POWER WEIGHTED AKASH DISTRIBUTION WITH PROPERTIES AND APPLICATIONS

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Abstract

In In this paper power weighted Akash distribution (PWAD) which includes weighted Akash distribution (WAD), power Akash distribution (PAD) and Akash distribution as particular cases has been proposed and investigated. Its moments, hazard rate function and mean residual life function have been discussed. Method of maximum likelihood estimation has been discussed for estimating the parameters of the distribution. Applications of the proposed distribution to two real lifetime datasets have been presented and compared with other one parameter, two-parameter and three-parameter well-known lifetime distributions.

Keywords: Akash distribution, Weighted Akash distribution, Power Akash distribution, Hazard rate function, stochastic ordering, Maximum Likelihood estimation, Applications.

I. Introduction

Shanker and Shukla [1] proposed a two-parameter weighted Akash distribution (WAD) having parameters θ and α and defined by its probability density function (pdf) and cumulative distribution function (cdf)

$$f_1(y;\theta,\alpha)=\frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)}\frac{y^{\alpha-1}}{\Gamma(\alpha)}(1+y^2)e^{-\theta y};y>0,\theta>0,\alpha>0 \tag{1.1}$$

$$F_1(y;\theta,\alpha)=1-\frac{[\theta^2+\alpha(\alpha+1)]\Gamma(\alpha,\theta y)+(\theta y)^{\alpha}(\theta y+\alpha+1)e^{-\theta y}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)} \tag{1.2}$$

where $\Gamma(\alpha)$ and $\Gamma(\alpha,z)$ are the complete gamma function and the upper incomplete gamma function defined as

$$\Gamma(\alpha)=\int_0^{\infty}e^{-t}t^{\alpha-1}dt\,;\alpha>0 \tag{1.3}$$

$$\Gamma(\alpha, z) = \int_z^{\infty} e^{-y} t^{\alpha-1} dt; \alpha > 0, z \geq 0 \tag{1.4}$$

Its structural properties including moments, hazard rate function, mean residual life function, estimation of parameters and applications for modeling survival time data has been discussed by Shanker and Shukla [1]. Shanker and Shukla [2] discussed various moments based properties including coefficient of variation, coefficient of skewness, coefficient of kurtosis and index of dispersion of weighted Akash distribution and its applications to model lifetime data from biomedical sciences and engineering.

Shanker and Shukla [2] proposed a power Akash distribution (PAD) having parameters θ and α and defined by its pdf and cdf

$$f_2(y; \theta, \beta) = \frac{\beta\theta^3}{(\theta^2+2)} \left(1 + y^{2\beta}\right) y^{\beta-1} e^{-\theta y^{\beta}}; y > 0, \theta > 0, \beta > 0 \tag{1.5}$$

$$F_2(y; \theta, \beta) = 1 - \left[1 + \frac{\theta y^{\beta}\left(\theta y^{\beta} + 2\right)}{\theta^2 + 2}\right] e^{-\theta y^{\beta}}; y > 0, \theta > 0, \beta > 0 \tag{1.6}$$

Note that the PAD is a convex combination of Weibull (α, θ) and a generalized gamma $(2, \alpha, \theta)$distribution with mixing proportion$\frac{\theta^2}{\theta^2+2}$. Shanker and Shukla [1] has discussed the properties of PAD including the shapes of the density, hazard rate functions, moments, skewness and kurtosis measures, estimation of parameters using maximum likelihood estimation and application to model a real lifetime data from engineering. Recall that WAD and PAD reduces to Akash distribution at $\alpha = 1$, and $\beta = 1$ respectively. The Akash distribution proposed by Shanker [3] is defined by its pdf and cdf

$$f_3(y; \theta) = \frac{\theta^3}{\theta^2 + 2}(1 + y^2) e^{-\theta y}; y > 0, \theta > 0 \tag{1.7}$$

$$F_3(y; \theta) = 1 - \left[1 + \frac{\theta y(\theta y + 2)}{\theta^2 + 2}\right] e^{-\theta y}; y > 0, \theta > 0 \tag{1.8}$$

Shanker [3] has discussed its various statistical and mathematical properties including shapes of the density. Moments and moments based measures, hazard rate function, mean residual life function, stochastic ordering, mean deviations, order statistics, Bonferroni and Lorenz curves, Renyi entropy measure, stress-strength reliability, estimation of parameter using both the method of moment and the maximum likelihood estimation and application to model lifetime data from engineering and biomedical sciences.

In the present paper, a three - parameter power weighted Akash distribution which includes Akash distribution, WAD, and PAD as particular cases, has been proposed and discussed. Its raw moments have been given. The survival function and the hazard rate function of the distribution have been derived and their shapes have been discussed for varying values of the parameters. The estimation of its parameters has been discussed using maximum likelihood method. Finally, the goodness of fit and the applications of the distribution have been explained through two real lifetime datasets and the fit has been compared with other one parameter, two-parameter and three-parameter lifetime distributions.

II. Power weighted Akash distribution

Assuming the power transformation $X = Y^{\frac{1}{\beta}}$ in (1.1), the pdf of the random variable Xcan be obtained as

$$f_4(x;\theta,\alpha,\beta) = \frac{\beta\theta^{\alpha+2}}{\theta^2+\alpha^2+\alpha}\frac{x^{\beta\alpha-1}}{\Gamma(\alpha)}\left(1+x^{2\beta}\right)e^{-\theta x^{\beta}};x > 0,\theta > 0,\alpha > 0,\beta > 0 \tag{2.1}$$

We would call the distribution in (2.1) as the power weighted Akash distribution (PWAD). It can be easily verified that the WAD(θ,α) in (1.1), PAD(θ,β) in (1.5) and Akash(θ)in (1.7) are the special cases of PWAD for $(\beta = 1)$,$(\alpha = 1)$and $(\alpha = \beta = 1)$, respectively.

