OPTIMAL ECONOMIC AGE REPLACEMENT MODELS FOR NON-REPAIRABLE SYSTEMS WITH SUDDEN BUT NON-CONSTANT FAILURE RATE

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Abstract

Proper maintenance of non-repairable systems is essential for optimum utilization of systems to prevent lost production runs, cost inefficiencies, defective output which leads to customer dissatisfaction and unavailability of the facility for future use. This work proposes new preventive replacement maintenance models with constant-interval preventive replacement time with associated cost of replacement maintenance. Improved results of economic values with respect to optimal replacement time at minimum cost were obtained for radio transmitter system with sudden but non-constant failure rate when compared to some existing models. Other parameters and maintenance probabilities of the system were also obtained including; reliability, hazard rate and availability to ascertain the operational condition of the system.

Keywords: *Birnbaum-Saunders distribution, Radio Transmitter Systems, Reliability, Replacement model, Availability, Optimum replacement time and cost*.

I. Introduction

The failure behaviour in time of a system could be examined by the failure rate of the system. The failure rate is also a function of time, which is also known as the hazard rate. In reliability studies, systems can be classified into two main categories which are repairable and non-repairable system. A repairable system is one which can be restored to satisfactory operation by any maintenance action, including parts replacement or changes to adjustable settings. Examples of such systems are mechanical systems like the generators, grinding machines, welding machines, etc. while a non-repairable system is one in which its component or the entire system is always replaced during any form of maintenance. Examples of these systems are mainly electronic systems like stabilizers, refrigerators, transmitters and many more. It is characterized by the failure time distribution such as the cumulative distribution function of its time to failure, whereas a repairable system's behavior is described by a stochastic point process, and so must be characterized differently, for example by using the rate of occurrence of failures (ROCOF) or the expected number of failures for a given time period.

According to [1] and [2], maintenance can be defined as "the combination of all technical and associated administrative actions intended to retain an item or system, or restore it to a state in which it can perform its required function." Maintenance does not only improve the cost-efficiency of operating a system but it can also significantly reduce the probability of catastrophic failure of the system, [3]. Therefore, maintenance managers must plan the maintenance actions, so that a balance is achieved between the expected benefits and corresponding expected potential consequences, [4]. On the contrary, poor maintenance of production facilities can result in defective end-product and customer dissatisfaction, lost production runs, cost inefficiencies, and sometimes, unavailability of the facility for future use, [5]. It was further observed by [6] that "facility maintenance is the effort in connection with different technical and administrative action to keep a physical asset, or restore it to a condition where it can perform a require function". Maintenance can be classified according to its type and its degree: corrective maintenance (CM) and preventive maintenance (PM). Corrective maintenance are all maintenance actions performed after a system has failed with the objective of restoring its functionality while preventive maintenance refers to planned maintenance actions performed while the system is operational with the objective of maintaining the system over a desired operational time horizon by preventing or delaying failures. For instance, in an age replacement maintenance policy a unit is replaced at failure (CM) or at PM time, T where T is a constant, [7] and [8]. The principle of age replacement model was applied by [9] in developing a periodic replacement policy for a two-unit system with failure rate interaction between units.

Preventive maintenance policies are perhaps one of the most studied maintenance policies in the literature, [10]. Furthermore, the expected replacement costs per unit time and the age that minimizes this value was used to provide the optimal maintenance interval, [11]. These fixed time frames are established ahead of time and remain in place regardless of when actual failures occur. This means that if a failure occurs just before reaching the fixed time frame, the unit will be replaced both at the failure and immediately again at the time interval. Replacement models of expected cost rates and optimal replacement times were obtained by [12] for a required availability level and were optimized. A theory for non-random preventive replacement (and corrective replacement only for units with exponential failure) and modification of age replacement was established for the situation when the life cycle of the unit is a random variable with probability distribution. The history and development of replacement model from the earliest work to the general replacement model was considered by [13]. They combined age and random replacement models and treated replacement first, replacement last, replacement overtime, replacement overtime first and replacement overtime last. These made up the general replacement models with n replacement times which were obtained by formulating the distributions of replacement times with n variables. The performances of seven optimization models of age replacement policy were evaluated by [14]. The performances were evaluated from perspectives; cost (or availability) and reliability. Furthermore, three performance measures that correspond to cost, reliability and overall performance, respectively, were developed for evaluating the performances of the models. A survey on age replacement model involving minimal repair was conducted by [15] by considering a parallel-series system with two subsystems. Age replacement models (involving minimal repair) that determine the optimal replacement time of the parallel-series system based on two different policies (Policy 1 and Policy 2) were formulated and compared using numerical example.

