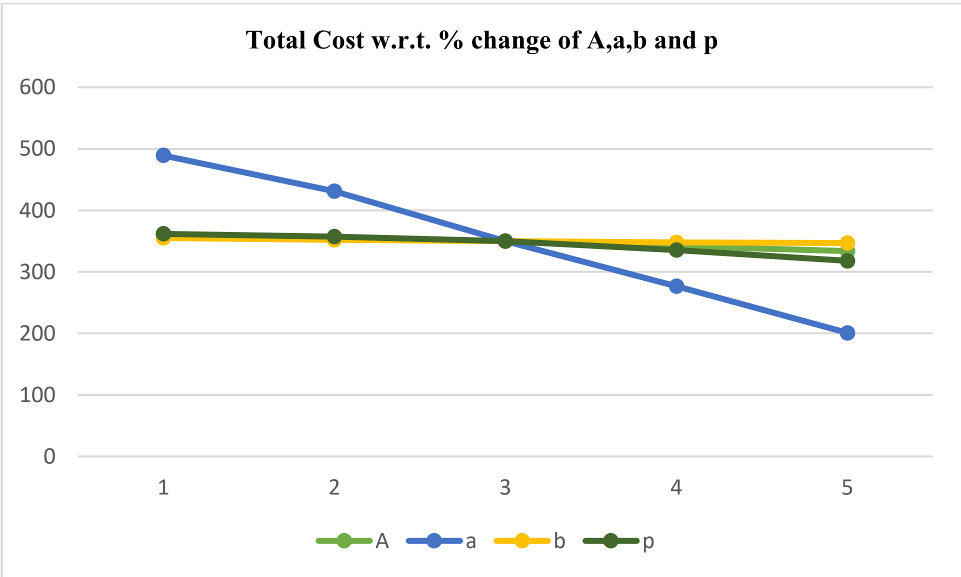

Figure 6. Total Cost v/s change in parameters

VII. Observations

In the present study, author have calculated the sensitivity analysis based on the parameters used in the study. Author have made changes in the parameters by (–50%, –25%, 0%, 25%, 50%). Some important inferences drawn from Table -1 and Figures 2 to 5 are as follows:

(i) Table No 1 shows that as the value of A goes from -50% to + 50%, the value of t_1,T, I_{max} , and total cost also increases.

(ii) Table No 1 shows that as the value of a goes from -50% to + 50%, the value of T and I_{max} decreases while the value of $t_1$and total cost increases rapidly.

(iii) Table No 1 shows that as the value of b goes from -50% to + 50%, the value of T total cost increases while the value of t_1, T and I_{max} decrease rapidly.

(iv) Table No 1 shows that as the value of p goes from -50% to + 50%, the value of T total cost increases while the value of t_1, T ,and I_{max} decreases.

As per above analysis it is obvious that it becomes a new tool to take initial deterioration and it is also observed that we have maximize the average total profit.

VIII. Conclusions

In the present study, author analyzed some important facts related to maximization of total cost related to inventory models and developed an inventory control production-based model for deteriorating items to reduce total cost. Also, the model has been developed by assuming the rate of demand depends on in-hand stock and holding cost is time-dependent.

In this model, author have assumed that produced items that have been partially degraded, customers have been discounted on their selling price, and products that were completely damaged or deteriorated have been discarded which is a more realistic assumption and helps to improve the profit. Also, in the present model, author have assumed that for the items having low life deterioration take place at the very initial stage of the cycle and developed this model under the effect of deterioration when production is going on. To validate the optimality, a numerical example has been taken and explained in the model. Sensitivity between the ranges -50% to 50% of different parameters have been carried out to check the deviation. The assumption of taking initial time-based deterioration makes this study more effective. Managerial aspect of the study is that this paper provide a good platform for the research scholars to use this study to investigate various changes in the deterioration perimeter and formulate new study.

IX. Future scope of the study

Present study may be further expanded by making some more changes in the main parameters of the study like this model can also be developed under a shortage, trade credits, and some price discount on the in-hand inventory may also be assumed.

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An Inventory Model with Quantity Dependent Trade Credit for Stock and Price Dependent Demand, Variable Holding Cost and Partial Backlogging

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Abstract

Here, the modelling of an inventory system for price and stock reliant demand with the combination of quantity discount and credit limit policy has been presented. Price and stock level are the key sources that always affect the demand of any product. In present study the cost of holding is considered as a time varying function. Vendors usually offer different policies or discounts to attract more customers. Different possible cases for offered trade credit period are discussed in the model. Shortages with partial backlogging are considered here in the development of the model. The different possible cases in this model is exemplified numerically with the help of software Mathematica 11.3 and a sensitivity analysis with respect to distinct system parameters is also presented.

Keywords: Inventory Model, Quantity Discount, Deterioration, Stock and Price Dependent, Demand, Trade Credit, Partial Backordering, Variable Holding Cost, Shortages.

I. Introduction

The aim of the present paper is to develop an inventory model for deteriorating items using price and stock dependent demand. Most of the previous inventory models were developed by assuming constant, time dependent, stock dependent and many different demand patterns. In general, there are more marketing schemes and strategies that influence the market demand. Available stock and selling price are the main factors that affect the demand of customers.

Deterioration is another important factor whose role in the construction of an inventory model is very useful. Deterioration can be defined as the reduction in the original quality and value of the product in any term. Mostly, all available products deteriorate, the only difference is that in some products the rate of deterioration is large and in some it is very low. So, for more accuracy in result, the deterioration should be considered in the modeling of inventory models.

Some researchers worked on inventory models without considering shortages. It cannot be predicted that the stocked units will always be enough to satisfy the demand of all the customers. So, shortage is also an important concept in inventory model that should be taken into account. In the construction of an inventory model the assumption that during stock out the occurring shortages are either completely lost or completely backlogged, is not realistic. So there will be a partial backlogging of the demand during stock out.

Further trade credit period is the useful incentive policy to attract more customers. In this time period vendor allows a certain time limit to retailer to pay all his dues. If the retailer pays all his dues before the credit limit then there will be no interest otherwise interest will be charged on unpaid amount. Retailer can increase his profit by earning interest on sales revenue.

In today's high competitive market, vendors usually offer new schemes or policies to promote their business. In the present paper quantity discount policy is considered with trade credit period and both of these works as a promotional tool for the business. To make the study more realistic and to improve the efficiency of the model, holding cost is also taken as a linear function of time.

II. Literature Review

Skouri and Papakristos [1] proposed an inventory model with quantity discount policy using price dependent demand. Chang [2] introduced an inventory model based on price-dependent demand under quantity and freight discounts. Alfares [3] developed an inventory policy under stock level dependent demand, time varying holding cost and quantity discount. Tripathi et al [4] investigated a partial backordering inventory model for deteriorating items under quantity discount scheme.

Geetha and Udayakumar [5] proposed a non-instantaneous deteriorating model for price and advertisement dependent demand with partial backorder. Sanni and Chigbu [6] developed a three-parameter Weibull distribution deteriorating inventory model under stock level dependent demand with shortage backordering. Li and Teng [7] introduced pricing and lot-sizing strategies for perishable products when demand depends on stock level, selling price, product freshness and reference price. Rastogi et al. [8] developed an inventory model for non-instantaneous deteriorating items with price sensitive demand and partial backlogging. Shaikh et al. [9] studied an EOQ model for decaying products using time dependent demand under shortage backordering and trade credit. Tayal et al. [10] presented deteriorating inventory model for two level of shortage using stock dependent demand and fractional backlogging. Rani et al. [11] studied a green supply chain inventory model for decaying items with credit period dependent demand. Handa et al. [12] worked on an inventory model under trade credit policy and shortages in which stock level plays a major role for demand.

In present paper holding cost is taken as a linear function of time. Jaggi [13] proposed a non-instantaneous deteriorating inventory model with variable holding cost in which demand depends upon price, and holding cost is taken as a variable. Tayal et al. [14] introduced an EPQ model with exponential demand rate and time dependent holding cost. Rastogi et al. [15] developed a deteriorating inventory model for price sensitive demand, linear holding cost and trade credit period. Aggarwal et al. [16] proposed an inventory model for price dependent demand, linear holding cost and partial backlogging under inflation.

Skouri et al. [17] formulated an inventory model with Weibull distribution deterioration and ramp type demand rate. Dutta and Kumar [18] studied a deteriorating inventory model in which demand and holding cost is considered as a function of time and permitted shortages are partially backlogged. Mahapatra et al. [19] introduced a model for deteriorating items based on reliability dependent demand under partial backlogging. Singh et al. [20] worked on replenishment policy for decaying items with partial backordering under credit financing and inflation.

Patra [21] investigated effect of inflation and time value of money for two warehouse inventory model under shortages. Bhojak and Gothi [22] introduced Weibull distributed deteriorating inventory model for time reliant demand with deficiency and backordering. Singh and Sharma [23] proposed a reverse logistic supply chain inventory model for imperfect production/remanufacturing with partial backordering and inflation. Kumar et al. [24] studied the effect of preservation and learning on partial backordering inventory model for deteriorating items with the effect of Covid-19 pandemic.

Kumar et al. [25] worked on an inventory model for two-level storage under the effect of learning and inflation. Wang et al. [26] studied a supply chain inventory model for decaying products under seller's optimal credit financing. Singh et al. [27] formulated an inventory model under preservation technology using stock dependent demand with credit financing. Shaikh [28] introduced a deteriorating inventory model based on price and advertisement dependent demand under shortage backordering and mixed type of trade credit. Sundararajan and Uthayakumar [29] formulated an optimal inventory policy with promotional efforts and backordering of shortages under trade credit period. Mishra and Talati [30] studied quantity discount inventory policy in which demand depends upon the frequency of advertisement and stock with preservation and backordering of shortages.

This study represents an inventory model considering variable demand, quantity discount and partial backlogging. To make the study more realistic, holding cost is taken as the function of time. Different cases for allowed trade credit period are also elaborated in the model. To improve the efficiency of the model numerical example for different cases and sensitivity analysis for distinct value of parameters have been discussed.