It can be easily verified that PWAD is a convex combination of generalized gamma distribution (GGD) having parameters (θ,α,β)proposed by Stacy (1962) and GGD having parameters $(\theta,\alpha+2,\beta)$.

We have

$$f_4(x;\theta,\alpha,\beta) = pg_1(x;\theta,\alpha,\beta) + (1-p)g_2(x;\theta,\alpha+2,\beta),$$

where

$$p = \frac{\theta^2}{\theta^2+\alpha^2+\alpha}$$
$$g_1(x;\theta,\alpha,\beta) = \frac{\beta\theta^{\alpha}}{\Gamma(\alpha)}x^{\beta\alpha-1}e^{-\theta x^{\beta}};x > 0,\theta > 0,\alpha > 0,\beta > 0$$
$$g_2(x;\theta,\alpha,\beta) = \frac{\beta\theta^{\alpha+2}}{\Gamma(\alpha+2)}x^{\beta(\alpha+2)-1}e^{-\theta x^{\beta}};x > 0,\theta > 0,\alpha > 0,\beta > 0.$$

Graphs of density function of PWAD for varying values of parameters θ,αandβ have been drawn and presented in figure 1. It is clear that the natures of PWAD are decreasing, positively skewed, negatively skewed, platykurtic, mesokurtic and leptokurtic for varying values of parameters and hence it can be applied to model lifetime datasets of various natures. It is observed that pdf is increasing for increased value of θ and its pdf is increasing vastly as increased value of θ,αandβrespectively. However, role of α on the shape of the graph more as compared to other parameters.

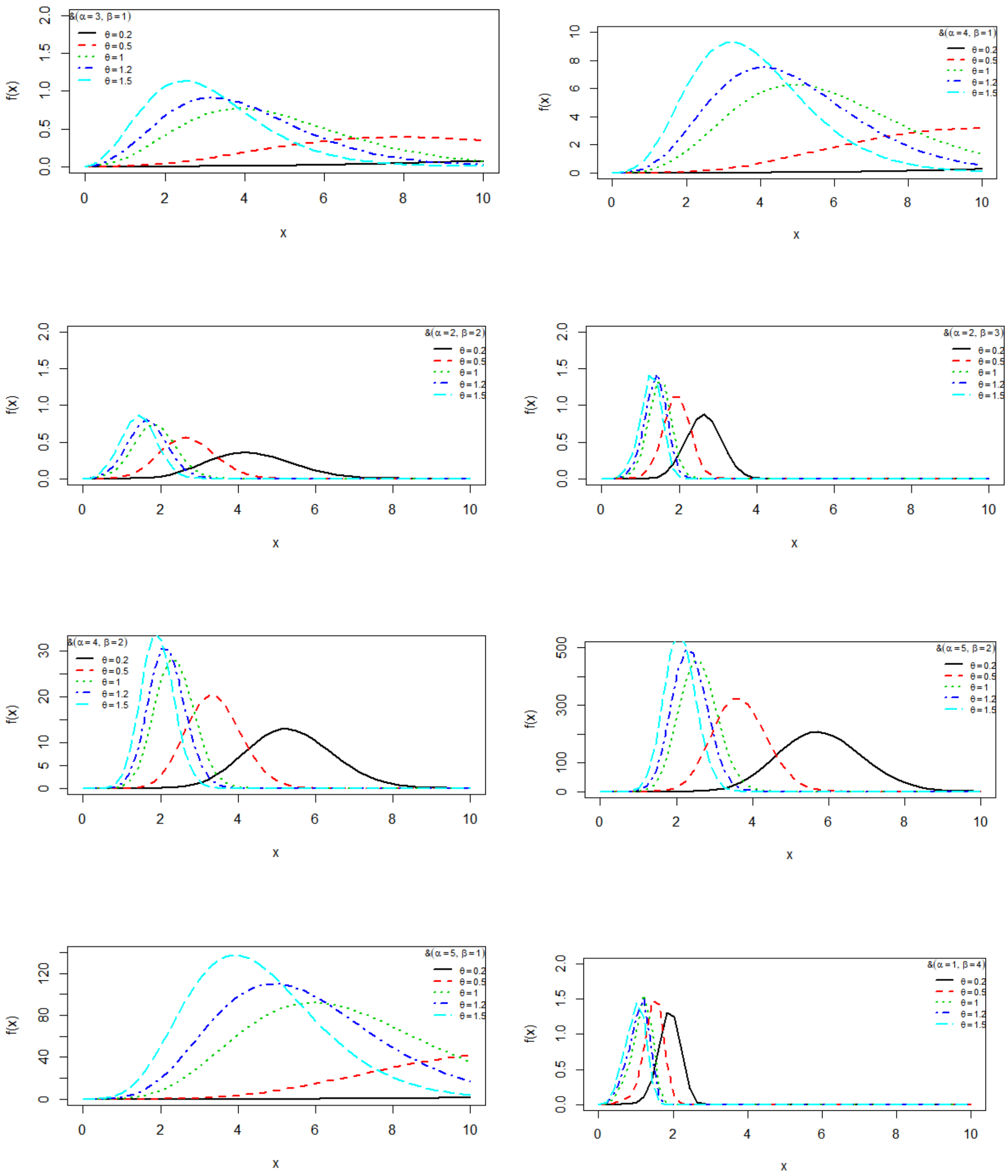

Figure. 2: Graphs of the probability density function of PWAD for varying values of parametersθ, αandβ

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III. Reliability Measures

Survival function and Cumulative distribution Function

The survival function $S(x;\theta,\alpha,\beta)$of PWAD can be obtained as

$$S(x;\theta,\alpha,\beta)=P(X>x)=\int_x^\infty f_4(t;\theta,\alpha,\beta)\,dt$$

$$=\frac{\beta\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}\int_x^\infty t^{\beta\alpha-1}\left(1+t^{2\beta}\right)e^{-\theta t^\beta}dt$$

$$=\frac{\beta\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}\left[\int_x^\infty e^{-\theta t^\beta}t^{\beta\alpha-1}\,dt+\int_x^\infty e^{-\theta t^\beta}t^{\beta\alpha+2\beta-1}\,dt\right]$$

