Bayesian and Non-Bayesian Inference of Exponentiated Moment Exponential Distribution with Progressive Censored Samples

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Abstract

In this paper, a progressive type-II censoring strategy is used to estimate the parameters, reliability and hazard rate functions of the exponentiated moment exponential distribution. The maximum likelihood and Bayesian techniques have been used to estimate the proposed estimators. Gamma (informative) and uniform (non-informative) priors are taken into account under the squared error loss function to produce the Bayesian estimators. The highest posterior density interval estimations and the 95% approximate confidence intervals along with coverage probability are calculated. In order to evaluate the effectiveness of estimates produced by the Metropolis-Hastings sampling algorithms, we provide a numerical research. According to the study's findings, the Bayes estimates under informative priors are typically more accurate than other estimates.

Key Words: Exponentiated moment exponential, gamma prior, credible interval, Metropolis-Hastings, Progressive censorning

1. Introduction

Censoring is widely used in reliability data analysis and other practical life-testing investigations. It becomes apparent when precise failure times for a subset of the test units used in an experiment are observed. The experimenter frequently runs into incomplete data in this scenario. Typical censoring systems include type I censoring (T1C) and type II censoring (T2C). The units can only be expelled after the conclusion of the experiment, which is a major drawback of TIC and T2C methods. In a more open-ended censoring technique known as progressive censoring (PC), units are designated to be discarded from the test at times other than the eventual termination time point. The remaining units are then tested again while being observed. To learn more, visit Balakrishnan [1].

Progressive T2C (PT2C) is the major topic of this research project. Let's assume that n identical items are used in the experiment, and that the PC scheme R is pre-fixed so that, after the first failure, R_1 surviving items are ejected from remaining live $(n-1)$ items, R_2 surviving items are ejected from remaining live $(n-R_1-2)$ items, and so on. After m^{th} failure, this procedure is maintained until all $R_m = n-m-R_1-...R_{m-1}$ remaining objects are expelled (see Hofmann et al [2]). Therefore, a PT2C procedure consists of m and $R_1, R_2, ..., R_m$ such that $\sum_{i=1}^{m} R_i + m = n$. Note that, if $R_1 = R_2 = \ldots = R_m = 0$ then the PT2C provides complete sampling and if $R_1 = R_2 = \ldots = R_{m-1} = 0$ and $R_m = n-m$ then PT2C yields T2C scheme (see Krishna and Kumar [3]).

According to PT2C samples, the likelihood function of random variable X (Balakrishnan and Aggarwala [4]) is supplied as follows .

$$L(\theta) = C\prod_{i=1}^{m} f(x_{(i)})\left[1-F(x_{(i)})\right]^{R_i}, \tag{1}$$

where $C = n(n-R_1-1)\ldots(n-R_1-R_2-\cdots-R_{m-1}-m+1)$. Some important literature regarding the estimation studies under PT2C scheme can be found in Wu [5] , Ng [6], Dey et al. [7], Hassan et al. [8], EL-Sagheer [9], Noor et al. [10], Alshenawy et al. [11], and Shrahili et al. [12].

Moment distributions are essential in probability theory and several economic, reliability, and biological studies, as well as other areas of mathematics and statistics. Some of the fundamental features of the moment exponential (ME) distribution were studied and suggested by Dara and Ahmad [13]. The version of the ME distribution that includes an additional shape parameter is known as the exponentiated ME (EME) distribution, and it is frequently employed in reliability research. Hasnain et al. [14] suggested several EME distribution features, including conditional-based characterisation, explored maximum likelihood (ML) estimators, and fitted it to actual data sets. Compared to the ME distribution and exponentiated exponential (EE) distribution, the EME distribution is more adaptable when fitting data. As described by Hasnain et al. [14] the EME distribution's cumulative distribution function (CDF), is

$$F(x;\alpha,\beta) = \left[1-\psi\right]^{\alpha}; \quad x,\beta,\alpha > 0, \tag{2}$$

where $\psi = (1+x\beta^{-1})e^{-x/\beta}$, β is scale parameter and α is shape parameter. The probability density function (PDF) of the EME distribution is

$$f(x;\alpha,\beta) = \alpha\beta^{-2}\left[1-\psi\right]^{\alpha-1} xe^{-x/\beta}; \qquad x,\beta,\alpha > 0. \tag{3}$$

For $\beta = 1$, the CDF (2) gives the CDF of one parameter EE distribution (Gupta and Kundu [15]). Also, for $\alpha = 1$, the CDF (2) gives the CDF of ME distribution. The reliability function (RF) and hazard rate function (HRF) related to (3) are defined as:

$$R(x) = 1-\left[1-\psi\right]^{\alpha}, \qquad h(x) = \frac{\alpha xe^{-x/\beta}\left[1-\psi\right]^{\alpha-1}}{\beta^2\left(1-\left[1-\psi\right]^{\alpha}\right)}.$$

Plots of the PDF and HRF of the EME distribution are displayed in Figure 1. It is evident that different parameter values result in varied forms for the PDF for the EME distribution. The distribution may alternatively be characterised as favourably skewed to right and uni-modal. It is clear that the EME distribution's HRF has an increasing trend.

