The Transmuted Weibull Frechet Distribution: Properties and Applications

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Abstract

The behaviour of everyday real life processes played a greater role in distribution theory. Thus, this article proposes a transmuted Weibull Frechet (TWFr) distribution for modeling real life datasets. Of most important, the statistical properties of the TWFr distribution such as the hazard, survival functions, order statistic, quantile, odd, cumulative functions were derived and examined. A simulation study to examine the performance of the TWFr distribution was also conducted. A glass fiber data and breaking stress of carbon data real life application were used to showcase the performance of the proposed model. The results showed that the TWFr distribution competes favourably well with other types of continuous distributions in the Frechet family of distributions.

Keywords: Frechét distribution, Hazard rate function, Order statistics, Transmutation, Weibull distribution.

1. Introduction

Modeling the distributions of real life processes poses greater challenges despite the numerous distributions that have been proposed in literature. However, there is a growing interest in developing newer classes of classical univariate distributions for modeling variety of data sets that arise from our daily scenarios. Thus, it becomes necessary to model these processes either by compounding one or more distributions to address these complex situations.

The Weibull distribution proposed by a famous statistician called Weibull in 1951 [27] has a wide range of applications in modeling failure time processes, lifetime processes, mechanical and electrical systems. More so, the Weibull distribution has been found to be better for modeling the minimum of large number of independent positive random variables in extreme value theory. On the other hand, the Frechet distribution is a special case of the Weibull distribution used to model extreme value scenarios like earthquakes, horse racing, floods, rainfall, wind speed, queues in supermarkets and sea waves (see [2]). The Frechet distribution has been widely used to model extreme value scenarios because of its stochastic phenomena. However, the Frechet distribution is used for modeling maximum of a large number of independent random variables from a particular class of distributions ([1]). Hence, because of its usefulness, improving the flexibility of the Frechet distribution becomes necessary by adding a transmuted parameter that can reflect the true characteristics of the data set(see [13]).

Several statistical distributions have been proposed in literature. For example, [2] proposed the Weibull Frechet distribution, [12] proposed the generalized odd Weibull generated family of distributions. Recently, [26] proposed the gamma extended Frechet distribution. [17] proposed the beta Frechet distribution. [16] proposed the exponentiated Frechet distribution. [14] proposed Kumaraswamy Frechet distribution. The generalized transmuted Frechet (GTFr) distribution was proposed in [18]. [14] estimated the Frechet type 11 parameters and [23] proposed a new generalization of the Frechet distribution. In this study, the transmuted Weibull Frechet distribution is introduced.

A random variable X with a scale parameter α, and β, τ, ν the shape parameters has a Weibull Frechet distribution if the cumulative density function is given as

$$G(x) = 1 - exp\Big(-\tau\Big\{exp\Big[(\frac{\alpha}{x})^{\beta}\Big] - 1\Big\}^{-\nu}\Big). \tag{1}$$

The corresponding probability density function is given as

$$\begin{aligned} g(x) =& \tau\nu\beta\alpha^{\beta}x^{-\beta-1}exp\Big[-\nu(\frac{\alpha}{x})^{\beta}\Big]\Big\{1 - exp\Big[-(\frac{\alpha}{x})^{\beta}\Big]\Big\}^{-\nu-1} \\ &\times exp\Big(-\tau\Big\{exp\Big[(\frac{\alpha}{x})^{\beta}\Big] - 1\Big\}^{-\nu}\Big). \end{aligned} \tag{2}$$

However, the pdf and cdf of a random variable X can be transmuted with a transmutation parameter $|\lambda| \leq 1$ as

$$f(x;\lambda) = g(x)\Big[1 + \lambda - 2\lambda G(x)\Big], \tag{3}$$

and

$$F(x;\lambda) = \Big(1 + \lambda\Big)G(x) - \lambda\Big[G(x)\Big]^2 \tag{4}$$

with $G(x)$ and $g(x)$ as the cdf and pdf of the baseline/parent distribution respectively.

This article is organized as follows: The introduction was given in section 1, Section 2 is the formulation of the transmuted Weibull Frechet distribution. Section 3 discussed the maximum likelihood of model parameters. In Section 4, we derived some properties of the TWFr distribution. Section 5 is the simulation study and real life application to validate the proposed model and Section 6 is the conclusion.

2. The Transmuted Weibull Frechet Distribution

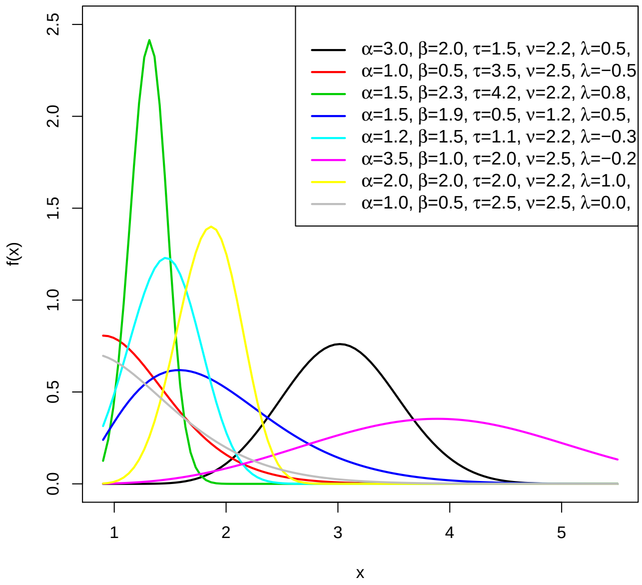
Let X be random variable. Then, the pdf of the TWFr is defined as

