Inverse Weibull-Burr III Distribution with Properties and Application Related to Survival Rates in Animals

Aijaz Ahmad*

•

Department of Mathematics, Bhagwant University, Ajmer, India
aijazahmad4488@gmail.com

Mujamil Jallal

•

Department of Mathematics, Bhagwant University, Ajmer, India
muzamiljallal@gmail.com

I. H. Dar

•

Department of Statistics, University of Kashmir, Srinagar, India
ishfaqqh@gmail.com

Rajnee Tripathi

•

Department of Mathematics, Bhagwant University, Ajmer, India
rajneetripathi@hotmail,com

Abstract

The objective of this study is to develop an extension of the Burr-III distribution which is achieved by adopting the inverse Weibull-G family of distribution and is referred as inverse Weibull-Burr III distribution (IWB-III) to evaluate complicated data. Different structural characteristics of the suggested distribution have been determined and analysed. Distinct plots depict the behaviour of the probability density function (pdf) and the cumulative distribution function (cdf). The maximum likelihood estimation method is applied to estimate the stated distribution parameters. To assess and investigate the efficacy of estimators in terms of bias, variance, and mean square error (MSE), a simulation study was conducted. Lastly, the effectiveness of the stated distribution is proven by an actual data set relevant to survival rates in animals.

Keywords: Inverse Weibull-G family; Burr-III distribution; moments, Renyi entropy; simulation; maximum likelihood estimation.

1. Introduction

Over numerous decades, academics have been attempting to develop a number of novel distributions to satisfy certain realistic demands. The rationale is that conventional distributions have generally been shown to lack fit in actual applications, such as medicinal research, engineering, hydrology, environmental science, and many more. In particular, the objective of creating novel distributions or generalisations is to construct adaptable statistical models effective at dealing with complicated real-world data. This adaptability may be obtained in a straightforward manner by introducing new parameters to the standard distribution.

The Weibull distribution has been utilised in a variety of disciplines and applications. The hazard function of the Weibull distribution can only be monotonic in nature. As a result, it can not be employed to simulate lifespan data with a bathtub-shaped hazard function.

Let X be a random variable that follows the Weibull distribution with parameters α and β. Then its probability density function (pdf) is defined as

$$\psi(x, \alpha, \beta) = \alpha^{\beta} \beta x^{\beta-1} e^{-\alpha^{\beta} x^{\beta}}; \quad x > 0, \alpha, \beta > 0$$

The transformation $T = \frac{1}{X}$, yields the inverse of the Weibull distribution. As a result, the probability density function (pdf) of the inverse Weibull distribution assumes the following structure.

$$h(t, \alpha, \beta) = \alpha^{\beta} \beta t^{-(\beta+1)} e^{-\alpha^{\beta} t^{-\beta}}; \quad t > 0, \alpha, \beta > 0 \tag{1}$$

In this work, we construct the inverse Weibull-Burr III distribution, which is an expansion of the Burr-III distribution. Burr, I.W [4] advocated a family of twelve cumulative distribution functions for simulating lifespan data. Burr-type III and Burr-type XII distributions were two prevalent members of the family. The Burr-III distribution has been studied thoroughly and employed in a range of aspects of research. Daniyal et al [9], Al-Dayian et al [10] and B.A. para et al [6] provide further information on the characteristics of the Burr-III distribution. The probability density function (pdf) of Burr-III distribution is stated as.

$$g(y, \theta, \lambda) = \theta \lambda y^{-\theta-1} \left(1 + y^{-\theta}\right)^{-\lambda-1}; \quad y > 0, \theta, \lambda > 0 \tag{2}$$

The associated cumulative distribution function (cdf) of equation (1.2) is given as

$$G(y, \theta, \lambda) = \left(1 + y^{-\theta}\right)^{-\lambda}; \quad y > 0, \theta, \lambda > 0 \tag{3}$$

In recent decades, researchers have concentrated on discovering novel generators from continuous conventional distributions. As an outcome, the resulting distribution enhances the efficacy and adaptability of data analysis. The following are some generated families of distribution: the beta-G family of distribution investigated by Eugene et al [11], the gamma-G family by Zagrofos and Balakrishana [13], the kumaraswamy-G family by Cordeiro et al [8], the transformedtransformer(T-X) by Alzaatrh et al [1], the Weibull-G by Bourguignon et al [3],Brito et al. [5] created the Topp-Leone odd log-logistic family of distributions, Morad Alizadeh et al. [12] constructed the Gompertz-G distribution family, and Amal S. Hassan et al. [2] established the inverse Weibull-G distribution.

T-X family of distributions defined by Alzaatreh et al [1] is given by

$$F(y) = \int_0^{W[G(y)]} r(t)dt \tag{4}$$

Where $r(t)$ be the probability density function of a random variable $T$and $W[G(y)]$ be a function of cumulative density function of random variable Y.

Suppose $G(y, \phi)$,denotes the baseline cumulative distribution function, which depends on parameter vector ϕ. Now using T-X approach, the cumulative distribution function $F(y)$ of inverse Weibull generator (IWG) can be derived by replacing $r(t)$ in equation (1.4) with (1.1) and $W[G(y)] = \frac{G(y,\phi)}{\bar{G}(y,\phi)}$, where $\bar{G}(y,\phi) = 1 - G(y,\phi)$ which follows

$$F(y,\phi) = \int_0^{\frac{G(y,\phi)}{\bar{G}(y,\phi)}} \alpha^\beta \beta t^{-(\beta+1)} e^{-\alpha^\beta t^{-\beta}} dy$$
$$= e^{-\alpha^\beta \left(\frac{G(y,\phi)}{\bar{G}(y,\phi)}\right)^{-\beta}} \quad ; y > 0, \alpha, \beta, \phi > 0 \tag{5}$$

where $\bar{G}(y,\phi) = 1 - G(y,\phi)$
The associated pdf of (1.5) is given as

$$f(y,\alpha,\phi) = \alpha^\beta \beta g(y,\phi) \frac{(G(y,\phi))^{-\beta-1}}{(\bar{G}(y,\phi))^{-\beta+1}} e^{-\alpha^\beta \left(\frac{G(y,\phi)}{\bar{G}(y,\phi)}\right)^{-\beta}} \quad ; y > 0, \alpha, \beta, \phi > 0 \tag{6}$$

In addition, Reliability function denoted as $\bar{F}(y,\phi)$, hazard rate function denoted as $h(y,\phi)$ and cumulative hazard rate function denoted as $H(y,\phi)$ are respectively given as

$$\bar{F}(y,\phi) = 1 - e^{-\alpha^\beta \left(\frac{G(y,\phi)}{\bar{G}(y,\phi)}\right)^{-\beta}}$$
$$h(y,\phi) = \frac{\alpha^\beta \beta g(y,\phi) \frac{(G(y,\phi))^{-\beta-1}}{(\bar{G}(y,\phi))^{-\beta+1}} e^{-\alpha^\beta \left(\frac{G(y,\phi)}{\bar{G}(y,\phi)}\right)^{-\beta}}}{1 - e^{-\alpha^\beta \left(\frac{G(y,\phi)}{\bar{G}(y,\phi)}\right)^{-\beta}}}$$
$$H(y,\phi) = -\, ln[\bar{F}(y,\phi)] = -ln\left\{1 - e^{-\alpha^\beta \left(\frac{G(y,\phi)}{\bar{G}(y,\phi)}\right)^{-\beta}}\right\}$$