III. Assumptions and notations

$I(t)$ level of inventory at any time t

d, β, γ coefficients of demand

Q_1 initial stock level

Q_2 back order quantity during stock out

$\phi(\eta)$ rate of backlogging

k rate of deterioration

η waiting time up to next arrival

T cycle time

u_1 time at which level of inventory becomes zero

h_a, h_b parameters of holding cost

s_r per unit shortage cost

λ per unit deterioration cost

l_r lost sale cost per unit

c purchasing cost per unit

A per order ordering cost

p selling price per unit

$U.T.P_x.$ unit time profit

M allowed trade credit period

I_c rate of interest charged

I_e rate of interest earned

These following are the assumptions used here:

- Products considered in this model are of deteriorating nature.

- Demand rate is a function of price and stock and is given by $D=(d-\beta p+\gamma I_1(t))$
- No replacement policy is allowed for deteriorating products in whole cycle.
- The system allows shortages and partial backlogging.
- Deterioration rate is constant.
- In the model all-units quantity discounts and the length of credit periods are defined as follow:

$$M_i = \begin{cases} M_1 & 1 \le X_1 < Y_1 \\ M_2 & Y_1 \le X_2 < Y_2 \\ M_3 & Y_2 \ge X_3 \end{cases}$$

where X_1, X_2, X_3 denote the boundaries of quantity in units and $M_1 > M_2 > M_3$.

- Backlogging rate is assumed as a function of waiting time.
- Holding cost is considered as a linear function of time i.e., $(h_a + h_b(t))$.
- Trade credit is allowed for different time period.
-

4. Mathematical Modelling

Figure 1. Represents the behaviour of inventory system with respect to time. Here Q_1 denotes the initial inventory level at $t=0$. The level of inventory depletes in the interval $[0,u_1]$ due to demand and deterioration. At $t=u_1$, inventory reaches to zero level and after that shortages occur. The depletion of the inventory is represented by the following Fig 1.

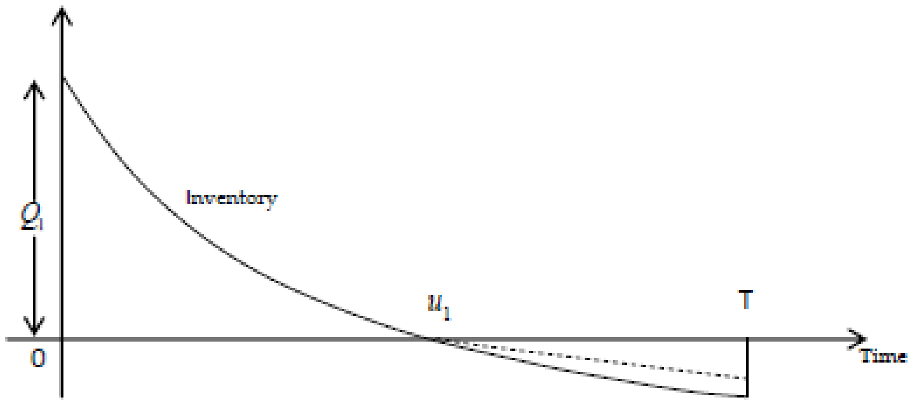

Figure 1 *Inventory time graph of system*

Inventory system can be represented by the following differential equations:

$$\frac{dI_1}{dt} + kI_1 = -(d-\beta p+\gamma I_1(t)) \quad 0 \le t \le u_1 \tag{1}$$

$$\frac{dI_2}{dt} = -(d-\beta p) \; .... \; u_1 \le t \le T \tag{2}$$

Boundary equations are given as follow:

$$I_1(u_1) = I_2(u_1) = 0 \tag{3}$$

Solution of the equations (1) and (2) are given by

$$I_1(t) = [(u_1 - t) + (k+\gamma)(u_1^2 - t^2)]e^{-(k+\gamma)t} \quad 0 \le t \le u_1 \tag{4}$$

$$I_2(t) = (d-\beta p)(u_1 - t) \quad u_1 \le t \le T \tag{5}$$

V Associated Costs

V.I. Ordering Cost

Ordering cost per order of the system is as follow:

$$O.C_x. = A \tag{6}$$

V.II. Purchasing Cost

If c is the purchasing cost per unit and Q_1, Q_2 are the ordering quantity and backordered quantity respectively then purchasing cost of the system will be:

$P.C_x. = \{Q_1 + Q_2\}c$

Where

$$Q = I(0) = (d - \beta p)(u_1 + (k+\gamma)\frac{u_1^2}{2}) \tag{7}$$

Here I(0) denotes the initial inventory level at the starting of the cycle and Q2 is the backordered quantity during stock out of the inventory.

$$Q_2 = \int_{u_1}^{T} (d - \beta\gamma)\phi(\eta)dt$$

$$Q_2 = \frac{(d-\beta p)(T^2 - u_1^2)}{2T} \tag{8}$$

Hence, the purchasing cost of the system will be

$$P.C_x = \{(u_1 + (k+\gamma)\frac{u_1^2}{2}) + \frac{(T^2 - u_1^2)}{2T}\}(d - \beta p)c \tag{9}$$

V.III. Holding Cost

Holding cost is considered in the duration of positive inventory. It is a linear function of time and is given by:

$$H.C_x = \int_0^{u_1} (h_a + h_b t) I_1(t) dt$$

$$H.C_x. = (d-\beta p)\{h_a(\frac{u_1^2}{2} + (k+\gamma)\frac{u_1^3}{6} - (k+\gamma)^2\frac{u_1^4}{8})) + h_b(\frac{u_1^3}{6} + (k+\gamma)\frac{u_1^4}{24} - (k+\gamma)^2\frac{u_1^5}{15})\} \tag{10}$$

V.IV. Shortage Cost

In the inventory system shortage occurs during the stock out of inventory. Shortage cost of the system will be as follow:

$$S.C_x. = s_r\int_{u_1}^{T}(d - \beta p)dt$$

$$S.C_x. = s_r(d-\beta p)(T - u_1) \tag{11}$$

V.V. Lost Sale Cost

In the inventory system lost sale cost occurs when some customers fulfil their demand from other places, during the stock out conditions.

$$L.S.C_x. = l_r\int_{u_1}^{T}(d-\beta p)(1-\phi(\eta))dt$$

$$L.S.C_x. = l_r\{\frac{(d-\beta p)(T-v)^2}{2T}\} \tag{12}$$

V.VI Deterioration cost

Deterioration cost is considered for those products that are deteriorated or decayed in the system. The deterioration cost for the system is as follow:

$$D.C_x. = \lambda\{I_1(0) - \int_0^{u_1}(d-\beta p)dt\}$$

$$D.C_x. = \lambda(d-\beta p)\{(k+\gamma-1)\frac{u_1^2}{2} - (k+\gamma)\frac{u_1^3}{6} + (k+\gamma)^2\frac{u_1^4}{8}\} \quad (13)$$

V.VII Sales revenue

Sales revenue:

$S.R_x. = (Q_1 + Q_2)p$

Hence, the sale revenue of the system is given by

$$S.R_x = \{(u_1 + (k+\gamma)\frac{u_1^2}{2}) + \frac{(T^2 - u_1^2)}{2T}\}(d-\beta p)p \quad (14)$$

VVIII.Permissible delay

Two cases for allowed trade credit period are given as follow:

Case 1: When $M \geq u_1$

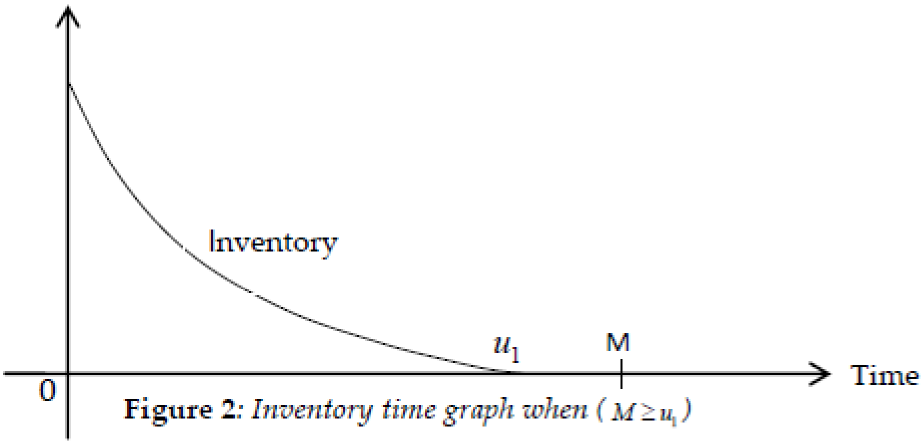

Figure 2: *Inventory time graph when* ($M \geq u_1$)

For this case retailer has an adequate amount of funds to clear up all his dues since the credit limit period is more than the time of positive inventory.

Interest charged in this case will be:

$I.C_1 = 0 = 0$

Interest earned in the duration of [0, M] is given by:

$$IE_1. = pI_e\int_0^{u_1}(d-\beta p+\gamma I(t))tdt + (M-u_1)\int_0^{u_1}(d-\beta p+\gamma I(t))dt$$

$$= pI_e(d-\beta p)\{(\frac{u_1^{\ 2}}{2} + \gamma(\frac{u_1^{\ 3}}{3} + (k+\gamma)\frac{u_1^{\ 4}}{24} - \frac{(k+\gamma)^2}{15}u_1^{\ 5}) + (M-u_1)(u_1 + \gamma(\frac{u_1^{\ 2}}{2} + (k+\gamma)\frac{u_1^{\ 3}}{6} - \frac{(k+\gamma)^2}{8}u_1^{\ 4})\} \quad (15)$$

Case 2: When $M < u_1$

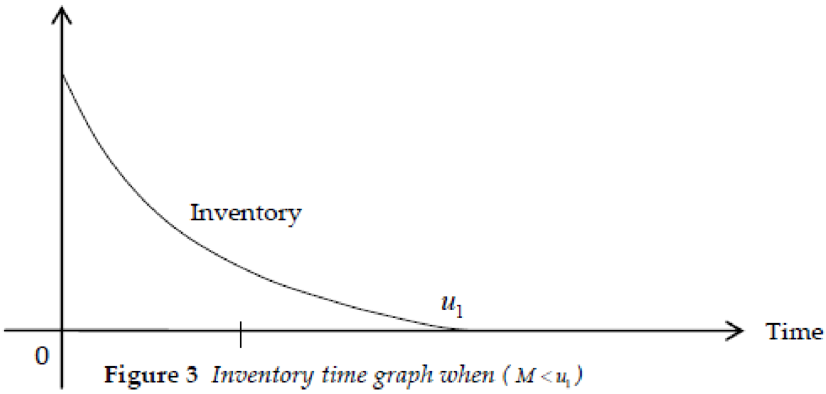

Figure 3 *Inventory time graph when* ($M < u_1$)

For this case the retailer has to settle all his payment before zero stock. On the basis of interest earned and interest charged, following two cases arise.