$$\begin{aligned} f_{TWFr}(x) =& \tau\nu\beta\alpha^{\beta}x^{-\beta-1}exp\Big[-\nu(\frac{\alpha}{x})^{\beta}\Big]\Big\{1 - exp\Big[-(\frac{\alpha}{x})^{\beta}\Big]\Big\}^{-\nu-1} \\ &\times exp\Big(-\tau\Big\{exp\Big[(\frac{\alpha}{x})^{\beta}\Big] - 1\Big\}^{-\nu}\Big) \\ &\times \Big[1 - \lambda + 2\lambda exp\Big(-\tau\Big\{exp\Big[(\frac{\alpha}{x})^{\beta}\Big] - 1\Big\}^{-\nu}\Big)\Big]. \end{aligned} \tag{5}$$

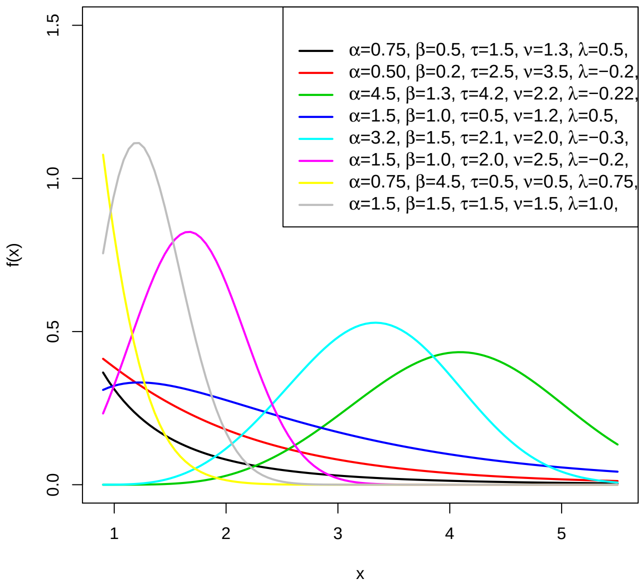
The corresponding cdf is given as

$$\begin{aligned} F_{TWFr}(x) =& \Big(1 + \lambda\Big)\Big(1 - exp\Big(-\tau\Big\{exp\Big[(\frac{\alpha}{x})^{\beta}\Big] - 1\Big\}^{-\nu}\Big)\Big) \\ &- \lambda\Big[1 - exp\Big(-\tau\Big\{exp\Big[(\frac{\alpha}{x})^{\beta}\Big] - 1\Big\}^{-\nu}\Big)\Big]^2. \end{aligned} \tag{6}$$

Figure 1 shows the plot of the pdf for TWFr distribution with different parameters values. The plot of the TWFr distribution shows that it could be increasing, decreasing and skewed to the right and left depending on the values of the parameters.



(a)



(b)

Figure 1 *The plots of the TWFr pdf for some parameter values*

The reliability or survival function (rf) of the random variable X is given as

$$\begin{aligned} rf_{TWFr}(x) =& exp\left(-\tau\left\{exp\left[\left(\frac{\alpha}{x}\right)^{\beta}\right]-1\right\}^{-\nu}\right)\left(1-\lambda\right) \\ &+\lambda\left(exp\left(-\tau\left\{exp\left[\left(\frac{\alpha}{x}\right)^{\beta}\right]-1\right\}^{-\nu}\right)\right)^{2}. \end{aligned} \tag{7}$$

Its hazard rate function (hrf) is given as

$$\begin{aligned} hr_{TWFr}(x) =&\tau\nu\beta\alpha^{\beta}x^{-\beta-1}exp\left[-\nu(\frac{\alpha}{x})^{\beta}\right]\left\{1-exp\left[-(\frac{\alpha}{x})^{\beta}\right]\right\}^{-\nu-1} \\ &\times exp\left(-\tau\left\{exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}^{-\nu}\right) \\ &\times \left[1-\lambda+2\lambda exp\left(-\tau\left\{exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}^{-\nu}\right)\right] \\ &\times \left\{exp\left(-\tau\left\{exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}^{-\nu}\right)\left(1-\lambda\right)+\lambda\left(exp\left(-\tau\left\{ \right.\right.\right.\right. \\ &\left.\left.\left.\left. exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}^{-\nu}\right)\right)^{2}\right\}^{-1}. \end{aligned} \tag{8}$$

Figure 2 shows the plot for the hazard rate function of the TWFr distribution. The plot shows that the TWFr model is decreasing and bathtub depending on the values of the associated parameters.