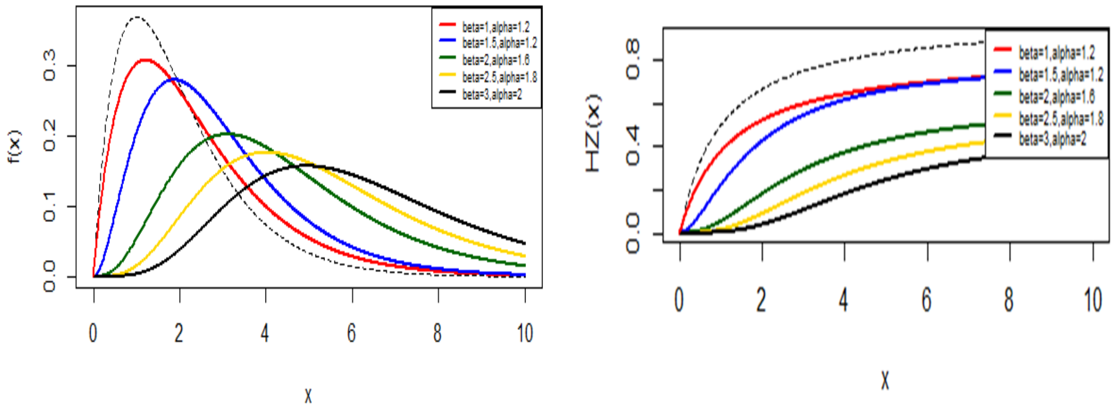


Figure 1: PDF and HRF plots of the EME distribution

Different approaches to estimating the PDF and CDF of the EME distribution were provided by Tripathi et al. [16]. The ML and Bayesian techniques developed by Fatima and Ahmad [17] have been taken into consideration when discussing the parameter estimators of the EME distribution. Akhter et al. [18] provided explicit algebraic equations that are generated from the EME distribution for both single and product moments of order statistics. Additionally, they used a full sample as well as a T2C sample to identify the best linear unbiased estimators based on these moments. Some generalizations of EME distribution may be found in Iqbal et al. [19], Ahmadini et al. [20] and Shrahili et al. [21].

The RF, HRF, and parameter estimators of the EME utilising ML and Bayesian techniques are addressed in the current study. Both the Bayesian credible intervals (BCIs) and the approximate confidence intervals (ACIs) are built using the PT2C data. This document can be constructed as shown below. Section 2 deals with ML estimators and the ACIs of parameters, RF and HRF. Sections 3 explore Bayesian estimate under informative (IF) and non-informative (NIF) priors. Sections 4 and 5, respectively, provide numerical illustrative studies and a conclusion.

2. Maximum Likelihood Procedure

Here, using PT2C data, we obtain the ML estimators of the parameters, RF, and HRF of the EME distribution. In addition, the ACIs for the RF, HRF and the parameters β and α are built.

Let $x_{(1)}, x_{(2)}, ..., x_{(m)}$ be the observed PT2C random samples extracted from the EME distribution. Based on (1), then the likelihood function of the EME distribution takes the following form:

$$L(\underline{x}|\beta,\alpha) \propto \prod_{i=1}^{m} \alpha\beta^{-2}\left(1-\psi_i\right)^{\alpha-1} x_i e^{-x_i/\beta}\left[1-\left(1-\psi_i\right)^{\alpha}\right]^{R_i}, \tag{4}$$

where ψ_i=$(1+x_i\beta^{-1})e^{-x_i/\beta}$ and we write $x_{(i)}=x_i$ for simplified form. The logarithm of (4), say ℓ= $\log L(\underline{x}|\beta,\alpha)$ becomes:

$$\ell \propto m\ln\alpha - 2m\ln\beta + \sum_{i=1}^{m}\ln x_i - \frac{1}{\beta}\sum_{i=1}^{m} x_i + (\alpha-1)\sum_{i=1}^{m}\ln\left(1-\psi_i\right) + \sum_{i=1}^{m} R_i \ln\left[1-\left(1-\psi_i\right)^{\alpha}\right]. \tag{5}$$

The first derivative of (5) via β and α are given as:

$$\frac{\partial\ell}{\partial\beta} = \frac{-2m}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{m} x_i - (\alpha-1)\sum_{i=1}^{m}\frac{\varpi_i}{(1-\psi_i)} + \alpha\sum_{i=1}^{m} R_i \frac{\left(1-\psi_i\right)^{\alpha-1} x_i\,(\psi_i - e^{-x_i/\beta})}{\beta^2\left[1-\left(1-\psi_i\right)^{\alpha}\right]},$$

$$\frac{\partial\ell}{\partial\alpha}=\frac{m}{\alpha}+\sum_{i=1}^{m}\ln\left(1-\psi_i\right)-\sum_{i=1}^{m}R_i\frac{\left(1-\psi_i\right)^{\alpha}\ln\left(1-\psi_i\right)}{1-\left(1-\psi_i\right)^{\alpha}},$$

where $\varpi_i=x_i\beta^{-2}(\psi_i-e^{-x_i/\beta})$. The estimator of β and α is the solution of the first derivative of $\partial\ell/\partial\beta\big|_{\beta=\hat{\beta}}=0$ and $\partial\ell/\partial\alpha\big|_{\alpha=\hat{\alpha}}=0$. Numerical iterative approach may be used to calculate the estimator of β and α for the specified values of ($m,\underline{R},\underline{x}$). Additionally, the invariance feature of the ML method is used to evaluate $R(x)$ and $h(x)$ as below

$$\hat{R}(x)=1-\left[1-\psi\right]^{\hat{\alpha}},\qquad \hat{h}(x)=\alpha x\hat{\beta}^{-2}e^{-x_i/\hat{\beta}}\left[1-\psi\ \right]^{\hat{\alpha}-1}\left(1-\left[1-\psi\right]^{\hat{\alpha}}\right)^{-1}.$$