Case 2.1: When $M < u_1$ and

$$pD[0,M] + IE_{2.1}[0,M] \geq cI(0) \tag{16}$$

For this case retailer has enough money to settle all his payments.
Interest charged is given by

$$I.C_{2.1} = 0 \tag{17}$$

Interest earned in the duration [0, M] is given by

$$I.E_{2.1} = pI_e \int_0^M (d - \beta p + \gamma I(t)) t dt$$

$$= pI_e (d - \beta p)\{\frac{M^2}{2} + \gamma(\frac{M^3}{6} + (k+\gamma)\frac{M^3}{24} - \frac{(k+\gamma)^2}{15} M^5\} \tag{18}$$

Case 2.2: When $M < u_1$ and

$$pD[0,M] + IE_{2.2}[0,M] < cI(0) \tag{19}$$

For this case retailer has not enough money to settle all his payments.
Interest earned in the duration [0, M] is given by

$$I.E_{2.2} = pI_e \int_0^M (d - \beta p + \gamma I(t)) t dt$$

$$I.E_{2.2} = pI_e (d - \beta p)\{\frac{M^2}{2} + \gamma(\frac{M^3}{6} + (k+\gamma)\frac{M^3}{24} - \frac{(k+\gamma)^2}{15} M^5\} \tag{20}$$

Interest charged on unpaid amount is given by

$$I.C_{2.2} = B.I_c \tag{21}$$

where B is given by:

$$B = cE_1(0) - \{pD[0,M] + IE_{2.2}[0,M]\}$$

$$B = I_c (d - \beta p)\{c(u_1 + (k+\gamma)\frac{u_1^2}{2}) - (pI_e(\frac{M^2}{2} + \gamma(\frac{M^3}{6} + (k+\gamma)\frac{M^3}{24} - \frac{(k+\gamma)^2}{15} M^5))$$
$$+ p(u_1 + \gamma(\frac{u_1^2}{2} + (k+\gamma)\frac{u_1^3}{6} - \frac{(k+\gamma)^2}{8} u_1^4)\}$$

V.IX. Unit Time Profit

$$U.T.P_x = \frac{1}{T}\{S.R_x. - P.C_x. - H.C_x. - D.C_x. - L.S.C_x. - S.C_x. - O.C_x. - IC + IE\} \tag{22}$$

VI. Numerical Example

Case 1: When $M \geq u_1$

A=500 per/order, c=35 Rs./unit, d=200 units, k=0.001, T=30 days, M=23 days, β =2.2, γ =0.1 Rs./unit, l_r =18 Rs./unit, s_r =15 Rs./unit, h_a =0.8 Rs./unit, h_b =0.45 Rs./unit, λ =18 Rs./unit I_e =0.02,
After solving this model with the help of corresponding parameters optimal value of
p =46.7032 Rs, u_1 =21.7042 Rs. and $U.T.P_x$ =8146.47 Rs.
The behavior of the system for $U.T.P_x$ is given by the figure 4 with the help of Mathematica 11.3.

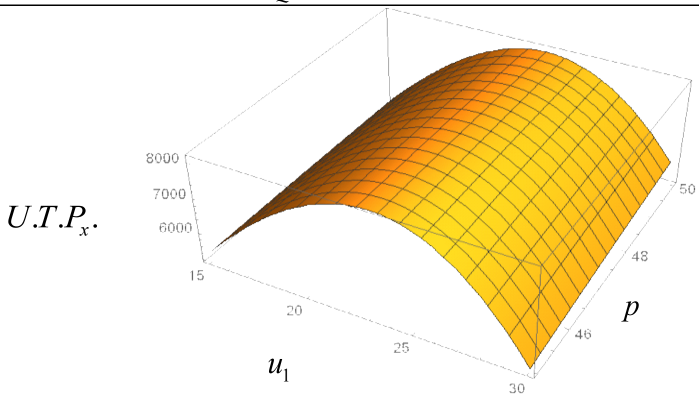

Fig. 4: *Optimality of the system for case 1*

Case 2.1: When $M < u_1$ and

$pD[0,M] + IE_{2.1}[0,M] \geq cI(0)$:

A=500 per/order, c=35 Rs./unit, d=200 units, k=0.001, T=30 days, M=20 days, β =2.2, γ =0.1 l_r =18 Rs./unit, s_r =15 Rs./unit, h_a =0.8 Rs./unit, h_b =0.45 Rs./unit, λ =18 Rs./unit, I_e =0.02,

After solving this model with the help of corresponding parameters optimal value of p =47.8656 Rs., u_1 =23.2287 days and $U.T.P_x$ =8155.1 Rs.

The behavior of the system for $U.T.P_x$ is given by the figure 5 with the help of Mathematica 11.3.

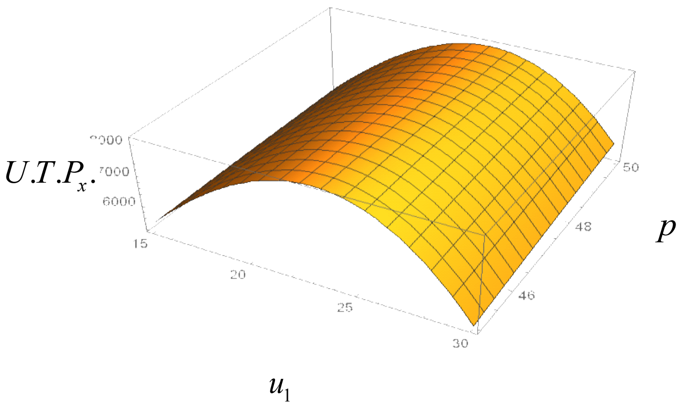

Fig. 5: *Optimality of the system for case 2.1*

Case 2.2: When $M < u_1$ and

$pD[0,M] + IE_{2.2}[0,M] < cI(0)$:

A=500 per/order, c=35 Rs./unit, d=200, k=0.001, T=30, M=17 days, β =2.2, γ =0.1, l_r =18 Rs./unit, s_r =15 Rs./unit, h_a =0.8 Rs./unit, h_b =0.45 Rs./unit, λ =18 Rs./unit I_e =0.02, I_c =0.03,

After solving this model with the help of corresponding parameters optimal value of p =48.2336, u_1 =23.1396 days and $U.T.P_x$ =8066.84 Rs

The behavior of the system for $U.T.P_x$ is given by the figure 6 with the help of Mathematica 11.3.

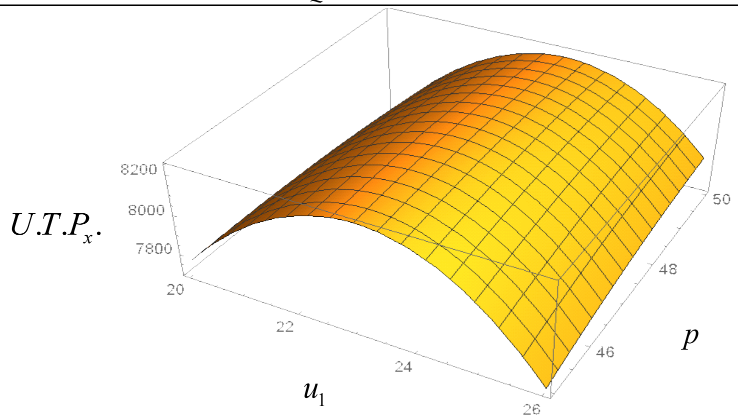

Fig. 6: *Optimality of the system for case 2.2*

The Algorithm

The solution procedure, to maximize the $U.T.P_x.$ for optimal ordering quantity is given as follows:
Here U.T.P. is the function of two variables 'u1' and 'p'. So, to maximize U.T.P. we put the partial derivatives

$\frac{\partial U.T.P_x.(u_1,p)}{\partial u_1}=0$ and $\frac{\partial U.T.P_x.(u_1,p)}{\partial p}=0$

After solving these equations, system gives the optimal value of v and p

$\frac{\partial^2 U.T.P_x.(u_1,p)}{\partial^2 u_1}<0$, $\frac{\partial^2 U.T.P_x.(u_1,p)}{\partial^2 p}<0$

provided

$$(\frac{\partial^2 U.T.P_x.(u_1,p)}{\partial^2 u_1})\,(\frac{\partial^2 U.T.P_x.(u_1,p)}{\partial^2 p}) - (\frac{\partial^2 U.T.P_x.(u_1,p)}{\partial u_1 \partial p})^2 > 0$$

Find the value of u_1 and p for every credit period length.
Calculate X_i i.e., ordering quantity for every value of u_1 and p.
Calculate a valid quantity $X_i = X^*$.
Find out the unit time profit for this X^* .
Evaluate $U.T.P_x.$ for all given credit period lengths and also for the value which is greater than X_i^* .

Quantity Discount Approach

Model is demonstrated numerically for the different trade credit period with the help of Mathematica 11.3.

$$M_i = \begin{cases} 23\ days & 1 \le X_1 < 2700 \\ 20\ days & 2700 \le X_2 < 5500 \\ 17\ days & X_3 \ge 5500 \end{cases}$$

With respect to these credit periods and the values of parameters discussed above, optimal ordering quantity i.e., $X^* = 5348.81$ units, $p = 47.8659$ Rs. and u_1 =23.2287 days.

And U.T.Px = 8155.1 Rs. Also $(U.T.P_x.)_{Y_2} = 8099.45$ Rs., Clearly

$(U.T.P_x.)_{X^*} > (U.T.P_x.)_{Y_2}$

$X^* = 5348.81$ is the optimal value of ordering quantity

$(U.T.P_x.)_{X^*} = 8155.1$ Rs. is the optimal value of unit time profit for the system.

Also, optimal value of u_1 =23.2287 days and $p = 47.8659$ Rs.

VII. Sensitivity Analysis

Sensitivity analysis for distinct parameters is specified as follows. In this the effect of different system parameters on unit time profit is calculated to check the stability of the system.