It is on this premise that this work seeks new perspectives by formulating new preventive replacement models characterized by failure and hazard distribution functions of the system that would provide optimal replacement time and minimum cost of maintenance.

The remainder of this paper shall consider methods in section II on the concept of formulating and optimizing age preventive replacement models and its application to the maintenance of radio transmitter system. Section III deals with numerical analysis based on the failure time distribution and maintenance probabilities of the radio transmitter system. Results would also be discussed in this section. The paper is concluded in section IV.

II. Methods

1 Formulation of Preventive Replacement Model

In developing a replacement model, the decision criterion is defined by $E[c(t)]$, which is the expected cost/cycle time of replacing a part of the system in cycle period (0,t). It was shown in [16] that the expected number of failures occurring in the cycle period (0,t) is equal to the probability of occurrence of failures before time, t, denoted by F(t). The number of failures occurring during the period (0,t) is defined as N(t), which is a discrete random variable. Its probability distribution function is defined as;

$$P[N(t) = n] = G(n); n = 0,1,2, \dots$$

Its expected value is then equal to;

$$E[N(t)] = \sum_{N(t)=0}^{N(t)=n} N(t) \times G[N(t)]$$

Where $G[N(t)]$ is the failure distribution function of N(t) occurring in the period (0, t). It is assumed that each interval is made as short as the need may be so that the probability of having more than one failure is negligible. In this situation, the probability of having two failures is small compared to having a single failure and so on. That is;

$$P[N(t) = 2] < P[N(t) = 1]$$

and the probability of having three failures is smaller compared to having two failures;

$$P[N(t) = 3] < P[N(t) = 2], \text{etc.}$$

Therefore,

$$P[N(t) = 1] < P[N(t) = 2] > \cdots$$

In the case of preventive replacement at τ, the expected number of failures in the period (0, τ), denoted by $E[N(\tau)]$can be estimated from the following;

$$G(1) = P[N(t) = 1] \approx F(\tau) and G(0) = P[N(t) = 0 \approx 1 - F(\tau)]$$

Then,

$$E[N(\tau)] = \sum_{n=0}^{\infty} n \times G(n) = 0 \times [1 - F(\tau)] + 1 \times F(\tau)$$

$$E[N(\tau)] = F(\tau) \tag{1}$$

Therefore, $E[N(\tau)]$ is equal to the probability of occurrence of a failure before time, τ and F(τ) is the cumulative failure function. Let C_{fr} be the total cost of failure replacement, C_{pr} be the total cost of preventive replacement maintenance and τ is the replacement time. The total expected cost per cycle for preventive replacement maintenance at replacement time, τ is defined as;

$$E[N(\tau)] = \frac{total\ expected\ cost}{replacement\ time} = \frac{C_{pr} + C_{fr}\{N(\tau)]}{\tau} \tag{2}$$

Therefore;

$$E[C(\tau)] = \frac{C_{pr} + C_{fr} F(\tau)]}{\tau} \tag{3}$$

Also, [17] stated that the mean number of failures occurring during the cycle (0,τ] is equal to the cumulative hazard rate at time, τ using the concept of non-homogeneous Poisson process. Hence, the expected cost per unit time for preventive replacement maintenance is given by;

$$E[C(\tau)] = \frac{C_{pr}+C_{fr}H(\tau)]}{\tau} \quad (4)$$

Where H(τ) is the cumulative hazard function. The objective is to determine the time necessary for preventive replacement maintenance in order to minimize the total expected replacement cost per unit time.