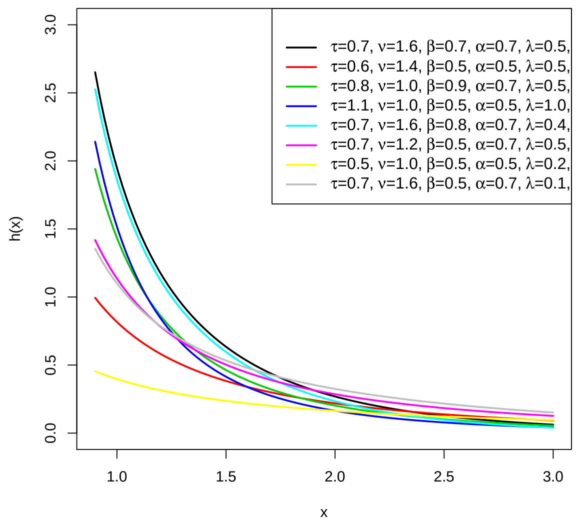


Figure 2 *The plots of the TWFr hrf for some parameter values*

The cumulative hazard rate function (chrf) of the TWFr model is given as

$$\begin{aligned} chrf_{TWFr}(x) =& -\ln\left\{exp\left(-\tau\left\{exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}^{-\nu}\right)\left(1-\lambda\right)\right. \\ &\left.+\lambda\left(exp\left(-\tau\left\{exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}^{-\nu}\right)\right)^{2}\right\}. \end{aligned} \tag{9}$$

3. Parameter Estimation of the Transmuted Weibull Frechet Distribution

Let X be random variable with TWFr distribution function. Then, the log-likelihood ℓ of the distribution for parameter vector $(\lambda, \beta, \tau, \nu, \alpha)^T$ is given as,

$$\ell = n\log(\tau\nu\beta\alpha^{\beta}) + s + m + z + p \tag{10}$$

$$\frac{\partial \ell}{\partial \lambda} = p'_{\lambda} = 0, \tag{11}$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{k'_{\alpha}}{k} - s'_{\alpha} + m'_{\alpha} - z'_{\alpha} + p'_{\alpha} = 0, \tag{12}$$

$$\frac{\partial \ell}{\partial \tau} = \frac{n}{\tau} - z'_{\tau} + p'_{\tau} = 0, \tag{13}$$

$$\frac{\partial \ell}{\partial \nu} = \frac{n}{\nu} - s'_{\nu} + m'_{\nu} - z'_{\nu} + p'_{\nu} = 0, \tag{14}$$

and

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \frac{k'_{\beta}}{k} - s'_{\beta} + m'_{\beta} - z'_{\beta} + p'_{\beta} = 0. \tag{15}$$

where $'$ denotes partial derivative and subscript the respective parameter and

$$k = \alpha^{\beta}; s = \sum_{i=1}^{n} \left[-\nu (\frac{\alpha}{x})^{\beta} \right]; m = \sum_{i=1}^{n} \log \left\{ 1 - exp \left[-(\frac{\alpha}{x})^{\beta} \right] \right\}^{-\nu-1};$$

$$z = \sum_{i=1}^{n} \left(-\tau \left\{ exp \left[(\frac{\alpha}{x})^{\beta} \right] - 1 \right\}^{-\nu} \right);$$

$$p = \sum_{i=1}^{n} \log \left[1 - \lambda + 2\lambda exp \left(-\tau \left\{ exp \left[(\frac{\alpha}{x})^{\beta} \right] - 1 \right\}^{-\nu} \right) \right].$$

The Equations (11), (12), (13), (14) and (15) are nonlinear and can not be easily obtained in closed form. Thus, the solutions to the parameter vector are obtained numerically using the Newton-Raphson algorithm and with various statistical and mathematical softwares like R, Mathematical, Maple and Matlab.

4. Some Statistical Properties of the Transmuted Weibull Frechet Distribution

This section investigates some statistical properties of the TWFr distribution. These include, quantile and random number generation and order statistics.

4.1. Quantile Function and Median

Let X be a random variable such that $X \sim TWFr(\alpha, \lambda, \beta, a, b)$. Then, the quantile function of X for $u \in (0, 1)$ is real solution of the following equation given as

$$Q(u) = F^{-1}(x). \tag{16}$$

Thus,

$$x_u = \alpha \left[\log \left\{ 1 + \left[(-\tau^{-1}) \log(1 - \phi(u)) \right]^{-\frac{1}{\nu}} \right\} \right]^{-\frac{1}{\beta}} \tag{17}$$

$$0 < u < 1$$

where

$$\phi(u) = \begin{cases} \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda}, & \text{if } \lambda < 0, \\ \frac{(1+\lambda) + \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda}, & \text{if } \lambda > 0, \\ u, & \text{otherwise } \lambda = 0. \end{cases}$$

By setting $u = 0.5$ in Equation (17), we have the median (M) of X as

$$M = \alpha\left[\log\left\{1+\left[(-\tau^{-1})\log(1-\phi(0.5))\right]^{-\frac{1}{\nu}}\right\}\right]^{-\frac{1}{\beta}}, \tag{18}$$

with

$$\phi(u) = \begin{cases} \frac{(1+\lambda)-\sqrt{(1+\lambda)^2-2\lambda}}{2\lambda}, & \text{if } \lambda < 0, \\ \frac{(1+\lambda)+\sqrt{(1+\lambda)^2-2\lambda}}{2\lambda}, & \text{if } \lambda > 0, \\ 0.5, & \text{otherwise } \lambda = 0. \end{cases}$$