Case 1: When $M \geq u_1$

Table 1: Variation in optimal solution for demand parameter (d):

% change in (d)	(d)	u_1	p	$U.T.P_x.$
-20%	160	21.3076	37.3959	5141.32
-15%	170	21.4102	39.7209	5826.7
-10%	180	21.5104	42.0471	6555.9
-5%	190	21.6084	44.3746	7329.19
0%	200	21.7042	46.7032	8146.47
5%	210	21.7979	49.0328	9007.97
10%	220	21.8896	51.3634	9913.83
15%	230	21.9793	53.6947	10864.2
20%	240	22.0672	56.0268	11859.1

Table 2: Variation in optimal solution for shortage parameter (β):

% change in (β)	(β)	u_1	p	$U.T.P_x.$
-20%	1.76	22.1533	58.3596	10315.7
-15%	1.87	22.0261	54.9293	9675.04
-10%	1.98	21.9097	51.8814	9107.39
-5%	2.09	21.8028	49.1555	8600.98
0%	2.2	21.7042	46.7032	8146.47
5%	2.31	21.6130	44.4855	7736.29
10%	2.42	21.5284	42.4702	7364.30
15%	2.53	21.4497	40.6310	7025.42
20%	2.64	21.3762	38.9457	6715.43

Table 3: Variation in optimal solution for lost sale cost parameter (γ):

% change in (γ)	(γ)	u_1	p	$U.T.P_x.$
-20%	0.08	28.3126	37.3189	13184.9
-15%	0.085	26.1631	40.2893	11471.9
-10%	0.09	24.4238	42.7617	10127.1
-5%	0.095	22.9620	44.8712	9041.34
0%	0.1	21.7042	46.7032	8146.47
5%	0.105	20.6039	48.3159	7396.91
10%	0.11	19.6293	49.7510	6760.70
15%	0.115	18.7576	51.0395	6214.61
20%	0.12	17.9717	52.2053	5741.37

Table 4: Variation in optimal solution for deterioration cost parameter (l_r):

% change in (l_r)	(l_r)	u_1	p	$U.T.P_x.$
-20%	14.4	21.6753	46.6497	8159.91
-15%	15.3	21.6827	46.6631	8156.54
-10%	16.2	21.6899	46.6765	8153.17
-5%	17.1	21.6970	46.6899	8149.82
0%	18	21.7042	46.7032	8146.47
5%	18.9	21.7113	46.7165	8143.12
10%	19.8	21.7184	46.7298	8139.79
15%	20.7	21.7255	46.7430	8136.43
20%	21.6	21.7326	46.7562	8133.14

Table 5: Variation in optimal solution for deterioration parameter (s_r):

% change in (s_r)	(s_r)	u_1	p	$U.T.P_x.$
-20%	12	21.6131	46.4286	8227.84
-15%	12.75	21.6359	46.4977	8207.37
-10%	13.5	21.6588	46.5665	8186.98
-5%	14.25	21.6815	46.6350	8166.68
0%	15	21.7042	46.7032	8146.47
5%	15.75	21.7268	46.7712	8126.34
10%	16.5	21.7494	46.8389	8106.30
15%	17.25	21.7719	46.9063	8086.34
20%	18	21.7943	46.9735	8066.47

Table 6: Variation in optimal solution for interest earned parameter (I_e):

% change in (I_e)	(I_e)	u_1	p	$U.T.P_x.$
-20%	0.016	21.9530	46.8795	8019.16
-15%	0.017	21.8887	46.8323	8050.30
-10%	0.018	21.8258	46.7873	8081.91
-5%	0.019	2.4643	46.7443	8113.97
0%	0.02	21.7042	46.7032	8146.47
5%	0.021	21.6453	46.6640	8179.39
10%	0.022	21.5877	46.6265	8212.72
15%	0.023	21.5306	46.5906	8246.49
20%	0.024	21.4760	46.5563	8280.55

Case 2: When M $M < u_1$

Table 7: Variation in optimal solution for demand parameter (d):

% change in (d)	(d)	u_1	p	$U.T.P_x.$
-20%	160	22.446	38.1208	5090.59
-15%	170	22.6416	40.5444	5784.96
-10%	180	22.8372	42.9764	6526.76
-5%	190	23.0329	45.4168	7316.64
0%	200	23.2287	47.8656	8155.1
5%	210	23.4247	50.3228	9042.74
10%	220	23.6209	52.7883	9980.14
15%	230	23.8175	55.2622	10967.9
20%	240	24.0146	57.7445	12006.8

Table 8: Variation in optimal solution for shortage parameter (β):

% change in (β)	(β)	u_1	p	$U.T.P_x.$
-20%	1.76	24.2121	60.2352	10474.1
-15%	1.87	23.9218	56.5753	9782.19
-10%	1.98	23.6646	53.3374	9174.07
-5%	2.09	23.435	50.4524	8635.46
0%	2.2	23.2287	47.8656	8155.1
5%	2.31	23.0422	45.5332	7724.07
10%	2.42	23.8728	43.4195	7335.16
15%	2.53	22.7182	41.495	6982.49
20%	2.64	22.5764	39.7356	6661.24

Table 9: Variation in optimal solution for lost sale cost parameter (γ):

% change in (γ)	(γ)	u_1	p	$U.T.P_x.$
-20%	0.08	31.417	40.0413	14157.7
-15%	0.085	28.5285	42.1984	12079.8
-10%	0.09	26.394	44.2577	10483.8
-5%	0.095	24.6733	46.1438	9206.48
0%	0.1	23.2287	47.8656	8155.1
5%	0.105	21.9865	49.4465	7270.26
10%	0.11	20.9008	50.9116	6511.38
15%	0.115	19.9409	52.2844	5849.5
20%	0.12	19.0845	53.5868	5263.38

Table 10: Variation in optimal solution for deterioration cost parameter (l_r):

% change in (l_r)	(l_r)	u_1	p	$U.T.P_x$.
-20%	14.4	23.1993	47.8156	8163.83
-15%	15.3	23.2067	47.8282	8161.64
-10%	16.2	23.214	47.8407	8159.45
-5%	17.1	23.2214	47.8532	8157.28
0%	18	23.2287	47.8656	8155.1
5%	18.9	23.236	47.8780	8152.93
10%	19.8	23.2432	47.8904	8150.77
15%	20.7	23.2505	47.9028	8148.61
20%	21.6	23.2577	47.9151	8146.46

Table 11: Variation in optimal solution for deterioration parameter (s_r):

% change in (s_r)	(s_r)	u_1	p	$U.T.P_x$.
-20%	12	23.1130	47.588	8219.98
-15%	12.75	23.1420	47.6579	8203.62
-10%	13.5	23.1710	47.7275	8187.68
-5%	14.25	23.1999	47.7968	8171.68
0%	15	23.2287	47.8656	8155.1
5%	15.75	23.2574	47.9341	8139.12
10%	16.5	23.2861	48.0023	8123.23
15%	17.25	23.3147	48.0701	8107.43
20%	18	23.3432	48.1376	8091.72

Table 12: Variation in optimal solution for interest earned parameter (I_e):

% change in (I_e)	(I_e)	u_1	p	$U.T.P_x$.
-20%	0.016	23.2312	47.8977	8044.48
-15%	0.017	23.2306	47.8896	8072.14
-10%	0.018	23.230	47.8816	8099.79
-5%	0.019	23.2293	47.8736	8127.45
0%	0.02	23.2287	47.8656	8155.1
5%	0.021	23.2281	47.8577	8182.76
10%	0.022	23.2274	47.8499	8210.42
15%	0.023	23.2268	47.8421	8238.07
20%	0.024	23.2262	47.8344	8265.73

VIII. Observations

- Table 1 and 7 represent the effect of demand d on critical time u_1, selling price p and on $U.T.P_x$. It is observed that with an increment in d, there is also an increment in u_1, p and in $U.T.P_x$.
- Table 2 and 8 show the effect of β on u_1, p and on $U.T.P_x$, it is observed that after an increment in β, a pattern of decrement is observed in u_1, p and in $U.T.P_x$ in both the tables.
- Table 3 and 9 list the variation in γ and its effect on u_1, p and $U.T.P_x$, it is observed that as the value of γ increases the values of u_1 and $U.T.P_x$ decreases, while the value of p in both the tables increases.
- Table 4 and 10 represent the effect of l_r on u_1, p and on $U.T.P_x$, it is observed that after an increment in l_r, some increment in u_1 and p is observed while some decrement in $U.T.P_x$ in both the tables is detected.
- Table 5 and 11 list the variation in parameter s_r on optimal value of u_1, p and on $U.T.P_x$. It is observed that after an increment in s_r, some increment in u_1 and p while some decrement in $U.T.P_x$ in both the tables are detected.
- Table 6 and 12 represent the effect of I_e on u_1, p and on $U.T.P_x$, it is observed that after an increment in I_e, some decrement in u_1 and p while some increment in $U.T.P_x$ in both the tables are detected.

IX. Conclusion

Present paper considers an inventory model for price and stock-dependent demand under some real-life situations like variable holding cost and credit financing policies. In today's period when there is the high competition in the market, vendors usually offer new schemes or policies to customers to promote their business. In present study quantity discount policy is also applied because it works as an incentive and a promotional tool for any business. Shortages are allowed with partial backordering. To improve the efficiency of the model numerical examples for different cases and sensitivity analysis for distinct parameters have been discussed with the help of Mathematica 11.3. This Model further can be extended for different demand patterns, deterioration, backlogging rate and also for different realistic approaches like preservation, inflationary environment and green supply chain.

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An Optimal Resource Allocation Model Considering Two-Phase Software Reliability Growth Model with Testing Effort and Imperfect Debugging

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Abstract

This study aims at investigating an optimal resource plan in order to minimize the software costs in the debugging and testing phases. We have proposed a resource allocation model to address testing efforts, imperfect debugging and change-point. We have considered two cases: In the first case, the software debugging cost is kept constant, and in this case, the optimal policy follows a bang-bang structure, which means investing entirely in the testing phase, followed by investing fully in the debugging stage. In the second case, we have taken the debugging cost in quadratic form. We have validated our model with the experimental data, and the results reveal that the presented model is reasonably accurate. We have also discussed the optimal resource allocation problems under certain conditions and examine the parameters' behavior in the model and obtain the variations in the total cost. This study provides a detailed optimal control theory-based testing resource allocation policy, which is supported by numerical examples.