In addition, we get the observed information matrix, say $I(\hat{\alpha},\hat{\beta})$, to build ACIs. The multivariate normal distribution $N_2(0,I^{-1}(\hat{\alpha},\hat{\beta}))$ is used to create ACIs for the parameters β and α with the usual regularity requirements. Based on the asymptotic normality criteria of the ML, the two-sided $100(1-\varepsilon)\%$ACI for parameter β and α is

$$\text{AsyCI\_Upper}=\hat{\beta}+Z_{\varepsilon/2}\sqrt{\text{var}(\hat{\beta})}\text{ and }\hat{\alpha}+Z_{\varepsilon/2}\sqrt{\text{var}(\hat{\alpha})},$$
$$\text{AsyCI\_Lower}=\hat{\beta}-Z_{\varepsilon/2}\sqrt{\text{var}(\hat{\beta})}\text{ and }\hat{\alpha}-Z_{\varepsilon/2}\sqrt{\text{var}(\hat{\alpha})},$$
$$\text{AIL}=\text{AsyCI\_Upper}-\text{AsyCI\_Lower},$$

where $Z_{\varepsilon/2}$ is the right tail probability's percentile $\varepsilon/2$ for the standard normal distribution. Once more, an R-based numerical method is offered to get the variance-covariance matrix. Also, the $100(1-\varepsilon)\%$ACI for $R(x)$ and $h(x)$ are given by $\hat{R}(x)\ \pm Z_{\varepsilon/2}\sqrt{\tilde{\text{var}}(\hat{R}(x))}$, $\hat{h}(x)\ \pm Z_{\varepsilon/2}\sqrt{\tilde{\text{var}}(\hat{h}(x))}$.

3. Bayesian Estimators

Here, Bayesian estimator of the parameters, RF and HRF of the EME distribution in case of IF and NIF priors under squared error (SE) loss function. Firstly, consider β and α have a gamma distribution with parameters (*a, b*) and (*c, d*) respectively. Assuming that β and α are independently distributed, the joint prior distribution of β and α is given by:

$$h_{1,2}(\beta,\alpha|\underline{x})=\frac{b^a d^c}{\Gamma(a)\Gamma(c)}\beta^{a-1}\alpha^{c-1}e^{-b\beta-d\alpha},$$

where *a*, *b*, *c* and *d* are chosen to reflect the prior knowledge about the unknown parameters (the criteria to select the hyper-parameter values is discussed in Section 3.1). The joint posterior distribution of parameters β and α is defined as:

$$\pi_2(\beta,\alpha|\underline{x})=\frac{\beta^{a-3}\alpha^{c}e^{-b\beta-d\alpha}\prod_{i=1}^{m}(1-\psi_i)^{\alpha-1}x_ie^{-x_i/\beta}\left\{1-(1-\psi_i)^{\alpha}\right\}^{R_i}}{\int_0^\infty\int_0^\infty\beta^{a-3}\alpha^{c}e^{-b\beta-d\alpha}\prod_{i=1}^{m}(1-\psi_i)^{\alpha-1}x_ie^{-x_i/\beta}\left\{1-(1-\psi_i)^{\alpha}\right\}^{R_i}d\beta\,d\alpha}.$$

Hence, the marginal posterior distributions of β and α take the following forms:

$$h_1(\beta|x) = k_1^{-1}\ \beta^{a-3}e^{-b\beta}\int_0^\infty \alpha^c e^{-d\alpha}\prod_{i=1}^m (1-\psi_i)^{\alpha-1} x_i e^{-x_i/\beta}\left\{1-(1-\psi_i)^\alpha\right\}^{R_i}\ d\alpha\ ,$$

$$h_2(\alpha|\underline{x}) = k_1^{-1}\ \alpha^c e^{-d\alpha}\int_0^\infty \beta^{a-3}e^{-b\beta}\prod_{i=1}^m (1-\psi_i)^{\alpha-1} x_i e^{-x_i/\beta}\left\{1-(1-\psi_i)^\alpha\right\}^{R_i}\ d\beta,$$

where $$k_1 = \int_0^\infty\int_0^\infty \beta^{a-3}\alpha^c e^{-b\beta-d\alpha}\prod_{i=1}^m (1-\psi_i)^{\alpha-1} x_i e^{-x_i/\beta}\left\{1-(1-\psi_i)^\alpha\right\}^{R_i}\ d\beta\ d\alpha.$$

The Bayesian estimator of β and α, expressed by $\tilde{\beta}$ and $\tilde{\alpha}$, are obtained as follows:

$$\tilde{\beta} = k_1^{-1}\int_0^\infty\int_0^\infty \beta^{a-2}\alpha^c e^{-b\beta-d\alpha}\prod_{i=1}^m (1-\psi_i)^{\alpha-1} x_i e^{-x_i/\beta}\left\{1-(1-\psi_i)^\alpha\right\}^{R_i}\ d\beta\ d\alpha\ ,$$

$$\tilde{\alpha} = k_1^{-1}\int_0^\infty\int_0^\infty \beta^{a-3}\alpha^{c+1} e^{-b\beta-d\alpha}\prod_{i=1}^m (1-\psi_i)^{\alpha-1} x_i e^{-x_i/\beta}\left\{1-(1-\psi_i)^\alpha\right\}^{R_i}\ d\beta\ d\alpha.$$