2 The propose age replacement models

Let C_r be the replacement maintenance costs for each failed unit (RM), let C_p be the preventive maintenance cost for each non-failed unit, where $C_p < C_r$. Also, let $N_1(t)$ denote the number of failures in the interval (0, t] and $N_2(t)$ denote the number of non-failed units that are preventively maintained in the interval (0, t]. Therefore, the expected cost during (0, t] is expressed as:

$$\hat{C}(T) = C_r E\{N_1(t)\} + C_p E\{N_2(t)\}$$

Based on the renewal reward theorem, [18], the expected cost per unit time (expected cost rate) for an infinite time span is;

$$\hat{C}(T) \equiv \lim_{T\to\infty} \frac{\hat{C}(T)}{t} = \frac{Expected\ cost\ of\ one\ cycle}{mean\ time\ of\ one\ cycle}$$

Let **T**, $(0 < T \leq \infty)$ be the time for a planned replacement of a component with failure time, τ; the expected cost on a cycle as obtained in [19] is expressed as;

$$C_r P(\tau \leq T) + C_p P(\tau > T) = C_r F(T) + C_p F^{/}(T)$$

where $$F^{/}(T) = 1 - F(T)$$

The mean time of a cycle is denoted by τ, where a cycle refers to the interval from the start of the system to the completion of repair maintenance action or replacement maintenance. Hence, our propose expected cost function is given in (5). An optimal policy can be found by obtaining the value of τ that minimizes this cost function.

$$\hat{C}(T) = \frac{C_r F(T) + C_p F^{/}(T)}{\tau} \quad (5)$$

Similarly, if we also assume that the mean number of failures occurring during the cycle (0, τ] is equal to the cumulative hazard rate at time, τ using the concept of non-homogeneous Poisson process as in [17]; the expected cost per unit time for preventive replacement maintenance is given by;

$$\hat{C}(T) = \frac{C_r F(T) + C_p H^{/}(T)}{\tau} \quad (6)$$

2.1 Minimization of the expected cost functions:

Taking the partial derive of (2) with respect to τ yields;

$$\tau^* = \frac{\frac{C_{pr}}{C_{fr}} + E[N(\tau)]}{\frac{d}{d\tau}E[N(\tau)]} \quad (7)$$

To obtain τ^* for the models in (3) and (4), we respectively have;

$$E[N(\tau)] = \frac{d}{d\tau}E[N(\tau)] = \frac{d}{d\tau}F(\tau) = f(t)$$

$$\tau^* = \frac{\frac{C_{pr}}{C_{fr}} + F(\tau)}{f(t)} \quad (8)$$

and

$$E[N(\tau)] = H(t); \frac{d}{d\tau}E[N(\tau)] = \frac{d}{d\tau}H(\tau) = h(t)$$

$$\therefore \tau^* = \frac{\frac{C_{pr}}{C_{fr}} + H(\tau)}{h(t)} \tag{9}$$

Similarly, we also obtain τ^* respectively from (5) and (6), as;

$$\tau^* = \frac{F(\tau)}{f(t)} + \frac{c_p}{(c_r - c_p)f(T)} \tag{9}$$

$$\tau^* = \frac{H(\tau)}{h(t)} + \frac{c_p}{(c_r - c_p)h(T)} \tag{10}$$

3. Limiting Availability of a System

Availability is the probability that a system will work as required during a particular period of time.