However, the 25^{th} and 75^{th} percentile for the random variable X is obtained as

$$Q_1 = \alpha\left[\log\left\{1+\left[(-\tau^{-1})\log(1-\phi(0.25))\right]^{-\frac{1}{\nu}}\right\}\right]^{-\frac{1}{\beta}}, \tag{19}$$

with

$$\phi(u) = \begin{cases} \frac{(1+\lambda)-\sqrt{(1+\lambda)^2-\lambda}}{2\lambda}, & \text{if } \lambda < 0, \\ \frac{(1+\lambda)+\sqrt{(1+\lambda)^2-\lambda}}{2\lambda}, & \text{if } \lambda > 0, \\ 0.25, & \text{otherwise } \lambda = 0. \end{cases}$$

$$Q_3 = \alpha\left[\log\left\{1+\left[(-\tau^{-1})\log(1-\phi(0.75))\right]^{-\frac{1}{\nu}}\right\}\right]^{-\frac{1}{\beta}}, \tag{20}$$

with

$$\phi(u) = \begin{cases} \frac{(1+\lambda)-\sqrt{(1+\lambda)^2-3\lambda}}{2\lambda}, & \text{if } \lambda < 0, \\ \frac{(1+\lambda)+\sqrt{(1+\lambda)^2-3\lambda}}{2\lambda}, & \text{if } \lambda > 0, \\ 0.75, & \text{otherwise } \lambda = 0. \end{cases}$$

4.2. Reversed Hazard Function and Odds Functions

The reversed hazard function (rhf) of the TWFr distribution is given as

$$\begin{aligned} rhf_{TWFr}(x) =& \tau\nu\beta\alpha^{\beta}x^{-\beta-1}exp\left[-\nu(\frac{\alpha}{x})^{\beta}\right]\left\{1-exp\left[-(\frac{\alpha}{x})^{\beta}\right]\right\}^{-\nu-1} \\ & \times exp\left(-\tau\nu\left\{exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}\right) \\ & \times\left[1-\lambda+2\lambda exp\left(-\tau\nu\left\{exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}\right)\right] \\ & \times\left\{1+exp\left(-\tau\left\{exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}^{-\nu}\right)\left(\lambda-1\right)\right. \\ & \left.-\lambda\left(exp\left(-2\tau\left\{exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}^{-\nu}\right)\right)\right\}^{-1}. \end{aligned} \tag{21}$$

The Odd function that corresponds to the TWFr distribution is given as

$$\begin{aligned} O_{TWFr}(x) =&1+\Bigg\{exp\Bigg(-\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg]-1\Bigg\}^{-\nu}\Bigg)\Bigg(1-\lambda\Bigg) \\ &+\lambda\Bigg(exp\Bigg(-2\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg]-1\Bigg\}^{-\nu}\Bigg)\Bigg)\Bigg\}^{-1}. \end{aligned} \tag{22}$$

4.3. Distribution of the Order Statistics

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the $f_{TWFr}(x)$ distribution and $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$be the corresponding order statistics, Then, probability density function of the kth order statistics X_k, say $f_k(x)$ is given as

$$f_k(x) = \frac{n!}{(k-1)!(n-k)!}\Bigg[F_{TWFr}(x)\Bigg]^{k-1} f_{TWFr}(x)\Bigg[1-F_{TWFr}(x)\Bigg]^{n-k}. \tag{23}$$

$$-\infty < x < \infty.$$

On substituting into Equation (23), we have

$$\begin{aligned} f_k(x) =&\frac{n!}{(k-1)!(n-k)!} \\ &\times \tau\nu\beta\alpha^{\beta}x^{-\beta-1}exp\Bigg[-\nu(\frac{\alpha}{x})^{\beta}\Bigg]\Bigg\{1-exp\Bigg[-(\frac{\alpha}{x})^{\beta}\Bigg]\Bigg\}^{-\nu-1} \\ &\times exp\Bigg(-\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg]-1\Bigg\}^{-\nu}\Bigg) \\ &\times \Bigg[1-\lambda+2\lambda exp\Bigg(-\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg]-1\Bigg\}^{-\nu}\Bigg)\Bigg] \\ \times&\Bigg[\Bigg(1+\lambda\Bigg)\Bigg(1-exp\Bigg(-\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg]-1\Bigg\}^{-\nu}\Bigg)\Bigg) \\ &-\lambda\Bigg[1-exp\Bigg(-\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg]-1\Bigg\}^{-\nu}\Bigg)\Bigg]^2\Bigg]^{k-1} \\ &\times \Bigg[1-\Bigg(1+\lambda\Bigg)\Bigg(1-exp\Bigg(-\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg]-1\Bigg\}^{-\nu}\Bigg)\Bigg) \\ &-\lambda\Bigg[1-exp\Bigg(-\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg]-1\Bigg\}^{-\nu}\Bigg)\Bigg]^2\Bigg]^{n-k}. \end{aligned} \tag{24}$$

The minimum and maximum order statistics are obtained when $k = 1$ and $k = n$ respectively.

4.4. Simulation Study

A simulation is performed to examine the flexibility and efficiency of the TWFr distribution. Tables 1 and 2 show the simulation results for different values of parameters. The simulation was performed as follows:

- Data were generated using

$$x_u = \alpha\Bigg[\log\Bigg\{1+\Bigg[(-\tau^{-1})\log(1-\phi(u))\Bigg]^{-\frac{1}{\nu}}\Bigg\}\Bigg]^{-\frac{1}{\beta}} \quad 0 < u < 1.$$

with

$$\phi(u)=\begin{cases}\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda u}}{2\lambda}, & \text{if } \lambda<0,\\ \\ \frac{(1+\lambda)+\sqrt{(1+\lambda)^2-4\lambda u}}{2\lambda}, & \text{if } \lambda>0,\\ \\ u, & \text{otherwise } \lambda=0.\end{cases}$$

- The values of the parameters were purportedly set as $\alpha = 1.5$, $\beta = 0.5$,$\tau = 1.0$, $\nu = 1.0$ and $\lambda = 0.5$, $\alpha = 1.5$, $\beta = 0.5$,$\tau = 1.0$, $\nu = 1.0$ and $\lambda = -0.5$.
- The sample sizes were taken as $n = 10, 50, 150, 350, 400$ and 500.
- Each sample size was replicated 5000 times.