The Bayesian estimators of $R(x)$ and $h(x)$ are given by:

$$\tilde{R}(x) = k_1^{-1}\int_0^\infty\int_0^\infty\left(1-\left[1-(1+x\beta^{-1})e^{-x/\beta}\right]^\alpha\right)\beta^{a-3}\alpha^c e^{-b\beta-d\alpha}\prod_{i=1}^m (1-\psi_i)^{\alpha-1} x_i e^{-x_i/\beta}\left\{1-(1-\psi_i)^\alpha\right\}^{R_i} d\alpha\, d\beta, \tag{6}$$

$$\tilde{h}(x) = k_1^{-1}\int_0^\infty\int_0^\infty \frac{xe^{-x/\beta}\left[1-(1+x\beta^{-1})\,e^{-x/\beta}\right]^{\alpha-1}}{\left(1-\left[1-(1+x\beta^{-1})e^{-x/\beta}\right]^\alpha\right)}\ \beta^{a-5}\alpha^{c+1}e^{-b\beta-d\alpha}\prod_{i=1}^m (1-\psi_i)^{\alpha-1} x_i e^{-x_i/\beta}\left\{1-(1-\psi_i)^\alpha\right\}^{R_i} d\alpha\, d\beta. \tag{7}$$

The above Bayesian estimators $\tilde{\beta},\tilde{\alpha},\tilde{R}(x)$ and $\tilde{h}(x)$ are not in closed forms but can be evaluated numerically for the given values of $a, b, c, d, n, m, \underline{x}$ and $\underline{R}$.
Secondly, assuming the prior of parameters β and α, denoted by $g_1(\beta)$ and $g_2(\alpha)$ has the uniform (NIF) prior distribution. The joint prior for parameters β and α, represented by $g_{1,2}(\alpha,\beta)$, assuming independent of priors, is

$$g_{1,2}(\alpha,\beta|\underline{x}) = (\alpha\beta)^{-1}, \quad 0<\beta,\ \alpha<1.$$

The joint posterior density of β and α given the data $\underline{x}$ is given by:

$$\pi_1(\beta,\alpha|\underline{x}) = k_2^{-1}\beta^{-3}\prod_{i=1}^m (1-\psi_i)^{\alpha-1} x_i e^{-x_i/\beta}\left\{1-(1-\psi_i)^\alpha\right\}^{R_i}.$$

where $$k_2 = \int_0^\infty\int_0^\infty \beta^{-3}\prod_{i=1}^m (1-\psi_i)^{\alpha-1} x_i e^{-x_i/\beta}\left\{1-(1-\psi_i)^\alpha\right\}^{R_i}\ d\beta\ d\alpha.$$

Hence, the marginal posterior distributions of β and α take the following forms:

$$g_1(\beta|\underline{x}) = \int_\alpha \pi_1(\beta,\alpha|\underline{x})\ d\alpha = k_2^{-1}\beta^{-3}\int_0^\infty\prod_{i=1}^m (1-\psi_i)^{\alpha-1}\left\{1-(1-\psi_i)^\alpha\right\}^{R_i}\ \ x_i e^{-x_i/\beta} d\alpha,$$

$$g_2(\alpha|\underline{x}) = \int_\beta \pi_1(\beta,\alpha|\underline{x})\ d\beta = k_2^{-1}\int_0^\infty \beta^{-3}\prod_{i=1}^m (1-\psi_i)^{\alpha-1}\left\{1-(1-\psi_i)^\alpha\right\}^{R_i}\ x_i e^{-x_i/\beta}\ d\beta,$$

The Bayesian estimator of β and α, denoted by $\ddot{\beta}$ and $\ddot{\alpha}$, are obtained as follows:

$$\ddot{\beta} = E(\beta|\underline{x}) = k_2^{-1}\int_0^\infty\int_0^\infty \beta^{-2}\prod_{i=1}^{m}(1-\psi_i)^{\alpha-1}x_i e^{-x_i/\beta}\left\{1-(1-\psi_i)^{\alpha}\right\}^{R_i}\,d\beta\,d\alpha, \quad (8)$$

$$\ddot{\alpha} = E(\alpha|\underline{x}) = k_2^{-1}\int_0^\infty\int_0^\infty \alpha\beta^{-3}\prod_{i=1}^{m}(1-\psi_i)^{\alpha-1}x_i e^{-x_i/\beta}\left\{1-(1-\psi_i)^{\alpha}\right\}^{R_i}\,d\beta\,d\alpha. \quad (9)$$

The Bayesian estimator of $R(x)$ and $h(x)$ are given by:

$$\ddot{R}(x) = \int_0^\infty\int_0^\infty\left(1-\left[1-(1+x\beta^{-1})\,e^{-x_i/\beta}\right]^{\alpha}\right)\beta^{-3}\prod_{i=1}^{m}(1-\psi_i)^{\alpha-1}x_i e^{-x_i/\beta}\left\{1-(1-\psi_i)^{\alpha}\right\}^{R_i}d\alpha\,d\beta, \quad (10)$$

$$\ddot{h}(x) = k_2^{-1}\int_0^\infty\int_0^\infty\frac{\alpha x e^{-x/\beta}\left[1-(1+x\beta^{-1})\,e^{-x_i/\beta}\right]^{\alpha-1}}{\beta^2\left(1-\left[1-(1+x\beta^{-1})e^{-x_i/\beta}\right]^{\alpha}\right)}\beta^{-3}\prod_{i=1}^{m}(1-\psi_i)^{\alpha-1}x_i e^{-x_i/\beta}\left\{1-(1-\psi_i)^{\alpha}\right\}^{R_i}\,d\alpha\,d\beta. \quad (11)$$

The above Bayes estimates $\ddot{\beta},\ddot{\alpha},\ddot{R}(x)$ and $\ddot{h}(x)$ are assessed numerically for the given values of $n,m,\underline{x}$ and $\underline{R}$. Integrals (8)–(11) are very hard to be solved analytically, so the Metropolis-Hastings (MH) algorithm will be used to solve these integrals.