Keywords: Software reliability growth model (SRGM), Change point, Testing effort, Optimal control.

1. Introduction

Computers are used in diverse areas, and with recent advances in computation, software related issues have emerged as one of the primary areas of concern. Hence, reliable software product demand has increased in the market. Software reliability models can be effectively utilized to generate quantified measures during the software development phase. SRGM attempts to tally between defect detection data and estimated residual defects with time. Therefore, software reliability is crucial in developing software and its quality. Specification, design, programming, testing-and-debugging are the four stages involved in any software development process. During the software development process, the testing-and-debugging phase is a key and expensive phase of the software development life cycle (SDLC).

The probability when the software will not result in system failure for a specific time and under specific conditions is known as software reliability (see, e.g., [1]). Further, software reliability and software cost must be up to the expectation level of users satisfaction. During the testing phase of SDLC, testing and debugging are the two main activities to be performed by the testing team, and there is always a trade-off between cost and reliability. The software testing team has to understand the variance of the software reliability and the instantaneous testing costs. Thus, software reliability, cost, and release time are important aspects of software development.

Research activities on software reliability growth models have been conducted over the past five decades. In general, the SRGMs have been proposed under consideration with the non-homogeneous Poisson process (NHPP) since software faults are associated with discrete time scales. Various SRGMs [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] have been proposed on different assumptions and applied to different situations. Many researchers have proposed SRGMs considering different scenarios; some researchers have proposed SRGM considering perfect debugging [2, 3, 5]; some researchers considered imperfect debugging [4, 6, 7]. Huang et al. [18] have discussed an SRGM, which incorporates debugging time-lag and fault dependency.

Many researchers proposed SRGMs incorporating testing efforts. Yamada et al. [19], and Musa et al. [20] proposed a novel SRGM. The model discussed in [19] describes the relationship between the amount of testing effort and the number of software errors detected. Kapur et al. [21], and Chang et al. [22] have proposed SRGMs, which incorporate the concept of testing-effort function. Generally, CPU hours are considered as testing efforts. Huang et al. [23], and Kapur et al. [24] represented a model in which the consumption rate of testing resource expenditures with the different testing-effort functions was discussed. Jin and Jin [25] used an S-shaped curve to describe software testing efforts. Huang et al. [26, 27] proposed SRGMs with logistic testing-effort function.

In software reliability studies, the initial number of faults contained are not known. One has to carry out the program in a particular environment. This helps to improve the quality of the program by correcting the faults. SRGMs proposed by Huang et al. [28] assumed that each failure caused by a fault is independent and random in time and occurs with the same distribution during the fault detection process. In SRGMs, the testing environment, testing strategy and resources are not necessarily the same throughout the development process. The testing environment is subjected to change with the learning process. Hence, the change-point problems play a vital role in software reliability growth modeling.

The change point is a point where the software testing team changes their testing strategy from one to another during the software development process due to the complexity of the program, testing facilities, and other random factors. The fault detection process is affected by different factors. The fault detection rate may change due to the increasing knowledge of the program and the testing strategies. Chang et al. [22] and Kapur et al. [29] discussed an SRGM with change point in fault detection rate. Shyur et al. [30] explained a stochastic software reliability growth model which incorporates imperfect debugging and change points.

Kapur et al. [31, 32] have discussed a model to assign the resources and minimize the total cost during the development period of SRGM under dynamic conditions. Kumar et al. [33, 34, 35] explained a resource allocation model for fault detection and fault correction process. They assumed detection and correction efforts to be independent. However, activities such as detection and correction may have budgetary constraints. Yonghua et al. [36, 37] proposed a model which incorporates a resource allocation plan to minimize the testing cost of the software. Kumar et al. [38] discussed resource allocation model for a multi-release SRGM to minimize the testing cost under dynamic conditions.

In the present work, we incorporate testing effort, imperfect debugging, and change points. These features are concomitant with the general SRGM. Yamada et al. [39] proposed an SRGM to minimize the total expenditure under static conditions. A problem emerges when the development process is carried out under dynamic conditions. The fault detection and fault correction process relies upon the operating environment and the quality of resources utilized. Experimental data analysis is carried out to evaluate the change point. We have used the failure increasing rate function to find the change point. We also study the optimization problem for optimal resource allocation for different conditions by examining the behavior of the model parameters and obtain variations in total cost. To do so, we have taken the quadratic form of the debugging function.

The remaining manuscript is structured as follows: We concluded Section 1 by providing notations used in the manuscript. In Section 2, we briefly discussed the model developed by Zhu et al. [40]. Section 3 deals with the model development, and we introduced an optimal control problem. In Section 4, the optimal policies are developed, and optimal solutions are given. Some theoretical results are shown in Section 5, and the optimal policies are discussed for two special cases. In Section 6, the change point is calculated with the help of the data, and the behavior of model parameters in the variation of the total cost is discussed. In Section 7, we conclude the paper with some possible research on this topic.

Notations

$[0, T]$	The complete life cycle of the software.
t_1	The change-point.
$m_1(t)$	The cumulative number of faults detected in phase-I by time t lies between 0 to t_1.
$m_2(t)$	The cumulative number of faults detected in phase-II by time t lies between t_1 and T.
$x_1(t)$	The number of faults detected at any point of time t in phase-I.
$x_2(t)$	The number of faults detected at any point of time t in phase-II.
$a_1(t)$	The total fault content function in phase-I.
a	The initial number of fault.
$a_2(t)$	The total fault content function in phase-II.
α	The fault introduction rate per detected fault.
b_1	The fault detection rate in phase-I.
b_2	The fault detection rate in phase-II.
c_1	The non-removable fault rate in phase-I.
c_2	The non-removable fault rate in phase-II.
$w_{11}(t)$	The testing effort in phase-I.
$w_{21}(t)$	The testing effort in phase-II.
$w_{12}(t)$	The debugging effort in phase-I.
$w_{22}(t)$	The debugging effort in phase-II.
$c_{11}(t)$	The debugging cost per unit at time t associated with the debugging effort $w_{12}(t)$ in phase-I.
$c_{21}(t)$	The debugging cost per unit at time t associated with the debugging effort $w_{22}(t)$ in phase-II.
$\tilde{c}_{11}$	The base cost of debugging in phase-I.
$\tilde{c}_{21}$	The base cost of debugging in phase-II.
c_{12}	The cost of testing per unit testing effort at time t in phase-I.
c_{22}	The cost of testing per unit testing effort at time t in phase-II.
$y(t)$	The observed cumulative number of failures by time t.
$y'(t)$	The failure increasing rate during time interval $(t, t + \Delta t)$.

2. Two-phase software reliability growth model

An administer SRGM is tested by software test personnel to detect and correct software faults during the development process. In realistic conditions, different types of software faults are found in SRGMs. Software test personnel do not remove all faults during the debugging process by applying the same testing effort. So different testing efforts are applied by software test personnel to remove different types of faults. Thus, to remove two types of fault, software test personnel used a two-phase debugging process. Zhu et al. [40] proposed a two-phase software reliability model under the following assumptions, which involved software fault dependency and imperfect debugging.

1. The error detection in SRGMs follows the non-homogeneous Poisson process.
2. The model considered imperfect debugging, and new faults are introduced into the program at each time.
3. Type-I and type-II software faults are defined. The type-I and type-II faults are detected and corrected during phase-I and phase-II, respectively.
4. The software development team cannot remove all the faults experienced in both phases.

5. Due to different software fault types, the fault detection and non-removable fault rates are different in phase I and II.

6. The time to debug a fault is negligible.

Using the above assumptions, the following two-phase SRGM is considered for the study.

Phase-I. The type-I faults, which are independent and easily detected, are detected and corrected in phase-I. The total number of software faults that cause failure in phase-I is proportional to the difference between the total number of detected faults and the total number of non-removable faults. The total number of detected faults that cause failure is the product of fault detection rate and the number of remaining type-I faults which are represented by the following differential equation

$$\frac{dm_1(t)}{dt} = b_1(t)\left[a_1(t) - m_1(t)\right] - c_1(t)m_1(t), \qquad 0 \leq t \leq t_1. \tag{1}$$

With a constant fault introduction rate α (> 0), new faults are introduced during the debugging phase due to imperfect debugging. Therefore, the fault present in phase-I at time t is given by

$$a_1(t) = a(1 + \alpha t). \tag{2}$$

In phase-I, let the non-removable fault rate is

$$c_1(t) = c_1, \;\; c_1 > 0, \tag{3}$$

with the initial condition $m_1 = 0$ at $t = 0$ for phase-I.

Phase-II. During this phase, type-II faults are detected after the completion of type-I faults detection and correction. No new and residue type-I faults are introduced in phase-II. The total number of software faults that cause failure is proportional to the difference between the total number of detected faults and non-removable faults. The total number of detected faults that cause failure is the product of fault detection rate, the ratio of the total number of reduced faults to the total number of failures experienced, and the number of remaining type-II faults. Therefore, we have the following differential equation

$$\frac{dm_2(t)}{dt} = b_2(t)\frac{m_2(t)}{a_2(t)}\left[a_2(t) - m_2(t)\right] - c_2(t)m_2(t), \;\; t_1 \leq t \leq T. \tag{4}$$

The cumulative number of fault present in phase-II is

$$a_2(t) = a_1(t_1) - m_1(t_1). \tag{5}$$

In phase-II, let the non-removable fault rate is

$$c_2(t) = c_2, \;\; c_2 > 0, \tag{6}$$

with the continuity condition between the two phases

$$m_2(t_1) = m_1(t_1). \tag{7}$$

3. Model development

The testing and debugging phase aims to detect and correct faults and make the software more reliable during the development process of the software. Software reliability is affected by the testing resources spent in the testing phase. We have modified the SRGM given in Section-2 without increasing the complexity. We have developed a model, which incorporates time-dependent testing effort with change point under imperfect debugging environments. Here, a model is constructed with concurrent detection and correction activities. Resources should be allocated optimally during software testing. Thus, a tactical plan is required for allocating resources optimally. The total resource in each phase is divided into two portions, i.e. testing effort and debugging effort. The mathematical expression for the resource allocation model is written as

$$w_{11}(t) + w_{12}(t) = 1, \quad 0 \le t \le t_1, \tag{8}$$

$$w_{21}(t) + w_{22}(t) = 1, \quad t_1 \le t \le T, \tag{9}$$

which is shown in Figure 1.