In the simulation study, we investigated the mean estimates (MEs), variance, biases and means squared errors (MSEs) of the MLEs.
The bias was calculated as (for $S = \alpha, \lambda, \beta, \tau, \nu$)

$$\hat{B}ias_S = \frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{S}_i - S\right).$$

Also, the MSE was obtained as

$$\hat{M}SE_S = \frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{S}_i - S\right)^2.$$

In Tables 1 and 2, the results of the Monte Carlo study show that the MSEs decay towards zero as the sample size increases which corroborates with the first-order asymptotic theory. The mean estimates of the TWFr distribution parameter estimates tend to the true parameter values as the sample size increases which also corroborates the fact that the asymptotic normal distribution provides an adequate approximation of the estimates.

5. Real-Life Applications

A breaking stress of carbon fibers and glass fiber real life datasets were used to examine the performance and flexibility of the model based on its test statistic. Several criteria were used to determine the distribution of the best fit: Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), and Hannan and Quinn Information Criteria (HQIC).

Table 1 *Simulation results for mean estimates, biases and root mean squared errors of* $\hat{\tau}, \hat{\nu}, \hat{\alpha}, \hat{\lambda} > 0$ *and* $\hat{\beta}$ *for the TWFr distribution.*

n	Parameter	ME	Bias	Variance	MSE
10	$\hat{\alpha}$	1.4627	-0.0373	0.1504	0.1518
	$\hat{\beta}$	0.6653	0.1653	0.0738	0.1011
	$\hat{\tau}$	1.0968	0.0968	0.1034	0.1128
	$\hat{\nu}$	0.9871	-0.0129	0.2505	0.2507
	$\hat{\lambda}$	0.3617	-0.1383	0.1493	0.1684
50	$\hat{\alpha}$	1.5383	0.0383	0.0618	0.0632
	$\hat{\beta}$	0.5326	0.0326	0.0167	0.0177
	$\hat{\tau}$	1.0924	0.0924	0.0485	0.0571
	$\hat{\nu}$	1.0037	0.0037	0.0924	0.0924
	$\hat{\lambda}$	0.4170	-0.0830	0.0690	0.0759
150	$\hat{\alpha}$	1.5540	0.0540	0.0279	0.0308
	$\hat{\beta}$	0.5068	0.0068	0.0050	0.0050
	$\hat{\tau}$	1.0529	0.0529	0.0240	0.0268
	$\hat{\nu}$	1.0012	0.0012	0.0322	0.0322
	$\hat{\lambda}$	0.4742	-0.0258	0.0417	0.0423
350	$\hat{\alpha}$	1.5528	0.0528	0.0142	0.0170
	$\hat{\beta}$	0.4983	-0.0017	0.0019	0.0019
	$\hat{\tau}$	1.0367	0.0367	0.0116	0.0129
	$\hat{\nu}$	1.0063	0.0063	0.0137	0.0137
	$\hat{\lambda}$	0.4960	-0.0040	0.0234	0.0234
400	$\hat{\alpha}$	1.5539	0.0539	0.0134	0.0163
	$\hat{\beta}$	0.4975	-0.0025	0.0016	0.0016
	$\hat{\tau}$	1.0334	0.0334	0.0102	0.0113
	$\hat{\nu}$	1.0072	0.0072	0.0115	0.0116
	$\hat{\lambda}$	0.5001	0.0001	0.0205	0.0205
500	$\hat{\alpha}$	1.5500	0.0500	0.0104	0.0129
	$\hat{\beta}$	0.4968	-0.0032	0.0013	0.0013
	$\hat{\tau}$	1.0276	0.0276	0.0079	0.0087
	$\hat{\nu}$	1.0071	0.0071	0.0091	0.0091
	$\hat{\lambda}$	0.5058	0.0058	0.0176	0.0176

The density functions considered include (for $x > 0$)

- Weibull Frechet: $f(x) = ab\beta\alpha^{\beta}x^{-\beta-1}exp\left[-b(\frac{\alpha}{x})^{\beta}\right]\left\{1-exp\left[-(\frac{\alpha}{x})^{\beta}\right]\right\}^{-b-1}$ $\times\, exp\left(-a\left\{exp[(\frac{\alpha}{x})^{\beta}]-1\right\}^{-b}\right)$;
- Exponentiated Frechect: $f(x) = \lambda\beta\alpha^{\beta}x^{-\beta-1}exp\left[-(\frac{\alpha}{x})^{\beta}\right]\left\{1-exp\left[-(\frac{\alpha}{x})^{\beta}\right]\right\}^{\lambda-1}$;
- Kumaraswamy Frechet: $f(x) = ab\beta\alpha^{\beta}x^{-\beta-1}exp\left[-a(\frac{\alpha}{x})^{\beta}\right]\left\{1-exp\left[-(\frac{\alpha}{x})^{\beta}\right]\right\}^{b-1}$;
- Beta Frechet: $f(x) = \frac{\beta\alpha^{\beta}x^{-\beta-1}}{B(a,b)}exp\left[-a(\frac{\alpha}{x})^{\beta}\right]\left\{1-exp\left[-(\frac{\alpha}{x})^{\beta}\right]\right\}^{b-1}$;
- Gamma Extended Frechet: $f(x) = \frac{a\beta\alpha^{\beta}x^{-\beta-1}}{\Gamma(b)}exp\left[-(\frac{\alpha}{x})^{\beta}\right]\left\{1-exp\left[-(\frac{\alpha}{x})^{\beta}\right]\right\}^{a-1}\Big(-log\{1-$