3.1 Hyper-Parameter Elicitation

This sub-section handled the elicitation of the hyper-parameter values in case of IP. These hyper-parameters of IP are obtained from ML estimators for β and α, by equating the mean and variance of $\hat{\beta}^i$ and $\hat{\alpha}^i$ with the mean and variance of the gamma distributions, where i=1,2,….,N and N is the number of samples available from the EME distribution. Thus,

$$\frac{1}{N}\sum_{i=1}^{N}\hat{\beta}^i = \frac{a}{b},\quad \frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{\beta}^i-\frac{1}{N}\sum_{i=1}^{N}\hat{\beta}^i\right)^2 = \frac{a}{b^2},\ \frac{1}{N}\sum_{i=1}^{N}\hat{\alpha}^i = \frac{c}{d},\quad \frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{\alpha}^i-\frac{1}{N}\sum_{i=1}^{N}\hat{\alpha}^i\right)^2 = \frac{c}{d^2}.$$

Hence, the estimated hyper-parameters are obtained as follows
:

$$a = \frac{\left(\frac{1}{N}\sum_{i=1}^{N}\hat{\beta}^i\right)^2}{\frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{\beta}^i-\frac{1}{N}\sum_{i=1}^{N}\hat{\beta}^i\right)^2},\ b = \frac{\frac{1}{N}\sum_{i=1}^{N}\hat{\beta}^i}{\frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{\beta}^i-\frac{1}{n}\sum_{i=1}^{N}\hat{\beta}^i\right)^2},\ c = \frac{\left(\frac{1}{N}\sum_{i=1}^{N}\hat{\alpha}^i\right)^2}{\frac{1}{N-1}\sum_{i=1}^{n}\left(\hat{\alpha}^i-\frac{1}{N}\sum_{i=1}^{N}\hat{\alpha}^i\right)^2},\ d = \frac{\frac{1}{N}\sum_{i=1}^{N}\hat{\alpha}^i}{\frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{\alpha}^i-\frac{1}{N}\sum_{i=1}^{N}\hat{\alpha}^i\right)^2}.$$

For more information (see Dey and Pradhan, 2014).

3.2 Bayesian Credible Intervals

Furthermore, the BCI of α and β denoted by $\tilde{\alpha}_{BCIF}$ and $\tilde{\beta}_{BCIF}$ is obtained under IF and NIF priors as follows:

$$\tilde{\beta}_{BCIF} = k_1^{-1}\int_L^U\int_0^\infty k_2^{-1}\beta^{a-2}\alpha^{c}e^{-b\beta-d\alpha}\prod_{i=1}^{m}(1-\psi_i)^{\alpha-1}x_i e^{-x_i/\beta}\left\{1-(1-\psi_i)^{\alpha}\right\}^{R_i}\,d\beta\,d\alpha = 0.95, \quad (12)$$

$$\tilde{\alpha}_{BCIF} = k_1^{-1}\int_L^U\int_0^\infty \beta^{a-3}\alpha^{c+1}e^{-b\beta-d\alpha}\prod_{i=1}^{m}(1-\psi_i)^{\alpha-1}x_i e^{-x_i/\beta}\left\{1-(1-\psi_i)^{\alpha}\right\}^{R_i}\,d\beta\,d\alpha = 0.95. \quad (13)$$

$$\ddot{\beta}_{BCNIF} = \int_L^U\int_0^\infty k_2^{-1}\beta^{-2}\prod_{i=1}^{m}(1-\psi_i)^{\alpha-1}x_i e^{-x_i/\beta}\left\{1-(1-\psi_i)^{\alpha}\right\}^{R_i}\,d\beta\,d\alpha = 0.95, \quad (14)$$

$$\ddot{\alpha}_{BCNIF} = \int_L^U \int_0^\infty k_2^{-1} \alpha \beta^{-3} \prod_{i=1}^{m} (1-\psi_i)^{\alpha-1} x_i e^{-x_i/\beta} \left\{1-(1-\psi_i)^{\alpha}\right\}^{R_i} \, d\beta \, d\alpha = 0.95. \tag{15}$$

Integrals (12)–(15) are very hard to be solved analytically, so the MH algorithm will be used to solve these integrals Similary, the BCI of $R(x)$ and $h(x)$ provided in (6), (7) under IP and the BCI of $R(x)$ and $h(x)$ provided in (10) and (11) under NIP are obtained using the above procedure.

4. Numerical Illustration

To determine ML estimates (MLEs) and Bayesian estimates (BEs) for parameters, RF and HRF under the PT2C scheme, a simulation study was conducted. Different sample sizes (n), effective failure sizes (m), and picking parameter values are taken into consideration. The R 3.6.1 software is used to complete the following stages.

1. Using the same technique as that provided by Balakrishnan and Sandhu [22], which includes the following, random samples $X_1, X_2, ..., X_n$ are produced from the EME distribution under PT2C samples:

i. Generate m independent and identically (iid) random numbers $W_1, W_2, ..., W_m$ from uniform distribution U(0,1).

ii. Set $V_i = W_i^{(1/i + R_m + R_{m-1} + ... + R_{m-i+1})}$ for $i = 1, 2, ..., m.$

iii. Set $U_i = 1 - V_m V_{m-1} \ldots V_{m-i+1}$ and for $i = 1, 2, ..., m$ .Then $U_1, U_2, ..., U_m$ is the PT2C sample from U(0, 1) distribution.

iv. Finally, set $X_i = F^{-1}(U_i)$ for $i = 1, 2, ..., m$, where $F^{-1}(.)$ is the inverse CDF of EME distribution consideration, then $X_1, X_2, ..., X_m$ are the required PT2C samples from EME distribution with censoring scheme $\underset{\sim}{R} = (R_1, R_2, ..., R_m)$.