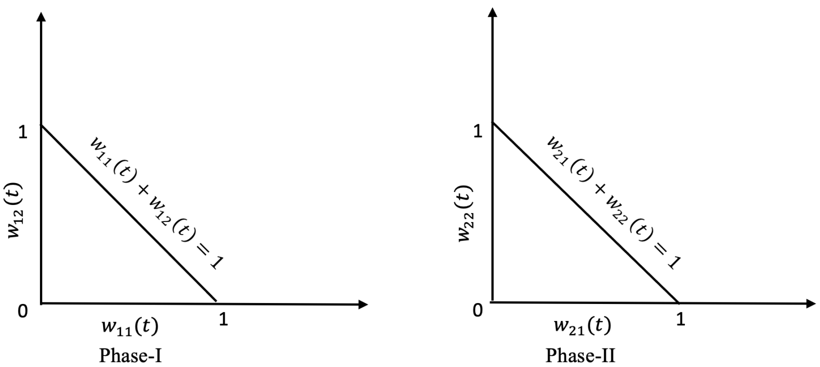

Figure 1: *Optimal resource allocation in Phase-I and Phase-II.*

Phase-I. We have assumed that the software development team detects the faults causing failure and corrects those faults during the testing phase to develop a new model. The total number of software faults that cause failure in phase-I per unit testing effort expenditure is proportional to the difference between the total number of detected faults and the total number of non-removable faults. The total number of detected faults that cause failure is the product of fault detection rate and the number of remaining type-I faults. All the detected faults are not possible to remove from software during the testing phase. Here, we have taken the testing effort in the fault detection process. We know that we should remove more faults if much testing effort is utilized in the testing phase. Some faults can't be removed. Based on the above assumptions and compatible with the idea of Zhu et al. [40], we have proposed the phase-I software reliability growth model as

$$x_1(t) = \frac{dm_1(t)}{dt} = w_{11}(t)\left[b_1(a_1(t) - m_1(t)) - c_1 m_1(t)\right], \qquad 0 \le t \le t_1. \tag{10}$$

Phase-II. In phase-II, the total number of software faults that cause failure per unit testing effort expenditure is proportional to the difference between the total number of detected faults and the total number of non-removable faults. The total number of detected faults that cause failure is the product of fault detection rate, the ratio of the total number of reduced faults to the total number of failures experienced and the number of remaining type-II faults. The following differential equation gives the phase-II model:

$$x_2(t) = \frac{dm_2(t)}{dt} = w_{21}(t)\left[b_2 \frac{m_2(t)}{a_2(t)}(a_2(t) - m_2(t)) - c_2 m_2(t)\right], \quad t_1 \le t \le T. \tag{11}$$

Optimal control problem. We aim to create a resource allocation plan to minimize the total testing expenditure. For the simplicity of the model, we have neglected all other costs except fault detection cost and correction cost over the finite planning period T. The proposed model incorporates testing effort as a control parameter. Then the problem can be described over the interval $[0, T]$ as follows:

$$\min\left[\int_0^{t_1}\Big\{c_{11}(t)x_1(t)+c_{12}w_{11}(t)\Big\}dt+\int_{t_1}^{T}\Big\{c_{21}(t)x_2(t)+c_{22}w_{21}(t)\Big\}dt\right], \tag{12}$$

subject to

$$x_1(t)=\frac{dm_1(t)}{dt} = w_{11}(t)\left[b_1(a_1(t)-m_1(t))-c_1m_1(t)\right], \quad 0\le t\le t_1, \tag{13}$$

$$x_2(t)=\frac{dm_2(t)}{dt} = w_{21}(t)\left[b_2\frac{m_2(t)}{a_2(t)}(a_2(t)-m_2(t))-c_2m_2(t)\right], \quad t_1\le t\le T, \tag{14}$$

with the conditions $m_1(0)=0$, and $m_2(t_1)=m_1(t_1)$.

4. Optimal policies and solution

To solve dynamic optimal control problem defined by equations (12)–(14), we shall use Pontryagin minimum principle [41]. Therefore, we first define the Hamiltonian function, which is given by

$$H(t)=\begin{cases}H_1(t), & 0\le t\le t_1\\ H_2(t), & t_1\le t\le T\end{cases} \tag{15}$$

where

$$H_1(m_1(t),\lambda_1(t),w_{11}(t),t) = c_{11}(t)x_1(t)+c_{12}w_{11}(t)+\lambda_1(t)x_1(t),\ 0\le t\le t_1,$$
$$H_2(m_2(t),\lambda_2(t),w_{21}(t),t) = c_{21}(t)x_2(t)+c_{22}w_{21}(t)+\lambda_2(t)x_2(t),\ t_1\le t\le T.$$

The necessary conditions within each time interval for an optimal solution are defined similar to [42]. The co-state variables (or adjoint variables) $\lambda_1(t)$ and $\lambda_2(t)$ are given by

$$\frac{d}{dt}\lambda_1(t) = -\frac{\partial H_1(m_1(t),\lambda_1(t),w_{11}(t),t)}{\partial m_1(t)},\quad 0\le t\le t_1, \tag{16}$$

$$\frac{d}{dt}\lambda_2(t) = -\frac{\partial H_2(m_2(t),\lambda_2(t),w_{21}(t),t)}{\partial m_2(t)},\quad t_1\le t\le T, \tag{17}$$

with terminal conditions $\lambda_2(T)=0$. Also, the following matching conditions are satisfied at $t=t_1$

$$\lambda_1(t_1) = \lambda_2(t_1), \tag{18}$$

$$H_1(m_1(t_1),\lambda_1(t_1),w_{11}(t_1),t_1) = H_2(m_2(t_1),\lambda_2(t_1),w_{21}(t_1),t_1), \tag{19}$$

and we have

$$H_1(m_1(t_1),\lambda_1(t_1),w_{11}(t_1),t_1) \le H_2(m_2(t_1),\lambda_2(t_1),w_{21}(t_1),t_1),\text{ if } 0=t_1<T, \tag{20}$$

$$H_1(m_1(t_1),\lambda_1(t_1),w_{11}(t_1),t_1) \ge H_2(m_2(t_1),\lambda_2(t_1),w_{21}(t_1),t_1),\text{ if } 0<t_1=T. \tag{21}$$

The control variables $w_{11}(t)$ and $w_{21}(t)$ are given by

$$\frac{\partial H_1(m_1(t),\lambda_1(t),w_{11}(t),t)}{\partial w_{11}(t)} = 0,\quad 0\le t\le t_1, \tag{22}$$

$$\frac{\partial H_2(m_2(t),\lambda_2(t),w_{21}(t),t)}{\partial w_{21}(t)} = 0,\quad t_1\le t\le T, \tag{23}$$

which will give from the equations (22) and (23)

$$w_{11}^*(t) = \frac{(c_1m_1(t)-b_1(a_1(t)-m_1(t)))(c_{11}(t)+\lambda_1(t))-c_{12}}{c_{11w_{11}}(t)b_1((a_1(t)-m_1(t))-c_1(t)m_1(t))},\qquad 0\le t\le t_1, \tag{24}$$

$$w_{21}^*(t) = \frac{(c_2m_2(t)-b_2(\frac{m_2(t)}{a_2(t)})(a_2(t)-m_2(t)))(c_{21}(t)+\lambda_2(t))-c_{22}}{c_{21w_{21}}(t)(b_2(\frac{m_2(t)}{a_2(t)})(a_2(t)-m_2(t))-c_2m_2(t))},\quad t_1\le t\le T, \tag{25}$$

and from equations (8) and (9), we obtain

$$w_{12}^*(t) = 1 - \frac{(c_1 m_1(t) - b_1(a_1(t) - m_1(t)))(c_{11}(t) + \lambda_1(t)) - c_{12}}{c_{11w_{11}}(t) b_1((a_1(t) - m_1(t)) - c_1(t) m_1(t))}, \quad 0 \le t \le t_1, \quad (26)$$

$$w_{22}^*(t) = 1 - \frac{(c_2 m_2(t) - b_2(\frac{m_2(t)}{a_2(t)})(a_2(t) - m_2(t)))(c_{21}(t) + \lambda_2(t)) - c_{22}}{c_{21w_{21}}(t)(b_2(\frac{m_2(t)}{a_2(t)})(a_2(t) - m_2(t)) - c_2 m_2(t))}, \quad t_1 \le t \le T. \quad (27)$$

5. Theoretical results

In this section, we have discussed optimal criteria for two-phase, continuous-time optimal control problems where the integration of these problems depends upon the change point. The qualitative results developed here are demonstrated in the following theorem.

Theorem 1. Let the change point t_1 occurs in the software development life cycle $[0, T]$. Then the following holds

(i) At $t_1 \in (0, T)$, the total cost of fault detection and correction in phase-I is equal to the total cost of fault detection and correction in phase-II.

(ii) At $t_1 = 0$, the total cost of fault detection and correction in phase-I is less than the total cost of fault detection and correction in phase-II.

(iii) At $t_1 = T$, the total cost of fault detection and correction in phase-I is greater than the total cost of fault detection and correction in phase-II.

Proof. Let $[0, T]$ is the planning period of SRGM and the change point $t_1 \in (0, T)$. Since, the Hamiltonian H_1 and H_2 are equal at the change point t_1 which is given by equation (19)

$$H_1(m_1(t_1), \lambda_1(t_1), w_{11}(t_1), t_1) = H_2(m_2(t_1), \lambda_2(t_1), w_{21}(t_1), t_1).$$

We obtain

$$c_{11}(t)x_1(t) + c_{12}w_{11}(t) + \lambda_1(t)x_1(t) = c_{21}(t)x_2(t) + c_{22}w_{21}(t) + \lambda_2(t)x_2(t),$$

and hence, we get

$$(c_{11}(t) + \lambda_1(t))x_1(t) + c_{12}w_{11}(t) = (c_{21}(t) + \lambda_2(t))x_2(t) + c_{22}w_{21}(t). \quad (28)$$

Therefore, in the equation (28), the total cost of fault detection and correction in phase-I is equal to the total cost of fault detection and correction in phase-II.