Table 2 *Simulation results for mean estimates, biases and root mean squared errors of $\hat{\tau}, \hat{\upsilon}, \hat{\alpha}, \hat{\lambda} < 0$ and $\hat{\beta}$ for the TWFr distribution.*

n	Parameter	ME	Bias	Variance	MSE
10	$\hat{\alpha}$	1.5887	0.0887	0.1859	0.1937
	$\hat{\beta}$	0.6626	0.1626	0.0532	0.0796
	$\hat{\tau}$	0.9814	-0.0186	0.1996	0.1999
	$\hat{\upsilon}$	0.9641	-0.0359	0.1531	0.1544
	$\hat{\lambda}$	-0.5970	-0.0970	0.7110	0.7204
50	$\hat{\alpha}$	1.5604	0.0604	0.0658	0.0694
	$\hat{\beta}$	0.5448	0.0448	0.0164	0.0184
	$\hat{\tau}$	0.9843	-0.0157	0.0538	0.0540
	$\hat{\upsilon}$	1.0009	0.0009	0.0683	0.0683
	$\hat{\lambda}$	-0.4541	0.0459	0.0995	0.1016
150	$\hat{\alpha}$	1.5587	0.0587	0.0284	0.0318
	$\hat{\beta}$	0.5117	0.0117	0.0060	0.0062
	$\hat{\tau}$	1.0013	0.0013	0.0224	0.0224
	$\hat{\upsilon}$	1.0134	0.0134	0.0272	0.0274
	$\hat{\lambda}$	-0.4521	0.0479	0.0363	0.0386
350	$\hat{\alpha}$	1.5561	0.0561	0.0146	0.0178
	$\hat{\beta}$	0.5010	0.0010	0.0024	0.0024
	$\hat{\tau}$	1.0100	0.0100	0.0100	0.0101
	$\hat{\upsilon}$	1.0158	0.0158	0.0114	0.0116
	$\hat{\lambda}$	-0.4636	0.0364	0.0168	0.0181
400	$\hat{\alpha}$	1.5582	0.0582	0.0130	0.0164
	$\hat{\beta}$	0.4998	-0.0002	0.0021	0.0021
	$\hat{\tau}$	1.0110	0.0110	0.0089	0.0091
	$\hat{\upsilon}$	1.0166	0.0166	0.0100	0.0103
	$\hat{\lambda}$	-0.4642	0.0358	0.0160	0.0173
500	$\hat{\alpha}$	1.5563	0.0563	0.0102	0.0134
	$\hat{\beta}$	0.4991	-0.0009	0.0017	0.0017
	$\hat{\tau}$	1.0109	0.0109	0.0074	0.0075
	$\hat{\upsilon}$	1.0159	0.0159	0.0078	0.0081
	$\hat{\lambda}$	-0.4650	0.0350	0.0129	0.0141

$exp\left[-(\frac{\alpha}{x})^{\beta}\right]\Big\}^{a}\Big)^{b-1};$

- Transmuted Frechet: $f(x) = \beta\alpha^{\beta}x^{-\beta-1}exp\left[-(\frac{\alpha}{x})^{\beta}\right]\left\{1+\lambda-2\lambda exp\left[-(\frac{\alpha}{x})^{\beta}\right]\right\};$
- Frechet: $f(x) = \lambda\alpha^{\lambda}x^{-\lambda-1}exp\left[-(\frac{\alpha}{x})^{\lambda}\right];$
- Alpha Power Inverse Weibull: $f(x) = \frac{\log(\alpha)}{(\alpha-1)}\lambda\beta exp(-\lambda x^{-\beta})\alpha^{exp(-\lambda x^{-\beta})};$
- Transmuted Rayleigh: $f(x) = \frac{x}{\alpha^2}exp\left(-\frac{x^2}{2\alpha^2}\right)\left(1-\beta+2\beta exp\left(-\frac{x^2}{2\alpha^2}\right)\right).$

5.1. Breaking Stress of Carbon fibres

The first data consist of 100 breaking stress of carbon fibres as used in [19] and [5]. It consists of 100 observations taken on breaking stress of carbon fibers (in Gba). Table 3 shows the test statistics. The dataset are as follow:

2.81, 2.77, 2.17, 2.83, 1.92, 1.41, ,1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 3.11,4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53,2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59,2.38, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69,1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12,1.89, 2.88, 3.68, 2.97, 1.36,3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19 2.82, 2.05, 3.65.