Assuming $u=t^\beta$, which gives $t=(u)^{\frac{1}{\beta}}$and $dt=\frac{1}{\beta}(u)^{\frac{1-\beta}{\beta}}du$, we get

$$S(x;\theta,\alpha,\beta)=\frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}\left[\int_{x^\beta}^\infty e^{-\theta u}u^{\alpha-1}\,du+\int_{x^\beta}^\infty e^{-\theta u}u^{\alpha+1}\,du\right]$$

$$=\frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}\left[\frac{\Gamma(\alpha,\theta x^\beta)}{\theta^\alpha}+\frac{e^{-\theta x^\beta}(\theta x^\beta+\alpha+1)(\theta x^\beta)^\alpha+\alpha(\alpha+1)\Gamma(\alpha,\theta x^\beta)}{\theta^{\alpha+2}}\right]$$

$$=\frac{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha,\theta x^\beta)+(\theta x^\beta)^\alpha(\theta x^\beta+\alpha+1)e^{-\theta x^\beta}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)},$$

where $\Gamma(\alpha,\theta x^\beta)$ is the upper incomplete gamma function defined as

$$\Gamma(\alpha,\theta x^\beta)=\int_{\theta x^\beta}^\infty y^{\alpha-1}e^{-y}dy;\alpha>0,\theta x^\beta>0.$$

It can be easily verified that at $(\beta=1)$,$(\alpha=1)$and $(\alpha=\beta=1)$ the survival function of PWAD reduce to the survival function of WAD, PAD and Akash distribution.

Thus the cdf of PWAD can be given by

$$F_4(x;\theta,\alpha,\beta)=1-S(x;\theta,\alpha,\beta)=1-\frac{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha,\theta x^\beta)+(\theta x^\beta)^\alpha(\theta x^\beta+\alpha+1)e^{-\theta x^\beta}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}$$

The natures of the cdf of PWAD for varying values of parameters θ,αandβ are shown in figure 2. From the figure 2, It is observed that distribution function is slightly increasing as increased value of θ.

Hazard Rate Function

The hazard rate function, $h(x;\theta,\alpha,\beta)$, of PWAD can be given by

$$h(x;\theta,\alpha,\beta)=\frac{f_4(x;\theta,\alpha,\beta)}{S(x;\theta,\alpha,\beta)}=\frac{\beta\theta^{\alpha+2}x^{\beta\alpha-1}(1+x^{2\beta})e^{-\theta x^\beta}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha,\theta x^\beta)+(\theta x^\beta)^\alpha(\theta x^\beta+\alpha+1)e^{-\theta x^\beta}}.$$

Graphs of $h(x;\theta,\alpha,\beta)$for varying values of parameters θ,αandβ are shown in figure 3. The graphs of $h(x;\theta,\alpha,\beta)$shows that it takes different shapes for varying values of parameters θ,αandβ and it is observed that hazard rate is increasing as increased value of θ,αandβrespectively.