To prove (ii), let the change-point be $t_1 = 0$, i.e. $t_1 \notin (0, T)$. Then we should skip directly to phase-II for fault detection and correction i.e. in phase-I, no-fault detection and correction are done. Hence, the total cost of fault detection and correction in phase-I is less than the total cost of fault detection and correction in phase-II.

Let the change-point be $t_1 = T$ i.e $t_1 \notin (0, T)$. Then we stick entirely in phase-I for fault detection and correction, i.e. in phase-II, no-fault detection and correction is done. Hence, the total cost of fault detection and correction in phase-I is greater than the total cost of fault detection and correction in phase-II. This complete the proof of part (iii). ■

Special cases. In general, at the initial time of the testing phase of SRGM, the debugging/fault correction cost is much higher as the uncertain nature of the errors. Later on, the fault intensity gradually decreases, and most of the expenditure is invested in the testing phase. Therefore the debugging cost gradually decreases with time. Here, we shall discuss two special cases to show how debugging/fault correction costs impact the optimal policies.

Case-I. Let the debugging/fault correction cost per unit for cumulative fault removed at time t before and after change point are constant, i.e.

$$c_{11}(t) = c_{11} \text{ and } c_{21}(t) = c_{21} \tag{29}$$

Then for constant debugging cost, the objective function (12) will take the following form

$$\min \left[\int_0^{t_1} \Big\{ c_{11}x_1(t) + c_{12}w_{11}(t) \Big\} dt + \int_{t_1}^{T} \Big\{ c_{21}x_2(t) + c_{22}w_{21}(t) \Big\} dt \right]$$

subject to

$$x_1(t) = \frac{dm_1(t)}{dt} = w_{11}(t)\left[b_1(a_1(t) - m_1(t)) - c_1 m_1(t)\right], \quad 0 \le t \le t_1 \tag{30}$$

$$x_2(t) = \frac{dm_2(t)}{dt} = w_{21}(t)\left[b_2\frac{m_2(t)}{a_2(t)}(a_2(t) - m_2(t)) - c_2 m_2(t)\right], \quad t_1 \le t \le T \tag{31}$$

with the conditions $m_1(0) = 0, m_2(t_1) = m_1(t_1), w_{11}(t) + w_{12}(t) = 1$ and $w_{21}(t) + w_{22}(t) = 1$. Then the Hamiltonian for the optimal control problem is

$$H_1(m_1(t), \lambda_1(t), w_{11}(t), t) = c_{11}x_1(t) + c_{12}w_{11}(t) + \lambda_1(t)x_1(t), \quad 0 \le t \le t_1$$
$$H_2(m_2(t), \lambda_2(t), w_{21}(t), t) = c_{21}x_2(t) + c_{22}w_{21}(t) + \lambda_2(t)x_2(t), \quad t_1 \le t \le T,$$

The adjoint variable $\lambda_1(t)$ and $\lambda_2(t)$ are given by

$$\frac{d}{dt}\lambda_1(t) = \dot{\lambda}_1(t) = w_{11}(t)(c_{11} + \lambda_1(t))(b_1 + c_1), \quad 0 \le t \le t_1 \tag{32}$$

$$\frac{d}{dt}\lambda_2(t) = \dot{\lambda}_2(t) = w_{21}(t)\{c_{21}(t) + \lambda_2(t)\}\{c_2 - b_2\left(1 - \frac{2m_2(t)}{a_2(t)}\right)\}, \quad t_1 \le t \le T \tag{33}$$

with the terminal condition $\lambda_2(T) = 0$, which on simplifying will give

$$\lambda_1(t_1) = \lambda_1(0) + \int_o^{t_1} w_{11}(t)(c_{11} + \lambda_1(t))(b_1 + c_1)dt \tag{34}$$

$$\lambda_2(t_1) = \int_{t_1}^{T} w_{21}(t)\{c_{21} + \lambda_2(t)\}\{b_2\left(1 - \frac{2m_2(t)}{a_2(t)}\right) - c_2\}dt. \tag{35}$$

The necessary conditions for optimality are

$$\frac{\partial H_1}{\partial w_{11}} = 0, \quad 0 \le t \le t_1 \tag{36}$$

$$\frac{\partial H_2}{\partial w_{21}} = 0, \quad t_1 \le t \le T \tag{37}$$

which will give

$$H_{1w_{11}} = (c_{11} + \lambda_1(t))(b_1(a_1(t) - m_1(t)) - c_1 m_1(t)) + c_{12}, \; 0 \le t \le t_1,$$

and

$$H_{2w_{21}} = (c_{21} + \lambda_2(t))(b_2\frac{m_2(t)}{a_2(t)}(a_2(t) - m_2(t)) - c_2 m_2(t)) + c_{22}, \quad t_1 \le t \le T.$$

where $H_{1w_{11}} = \frac{\partial H_1}{\partial w_{11}}$ and $H_{2w_{21}} = \frac{\partial H_2}{\partial w_{21}}$.

The Hamiltonian in equation (15) is linear in $w_{11}(t)$ and $w_{21}(t)$, respectively. So we have the following bang-bang solution for $w_{11}(t)$ and $w_{21}(t)$ to minimize the Hamiltonian. Therefore, we obtain $w_{11}(t)$ and $w_{21}(t)$ as

(i) If $0 \leq t \leq t_1$,

$$w_{11}^*(t) = \begin{cases} 1, & \text{if } H_{1w_{11}} > 0 \\ \text{undefined}, & \text{if } H_{1w_{11}} = 0 \\ 0, & \text{if } H_{1w_{11}} < 0. \end{cases} \tag{38}$$

If the marginal value of the testing effort $H_{1w_{11}}$ is positive, then maximum possible testing effort is $w_{11}(t) = 1$ because Hamiltonian H_1 is linear in control variable $w_{11}(t)$. If $H_{1w_{11}}$ is negative, then minimum testing effort is $w_{11}(t) = 0$ i.e. all effort should be allocated to debugging. If $H_{1w_{11}} = 0$, then the testing effort $w_{11}(t)$ need not be determined from the condition $H_{1w_{11}} = 0$.

(ii) If $t_1 \leq t \leq T$,

$$w_{21}^*(t) = \begin{cases} 1, & \text{if } H_{2w_{21}} > 0 \\ \text{undefined}, & \text{if } H_{2w_{21}} = 0 \\ 0, & \text{if } H_{2w_{21}} < 0. \end{cases} \tag{39}$$

Similarly, if the marginal value of the testing effort $H_{2w_{21}}$ is positive, then maximum possible testing effort is $w_{21}(t) = 1$ because Hamiltonian H_2 is linear in control variable $w_{21}(t)$. If $H_{2w_{21}}$ is negative, then minimum testing effort is $w_{21}(t) = 0$ i.e. all effort should be allocated to debugging. If $H_{2w_{21}} = 0$, then the testing effort $w_{21}(t)$ need not be determined from the condition $H_{2w_{21}} = 0$.

Case-II. Researchers have been proposed the effect of the experience curve in SRGM. Kapur et al. [31] used Pegels' form [43] as debugging cost function. In this case, we consider that the total cost per unit fault removed is a quadratic function of the debugging efforts, i.e.

$$c_{11}(t) = \tilde{c}_{11}(w_{12}(t))^2 \text{ and } c_{21}(t) = \tilde{c}_{21}(w_{22}(t))^2. \tag{40}$$

Then the Hamiltonian (15) will reduce to

$$\begin{aligned} H_1(m_1(t), \lambda_1(t), w_{11}(t), t) &= c_{11}(t)x_1(t) + c_{12}w_{11}(t) + \lambda_1(t)x_1(t), \quad 0 \leq t \leq t_1 \\ H_2(m_2(t), \lambda_2(t), w_{21}(t), t) &= c_{21}(t)x_2(t) + c_{22}w_{21}(t) + \lambda_2(t)x_2(t), \quad t_1 \leq t \leq T. \end{aligned}$$

From the necessary condition of optimality described by (22) and (23) , we obtain

$$w_{12}(t) = \sqrt{\frac{\lambda_1(t)(c_1 m_1(t) - b_1(a_1(t) - m_1(t))) - c_{12}}{\tilde{c}_{11}(b_1(a_1(t) - m_1(t)) - c_1 m_1(t))}}, \; 0 \leq t \leq t_1, \tag{41}$$

$$w_{22}(t) = \sqrt{\frac{\lambda_2(t)(c_2 m_2(t) - b_2(\frac{m_2(t)}{a_2(t)})(a_2(t) - m_2(t))) - c_{22}}{\tilde{c}_{21}(b_2(\frac{m_2(t)}{a_2(t)})(a_2(t) - m_2(t)) - c_2 m_2(t))}}, \; t_1 \leq t \leq T. \tag{42}$$

Therefore, the optimal policy for $w_{11}(t)$ and $w_{21}(t)$ using equations (8) and (9), can be expressed as

$$w_{11}(t) = 1 - \sqrt{\frac{\lambda_1(t)(c_1 m_1(t) - b_1(a_1(t) - m_1(t))) - c_{12}}{\tilde{c}_{11}(b_1(a_1(t) - m_1(t)) - c_1 m_1(t))}}, \; 0 \leq t \leq t_1, \tag{43}$$

$$w_{21}(t) = 1 - \sqrt{\frac{\lambda_2(t)(c_2 m_2(t) - b_2(\frac{m_2(t)}{a_2(t)})(a_2(t) - m_2(t))) - c_{22}}{\tilde{c}_{21}(b_2(\frac{m_2(t)}{a_2(t)})(a_2(t) - m_2(t)) - c_2 m_2(t))}}, \; t_1 \leq t \leq T. \tag{44}$$

6. Numerical Analysis

In this section, different optimal policies have been discussed on the proposed model using a numerical example. This study aims to get some view into the result and study the impact of change in efforts on the model's total cost.

Our first objective is to find the change point. We have taken PL/I database application software data, given in Table 1, which was studied by Ohba [13] to get the change point of the model. Specifically, testing for the PL/I database application software ranges from week 1 to week 19, and a total of 328 errors are found. The discussed model in section-3 comprises two phases. Generally, when the testing is carried out, the software developers team knows the change point t_1. But here, we don't know the value of t_1. Therefore, we first need to determine the value of t_1 in the model validation.