1.1. Usefull Expansions

Applying Taylor series expansion to the exponential function of the pdf in equation (1.6) we have

$$e^{-\alpha^\beta \left(\frac{G(y,\phi)}{\bar{G}(y,\phi)}\right)^{-\beta}} = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \alpha^{s\beta} \left(\frac{G(y,\phi)}{\bar{G}(y,\phi)}\right)^{-\beta s} \tag{7}$$

substituting equation (1.7) in equation (1.6), we have

$$f(y,\phi) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \alpha^{\beta(s+1)} \beta g(y,\phi) \frac{(G(y,\phi))^{-\beta(s+1)-1}}{(\bar{G}(y,\phi))^{-\beta(s+1)+1}} \tag{8}$$
$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \alpha^{\beta(s+1)} \beta g(y,\phi) (G(y,\phi))^{-\beta(s+1)-1} (1 - G(y,\phi))^{\beta(s+1)-1} \tag{9}$$

using generalised binomial theorem, we have

$$(1-z)^{a-1} = \sum_{p=0}^{\infty} (-1)^p \binom{a-1}{p} z^p$$
$$(\bar{G}(y,\phi))^{-\beta(s+1)+1} = (1 - G(y,\phi))^{\beta(p+1)-1} = \sum_{p=0}^{\infty} (-1)^p \binom{\beta(p+1)-1}{p} (G(y,\phi))^p$$
$$= \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{s+p}}{s!} \binom{\beta(s+1)-1}{p} \alpha^{\beta(s+1)} \beta g(y,\phi) (G(y,\phi))^{p-\beta(s+1)-1}$$
$$= \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \zeta_{s,p} g(y,\phi) (G(y,\phi))^{p-\beta(s+1)-1} \tag{10}$$

where

$$\zeta_{s,p} = \frac{(-1)^{s+p}}{s!} \binom{\beta(s+1)-1}{p} \alpha^{\beta(s+1)} \beta$$

Now using equation (2) and (3) in equation (10), we obtain pdf of formulated distribution in mixture form, as follows

$$f(y,\alpha,\beta,\theta,\lambda)=\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\zeta_{s,p}\theta\lambda y^{-\theta-1}\left(1+y^{-\theta}\right)^{-\lambda-1}\left[(1+y^{-\theta})^{-\lambda}\right]^{p-\beta(s+1)-1}$$
$$;y>0,\alpha,\beta,\theta,\lambda>0 \qquad (11)$$

2. Inverse Weibull-Burr III Distribution

In this part, we'll investigate the inverse Weibull-Burr III distribution and look at aspects of its statistical characteristics. We derive the cumulative distribution function (cdf) of the given distribution using equation (3) in equation (5) as follows.

$$F(y,\alpha,\beta,\theta,\lambda)=e^{-\alpha^{\beta}((1+y^{-\theta})^{\lambda}-1)^{\beta}} \qquad (12)$$

Figure 1: Expounds some of possible layouts of the cdf of IWB-III distribution for distinct choice of parameters

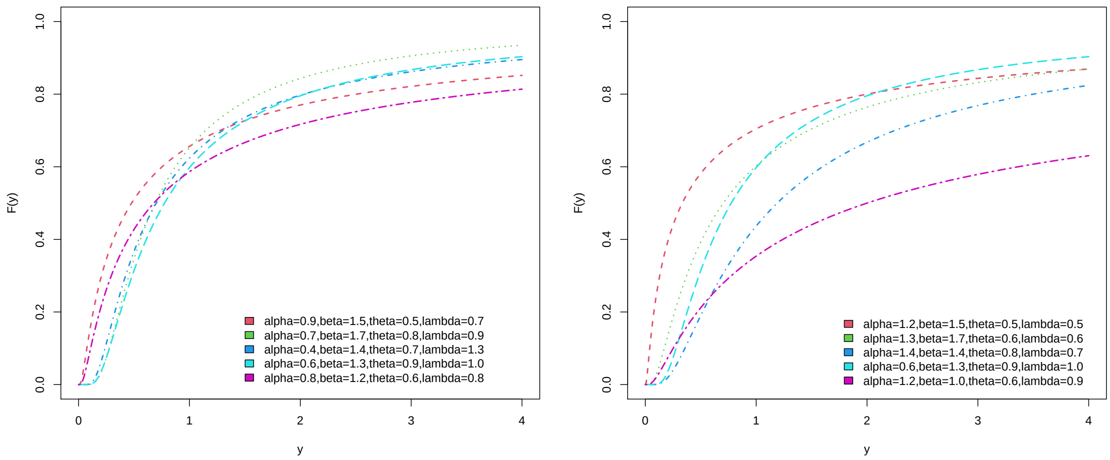

Figure 1: *plots of cdf for IWB-III distribution*

The associated pdf of (12) is given as

$$f(y,\alpha,\beta,\theta,\lambda)=\alpha^{\beta}\beta\theta\lambda y^{-\theta-1}(1+y^{-\theta})^{\lambda-1}((1+y^{-\theta})^{\lambda}-1)^{\beta-1}e^{-\alpha^{\beta}((1+y^{-\theta})^{\lambda}-1)^{\beta}} \qquad (13)$$
$$;\quad \alpha>0,\beta>0,\theta>0,\lambda>0$$

Figure 2: Expounds some of possible layouts of the pdf of IWB-III distribution for distinct choice of parameters

3. Reliability Measures of (IWB-III) Distribution

This section is focused on researching and developing distinct ageing indicators for the formulated distribution.