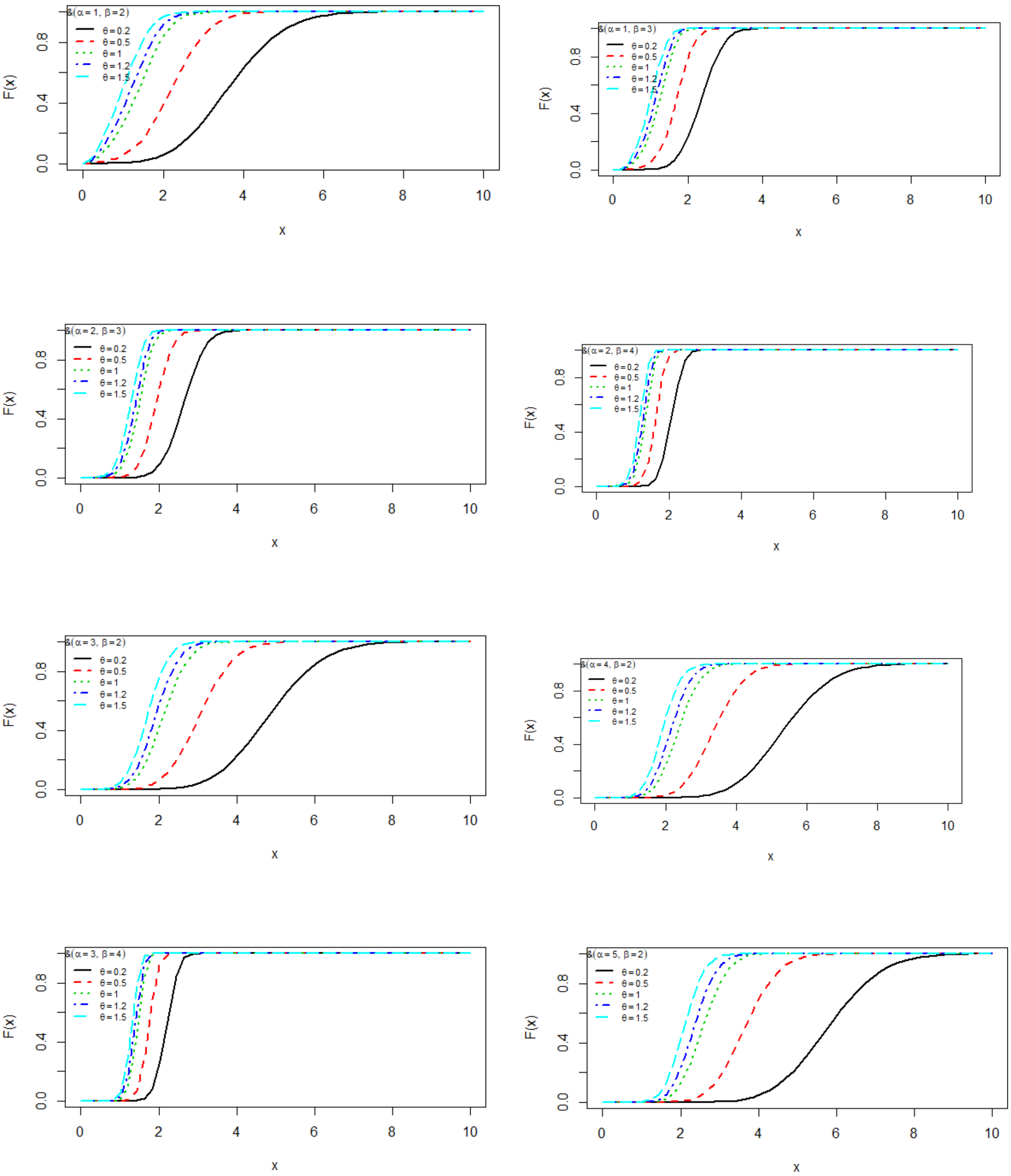

Figure 2: Graphs of the cdf of PWAD for varying values of parameters θ, α and β

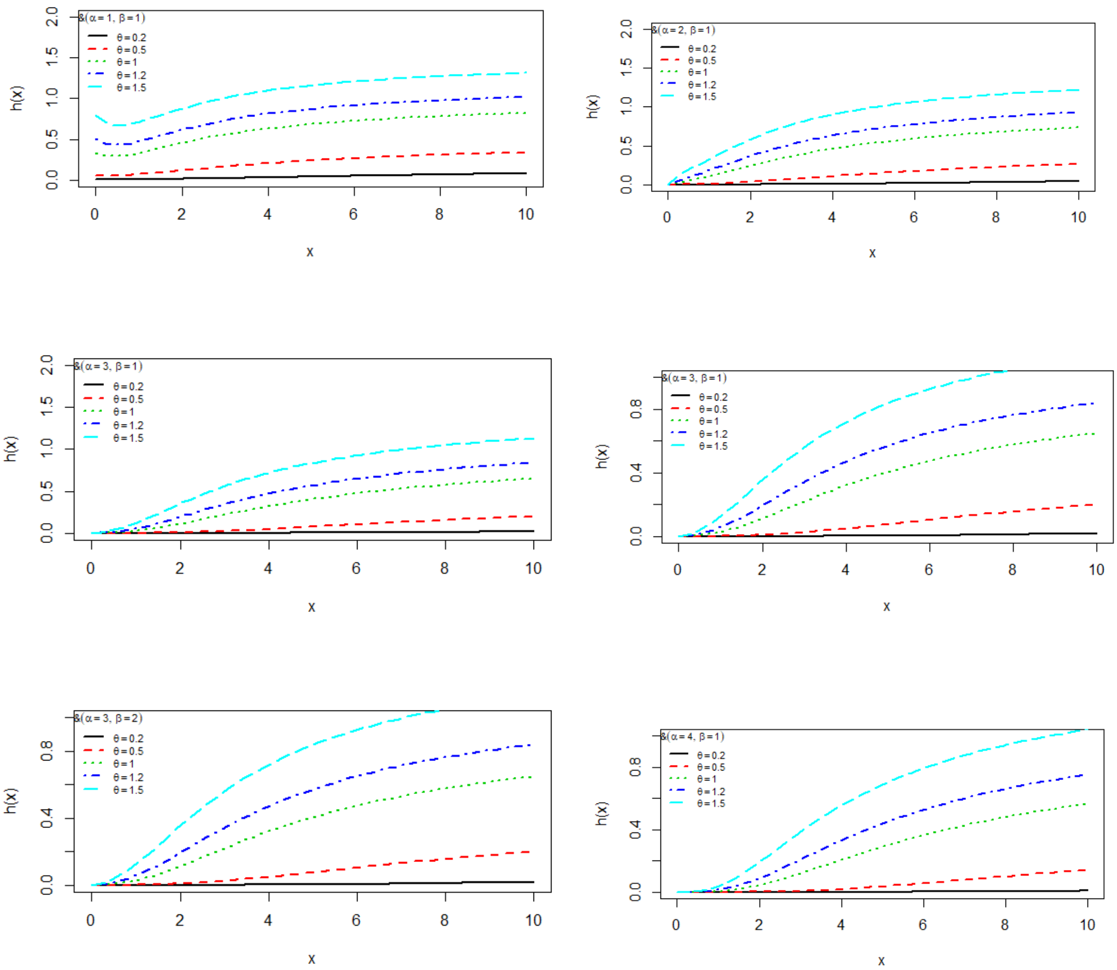

Figure 3: *Graphs of hazard rate function for varying values of parameters* θ, α and β.

Mean Residual Life Function

The mean residual life function, $m(x) = m(x;\theta,\alpha,\beta)$, of PWAD can be obtained as

$$m(x) = m(x;\theta,\alpha,\beta) = \frac{1}{S(x;\theta,\alpha,\beta)}\int_x^\infty t f_4(t;\theta,\alpha,\beta)\,dt - x$$

$$= \frac{\beta\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha,\theta x^\beta)+(\theta x^\beta)^\alpha(\theta x^\beta+\alpha+1)e^{-\theta x^\beta}}\int_x^\infty t^{\beta\alpha}\left(1+t^{2\beta}\right)e^{-\theta t^\beta}\,dt - x$$

$$= \frac{\beta\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha,\theta x^\beta)+(\theta x^\beta)^\alpha(\theta x^\beta+\alpha+1)e^{-\theta x^\beta}}\left[\int_x^\infty e^{-\theta t^\beta}t^{\beta\alpha}\,dt + \int_x^\infty e^{-\theta t^\beta}t^{\beta\alpha+2\beta}\,dt\right] - x$$