Let A(τ^*) denote the availability of a system at optimal time, τ^*

E[Up] is the expected uptime at optimal time, τ^*

E[Down] is the expected downtime at optimal time, τ^*

D_{pr} is the average downtime for preventive replacement

D_{fr} is the average downtime for failure replacement

R(τ^*) is the reliability at optimal time, τ^*

F(τ^*) is the cumulative failure at optimal time, τ^*

According to [17];

$$E[Up] = \int_0^\infty tf(t)dt + \tau^* R(\tau^*)$$

where

$$\int_0^\infty tf(t)dt = \tau^* f(\tau^*) - 0 \times f(0) = \tau^* \times f(\tau^*)$$

$$E[Down] = D_{pr}F(\tau^*) + D_{fr}R(\tau^*)$$

where

$$D_{fr} = \frac{\sum_{i=1}^{n} D_i}{n} \text{ and } D_{pr} = \frac{\sum_{j=1}^{m} D_i}{m}$$

$$A(\tau^*) = \frac{E(UP)}{E(Up) + E(Down)} = \frac{\tau^* f(\tau^*) + \tau^* R(\tau^*)}{\tau^* f(\tau^*) + \tau^* R(\tau^*) + \left(D_{pr}F(\tau^*) + D_{fr}R(\tau^*)\right)} \tag{12}$$

4. Application of Replacement models to the maintenance of radio transmitter system

4.1 The Birnbaum-Saunders (Fatigue Life) Failure Distribution

The Birnbaum Sanders (BS) distribution has appeared in several different contexts, with varying derivations. It was given by Fletcher in 1911, and was formally obtained by [20]. However, it was the derivation by [21] that brought the usefulness of this distribution into a clear focus. Authors in [22] introduced a two-parameter lifetime distribution to model fatigue life of a metal, subject to cyclic stress by making a monotone transformation on the standard normal random variable. Consequently, the distribution is also sometimes referred to as the fatigue-life distribution. Since then, extensive work has been done on this model providing different interpretations, constructions, generalizations, inferential methods, and extensions to bivariate and multivariate cases, [23]. Its application is sought in this work as the best fit probability model to provide parameters estimates and optimal probabilities for inter-failure times of electronic systems with radio transmitter system as a case study. These estimates would be used to obtain optimal replacement policies in existing age preventive replacement maintenance models as well as our propose class of models.

The radio transmitter is a complex electronic device which major components are integrated circuits, diodes and fuses which are replaced after each failure. Hence, it is a non-repairable system. It fails suddenly but at a non-constant rate. The inter-failure times of the transmitter system was modeled as the Birnbaum-Saunders distribution having a chi-square best fit of rank 1 using Easyfit (5.6) software.

4.1.1 Failure and Cumulative Failure Distributions of the Two-Parameter Birnbaum-Saunders distribution

Consider a material that continually undergoes cycles of stress loads. During each cycle, a dominant crack grows towards a critical length that will cause failure. Under repeated application of **n** cycles of loads, the total extension of the dominant crack can be written as; $W_n = \sum_{j=1}^{n} Y_j$. Let Q be an integer-valued non-negative random variable denoting the smallest number of cycles at which W_n exceeds a critical value ω, which the failure of the material occurs. Clearly, $P(Q \leq n) = P(W_n \geq \omega)$ and this implies that;

$$P(Q \leq n) = 1 - P\left(\sum_{j=1}^{n} Y_j \leq \omega\right) \tag{13}$$

Since Y_j's are assumed to have mean, μ and variance, σ^2, thus the Y_j's can be standardized to give

$$P(Q \leq n) = 1 - P\left(\frac{\sum_{j=1}^{n}(Y_j - \mu)}{\sigma\sqrt{n}} \leq \frac{\omega - n\mu}{\sigma\sqrt{n}}\right)$$

Another assumption is that the Y_j's are independent. Also if **n** is large (a criterion easy to satisfy in fatigue studies), the central limit theorem applies. Hence, by the symmetry of the normal distribution,

$$P(Q \leq n) = 1 - \emptyset\left(\frac{\omega - n\mu}{\sigma\sqrt{n}}\right) = \emptyset\left(\frac{n\mu}{\sigma\sqrt{n}} - \frac{\omega}{\sigma\sqrt{n}}\right); \text{where} = \emptyset(X) = \int_{-\infty}^{x} \frac{e^{\frac{x^2}{2}}}{\sqrt{2\pi}} ds$$