Table 3 *Performance rating of the TWFr distribution with breaking stress of carbon fibres dataset*

Model	Parameter MLEs(Std. Errors)	AIC	CAIC	BIC	HQIC	-2ℓ
Transmuted Weibull Frechét	$\hat{\tau} = 86.6226(41.4818)$ $\hat{\upsilon} = 0.4167(0.2361)$ $\hat{\alpha} = 3.7977(1.3096)$ $\hat{\lambda} = 0.6301(0.1704)$ $\hat{\beta} = 0.4463(0.1260)$	**292.0566**	**292.4776**	**302.4773**	**296.2740**	**282.0566**
Weibull Frechét	$\hat{\alpha} = 0.6942(0.363)$ $\hat{\beta} = 0.6178(0.284)$ $\hat{a} = 0.0947(0.456)$ $\hat{b} = 3.5178(2.942)$	294.6000	295.0211	305.0207	298.8174	286.6000
Exponentiated Frechect	$\hat{\alpha} = 69.1489(57.349)$ $\hat{\beta} = 0.5019(0.0800)$ $\hat{\lambda} = 145.3275(122.824)$	295.7000	295.8237	300.9103	297.8087	291.7000
Kumaraswamy Frechét	$\hat{\alpha} = 2.0556(0.0710)$ $\hat{\beta} = 0.4654(0.0070)$ $\hat{a} = 6.2815(0.0630)$ $\hat{b} = 224.1800(0.1640)$	297.1000	297.5211	307.5207	301.3174	289.1000
Beta Frechét	$\hat{\alpha} = 1.6097(2.4980)$ $\hat{\beta} = 0.4046(0.1080)$ $\hat{a} = 22.0143(21.432)$ $\hat{b} = 29.7617(17.4790)$	311.1000	311.5211	321.5207	315.3174	303.1000
Gamma Extended Frechét	$\hat{\alpha} = 1.3692(1.3692)$ $\hat{\beta} = 0.4776(0.1330)$ $\hat{a} = 27.6452(14.1360)$ $\hat{b} = 17.4581(14.8180)$	312.0000	312.4211	322.4207	316.2174	304.0000
Alpha Power Inverted Weibull	$\hat{\alpha} = 109.8227(75.5562)$ $\hat{\beta} = 1.1138(0.2018)$ $\hat{\lambda} = 2.2803(0.1420)$	328.4842	328.7342	336.2997	331.6473	322.4842
Transmuted Frechét	$\hat{\alpha} = 1.9315(0.0971)$ $\hat{\beta} = 1.7435(0.0760)$ $\hat{\lambda} = 0.0819(0.1980)$	350.5000	350.7500	358.3155	353.6631	344.5
Frechét	$\hat{\alpha} = 1.8705(0.1120)$ $\hat{\lambda} = 1.7766(0.113)$	348.3000	348.4237	353.5103	350.4087	344.3000

Figures 3 and 4 show the empirical pdf and cdf for the breaking stress of carbon for the TWFr model.

5.2. Glass fibres data

The second data consist of 1.5 cm strengths of glass fibres obtained at the UK National Physical Laboratory. The data were used to compare the performance of the *TWFr* distribution as used in [25], [11], [3], [15], [24], [21], [5], [6], [4], [7], [22], [8], [9], [10], [20] and [28]. The observations are as follows:

1.53, 1.54, 1.55, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.61, 1.58, 1.59, 1.60, 1.61,0.55, 0.74,1.50, 1.50, 1.55, 1.52, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67,.

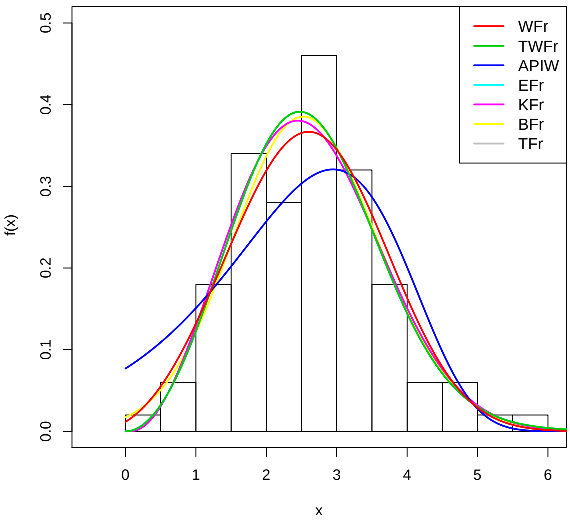


Figure 3 *The empirical cdfs of the TWFr density for the breaking stress of carbon*

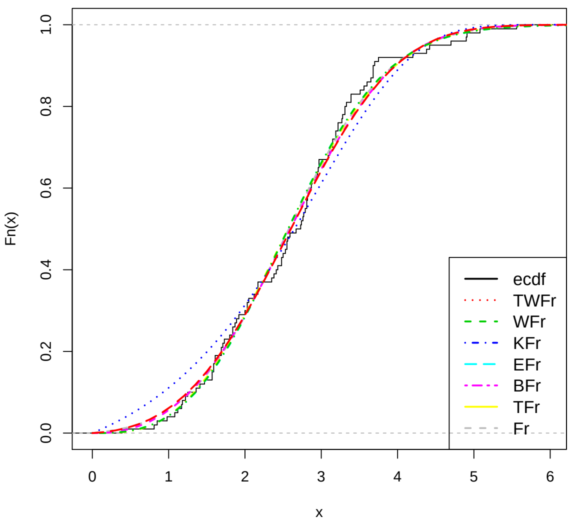


Figure 4 *The empirical pdfs of the TWFr density for the breaking stress of carbon*

The descriptive statistics of the glass fibers dataset are showed in Table 4. Table 5 shows the measure of comparison for the various distribution under consideration.

Table 4 *Descriptive statistics for the glass fibres dataset to 2 decimal points*

Mean	Median	Mode	St.D	IQR	Variance	Skewness	Kurtosis	$25^{th}P.$	$75^{th}P.$
1.51	1.59	1.61	0.32	0.31	0.11	-0.81	0.80	1.38	1.69

Figures 5 and 6 show the empirical pdf and cdf for the glass fiber data for the TWFr model.