2. Three different sampling schemes are considered as follows:

Scheme I: $R_1 = R_2 = \ldots = R_{m-1}$ and $R_m = n - m$ (T2C),

Scheme II: $R_m = n - m$, $R_2 = R_3 = \ldots = R_m = 0$ and

Scheme III: $R_1 = R_2 = (n - m) / 2$, $R_3 = R_4 = \ldots = R_m = 0$.

3. The parameters β and α are chosen with values; Case 1: $\beta = 0.5$, $\alpha = 1.5$ and Case 2: $\beta = 0.5$, $\alpha = 3$

4. With the mission time x = 0.8, the number of stages m, and the censoring strategy $\underset{\sim}{R} = (R_1, R_2, ..., R_m)$, various sample sizes of n=50, 100, and 150 are chosen. The method described by Dey et al. [23] is used to choose the hyper-parameters for gamma priors

5. To create samples from the posterior distributions, the MH approach is applied.

6. The biases, mean squared errors (MSEs), average lengths (AILs), and CPs for MLEs and BEs are computed for various sample sizes, with the number of repeated samples being 1000 samples

7. A portion of the results, which are lengthy numerically, are shown Tables 1–3 for MLEs and BEs under IF.

Figures 2–8 provide examples from the investigation.

Regarding the behaviour of various estimations, the following findings are found.

- All the precision measures for MLEs and BEs tend to decrease with sample sizes n and number of stages m, in majority of the cases. The sample size n and number of stages m both enhance the CPs of the HRF estimates.
- Figure 2 shows that the MSEs of $\tilde{\alpha}$ and $\tilde{\beta}$, obtain the least values across all schemes, and the MSEs of $\hat{\alpha}$ and $\hat{\beta}$ in Case 1 get the biggest values across all schemes.

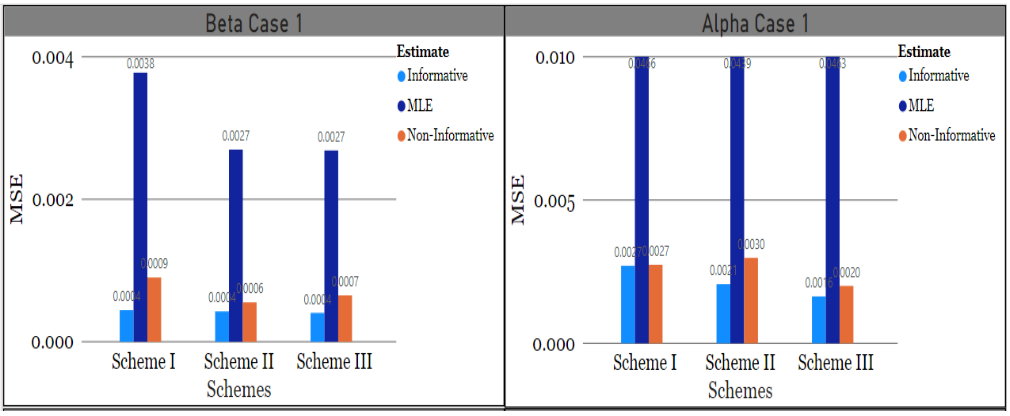


Figure 2: *MSEs for* α and β *estimates in Case 1 for all values of m*

- Figure 3 demonstrates that the MSEs of $\tilde{\alpha}$ and $\tilde{\beta}$ in Case 2 obtain the lowest values among all schemes, whereas the MSEs of $\hat{\alpha}$ and $\hat{\beta}$ obtain the highest values within all schemes

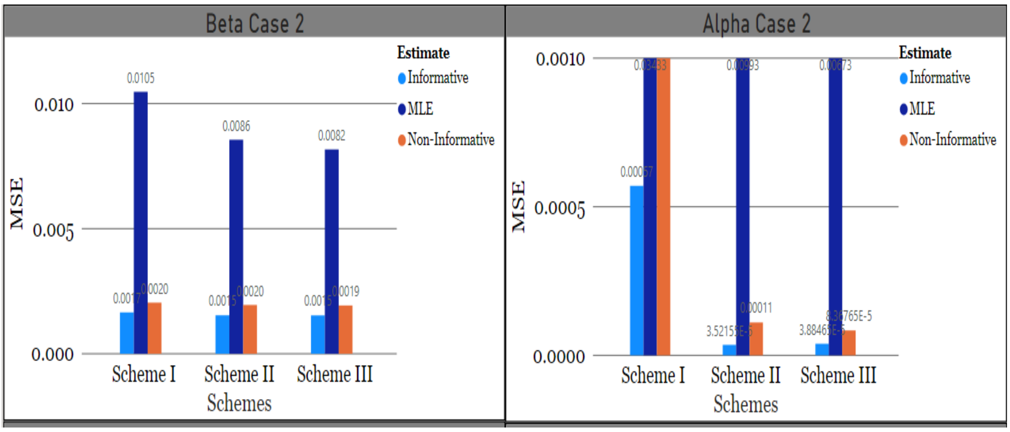


Figure 3: *MSEs for* α and β *estimates in Case 2 for all values of m*

- Regarding Case 1, in Figure 4, the MSEs of $\tilde{R}(x)$ and $\tilde{h}(x)$ in all schemes take the least value, while the MSEs of $\tilde{R}(x)$ and $\tilde{h}(x)$ receive the biggest value