The above derivation, which involved a non-negative integer-valued random variable Q, can be extended to continuous variables. Let **T**, a continuous non-negative random variable denote the time to failure of the material with a distribution function, F(t). If **T** is viewed as the continuous analog of Q, and **t** as a continuous analog of **n**, then,

$$F(t) = P(T \leq t) = P(Q \leq n) = \emptyset\left(\frac{\mu\sqrt{n}}{\sigma} - \frac{\omega}{\sigma\sqrt{n}}\right) \tag{14}$$

Replacing **n** by **t**, we have;

$$F(t) = \emptyset\left(\emptyset\left(\frac{\mu\sqrt{t}}{\sigma} - \frac{\omega}{\sigma\sqrt{t}}\right)\right)$$

Let

$$\alpha = \frac{\sigma}{\sqrt{\mu\omega}} \; and \; \beta = \frac{\omega}{\mu}$$

Then

$$F(t;\alpha,\beta) = \emptyset\left(\frac{1}{\alpha}\left[\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right]\right) \tag{15}$$

Equation (15) is the two parameter Birnbaum-Saunders failure cumulative distribution function with shape parameter α and scale parameter β. It follows that;

$$Z = \frac{1}{\alpha}\left[\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right] \tag{16}$$

Then, Eq (16) is distributed with mean 0 and variance 1 and that the probability density function of T is;

$$f(t;\alpha,\beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{t}{\beta}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right] exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta} + \frac{\beta}{t} - 2\right)\right]; t > 0; \alpha,\beta > 0 \tag{17}$$

4.1.2 The Mean and Variance of the two-parameter Birnbaum-Saunders Distribution using Monotone Transformation

The mean and variance of **T** can be found in the usual manner by integration. For ease of computation, however, the following alternative approach is adopted. Let a random variable **X** be normally distributed with mean 0 and variance $\alpha^2/4$. It follows that 2X is also normally distributed with mean 0, and variance α. Moreover, since **Z** has a unit normal distribution, αZ is distributed normally with mean 0 and variance α^2. Thus,

$$Z = \frac{2X}{\alpha} \Rightarrow \alpha Z = 2X$$

Hence, from Eq (16);

$$2X = \left(\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right) \tag{18}$$

Squaring both sides we have;

$$4X^2 = \left(\frac{T}{\beta} + \frac{\beta}{T} - 2\right) \times \beta T$$

$$T^2 + \beta^2 - 2T(\beta + 2\beta X^2) = 0 \tag{19}$$

The positive roots yield;

$$T = \beta\left(1 + 2X^2 + 2X(1 + X^2)^{\frac{1}{2}}\right) \tag{20}$$

where **T** is the Birnbaum-Saunders random variable.

The mean of T

$$E[T] = E\left[\beta\left(1 + 2X^2 + 2X(1 + X^2)^{\frac{1}{2}}\right)\right]$$

$$E[T] = \beta\left[E[1] + 2E[X^2] + 2[X]E(1 + X^2)^{\frac{1}{2}}\right] \tag{21}$$

But **X** follows a normal distribution with mean, $\mu = 0$ and variance, $\sigma = {\alpha^2}/{4}$; then we have that;

$$E[X^2] = var(X) = \frac{\alpha^2}{4} \tag{22}$$

$$\therefore E[T] = \beta\left(1 + \frac{2\alpha^2}{4}\right) \tag{23}$$

Variance of T

$$Var(T) = var\left(\beta\left(1 + 2X^2 + 2X(1 + X^2)^{\frac{1}{2}}\right)\right) = \beta^2\left[var(1) + var(2X^2) + var\left(2X(1 + X^2)^{\frac{1}{2}}\right)\right]$$

$$= \beta^2\left(var(1) + 4var(X^2) + 4var\left(X(1 + X^2)^{\frac{1}{2}}\right)\right) \tag{24}$$