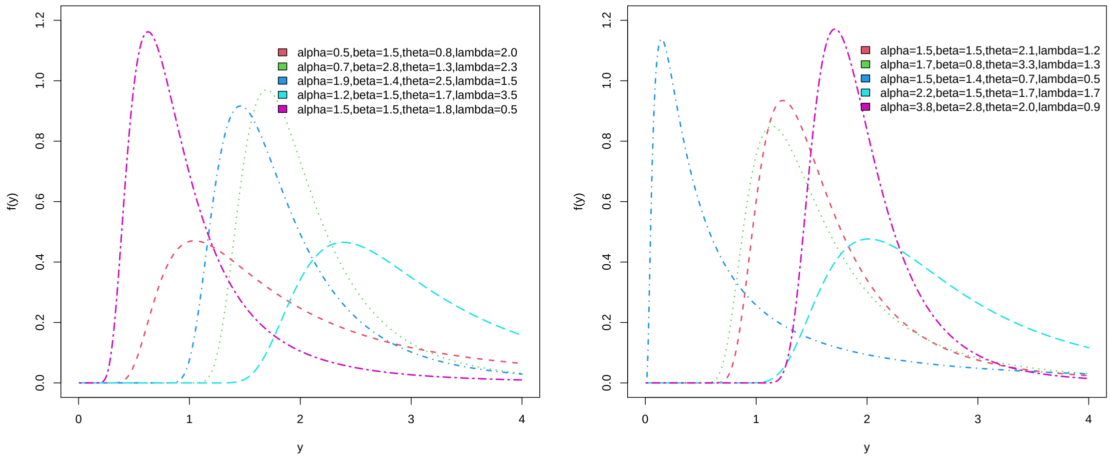

Figure 2: *plots of pdf for IWB-III distribution*

3.1. Survival function

Suppose Y be a continuous random variable with cdf $F(y)$.Then its Survival function which is also called reliability function is defined as

$$S(y) = p_r(Y > y) = \int_y^\infty f(y)dy = 1 - F(y)$$

Therefore, the survival function for IWB-III distribution is given as

$$\begin{aligned} S(y,\alpha,\beta,\theta,\lambda) =& 1 - F(y,\alpha,\beta,\theta,\lambda) \\ =& 1 - e^{-\alpha^\beta((1+y^{-\theta})^\lambda - 1)^\beta} \end{aligned} \tag{14}$$

3.2. Hazard rate function

The hazard rate function of a random variable y is denoted as

$$h(y,\alpha,\beta,\theta,\lambda) = \frac{f(y,\alpha,\beta,\theta\lambda)}{F(y,\alpha,\beta,\theta\lambda)} \tag{15}$$

using equation (12) and (13) in equation (15), then the hazard rate function of IWB-III distribution is given as

$$h(y,\alpha,\beta,\theta\lambda) = \frac{\alpha^\beta\beta\theta\lambda y^{-\theta-1}(1+y^{-\theta})^{\lambda-1}((1+y^{-\theta})^\lambda - 1)^{\beta-1}e^{-\alpha^\beta((1+y^{-\theta})^\lambda-1)^\beta}}{1 - e^{-\alpha^\beta((1+y^{-\theta})^\lambda-1)^\beta}} \tag{16}$$

Figure 3: Expounds some of possible layouts of the hazard function of IWB-III distribution for distinct choice of parameters

3.3. Cumulative hazard rate function

The cumulative hazard rate function of a random variable y is given as

$$H(y,\alpha,\beta,\theta,\lambda) = -\,ln[\bar{F}(y,\alpha,\beta,\theta,\lambda)] \tag{17}$$

using equation (12) in equation (17), then we obtain cumulative hazard rate function of IWB-III distribution

$$H(y,\alpha,\beta,\theta,\lambda) = -\,ln\left\{1 - e^{-\alpha^\beta((1+y^{-\theta})^\lambda-1)^\beta}\right\} \tag{18}$$

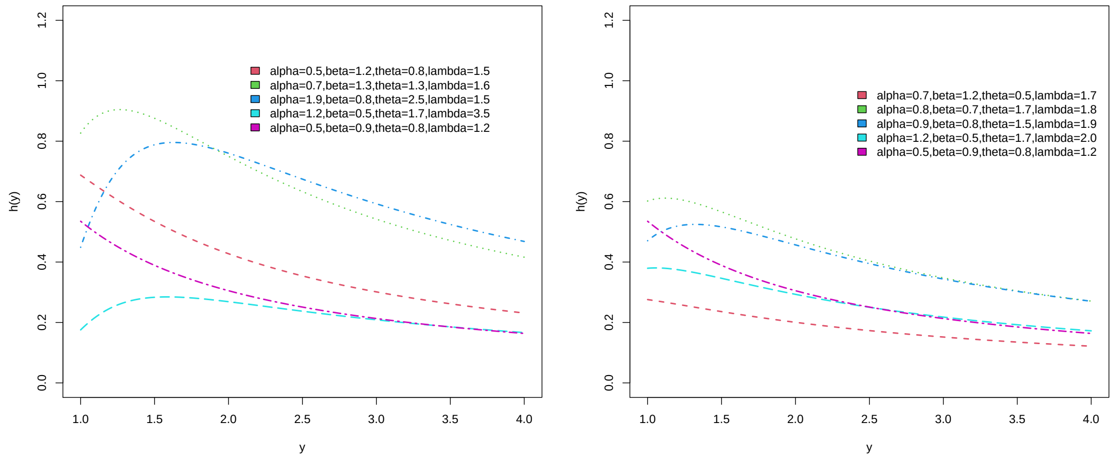

Figure 3: *plots of hazard function for IWB-III distribution*

3.4. Mean residual function

The mean residual lifetime is the predicted residual life or the average completion period of the constituent after it has exceeded a certain duration y. It is extremely significant in reliability investigations.
Mean residual function of random y variable can be obtained as

$$\begin{aligned} m(y,\alpha,\beta,\theta,\lambda) =& \frac{1}{S(y,\alpha,\beta,\theta,\lambda)} \int_y^\infty t f(t,\alpha,\beta,\theta,\lambda) dt - y \\ =& \frac{1}{\left\{1 - e^{-\alpha^\beta((1+y^{-\theta})^\lambda - 1)^\beta}\right\}} \sum_{s=0}^\infty \sum_{p=0}^\infty \zeta_{s,p} \theta\lambda \\ & \times \int_y^\infty t^{-\theta} \left(1+t^{-\theta}\right)^{-\lambda-1} \left[(1+t^{-\theta})^{-\lambda}\right]^{p-\beta(s+1)-1} dt - y \end{aligned}$$

Making substitution $(1+t^{-\theta})^{-\lambda} = z$, sothat $(1+y^{-\theta})^{-\lambda} \le z \le 1$, we have

$$m(y,\alpha,\beta,\theta,\lambda) = \int_{(1+y^{-\theta})^{-\lambda}}^1 z^{p-\beta(s+1)+\frac{1}{\lambda\theta}-1} (1 - z^{\frac{1}{\lambda}})^{-\frac{1}{\theta}} dz$$

After solving the integral, we get

$$B\left(1-(1+y^{-\theta})^{-1}, (p-\beta(s+1))\lambda + \frac{1}{\theta}, 1 - \frac{1}{\theta}\right)$$

Where $B(x,a,b) = \int_0^x u^{a-1}(1-u)^{b-1}$denotes incomplete beta function

4. Statistical Properties Of (IWB-III) Distribution

This section is devoted to derive and examine disttinct properties of IWB-III

4.1. Moments

Let y denotes a random variable, then the r^{th} moment of IWB-III is denoted as μ_r' and is given by

$$\begin{aligned}\mu_r' =& E(y^r) = \int_0^\infty y^r f(y,\alpha,\beta,\theta,\lambda)dy \\ =& \sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\zeta_{s,p}\theta\lambda\int_0^\infty y^r(1+y^{-\theta})^{-\lambda-1}\left[(1+y^{-\theta})^{-\lambda}\right]^{p-\beta(s+1)-1}dy\end{aligned}$$

Making substitution $(1+y^{-\theta})^{-\lambda}=z$, so that $0<z<1$, we have

$$\mu_r' = \sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\zeta_{s,p}\int_0^1 z^{p-\beta(s+1)+\frac{r}{\lambda\theta}-1}\left(1-z^{\frac{1}{\lambda}}\right)^{\frac{-r}{\theta}}dy$$

After solving the integral, we have

$$\mu_r' = \sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\zeta_{s,p}\lambda B\left((p-\beta(s+1))\lambda+\frac{r}{\theta},1-\frac{r}{\theta}\right)$$

Where $B(.)$ denotes incomplete beta function.