Table 5 *Performance rating of the TWFr distribution with glass fibres dataset*

Model	Parameter MLEs(Std. Errors)	AIC	CAIC	BIC	HQIC	-2ℓ
Transmuted Weibull Frechét	$\hat{\tau} = 0.5739(0.1877)$ $\hat{\nu} = 3.9560(2.2162)$ $\hat{\lambda} = 0.0300(0.0135)$ $\hat{\alpha} = 5.8185(4.2645)$ $\hat{\beta} = 1.2345(0.1734)$	**35.7214**	**36.7740**	**46.4370**	**39.9359**	**25.7214**
Weibull Frechét	$\hat{\alpha} = 0.3865(0.7990)$ $\hat{\beta} = 0.2436(0.2850)$ $\hat{a} = 1.4762(4.782)$ $\hat{b} = 16.8561(20.4850)$	39.0000	39.6896	47.5725	42.3716	31.0000
Beta Frechét	$\hat{\alpha} = 2.0518(0.9886)$ $\hat{\beta} = 0.6466(0.1630)$ $\hat{a} = 15.0756(12.057)$ $\hat{b} = 36.9397(22.649)$	68.6261	69.3157	77.1986	71.9977	69.300
Gamma Extended Frechét	$\hat{\alpha} = 1.6625(0.9520)$ $\hat{\beta} = 0.7421(0.197)$ $\hat{a} = 32.1120(17.3970)$ $\hat{b} = 13.2688(9.967)$	69.6237	70.3016	78.1098	72.9007	61.4503
Alpha Power Inverted Weibull	$\hat{\alpha} = 61.10(48.14)$ $\hat{\beta} = 0.78(0.16)$ $\hat{\lambda} = 3.80(0.30)$	82.5800	82.9900	89.0100	85.1100	76.5848
Frechét	$\hat{\alpha} = 1.2640(0.0589)$ $\hat{\lambda} = 2.8879(0.2340)$	97.7105	97.9045	102.0078	99.3560	93.6980
Transmuted Frechét	$\hat{\alpha} = 1.3068(0.034)$ $\hat{\beta} = 2.7898(0.1648)$ $\hat{\lambda} = 0.1298(0.2080)$	100.1009	100.5078	106.4897	102.5908	94.0893
Transmuted Rayleigh	$\hat{\alpha} = 1.0895(1.1e-08)$ $\hat{\beta} = 1.0e-10(1.7e-12)$	103.5818	103.7820	107.8680	105.2676	99.5818
Alpha Power Inverted Exponential	$\hat{\alpha} = 83.4497(79.2814)$ $\hat{\lambda} = 0.3137(0.0774)$	196.3253	196.5253	200.6116	198.0111	191.4580

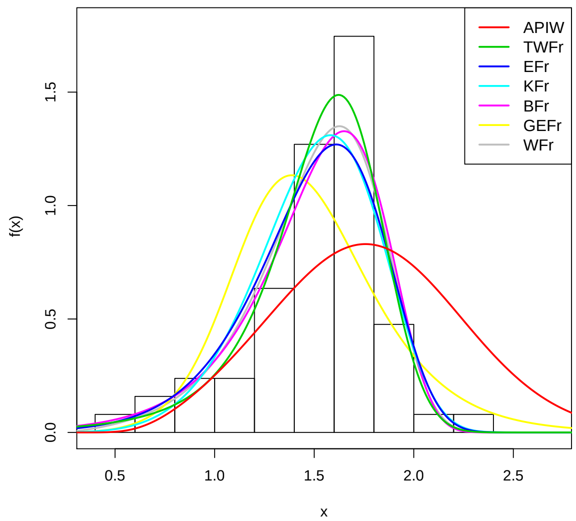


Figure 5 *The empirical cdfs of the TWFr density for the glass fiber data*

5.3. Discussion

The performance of a model is determined by the value that corresponds to the lowest Akaike Information Criteria (AIC) or the highest Log-likelihood value is regarded as the *best* model. In

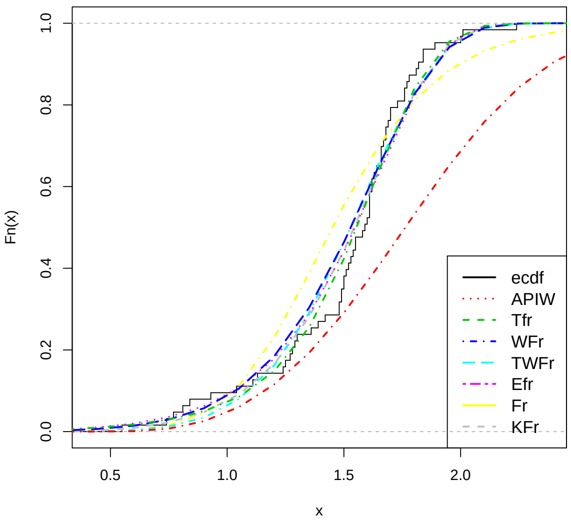


Figure 6 *The empirical pdfs of the TWFr density for the glass fiber data*

the two real life cases considered, the *TWFr* distribution has the lowest AIC values.

6. Conclusion

The Transmuted Weibull Frechet distribution has been successfully derived. Its expressions for the basic statistical properties which include the order statistics distribution, cumulative hazard function, reversed hazard function, quantile, median, hazard function, odds function have been successfully established. The shape of the distribution could be increasing (depending on the value of the parameters). An application of real life data shows that the TWFr distribution is a better competitor for some other families of distributions.

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