From the non-central moment of a normal distribution, we have that;

$$E(X^4) = \mu + 6\mu^2\sigma^2 + 3\sigma^4$$

But

$$\mu = 0, \sigma^2 = \frac{\alpha^2}{4}$$

Then,

$$E(X^4) = \frac{3\alpha^2}{16};\ E(X^2) = var(X) = \frac{\alpha^2}{4}$$

$$\therefore\ var(X^2) = \frac{3\alpha^2}{16} - \left(\frac{\alpha^2}{4}\right)^2 = \frac{2\alpha^4}{16} \tag{25}$$

Let

$$Y = X(1 + X^2)^{\frac{1}{2}}$$

$$Var\left(X(1+X^2)^{\frac{1}{2}}\right) = E\left(\left(X(1+X^2)^{\frac{1}{2}}\right)^2\right) - \left(E\left(X(1+X^2)^{\frac{1}{2}}\right)\right)^2$$

$$E\big(X^2(1+X^2)\big) = E(X^2+X^4) = E(X^2) + E(X^4)$$

$$Var\left(X(1+X^2)^{\frac{1}{2}}\right) = \frac{\alpha^2}{4} + \frac{3\alpha^2}{16} \tag{26}$$

Substitute (25) and (26) into (24), we have;

$$Var(T) = \beta^2\alpha^2\left(1+\frac{5\alpha^2}{4}\right) \tag{27}$$

4.1.3 Modified moment estimation for the two-parameter Birnbaum-Saunders Distribution

For the usual moment estimators in a two-parameter case, the first and second population moments are equated with the corresponding sample moments. In the case of modified moment estimation (MME), the expectation of the random variable is equated to the sample arithmetic mean and the expectation of the inverse of the random variable is equated to the sample harmonic mean. Let $\{t_1, t_2, t_3 \ldots\}$ be a random sample of size **n** from the Birnbaum–Saunders distribution with the probability density function as given in Eq (17). The sample arithmetic and harmonic means are defined by;

$$S = \frac{\sum_{i=1}^{n} t_i}{n} \tag{28}$$

$$r = \left(\frac{\sum_{i=1}^{n} {t_i}^{-1}}{n}\right)^{-1} \tag{29}$$

Therefore by MME;

$$S = E(T) \text{ and } r = E(T^{-1})$$

If **T** has a Birnbaum–Saunders distribution with parameters α and β, then T^{-1} also has a Birnbaum–Saunders distribution with the corresponding parameters α and β^{-1} respectively, [21]. Therefore, we readily have;

$$E(T^{-1}) = \beta^{-1}\left(1+\frac{\alpha^2}{2}\right)$$

$$Var(T^{-1}) = \alpha^2\beta^{-2}\left(1+\frac{5\alpha^2}{4}\right)$$

Hence;

$$S = \beta\left(1+\frac{\alpha^2}{2}\right) \tag{30}$$

$$r^{-1} = \beta^{-1}\left(1+\frac{\alpha^2}{2}\right) \tag{31}$$

$$\frac{S}{r^{-1}} = \frac{\beta\left(1+\frac{\alpha^2}{2}\right)}{\beta^{-1}\left(1+\frac{\alpha^2}{2}\right)} = \beta^2$$

$$\therefore \hat{\beta} = \sqrt{Sr} \tag{32}$$

By substituting (28) and (29) in (32), we obtain;

$$\hat{\beta} = \sqrt{\left(\frac{\sum_{i=1}^{n} t_i}{n}\right)\left(\left(\frac{\sum_{i=1}^{n} {t_i}^{-1}}{n}\right)^{-1}\right)} \tag{33}$$

Substituting (31) into (29), we have;

$$S = \sqrt{Sr}\left(1+\frac{\alpha^2}{2}\right)$$

$$\hat{\alpha} = \left\{2\left(\frac{S}{\sqrt{Sr}}-1\right)\right\}^{\frac{1}{2}} = \left\{2\left(\frac{S}{\hat{\beta}}-1\right)\right\} \tag{34}$$

4.1.4 Reliability Function of Birnbaum- Saunders Distribution

Let R(t) be the reliability function of the Birnbaum – Saunders distribution given as;

$$R(t) = 1 - F(t) = P(T > t); t > 0$$
$$= 1 - \int_0^t f(u)du = \int_t^\infty f(u)du$$