4.2. Moment generating function

suppose Y denotes a random variable follows IWB-III distribution. Then the moment generating function of the distribution denoted by $M_Y(t)$is given

$$\begin{aligned}M_Y(t) = E(e^{ty}) &= \int_0^\infty e^{ty}f(y,\alpha,\beta,\theta,\lambda)dy \\ &= \int_0^\infty\left(1+ty+\frac{(ty)^2}{2!}+\frac{(ty)^3}{3!}+....\right)f(y,\alpha,\beta,\theta,\lambda)dy \\ &= \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^\infty y^r f(y,\alpha,\beta,\theta,\lambda)dy \\ &= \sum_{r=0}^{\infty}\frac{t^r}{r!}E(y^r) \\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\frac{t^r}{r!}\zeta_{s,p}\lambda B\left((p-\beta(s+1))\lambda+\frac{r}{\theta},1-\frac{r}{\theta}\right)\end{aligned}$$

The characteristics function of the IWB-III distribution denoted as $\phi_Y(t)$ can be yeild by replacing $t=it$ where $i=\sqrt{-1}$

$$\phi_Y(t) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\frac{(it)^r}{r!}\zeta_{s,p}\lambda B\left((p-\beta(s+1))\lambda+\frac{r}{\theta},1-\frac{r}{\theta}\right)$$

4.3. Incomplete moments

The general expression for incomplete moments is given as

$$\begin{aligned}m(y) &= \int_0^y y^r f(y,\alpha,\beta,\theta,\lambda)dy \\ &= \sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\zeta_{s,p}\theta\lambda\int_0^y y^{r-\theta-1}\left(1+y^{-\theta}\right)^{-\lambda-1}\left[(1+y^{-\theta})^{-\lambda}\right]^{p-\beta(s+1)-1}dy\end{aligned}$$

Making substitution $\left(1+y^{-\theta}\right)^{-\lambda}=z$, so that $0 \le z \le \left(1+y^{-\theta}\right)^{-\lambda}$, we have

$$=\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\zeta_{s,p}\int_0^{(1+y^{-\theta})^{-\lambda}} z^{j-\beta(s+1)+\frac{r}{\theta\lambda}-1}\left(1-z^{\frac{1}{\lambda}}\right)^{-\frac{r}{\theta}}dy$$

After solving the integral, we get

$$m(y)=\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\zeta_{s,p}\lambda B\left(1-(1+y^{-\theta})^{-1};(p-\beta(s+1))\lambda+\frac{r}{\theta},1-\frac{r}{\theta}\right)$$

where $B(.)$ denotes the incomplete beta function.

4.4. Quantile function

The quantile function of a random variable Y, where $Y \sim IWB-III$ distribution can be obtained by inverting equation (12), we have

$$y_q=\left\{\left[1+\left(-\frac{1}{\alpha^{\beta}}log(q)\right)^{\frac{1}{\beta}}\right]^{\frac{1}{\lambda}}-1\right\}^{-\frac{1}{\theta}}$$

In particular, the median of the distribution can be obtained by setting $q=0.5$, we have

$$y_{0.5}=\left\{\left[1+\left(-\frac{1}{\alpha^{\beta}}log(0.5)\right)^{\frac{1}{\beta}}\right]^{\frac{1}{\lambda}}-1\right\}^{-\frac{1}{\theta}}$$

4.5. Random number generation

Suppose y denotes a random variable with pdf given in equation (2.1) . The random number of IWB-III distribution can be generated as

$$\begin{aligned} F(y)=&u \implies y=F^{-1}(u)\\ y=&\left\{\left[1+\left(-\frac{1}{\alpha^{\beta}}log(u)\right)^{\frac{1}{\beta}}\right]^{\frac{1}{\lambda}}-1\right\}^{-\frac{1}{\theta}}\end{aligned}$$

Where u is the uniform random variable defined in an open interval (0,1) .

4.6. Mean deviation about mean and median

The quantity of scattering in a population is evidently measured to some extent by the totality of the deviations.
Let Y be a random variable from IWB-III distribution with mean μ. Then the mean deviation from mean is defined as.

$$\begin{aligned} D(\mu)=&E\left(|Y-\mu|\right)\\ =&\int_0^{\infty}|Y-\mu|f(y)dy\\ =&\int_0^{\mu}(\mu-y)f(y)dy+\int_{\mu}^{\infty}(y-\mu)f(y)dy\\ =&2\mu F(\mu)-2\int_0^{\mu}yf(y)dy \end{aligned} \tag{19}$$

Now

$$\int_0^{\mu} yf(y)dy = \sum_{s=0}^{\infty}\sum_{p=0}^{\infty} \zeta_{s,p}\theta\lambda \int_0^{\mu} y^{-\theta-1}\left(1+y^{-\theta}\right)^{-\lambda-1}\left[\left(1+y^{-\theta}\right)^{-\lambda-1}\right]^{p-\beta(s+1)-1} dy$$

Making substitution $\left(1+y^{-\theta}\right)^{-\lambda} = z$, so that $0 \le z \le \left(1+\mu^{-\theta}\right)^{-\lambda}$, we have

$$\int_0^{\mu} yf(y)dy = \sum_{s=0}^{\infty}\sum_{p=0}^{\infty} \zeta_{s,p} \int_0^{(1+\mu^{-\theta})^{-\lambda}} z^{p-\beta(s+1)+\frac{1}{\theta\lambda}-1}\left(1-z^{\frac{1}{\lambda}}\right)^{-\frac{1}{\theta}} dz$$

After solving the integral, we get

$$\int_0^{\mu} yf(y)dy = \sum_{s=0}^{\infty}\sum_{p=0}^{\infty} \zeta_{s,p}\lambda B\left(1-(1+\mu^{-\theta})^{-1};(p-\beta(s+p))\lambda+\frac{1}{\theta},1-\frac{1}{\theta}\right) \tag{20}$$

Where $B(.)$ denotes incomplete beta function.
Substitute equation (20) in equation (19), we have

$$D(\mu) = \mu e^{-\alpha^{\beta}((1+\mu^{-\theta})^{\lambda}-1)^{\beta}} - \sum_{s=0}^{\infty}\sum_{p=0}^{\infty} \zeta_{s,p}\lambda B\left(1-(1+\mu^{-\theta})^{-1};(p-\beta(s+p))\lambda+\frac{1}{\theta},1-\frac{1}{\theta}\right)$$