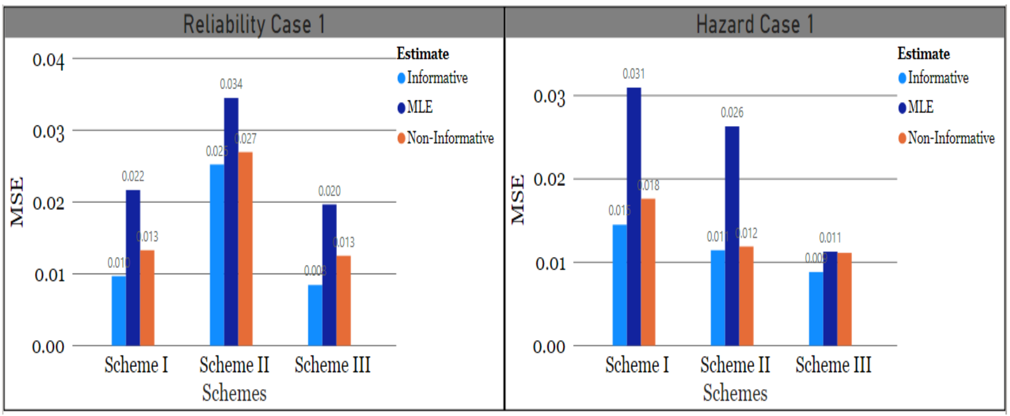


Figure 4: *MSEs for RF and HRF estimates in Case 1 for all values of m*

❖ The MSEs of $\tilde{R}(x)$ and $\tilde{h}(x)$, in all schemes, obtain the least values, as shown in Figure 5.

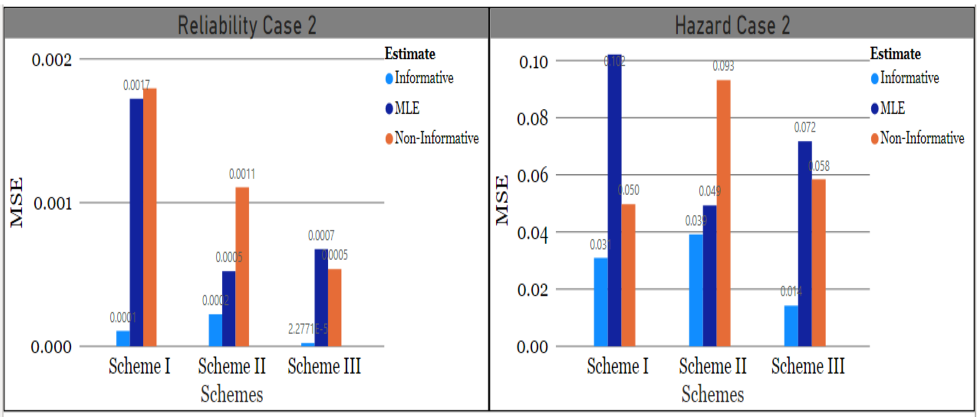


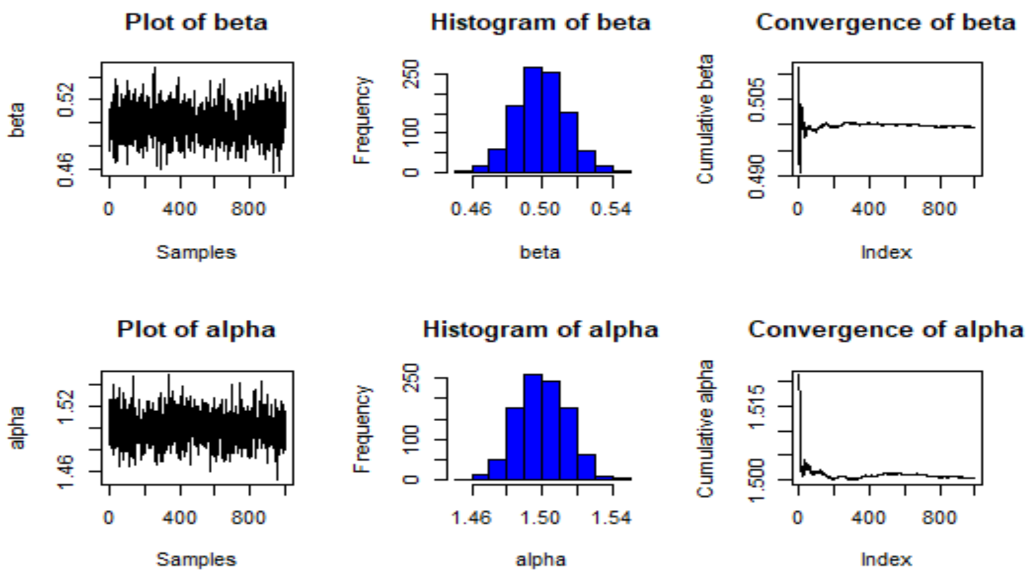
Figure 5: *MSEs for RF and HRF in Case 2 for all values of m*

❖ In most cases, it is possible to draw the conclusion that the MSEs of population parameters employing IF priors take the lowest values.

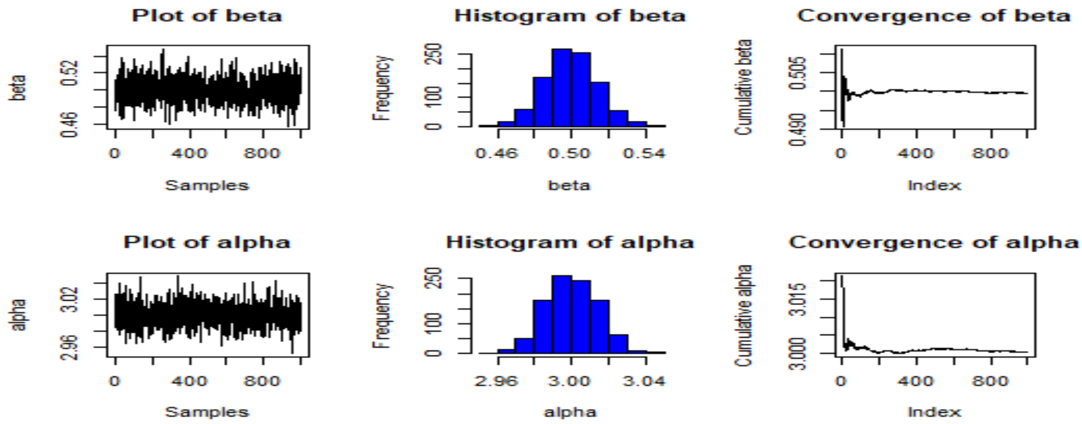
❖ The widths of the BCIs via IF priors are shorter than those of the MLEs and BEs under NIP priors in Case 1 (β =0.5, α =1.5).