Weeks	Cumulative execution time (CPU hours)	Cumulative number of detected faults	Weeks	Cumulative execution time (CPU hours)	Cumulative number of detected faults
1	2.45	15	11	26.23	233
2	4.9	44	12	27.67	255
3	6.86	66	13	30.93	276
4	7.84	103	14	34.77	298
5	9.52	105	15	38.61	304
6	12.89	110	16	40.91	311
7	17.1	146	17	42.67	320
8	20.47	175	18	44.66	325
9	21.43	179	19	47.65	328
10	23.35	206			

Table 1: *Data set (Failure data from PL/I database application [13])*

To determine the value of t_1, we shall use the software failure increasing rate concept which is given by

$$y'(t) = \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}. \tag{45}$$

For different value of Δt, the pattern of $y'(t)$ is shown in Figure 2. We know that $y'(t)$ increases until it reached its optimum due to the tester's growing fault correction experience. Afterwards, it starts to decline to stabilize the rate. A similar trend will be showing again for fault correction of another type of faults. From Figure 2, we conclude that the optimum value for failure increasing rate in phase-I is obtained at $t = 6$ and phase-II at $t = 9$. Therefore, we get the change point at $t_1 = 7$.

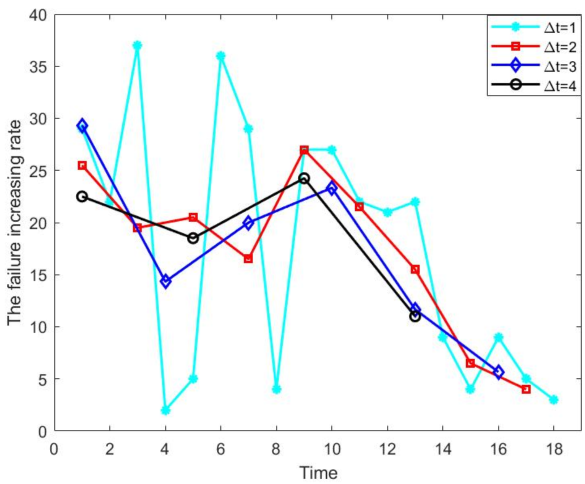

Figure 2: *Software failure rate with respect to time t.*

From the above analysis, we have concluded that the change point t_1 occurs at 7. For $t_1 = 7$, the software failure in phase-I and phase-II is defined when $t \in [0, 7]$ and $t \in [7, 19]$ respectively.

We discuss briefly the comparative analysis of the proposed SRGM with different testing efforts. The common criteria used to compare the model are the mean-squared error (MSE), the predictive-ratio risk (PRR) and the predictive power (PP), and the results are shown in Table 2.

The MSE measures the deviation between the predicted values with the actual data and is defined as:

$$MSE = \frac{1}{n} \sum_{i=1}^{n} (m(t_i) - y_i)^2,$$

where n is the number of observations in the model. The PRR measures the distance of model estimates from the actual data against the model estimate which is given by

$$PRR = \sum_{i=1}^{n} \left(\frac{m(t_i) - y_i}{m(t_i)} \right)^2,$$

whereas the PP measures the distance of model estimates from the actual data against the actual data and is defined as

$$PP = \sum_{i=1}^{n} \left(\frac{m(t_i) - y_i}{y_i} \right)^2.$$

	$w_{11}(t) = 0.5$, $w_{21}(t) = 0.6$	$w_{11}(t) = 0.6$, $w_{21}(t) = 0.7$	$w_{11}(t) = 0.7$, $w_{21}(t) = 0.8$	$w_{11}(t) = 0.8$, $w_{21}(t) = 0.9$
MSE	758.15	428.54	292.29	46.57
PRR	0.5001	0.3408	0.5849	0.1982
PP	0.4588	0.3881	0.3934	0.4087

Table 2: *Model parameters comparison criteria*

We have estimated different model parameters using Excel 2019. The estimated parameters (a, α, b_1, c_1, b_2 and c_2) and the hypothetical value of the other parameters are presented in Table 3.

Parameter	$w_{11}(t) = 0.5$, $w_{21}(t) = 0.6$	$w_{11}(t) = 0.6$, $w_{21}(t) = 0.7$	$w_{11}(t) = 0.7$, $w_{21}(t) = 0.8$	$w_{11}(t) = 0.8$, $w_{21}(t) = 0.9$
a	380	440	450	454
α	0.02	0.016	0.018	0.015
b_1	0.109	0.085	0.075	0.072
c_1	0.0573	0.0496	0.046	0.032
b_2	0.4889	0.4718	0.465	0.3354
c_2	0.0135	0.0644	0.0644	0.0185
$\tilde{c}_{11}$	500	950	1950	2200
$\tilde{c}_{21}$	1100	3800	15000	220000
c_{12}	8500	21000	25000	30000
c_{22}	5000	12000	13000	15000

Table 3: *Value of parameters for different $w_{11}(t)$ and $w_{21}(t)$.*

In this analysis, we have discussed the significance of allocation of testing efforts $w_{11}(t)$ and $w_{21}(t)$. Figure 3 describe the relationship between the cumulative number of faults removed versus time. It shows that the increase in the testing efforts $w_{11}(t)$ and $w_{21}(t)$ will increase the number of faults removed. It has been seen that when the value of $w_{11}(t)$ and $w_{21}(t)$ gradually increases, the rate of fault removal increases.

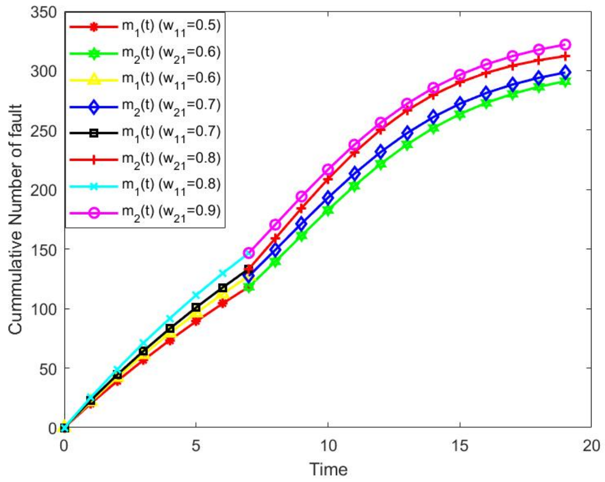

Figure 3: *Cumulative number of faults removed vs time*

The analysis is also done to show how the debugging cost impacts the future cost to remove one fault and also indicates that decreasing the debugging efforts $w_{12}(t)$ and $w_{22}(t)$ will lead to a depletion in the total debugging cost. The pattern is shown in Figure 4, which says that the co-state variable decreases with time and approaches zero.

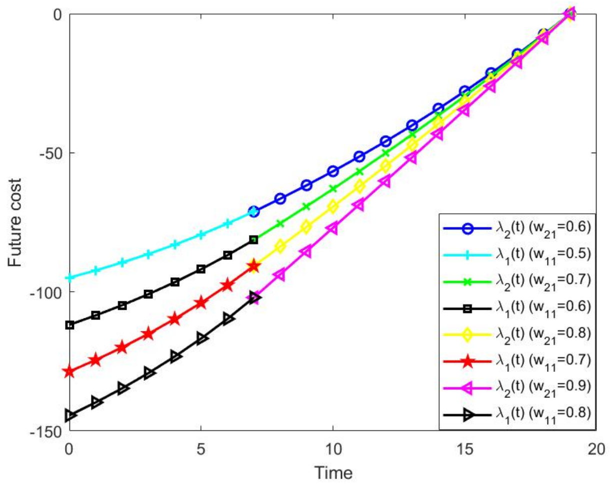

Figure 4: *Shadow cost vs time*

In the analysis, a reduction in the total debugging cost is observed when the debugging effort decreases and hence the rate of increase of the total expenditure is reduced. Also, it shows how the Hamiltonian (H) that is the sum of testing and debugging cost, starts decreasing after some time. In short, Hamiltonian represents the instantaneous total cost of the model at time t. The results are shown in Table 4, and the total developmental cost of the SRGM with different testing efforts are 93469, 247329, 357569 and 734065, respectively.

Time (Weeks)	Total Cost ($w_{11}(t) = 0.5$, $w_{21}(t) = 0.6$)	Total Cost ($w_{11}(t) = 0.6$, $w_{21}(t) = 0.7$)	Total Cost ($w_{11}(t) = 0.7$, $w_{21}(t) = 0.8$)	Total Cost ($w_{11}(t) = 0.8$, $w_{21}(t) = 0.9$)
1	4886	13516	18628	25407
2	4900	13532	18651	25413
3	4915	13550	18675	25418
4	4932	13569	18702	25425
5	4950	13590	18731	25431
6	4970	13612	18763	25437
7	4991	13637	18798	25444
8	5171	14009	23142	62883
9	5371	14329	23702	63850
10	5488	14419	23594	62797
11	5507	14270	22849	59895
12	5429	13910	21611	55575
13	5270	13393	20086	50399
14	5054	12786	18477	44925
15	4806	12150	16942	39606
16	4552	11536	15577	34744
17	4307	10974	14422	30501
18	4083	10482	13482	26925
19	3887	10065	12737	23990

Table 4: *Total Cost (Testing cost and debugging cost)*

7. Conclusions and future work

In this article, we have presented a resource allocation technique considering a two-phase software reliability growth model with testing effort, change-point, and imperfect debugging. Numerical simulations are done which support the accuracy of our proposed model. We have given an insight into how to find the change point. Generally, the software developers team knows the change point of real software testing. This paper aims to calculate the total cost of the software using two-phase SRGM. We have proposed the theoretical results using optimal control theory, and a different approach is used to allocate testing resources optimally. We can control the testing efforts when the company switches from one testing strategy to another. We observed from the graph of the future cost of detection that it eventually reaches zero with time. Moreover, from the graph of shadow cost of correction, we observed that, as time increases, the shadow cost decreases and tends to zero. The variation in cost with the change in various model parameters has also been depicted.

In contrast with the other studies, we proposed the theoretical results to find the change point from one testing strategy to another using optimal control theory. But in this paper, experimental data analysis is utilized to determine the change point. We have used the failure increasing rate function to find the change point. To propose a more realistic SRGM, more information is needed by software managers. In this direction, the stochastic model for fault detection and correction can be used for future work. We can also extend the proposed model by incorporating different fault content functions. All these issues may be part of further work.

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