Hence, R(t) is the probability that the item does not fail in the time interval (0,t], or, in other words, the probability that the item survives the time interval (0, t] and is still functioning at time t.
For the two parameter Birnbaum-Saunders distribution, we have;

$$R(t) = 1 - F(t; \alpha, \beta)$$

Recall Eq (15):

$$F(t; \alpha, \beta) = \emptyset \left\{ \frac{1}{\alpha} \left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}} \right) \right\}$$
$$R(t) = 1 - \emptyset \left\{ \frac{1}{\alpha} \left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}} \right) \right\}$$

Because of the symmetry of the normal distribution, we have that;

$$R(t) = \emptyset \left\{ \frac{1}{\alpha} \left(\sqrt{\frac{\beta}{t}} - \sqrt{\frac{t}{\beta}} \right) \right\} \quad (35)$$

4.1.5 Determination of Hazard Function, h(t)

The probability that an item will fail in the interval $(t, t + \Delta t)$ when we know that the item is functioning at time, **t** is;

$$P(t < T \le t + \Delta t / T > t) = \frac{P(t < T \le t + \Delta t)}{P(T > t)} = \frac{F(t + \Delta t) - F(t)}{R(t)}$$

By dividing this probability by the length of the time interval, $\boldsymbol{\Delta t}$, and letting $\boldsymbol{\Delta t \to 0}$, we get the failure rate function, h(t) of the item as;

$$h(t) = \lim_{\Delta t \to 0} \frac{P(t < T \le t + \Delta t / T > t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t} \times \frac{1}{R(t)} = \frac{f(t)}{R(t)}$$
$$\therefore R(t) = \frac{f(t)}{R(t)} \quad (36)$$

Note: According to [24], the density of Birnbaum-Saunders failure distribution can be written in a different form as;

$$f(t) = \frac{\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}} \right)}{2\alpha t} \times \emptyset \left\{ \frac{1}{\alpha} \left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}} \right) \right\}$$
$$Z = \frac{1}{\alpha} \left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}} \right)$$

$$\emptyset(Z) = \frac{exp\left(-\frac{Z^2}{2} \right)}{\sqrt{2\pi}}$$

Therefore,

$$R(t) = \emptyset(-Z)$$

and

$$h(t) = \frac{\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}} \right)}{2\alpha t} \times \left(\frac{\emptyset(Z)}{\emptyset(-Z)} \right) \quad (37)$$

4.1.6 The Cumulative Hazard function of Birnbaum-Saunders distribution

By definition;

$$f(t) = \frac{d}{dt}F(t) = \frac{d}{dt}\left(1 - R(t)\right) = -R(t)$$

Then

$$h(t) = \frac{R^{/}(t)}{R(t)} = -\frac{d}{dt}lnR(t)$$

Recall

$$\int h(t)\,dt = -lnR(t) = H(t)$$

$$H(t) = -lnR(t) \qquad (38)$$

III. Results

I. Estimation of the Birnbaum-Saunders Parameter of the Radio Transmitter System

The modified moment estimators as given in (33) and (34) were used to obtained estimated shape parameter, $\hat{\alpha} = 0.95701$ and scale parameter, $\hat{\beta} = 557.37$ for the inter-failure times of the radio transmitter which follows a two-parameter Birnbaum-Saunders distribution. Easyfit version 5.6 was used for the goodness-of fit test as well as the estimation of parameters.

II. Replacement models and probability functions of radio transmitter systems at respective optimum times

Optimal probability functions in Eqs (15) and (17), availability factor in Eq (12) and expected cost per cycle in Eqs (3) - (6) were obtained in Table 1 at the respective optimum values of the four replacement models under consideration.