Let Y be a random variable from IWB-III distribution with median M. Then the mean deviation from median is defined as.

$$\begin{aligned} D(M) =& E\left(|Y-M|\right) \\ =& \int_0^{\infty} |Y-M| f(y)dy \\ =& \int_0^{M} (M-y)\, f(y)dy + \int_M^{\infty} (y-M)\, f(y)dy \\ =& \mu - 2\int_0^{M} yf(y)dy \end{aligned} \tag{21}$$

NOW

$$\int_0^{M} yf(y)dy = \sum_{s=0}^{\infty}\sum_{p=0}^{\infty} \zeta_{s,p}\theta\lambda \int_0^{M} y^{-\theta-1}\left(1+y^{-\theta}\right)^{-\lambda-1}\left[\left(1+y^{-\theta}\right)^{-\lambda-1}\right]^{p-\beta(s+1)-1} dy$$

Making substitution $\left(1+y^{-\theta}\right)^{-\lambda} = z$, so that $0 \le z \le \left(1+M^{-\theta}\right)^{-\lambda}$, we have

$$\int_0^{M} yf(y)dy = \sum_{s=0}^{\infty}\sum_{p=0}^{\infty} \zeta_{s,p} \int_0^{(1+M^{-\theta})^{-\lambda}} z^{p-\beta(s+1)+\frac{1}{\theta\lambda}-1}\left(1-z^{\frac{1}{\lambda}}\right)^{-\frac{1}{\theta}} dz$$

After solving the integral, we get

$$\int_0^{M} yf(y)dy = \sum_{s=0}^{\infty}\sum_{p=0}^{\infty} \zeta_{s,p}\lambda B\left(1-(1+M^{-\theta})^{-1};(p-\beta(s+p))\lambda+\frac{1}{\theta},1-\frac{1}{\theta}\right) \tag{22}$$

Where $B(.)$ denotes incomplete beta function.
Substitute equation (22) in equation (21), we have

$$D(M) = \mu - 2\sum_{s=0}^{\infty}\sum_{p=0}^{\infty} \zeta_{s,p}\lambda B\left(1-(1+M^{-\theta})^{-1};(p-\beta(s+p))\lambda+\frac{1}{\theta},1-\frac{1}{\theta}\right)$$

5. Renyi Entropy

Let Y be a continuous random variable with probability density functionf $f(y)$. Then Renyi entropy is stated as

$$T_R(\rho) = \frac{1}{1-\rho} log \left[\int_0^\infty f^\rho(y) dy \right]$$

Where $\rho > 0$ and $\rho \neq 1$

$$\begin{aligned} T_R(\rho) =& \frac{1}{1-\rho} log \left[\int_0^\infty \left(\alpha^\beta \beta g(y,\phi) \frac{(G(y,\phi))^{-(\beta+1)}}{(\bar{G}(y,\phi))^{-(\beta-1)}} e^{-\alpha^\beta \left(\frac{G(y,\phi)}{\bar{G}(y,\phi)}\right)^{-\beta}} \right)^\rho dy \right] \\ =& \frac{1}{1-\rho} log \left[\alpha^{\beta\rho} \beta^\rho \int_0^\infty (g(y,\phi))^\rho \, (G(y,\phi))^{-\rho(\beta+1)} \, (1-G(y,\phi))^{\rho(\beta+1)} \, e^{-\alpha^\beta \rho \left(\frac{G(y,\phi)}{\bar{G}(y,\phi)}\right)^{-\beta}} dy \right] \end{aligned} \tag{23}$$

Using the expansion $e^{-kx} = \sum_{s=0}^\infty \frac{(-1)^s}{s!}(kx)^s$ in equation (5.1), we have

$$\begin{aligned} =& \frac{1}{1-\rho} log \left[\alpha^{\beta\rho} \beta^\rho \int_0^\infty (g(y,\phi))^\rho \, (G(y,\phi))^{-\rho(\beta+1)} \, (1-G(y,\phi))^{\rho(\beta+1)} \sum_{s=0}^\infty \frac{(-1)^s}{s!} (\alpha^{\beta\rho})^s \left(\frac{G(y,\phi)}{\bar{G}(y,\phi)} \right)^{-\beta s} dy \right] \\ =& \frac{1}{1-\rho} log \left[\sum_{s=0}^\infty \frac{(-1)^s}{s!} \alpha^{\beta(\rho+s)} \beta^\rho \rho^s \int_0^\infty (g(y,\phi))^s (G(y,\phi))^{-\beta(\rho+s)} (1-G(y,\phi))^{\beta(\rho-s)+\rho} dy \right] \end{aligned} \tag{24}$$

Using generalized bionomial expansion $(1-z)^{a-1} = \sum_{p=0}^\infty (-1)^p \binom{a-1}{p} z^p$ in equation (5.2), we have

$$T_R(\rho) = \frac{1}{1-\rho} log \left[\sum_{s=0}^\infty \sum_{p=0}^\infty \frac{(-1)^{s+p}}{s!} \binom{\beta(\rho-s)+\rho}{p} \alpha^{\beta(\rho+s)} (\beta s)^\rho \int_0^\infty (g(y,\phi))^\rho (G(y,\phi))^{p-\beta(\rho+s)} dy \right] \tag{25}$$

Using equation (12) and (13) in (25), we have

$$T_R(\rho) = \frac{1}{1-\rho} log \left[\sum_{s=0}^\infty \sum_{p=0}^\infty \zeta_{s,p} (\theta\lambda)^\rho \int_0^\infty y^{-\rho(\theta+1)} (1+y^{-\theta})^{-\rho(\lambda+1)} \left((1+y^{-\theta})^{-\lambda} \right)^{p-\beta(\rho+s)} dy \right]$$

Where

$$\zeta_{s,p} = \frac{(-1)^{s+p}}{s!} \binom{\beta(\rho-s)+\rho}{p} \alpha^{\beta(\rho+s)} (\beta s)^\rho$$

Making substitution $(1+y^{-\theta})^{-\lambda} = z, 0 < z < 1$, we have

$$T_R(\rho) = \frac{1}{1-\rho} log \left[\sum_{s=0}^\infty \sum_{p=0}^\infty \zeta_{s,p} (\theta\lambda)^{\rho-1} \int_0^1 z^{p-\beta(\rho+s)+\frac{\rho(\theta\lambda-1)+1}{\theta\lambda}-1} (1-z^{\frac{1}{\lambda}})^{\frac{\rho(\theta+1)}{\theta}-1} dy \right]$$

After solving the integral, we get

$$T_R(\rho) = \frac{1}{1-\rho} log \left[\sum_{s=0}^\infty \sum_{p=0}^\infty \zeta_{s,p} (\theta\lambda)^{\rho-1} \lambda B(m\lambda, n) \right]$$