Assuming $u = t^\beta$, which gives $t = (u)^{\frac{1}{\beta}}$and $dt = \frac{1}{\beta}(u)^{\frac{1-\beta}{\beta}}du$, we get

$$m(x) = m(x;\theta,\alpha,\beta) = \frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha,\theta x^\beta)+(\theta x^\beta)^\alpha(\theta x^\beta+\alpha+1)e^{-\theta x^\beta}}$$
$$\times\left[\int_{x^\beta}^\infty e^{-\theta u}u^{\alpha+\frac{1}{\beta}-1}\,du + \int_{x^\beta}^\infty e^{-\theta u}u^{\alpha+2+\frac{1}{\beta}-1}\,du\right] - x$$
$$= \frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha,\theta x^\beta)+(\theta x^\beta)^\alpha(\theta x^\beta+\alpha+1)e^{-\theta x^\beta}}$$
$$\times\left[\frac{\Gamma\left(\alpha+\frac{1}{\beta},\theta x^\beta\right)}{\theta^{\alpha+\frac{1}{\beta}}} + \frac{\Gamma\left(\alpha+2+\frac{1}{\beta},\theta x^\beta\right)}{\theta^{\alpha+2+\frac{1}{\beta}}}\right] - x$$
$$= \frac{\theta^2\Gamma\left(\alpha+\frac{1}{\beta},\theta x^\beta\right)+\Gamma\left(\alpha+2+\frac{1}{\beta},\theta x^\beta\right)}{\theta^{\frac{1}{\beta}}\left[(\theta^2+\alpha^2+\alpha)\Gamma(\alpha,\theta x^\beta)+(\theta x^\beta)^\alpha(\theta x^\beta+\alpha+1)e^{-\theta x^\beta}\right]} - x.$$

The behaviors of $m(x)$of PWAD for varying values of its parameters $\theta,\alpha$and$\beta$are shown in figure 4. It is observed from the figure 4 that overall mean residual value is decreasing as increased value of θ whereas other parameters are kept as constant; however mean residual is very much affected with value ofβ.

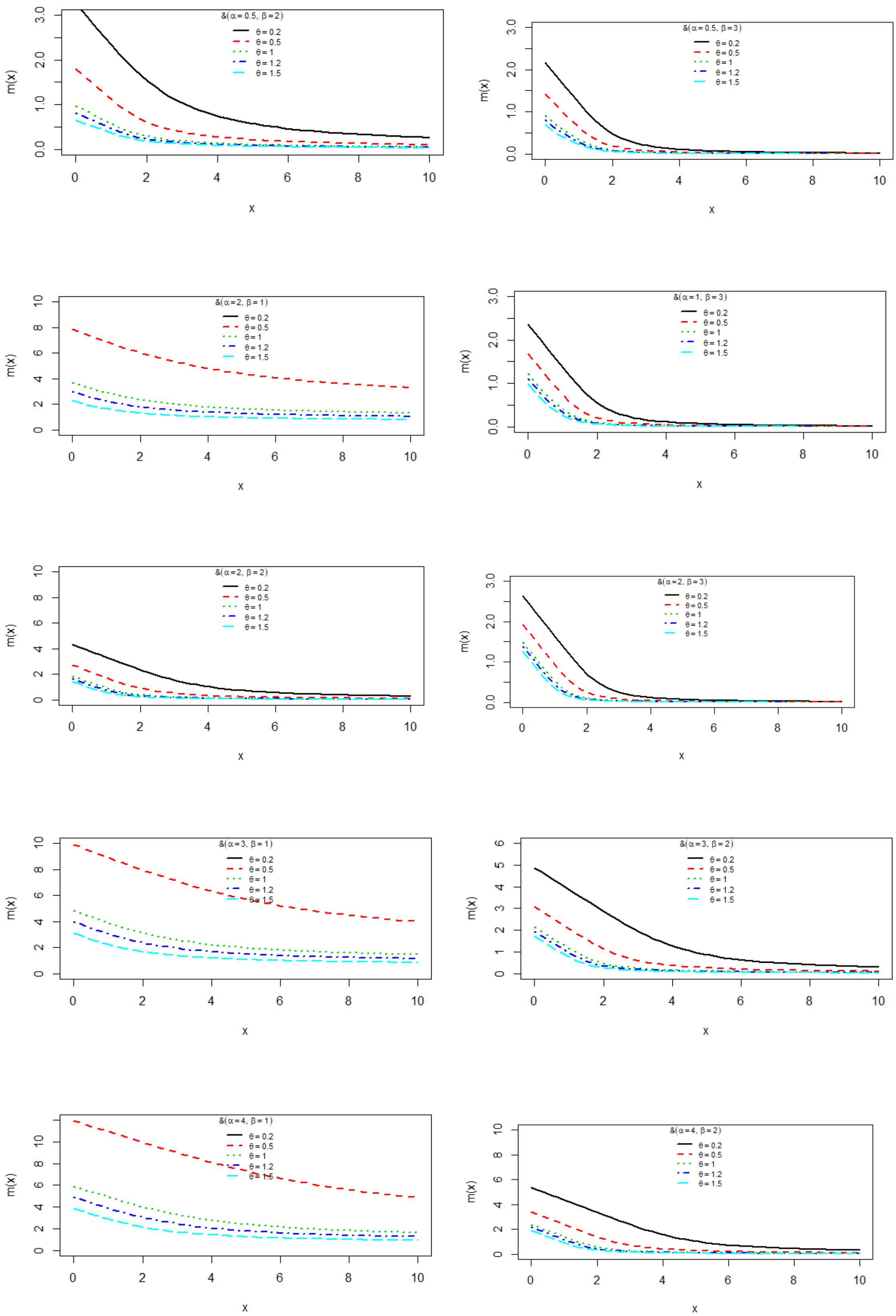

Figure 4: *Graphs of mean residual life function for varying values of parameters* θ, α *and* β.