Table 1: Optimal probabilities, availability and expected cost functions for replacement models

Probability function	Cumulative Failure-Based Replacement model **A** $\tau^* = 143$	Cumulative Hazard-Based Replacement model **B** $\tau^* = 137$	Proposed Replacement model **C** based on cumulative failures, $\tau^* = 153$	Proposed Replacement model **D** based on cumulative hazard, $\tau^* = 149$
$f(\tau^*)$	0.00115	0.001081	0.001376	0.001145
$F(\tau^*)$	0.06256	0.0560	0.093197	0.069341
$h(\tau^*)$	0.0012	0.0015	0.0015174	0.001231
$R(\tau^*)$	0.93744	0.94407	0.906803	0.930659
$A(\tau^*)$	0.9829	0.98	0.9597	0.95903
$E[C(\tau^*)]$	388	392	208	166

IV. Discussion

I. Estimated parameters of the Birnbaum-Saunders distribution

The estimates $\hat{\alpha} = 0.95701$ and $\hat{\beta} = 557.37$ are the respective shape and scale parameters of the Birnbaum-Saunders distribution. The shape of the failure density function and the hazard function are governed by α. Also, the failure density function is unimodal for all values of α. The scale parameter is also known as the median of the distribution. As α increases the hazard rate and the failure density function of the distribution becomes more skewed to the right.

II. Choice of Optimal Replacement Models

The propose replacement maintenance models by the authors yield improved results and are therefore considered the preferred economic optimal model for maintenance policy of the system due to the following reasons;

i. Improved optimal operational time estimate before replacement: The cumulative failure function-based replacement model **A** from [16] in column (2) of Table 1 yields an optimal replacement time of 143 hours versus our propose optimal replacement time of 153 hours based on model **C** with same parameters in column 4. Also, the cumulative hazard function-based replacement model **B** by [17] in column 3 of Table 1 shows that the radio transmitter system has an optimal replacement time of 137 hours versus our propose optimal replacement time of 149 hours from model **D** based on the parameters of the same kind in column 5.
ii. Improved expected minimum cost value for replacement maintenance: The cumulative failure function-based replacement model **A** from [16] in column (2) of Table 1 yields an expected minimum cost value of 388naira versus our propose expected minimum cost value of 208 naira from model **C** based on the same parameters. Also, the cumulative hazard function-based replacement model **B** by [17] in column 3 of Table 1 shows that the radio transmitter system has an expected minimum cost value of 396 naira versus our propose expected minimum cost value of 166 in model **D** based on the same parameters.
iii. Comparative chance of failure occurrence: The cumulative failure based function replacement model **A** from [16] in column (2) of Table 1 yields a 0.12% chance of failure occurrence versus a 0.15% chance of failure occurrence obtained from our propose model **C** of same kind. But, the cumulative hazard function-based replacement model **B** by [17] in column 3 of Table 1 yields a 0.15% chance of failure occurrence versus a lesser percentage of 0.12% chance of failure occurrence obtained from our propose model **D** of the same kind.

The results obtained in this study show that the failure distribution of Radio transmitter system used as a case study follows the Birnbaum-Sanders distribution with best fit parameters: $\hat{\alpha} = 0.95701$ and $\hat{\beta} = 557.37$. Hence, the Birnbaum-Sanders failure distribution is recommended as a good probability model which characterized the failure distribution of transmitter and similar systems. It is also clear from the results that the propose class of replacement maintenance models gives improved results than earlier models by [16] and [17]. Specifically, our second propose cumulative hazard function-based model yields the most economic cost (166 naira) at optimal time $\tau^* = 149$ with competing availability and reliability values and a smaller chance of failure occurrence before replacement maintenance. It is remarkable that the propose models are better in terms of optimal replacement time and minimum expected cost, with comparable variations in the probability of occurrence of failure in the cumulative failure function-based model and an improved result from our cumulative hazard-based replacement models. These provide good reasons for the choice of our propose hazard function-based model as the preferred model in particular and the propose class of models in general for the study.

Consequently, our propose preventive replacement maintenance models are the optimal economic models with respect to time and cost as vital economic factors in formulating replacement maintenance policies for the radio transmitter and similar systems.

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