Where $B(.)$ denotes beta function and $m = p - \beta(\rho+s) + \frac{\rho(\theta\lambda-1)+1}{\theta\lambda}, n = \frac{\rho(\theta+1)}{\theta}$

6. Order Statistics of (IWB-III) Distribution

Let us suppose $Y_1, Y_2, ..., Y_n$ be random samples of size n from IWB-III distribution with pdf $f(y)$ and cdf F(y). Then the probability density function of th k^{th} order statistics is given as

$$f_Y(k) = \frac{n!}{(k-1)!(n-1)!} f(y) \left[F(y)\right]^{k-1} \left[1 - F(y)\right]^{n-1} \tag{26}$$

Using equation (12) and (13) in equation (26), we have

$$\begin{aligned} f_Y(k) =& \frac{n!}{(k-1)!(n-1)!} \alpha^\beta \beta\theta\lambda y^{-\theta-1}(1+y^{-\theta})^{\lambda-1}((1+y^{-\theta})^\lambda - 1)^{\beta-1} e^{-\alpha^\beta((1+y^{-\theta})^\lambda-1)^\beta} \\ & \times \left[e^{-\alpha^\beta((1+y^{-\theta})^\lambda-1)^\beta}\right]^{k-1} \left[1 - e^{-\alpha^\beta((1+y^{-\theta})^\lambda-1)^\beta}\right]^{n-k} \end{aligned}$$

The pdf of the first order statistics Y_1 of IWB-III distribution is given by

$$\begin{aligned} f_Y(1) =& n\alpha^\beta \beta\theta\lambda y^{-\theta-1}(1+y^{-\theta})^{\lambda-1}((1+y^{-\theta})^\lambda - 1)^{\beta-1} e^{-\alpha^\beta((1+y^{-\theta})^\lambda-1)^\beta} \\ & \times \left[1 - e^{-\alpha^\beta((1+y^{-\theta})^\lambda-1)^\beta}\right]^{n-1} \end{aligned}$$

The pdf of the n^{th} order statistics Y_n of IWB-III distribution is given by

$$\begin{aligned} f_Y(1) = n\alpha^\beta \beta\theta\lambda y^{-\theta-1}(1+y^{-\theta})^{\lambda-1}((1+y^{-\theta})^\lambda - 1)^{\beta-1} e^{-\alpha^\beta((1+y^{-\theta})^\lambda-1)^\beta} \\ \left[e^{-\alpha^\beta((1+y^{-\theta})^\lambda-1)^\beta}\right]^{n-1} \end{aligned}$$

7. Maximum Likelihood Estimation of (IWB-III) Distribution

Let the random samples $y_1, y_2, y_3, ..., y_n$ are drawn from IWB-III distribution. The likelihood function of n observations is given as

$$L = \prod_{i=1}^{n} \left(\alpha^\beta \beta\theta\lambda y^{-\theta-1}(1+y^{-\theta})^{\lambda-1}((1+y^{-\theta})^\lambda - 1)^{\beta-1} e^{-\alpha^\beta((1+y^{-\theta})^\lambda-1)^\beta}\right)$$

The log-likelihood function is given as

$$\begin{aligned} l =& n\beta log(\alpha) + nlog(\beta) + nlog(\theta) + nlog(\lambda) - (\theta+1)\sum_{i=1}^{n} log y_i + (\lambda-1)\sum_{i=1}^{n} log(1+y_i^{-\theta}) \\ & + (\beta-1)\sum_{i=1}^{n} log\left((1+y_i^{-\theta})^\lambda - 1\right) - \alpha^\beta \sum_{i=1}^{n}\left((1+y_i^{-\theta})^\lambda - 1\right)^\beta \end{aligned} \tag{27}$$

The partial derivatives of the log-likelihood function with respect to α, β, θ and λ are given as

$$\frac{\partial l}{\partial \alpha} = \frac{n\beta}{\alpha} - \beta\alpha^{\beta-1} \sum_{i=1}^{n} \left((1+y_i^{-\theta})^{\lambda} - 1 \right)^{\beta} \tag{28}$$

$$\frac{\partial l}{\partial \beta} = nlog(\alpha) + \frac{n}{\beta} \sum_{i=1}^{n} \left((1+y_i^{-\theta})^{\lambda} - 1 \right) - \alpha^{\beta} \sum_{i=1}^{n} \left((1+y_i^{-\theta})^{\lambda} - 1 \right)^{\beta} log \left((1+y_i^{-\theta})^{\lambda} - 1 \right) - \alpha^{\beta} log(\alpha) \sum_{i=1}^{n} \left((1+y_i^{-\theta})^{\lambda} - 1 \right)^{\beta} \tag{29}$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} log(y_i) - (\lambda - 1) \sum_{i=1}^{n} \frac{y_i^{-\theta}}{(1+y_i^{-\theta})} log(y_i) - (\beta - 1)\lambda \sum_{i=1}^{n} \frac{\left(1+y_i^{-\theta}\right)^{\lambda-1}}{\left(1+y_i^{-\theta}\right) - 1} y_i^{-\theta} log(y_i) + \alpha^{\beta}\beta\lambda \sum_{i=1}^{n} \left((1+y_i^{-\theta})^{\lambda} - 1 \right)^{\beta-1} \left(1+y_i^{-\theta}\right)^{\lambda-1} y_i^{-\theta} log(y_i) \tag{30}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} log(1+y_i^{-\theta}) + (\beta+1) \sum_{i=1}^{n} \frac{\left(1+y_i^{-\theta}\right)^{\lambda}}{\left(1+y_i^{-\theta}\right) - 1} log(1+y_i^{-\theta}) - \alpha^{\beta}\beta \sum_{i=1}^{n} \left((1+y_i^{-\theta})^{\lambda} - 1 \right)^{\beta} \times \left(1+y_i^{-\theta}\right)^{\lambda} log(1+y_i^{-\theta}) \tag{31}$$

Clearly the equations (28),(29),(30) and (31), are non-linear equations which cannot be expressed in compact form and it is difficult to solve them explicitly for α, β, θ and λ .By applying the iterative methods such as Newton–Raphson method, secant method, Regula-falsi method etc. The MLE of the parameters denoted as $\hat{\xi}(\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\lambda})$ of $\xi(\alpha, \beta, \theta, \lambda)$ can be obtained by using the above methods.
For interval estimation and hypothesis tests on the model parameters, an information matrix is required. The 3 by 3 observed matrix is

$$I(\xi) = \frac{-1}{n} \begin{bmatrix} E\left(\frac{\partial^2 logl}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 logl}{\partial \alpha \partial \beta}\right) & E\left(\frac{\partial^2 logl}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 logl}{\partial \alpha \partial \lambda}\right) \\ E\left(\frac{\partial^2 logl}{\partial \beta \partial \alpha}\right) & E\left(\frac{\partial^2 logl}{\partial \beta^2}\right) & E\left(\frac{\partial^2 logl}{\partial \beta \partial \theta}\right) & E\left(\frac{\partial^2 logl}{\partial \beta \partial \lambda}\right) \\ E\left(\frac{\partial^2 logl}{\partial \theta \partial \alpha}\right) & E\left(\frac{\partial^2 logl}{\partial \theta \partial \beta}\right) & E\left(\frac{\partial^2 logl}{\partial \theta^2}\right) & E\left(\frac{\partial^2 logl}{\partial \theta \partial \lambda}\right) \\ E\left(\frac{\partial^2 logl}{\partial \lambda \partial \alpha}\right) & E\left(\frac{\partial^2 logl}{\partial \lambda \partial \beta}\right) & E\left(\frac{\partial^2 logl}{\partial \lambda \partial \theta}\right) & E\left(\frac{\partial^2 logl}{\partial \lambda^2}\right) \end{bmatrix}$$