IV. Moments

The rth moment about origin, $\mu_r{}'$ of PWAD (2.1) can be obtained as

$$\mu_r{}' = E(X^r) = \frac{\beta\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}\int_0^\infty x^{\beta\alpha+r-1}\left(1+x^{2\beta}\right)e^{-\theta x^\beta}dx$$

$$= \frac{\beta\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}\left[\int_0^\infty e^{-\theta x^\beta}x^{\beta\alpha+r-1}\,dx + \int_0^\infty e^{-\theta x^\beta}x^{\beta\alpha+2\beta+r-1}\,dx\right]$$

Assuming $u=\theta x^\beta$, which gives $x=\left(\frac{u}{\theta}\right)^{\frac{1}{\beta}}$ and $dx=\frac{1}{\theta\beta}\left(\frac{u}{\theta}\right)^{\frac{1-\beta}{\beta}}du$, we get

$$\mu_r{}' = \frac{\theta^{\alpha+1}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}\left[\int_0^\infty e^{-u}\left(\frac{u}{\theta}\right)^{\frac{\beta\alpha+r}{\beta}-1}du + \int_0^\infty e^{-u}\left(\frac{u}{\theta}\right)^{\frac{\beta\alpha+2\beta+r}{\beta}-1}du\right]$$

$$= \frac{\theta^{\alpha+1}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}\left[\frac{1}{\theta^{\alpha+\frac{r}{\beta}-1}}\int_0^\infty e^{-u}u^{\alpha+\frac{r}{\beta}-1}du + \frac{1}{\theta^{\alpha+2+\frac{r}{\beta}-1}}\int_0^\infty e^{-u}u^{\alpha+2+\frac{r}{\beta}-1}du\right]$$

$$= \frac{\theta^{\alpha+1}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}\left[\frac{\Gamma\left(\alpha+\frac{r}{\beta}\right)}{\theta^{\alpha+\frac{r}{\beta}-1}} + \frac{\Gamma\left(\alpha+2+\frac{r}{\beta}\right)}{\theta^{\alpha+2+\frac{r}{\beta}-1}}\right]$$

$$= \frac{\theta^{\alpha+1}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}\left[\frac{\theta^2\Gamma\left(\alpha+\frac{r}{\beta}\right)+\Gamma\left(\alpha+2+\frac{r}{\beta}\right)}{\theta^{\alpha+1+\frac{r}{\beta}}}\right]$$

$$= \frac{\theta^2\Gamma\left(\alpha+\frac{r}{\beta}\right)+\Gamma\left(\alpha+2+\frac{r}{\beta}\right)}{\theta^{\frac{r}{\beta}}(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)};\ r=1,2,3,\ldots \tag{4.1}$$

Thus the first four moments about origin of PWAD can be given by

$$\mu_1{}' = \frac{\theta^2\Gamma\left(\alpha+\frac{1}{\beta}\right)+\Gamma\left(\alpha+2+\frac{1}{\beta}\right)}{\theta^{\frac{1}{\beta}}(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}$$

$$\mu_2{}' = \frac{\theta^2\Gamma\left(\alpha+\frac{2}{\beta}\right)+\Gamma\left(\alpha+2+\frac{2}{\beta}\right)}{\theta^{\frac{2}{\beta}}(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}$$

$$\mu_3{}' = \frac{\theta^2\Gamma\left(\alpha+\frac{3}{\beta}\right)+\Gamma\left(\alpha+2+\frac{3}{\beta}\right)}{\theta^{\frac{3}{\beta}}(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}$$

$$\mu_4{}' = \frac{\theta^2\Gamma\left(\alpha+\frac{4}{\beta}\right)+\Gamma\left(\alpha+2+\frac{4}{\beta}\right)}{\theta^{\frac{4}{\beta}}(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}.$$

Using the relationship between moments about origin and central moments, central moments can be obtained. Since the expressions for central moments are complicated, central moments are not being given.

V. Maximum Likelihood Estimation

Suppose $(x_1, x_2, x_3, \ldots, x_n)$ be a random sample of size n from PWAD (2.1). The natural log likelihood function is thus obtained as

$ln\,L = \sum_{i=1}^n ln\,f_4\,(x_i;\theta,\alpha,\beta)$ $= n[ln\,\beta + (\alpha+2)\,ln\,\theta - ln(\theta^2+\alpha^2+\alpha) - ln\,\Gamma\,(\alpha)] + (\beta\alpha-1)\sum_{i=1}^n ln(x_i) + \sum_{i=1}^n ln\left(1+{x_i}^{2\beta}\right) - \theta\sum_{i=1}^n {x_i}^\beta$ The maximum likelihood estimates (MLEs) of parameters (θ,α,β) of PWAD are the solution of the following nonlinear log likelihood equations

$$\frac{\partial\,ln\,L}{\partial\theta} = \frac{n(\alpha+2)}{\theta} - \frac{2n\theta}{\theta^2+\alpha^2+\alpha} - \sum_{i=1}^n {x_i}^\beta = 0$$

$$\frac{\partial\,ln\,L}{\partial\alpha} = n\,ln\,\theta - \frac{n(2\alpha+1)}{\theta^2+\alpha^2+\alpha} - n\psi(\alpha) + \beta\sum_{i=1}^n ln(x_i) = 0$$

$$\frac{\partial\,ln\,L}{\partial\beta} = \frac{n}{\beta} + \alpha\sum_{i=1}^n ln\,x_i + \sum_{i=1}^n\frac{2{x_i}^{2\beta}\,ln(x_i)}{1+{x_i}^{2\beta}} - \theta\sum_{i=1}^n {x_i}^\beta\,ln(x_i) = 0$$

where $\bar{x}$ is the sample mean and $\psi(\alpha) = \frac{d}{d\alpha} ln\,\Gamma(\alpha)$ is the digamma function. These three natural log- likelihood equations do not seem to be solved directly because they cannot be expressed in closed forms. However, the MLE's of parameters (θ, α, β) can be obtained directly by solving the log likelihood equation using Newton-Raphson iteration method available in R –Software till sufficiently close estimates of parameters are obtained.