The elements of above information matrix can be obtain by differentiating equations (28),(29),(30) and (31) again partially. Under standard regularity conditions when $n \to \infty$ the distribution of $\hat{\xi}$ can be approximated by a multivariate normal $N(0, I(\hat{\xi})^{-1})$ distribution to construct approximate confidence interval for the parameters. Hence the approximate $100(1-\psi)\%$ confidence interval for α, β, θ and λ are respectively given by
$\hat{\alpha} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\alpha\alpha}^{-1}(\hat{\xi})}$, $\hat{\beta} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\beta\beta}^{-1}(\hat{\xi})}$, $\hat{\theta} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\theta\theta}^{-1}(\hat{\xi})}$ and $\hat{\lambda} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\lambda\lambda}^{-1}(\hat{\xi})}$

8. Simulation Analysis

The MLE'S, bias and mean quare error (MSE) were all addressed to simulation analysis. From IWB-III with N=1000, samples of size n=50,150,250,350 and 500 were obtained. The following expression has been used to produce random numbers.

$$y = \left\{ \left[1 + \left(-\frac{1}{\alpha^{\beta}} log(u) \right)^{\frac{1}{\beta}} \right]^{\frac{1}{\lambda}} - 1 \right\}^{-\frac{1}{\theta}}$$

Where u is uniform random numbers with $u \in (0,1)$. For various parameter combinations, simulation results have been achieved. The MLE's bias, and MSE values are calculated and presented in table 1 and 2. As the sample size increases, this becomes apparent that these estimates are relatively consistent and approximate the actual values of parameters. Interestingly, with all parameter combinations, the bias and MSE reduce as the sample size increases.

Table 1: *Average values of MLEs their corresponding MSEs and Bias for different parameter values* $\alpha = 0.6, \beta = 1.8, \theta = 1.7, \lambda = 0.9$

Sample size	Parameters	MLEs	Bias	MSE
50	α	0.96617	0.36617	0.13433
	β	0.78938	-1.01061	1.02160
	θ	2.28698	0.58698	0.36917
	λ	0.97504	0.07504	0.03660
150	α	0.96446	0.36446	0.13040
	β	0.78690	-1.00309	1.00635
	θ	2.28292	0.58292	0.34793
	λ	0.90522	0.00522	0.01064
250	α	0.95675	0.35675	0.14210
	β	0.79891	-1.00108	1.00229
	θ	2.18506	0.48506	0.24773
	λ	0.89007	-0.00992	0.00780
350	α	0.94957	0.34957	0.14120
	β	0.70135	-1.09864	0.99739
	θ	2.18423	0.47623	0.24659
	λ	0.87259	-0.02740	0.00611
500	α	0.94070	0.34070	0.14103
	β	0.70123	-1.99776	0.99560
	θ	2.18660	0.46960	0.24483
	λ	0.86856	-0.03143	0.00485

Table 2: *Average values of MLEs their corresponding MSEs and Bias for different parameter values* $\alpha = 0.9, \beta = 1.8, \theta = 1.3, \lambda = 1.1$

Sample size	Parameters	MLEs	Bias	MSE
50	α	0.96470	0.06470	0.00444
	β	0.78783	-1.01216	1.02473
	θ	1.55140	0.25140	0.08037
	λ	0.85433	-0.24566	0.08322
150	α	0.96430	0.064308	0.00430
	β	0.78731	-1.01268	1.00554
	θ	1.50394	0.20394	0.08034
	λ	0.77932	-0.32067	0.01114
250	α	0.96324	0.06324	0.00421
	β	0.77744	-1.02255	1.00523
	θ	1.40394	0.20294	0.08009
	λ	0.77879	-0.32120	0.00351
350	α	0.95853	0.04853	0.00330
	β	0.60026	-1.99973	0.99958
	θ	1.31691	0.11691	0.00448
	λ	0.76227	-0.33772	0.00345
500	α	0.94868	0.03868	0.00245
	β	0.50124	-1.99975	0.99758
	θ	1.22246	0.10246	0.00370
	λ	0.75697	-0.34302	0.00341

9. Data Aanalysis

This subsection evaluates a real-world data set to demonstrate the IWB-III distribution's applicability and effectiveness. The IWB-III distribution's adaptability is determined by comparing its efficacy to that of other analogous distributions such as new modified Weibull distribution (NMWD), modified Weibull distribution (MWD),Topp-Leone Burr distribution (TLBD), inverse Weibull distribution (IWD) and Burr-III distribution (B-IIID),inverse Rayleigh distribution (IRD),inverse Lindley distributon (ILD).
To compare the versatility of the explored distribution, we consider the criteria like AIC (Akaike information criterion), CAIC (Consistent Akaike information criterion), BIC (Bayesian information criterion) and HQIC (Hannan-Quinn information criterion). Distribution having lesser AIC, CAIC, BIC and HQIC values is considered better.

$$AIC = -2l + 2p, \qquad AICC = -2l + 2pm/(m-p-1), \qquad BIC = -2l + p(log(m))$$
$$HQIC = -2l + 2plog(log(m)) \qquad K.S = max_{1 \le j \le m}\left(F(x_j) - \frac{j-1}{m}, \frac{j}{m} - F(x_j)\right)$$

Where $'l'$ denotes the log-likelihood function,'p'is the number of parameters and'm'is the sample size.
Data set: Bjerkedel studied the survival rates (in days) of 72 guinea pigs treated with pathogenic turbercle bacteria [7]. The data are as follows
0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93,0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46,1.53, 1.59, 1.6, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.78,2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55, 2.54, 0.77.
The ML estimates with corresponding standard errors in parenthesis of the unknown parameters are presented in Table 4 and the comparison statistics, AIC, BIC, CAIC, HQIC and the goodness-of-fit statistic for the data set are displayed in Table 5.