VI. Applications

In this section, the applications and goodness of fit of the PWAD have been discussed for two real lifetime datasets. The fit is compared with one parameter lifetime distributions including exponential distribution, Lindley distribution proposed by Lindley [4] and studied by Ghitany et al [5], Akash distribution; two-parameter lifetime distributions including Weibull distribution introduced by Weibull [6], Gamma distribution, Generalized exponential distribution (GED) introduced by Gupta and Kundu [7], Power Lindley distribution (PLD) proposed by Ghitany et al [8], Shukla distribution (SD) proposed by Shukla and Shanker [9],Weighted Lindley distribution (WLD) introduced by Ghitany et al [10] and PAD and WAD and three-parameter lifetime distributions including generalized gamma distribution (GGD) introduced by Stacy[11] and generalized Lindley distribution (GLD) suggested by Zakerzadeh and Dolati [12]. Note that Shanker et al [13] and Shanker[14] have detail discussion on WLD and GLD regarding some important properties and applications for various lifetime data from engineering and biomedical sciences. The first dataset is the data reported by Efron[15] represents the survival times of a group of patients suffering from Head and Neck cancer disease and treated using a combination of radiotherapy and chemotherapy (RT+CT). The second dataset is the data which represents the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm and are available in Bader and Priest [16].

Table1 *The data set 1 reported by Efron [15] represent the survival times of a group of patients suffering from Head and Neck cancer disease and treated using a combination of radiotherapy and chemotherapy (RT+CT).*

12.20	23.56	23.74	25.87	31.98	37	41.35	47.38	55.46	58.36
63.47	68.46	78.26	74.47	81.43	84	92	94	110	112
119	127	130	133	140	146	155	159	173	179
194	195	209	249	281	319	339	432	469	519
633	725	817	1776						

Table2 *The following data set 2 represent the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm, Bader and Priest [16]*

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958
1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179
2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382
2.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554
2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726
2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585	

In order to compare the goodness of fit of these distributions for the two datasets, values of $-2\ln L$, AIC (Akaike information criterion), K-S Statistic (Kolmogorov-Smirnov Statistic) and p-value for two datasets have been computed The formulae for AIC and K-S Statistics are as follows: $AIC = -2\ln L + 2k$, and $K-S = \sup_x |F_n(x) - F_0(x)|$, where k being the number of parameters involved in the respective distributions, n is the sample size and $F_n(x)$is the empirical distribution function. The best distribution corresponds to the lower values of$-2\ln L$, AIC and K-S statistic.

Note that the estimates of parameters of the considered distributions are based on maximum likelihood estimates. In this paper, the initial values of the parameters for ML estimates of PWAD have been selected as$\theta = 1.5$,$\alpha = 0.5$ and$\beta = 1.5$for both dataset. In general, it has been observed that the initial values of the parameters can be taken as any positive real numbers, preferably from 0.5 to 5, for any dataset.

The pdf of the fitted distributions are presented in table 3. The ML estimates of parameters of the considered distributions for datasets 1 and 2 are presented in tables 4 and 5. The goodness of fit by K-S statistics for datasets 1 and 2 with considered distributions are presented in tables 6 and 7. The variance-covariance matrix of the parameters (θ, α, β) of PWAD for datasets 1 and 2 are presented in tables 8 and 9. It is obvious from the goodness of fit of the proposed distribution that in tables 4 and 5 it gives better fit than all considered distributions and competes well with GGD. Therefore, PWAD can be considered an important three-parameter lifetime distribution alternative to GGD and other lifetime distributions.

Table 3*: pdf of the fitted distributions*

Distributions	Pdf
Weibull	$f(x;\theta,\alpha) = \theta\alpha x^{\alpha-1}e^{-\theta x^{\alpha}}; x>0, \theta>0, \alpha>0$
Gamma	$f(x;\theta,\alpha) = \frac{\theta^{\alpha}}{\Gamma(\alpha)}e^{-\theta x}x^{\alpha-1}; x>0, \theta>0, \alpha>0$
PLD	$f(x;\theta,\alpha) = \frac{\alpha\theta^2}{(\theta+1)}x^{\alpha-1}(1+x^{\alpha})e^{-\theta x^{\alpha}}; x>0, \theta>0, \alpha>0$
WLD	$f(x;\theta,\alpha) = \frac{\theta^{\alpha+1}}{\theta+\alpha}\frac{x^{\alpha-1}}{\Gamma(\alpha)}(1+x)e^{-\theta x}; x>0, \theta>0, \alpha>0$
GED	$f(x;\theta,\alpha) = \theta\alpha\left(1-e^{-\theta x}\right)^{\alpha-1}e^{-\theta x}; x>0, \theta>0, \alpha>0$
SD	$f(x;\theta,\alpha) = \frac{\theta^{\alpha+1}}{\theta^{\alpha}+\Gamma(\alpha+1)}(1+x^{\alpha})e^{-\theta x}; x>0, \theta>0, \alpha\geq 0$
GGD	$f(x;\theta,\alpha,\beta) = \frac{\beta\theta^{\alpha}}{\Gamma(\alpha)}x^{\beta\alpha-1}e^{-\theta x^{\beta}}; x>0, \theta>0, \alpha>0, \beta>0$
GLD	$f(x;\theta,\alpha,\beta) = \frac{\theta^{\alpha+1}}{\theta+\beta}\frac{x^{\alpha-1}}{\Gamma(\alpha+1)}(\alpha+\beta x)e^{-\theta x}$
Lindley	$f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}; x>0, \theta>0$

Table 4: *Summary of the ML estimates of parameters for dataset 1*

Model	ML Estimates		
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
PWAD	11.8734	27.3026	**0.1804**
GLD	0.00473	0.05243	**5.07505**
GGD	11.25540	27.72340	**0.18220**
SD	0.00458	0.02380	
WAD	0.0090	0.0165	……..
PAD	0.16751	0.55764	……..
WLD	0.00531	0.21191	……
PLD	0.05301	0.68893	…….
GED	0.00482	1.09367	……
Gamma	0.00489	1.08501	……
Weibull	0.00710	0.92327	…….
Akash	0.01344	…….	……..
Lindley	0.00892	…….	……..
Exponential	**0.00447**	……	…….