Table 3: *Descriptive statistics for data set*

Min.	Max.	Ist Qu.	Med.	Mean	3rd Qu.	kurt.	Skew.
0.100	5.550	1.077	1.450	1.754	2.240	4.9139	1.3282

Table 4: *The ML Estimates (standard error in parenthesis) for data set*

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\gamma}$
IWB-IIID	189.54	0.1577	15.511	0.1618	...
	(172.96)	(0.0159)	(0.0977)	(0.0584)	...
NMWD	0.0010	0.2922	1.7967	0.0010	1.7941
	(0.0035)	(0.0940)	(5.0481)	(0.0014)	(0.1570)
AWD	0.0010	0.2924	1.7961	1.7962	...
	(0.0205)	(0.0152)	(0.1563)	(0.1573)	...
TLBD	0.484	2.3688	1.8033	...	...
	(0.2556)	(0.8929)	(0.9392)	...	...
IWD	1.1753	1.0402	...	...	...
	(0.0849)	(0.1110)	...	...	...
B-IIID	...	...	2.3189	1.8576	...
	...	....	(0.2144)	(0.2192)	....
IRD	0.9112	...	...	...	...
	(0.1073)	...	...	...	...
ILD	1.5540	...	...	...	...
	(0.1434)	...	...	...	...

It is observed from table 5 that IWB-IIID provides best fit than other competative models based on the measures of statistics, AIC, BIC, AICC, HQIC and K-S statistic. Along with p-values of each model.

Table 5: *Comparison criterion and goodness-of-fit statistics for data set*

Model	-l	AIC	AICC	BIC	HQIC	K.S statistic	p-value
IWB-IIID	93.007	194.01	194.61	203.12	197.64	0.0765	0.7933
NMWD	96.03	202.07	202.98	213.45	206.60	0.0983	0.4897
AWD	96.02	200.05	200.64	209.15	203.67	0.0982	0.4902
TLBD	97.60	201.21	201.56	208.04	203.93	0.3966	2.907e-10
IWD	117.32	238.65	238.82	243.20	240.46	0.1899	0.01109
B-IIID	97.608	199.21	199.38	203.76	201.02	0.1092	0.3565
IRD	161.85	325.71	325.77	327.99	326.61	0.4674	4.352e-14
ILD	118.93	239.86	239.92	242.14	240.77	0.3219	2.2e-16

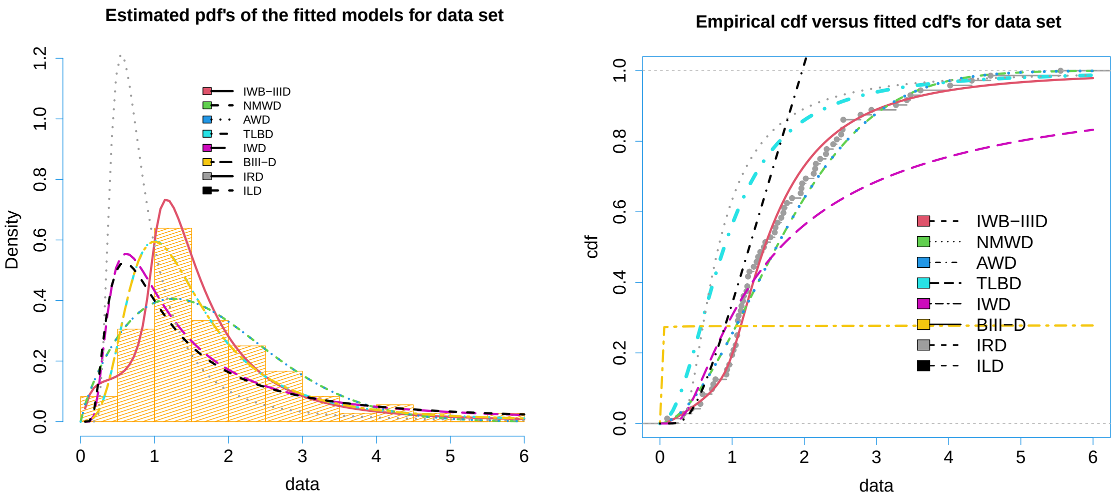

Figure 4: *fitted pdf's and cdf's for the data set*

10. Conclusion

In this work, we developed a novel flexible distribution known as the inverse Weibull-Burr III distribution. Numerous mathematical characteristics are determined for this distribution, including moments, moment generating functions, incomplete moments, order statistics, Renyi entropy, mean deviations, and reliability analysis. The maximum likelihood estimation approach was used to estimate the distribution's parameters. Ultimately, it has been demonstrated by employing a real-world data set that the stated distribution leads to a better fit than the comparable ones.

References

[1] A. Alzaatreh, C.Lee,F. Famoye . A new method for generating families of distributions. *Metron, 71 (2013),https://doi.org/10.1002/wics.1255*, 63-79.

[2] H. Amal and G.N. Said. The inverse Weibull-G familiy. *Journal of data science (2018)*, 723-742, DOI: 10.6339/JDS.201810-16(4).00004.

[3] M. Bourguingnon, R.B. Silva and G.M. Corderio. The Weibull-G family of probability distributions *Journal of data science,12 (2014), http://www.jds-online.com*, 53-68.

[4] I.W. Burr. Cummlative frequency function. *Annals of mathematical statistics,13 (1942),https://www.jstor.org/stable/2235756*, 215-232.

[5] E. Brito, G.M. Cordeiro, H.M Yousuf,M. Alizadeh,G.O Silva. The Topp-Leone odd log-logistic family of distributions *J stat comput simul*, *87(15) (2017), https://doi.org/10.1080/00949655.2017.1351972*, 3040-3058.

[6] A.P. Bilal, J.Suriya and J. Tariq. Generalization of Burr-III distribution. *International journal of modern mathematical science*, *13(3) (2015), https://www.ModernScientificPress.com/Journals/ijmms.aspx*,322-329.

[7] T. Bjerkedal. Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli. *American journal of Epidemiology*, *72(1) (1960)*,130-148.

[8] R. Corderio and G. Pulcini. A new family of generalized distributions. *Journal of statistical computation and simulation*, *81 (2011),https://doi.org/10.1080/00949650903530745*, 883-893.

[9] M. Daniyal and M. Aleem. On Mixture of Burr XII and Weibull Distributions. *Journal of Statistics and Probability*, *3(2) (2014)*, 251-267, DOI: 10.12785/jsap/030215.

[10] A. Danyian, G. Behairy, S.M. and A.A. EL-Helbawy. The Kumaraswamy-Burr type-III distribution: Properties and Estimation. *British journal of mathematics and computer science*, *14(2) (2016)*, 1-21, DOI: 10.9734/BJMCS/2016/19958.

[11] N. Eugene, C. Lee and F. Famoye. Beta-normal distribution and its applications. *Communication in statistics- theory and methods*,*31 (2002),https://doi.org/10.108/STA-120003130*, 497-512.

[12] A. Morad, M.C. Gauss. The Gompertz-G family of distributions *Journal of statistical theory and practice*,*11(1)(2017),https:doi.org/10.108/15598608.2016.1267668*, 179-207.

[13] K. Zagrafos andN. Balakrishnan. On families of beta-and generalized gamma-generated distributions and associated inference. *Statistical Methodology*, *6 (2009),https://doi.org/10.1016/j.stamet.2008.12.003*, 344-362.