Table 5: *Summary of the ML estimates of parameters of dataset 2*

Model	ML Estimates		
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
PWAD	0.2918	1.7049	**2.7229**
GLD	9.39076	22.71981	**4.77105**
GGD	0.30440	3.58610	**2.64830**
SD	5.9922	17.1611	
WAD	9.7584	22.2327	….
PAD	0.16964	3.06033	…..
WLD	9.62655	22.89383	…..
PLD	0.0500	3.8680	……
GED	2.03307	87.28471	……
Gamma	9.53843	23.38184	…..
Weibull	0.00558	5.33523	……
Akash	0.96472	……	…….
Lindley	0.65450	…….	…….
Exponential	**0.40794**	…….	…….

Table 6: *Summary of Goodness of fit by K-S Statistic for dataset 1*

Model	$-2\ln L$	AIC	K-S	p-value
PWAD	555.67	561.67	0.081	**0.921**
GLD	564.09	570.09	0.150	**0.248**
GGD	555.64	561.64	0.079	**0.921**
SD	564.00	568.00	0.147	**0.267**
WAD	580.32	584.32	0.219	**0.023**
PAD	559.10	563.10	0.108	**0.635**
WLD	565.91	569.91	0.161	**0.181**
PLD	560.78	564.78	0.118	**0.529**
GED	563.93	567.93	0.145	**0.280**
Gamma	564.10	568.10	0.149	**0.249**
Weibull	563.71	567.71	0.298	**0.005**
Akash	609.92	611.92	0.279	**0.001**
Lindley	579.16	581.16	0.219	**0.025**
Exponential	**564.01**	**566.01**	**0.145**	**0.282**

Table 6 represents the goodness of fit by K-S Statistic for data set-1, It is observed that AIC and P-value from K-S test were found almost minimum and maximum in comparison to all other included distributions respectively. Therefore, it may be concluded that PWAD is better fits than other included distributions except GGD (Generalized Gamma distribution). Hence, PWAD can be considered an important lifetime distribution for modeling lifetime data.

Table 7: *Summary of Goodness of fit by K-S Statistic for dataset 2*

Model	$-2\ln L$	AIC	K-S	p-value
PWAD	97.93	103.93	0.037	**0.999**
GLD	101.96	107.96	0.056	**0.979**
GGD	100.58	106.58	0.044	**0.999**
SD	184.35	188.35	0.290	**0.000**
WAD	99.95	103.95	0.057	**0.976**
PAD	98.02	102.02	0.038	**0.999**
WLD	100.04	104.04	0.058	**0.974**
PLD	98.12	102.12	0.044	**0.998**
GED	109.24	113.24	0.095	**0.558**
Gamma	100.07	104.07	0.058	**0.973**
Weibull	99.31	103.31	0.060	**0.964**
Akash	224.27	226.27	0.362	**0.000**
Lindley	238.38	240.38	0.401	**0.000**
Exponential	**261.73**	**263.73**	**0.448**	**0.000**

Table 7 represents the goodness of fit by K-S Statistic for data set-2, It is observed that AIC and P-value from K-S test were found almost minimum and maximum in comparison to almost all other included distributions respectively expect PAD and PLD. Therefore, it may be concluded that PWAD is a better fit than other included distributions except PAD (Power Akash distribution) and

PLD (Power Lindley distribution). Further, PWAD competing well with the considered distributions and hence can be an important distribution for lifetime data.

Table 8: *Variance-covariance matrix of the parameters θ, α and β of PWAD for dataset* **1**

$$\begin{array}{c} \\ \hat{\theta} \\ \hat{\alpha} \\ \hat{\beta} \end{array} \begin{array}{c} \hat{\theta}\hat{\alpha}\hat{\beta} \\ \begin{bmatrix} 3044.5355 & 4357.7480 & -4.9546 \\ 4357.7480 & 6246.2648 & -7.0633 \\ -4.9546 & -7.0633 & 0.0081 \end{bmatrix} \end{array}$$

Table 9: *Variance-covariance matrix of the parameters θ, α and β of PWAD for dataset 2*

$$\hat{\theta}\hat{\alpha}\hat{\beta} \qquad \begin{array}{c} \hat{\theta} \\ \hat{\alpha} \\ \hat{\beta} \end{array} \begin{bmatrix} -12.5447 & -27.6412 & 1.7898 \\ -27.6412 & -60.2546 & 3.9955 \\ 1.7898 & 3.9955 & -0.2499 \end{bmatrix}$$

VII. Conclusions

In the present paper a three-parameter power weighted Akash distribution (PWAD) ,of which two-parameter weighted Akash distribution (WAD), two-parameter power Akash distribution (PAD) and one parameter Akash distribution are particular cases, has been introduced and studied. Its moments, hazard rate function, mean residual life function and stochastic ordering have been discussed. Maximum likelihood estimation has been discussed for estimating the parameters of the distribution. The applications of the proposed distribution have been discussed through two real lifetime datasets. The goodness of fit test of the proposed distribution is a better model for lifetime data than the other well-known one parameter, two-parameter and three-parameter lifetime distributions.

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