❖ The CPs for BEs under IF priors are higher than the equivalent for MLEs and BEs under NIF priors.

❖ In Figure 6, for NIF prior, history graphs for various estimates of β and α are demonstrated. The plots of the parameter chains resemble a horizontal band without any discernible lengthy upward or downward trends, which are evidence of convergence.



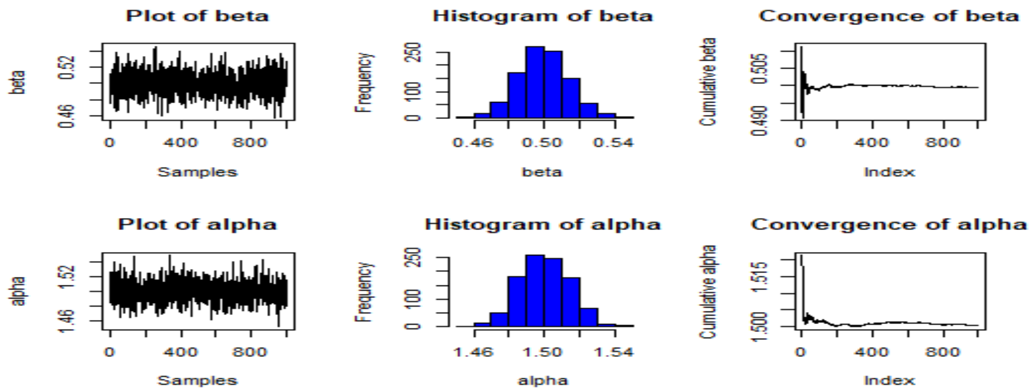
(a) $\ddot{\beta}$ and $\ddot{\alpha}$ at *n*=100, *m*= 50 for β =0.5, α =1.5



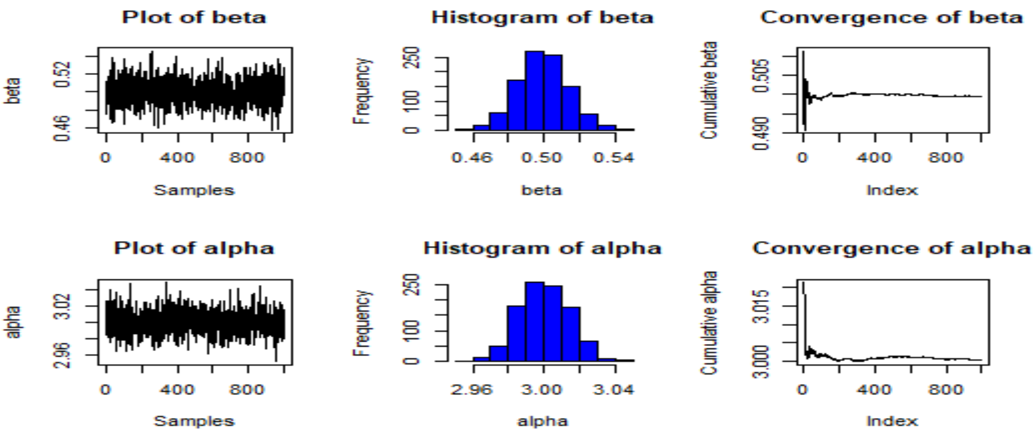
(b) *$\ddot{\beta}$ and $\ddot{\alpha}$ at n=100, m= 50 for β =0.5, $\alpha = 3$*

Figure 6: *Different BEs for β and α under NIF priors*

- In Figure 7, for IF priors, history graphs for various estimations of β and α are shown. The plots of the chains for the parameters resemble a horizontal band without any significant long-term rising or downward trends, which are signs of convergence



(a) $\tilde{\beta}$ and $\tilde{\alpha}$ at *n*=100, *m*= 50 for β =0.5, $\alpha = 1.5$



(b) *$\tilde{\beta}$ and $\tilde{\alpha}$ at n=100, m= 50 for β =0.5, $\alpha = 3$*

Figure 7: *Different Bayesian estimates for β and α under gamma priors*

Table 1: *MLEs and associated measures for* $\alpha, \beta, R(x)$ and $h(x)$ *in case 1*

Scheme I							
n	m	Estimate	Mean	Bias	MSE	AIL	CP
50	20	$\hat{\beta}$	0.490	0.010	0.017	0.511	95.6
		$\hat{\alpha}$	1.780	0.280	0.562	2.728	95.8
		$\hat{R}(x)$	0.842	0.169	0.029	0.299	95.0
		$\hat{h}(x)$	0.627	0.366	0.134	0.758	95.0
	30	$\hat{\beta}$	0.485	0.015	0.010	0.383	96.6
		$\hat{\alpha}$	1.734	0.234	0.471	2.531	96.2
		$\hat{R}(x)$	0.790	0.117	0.014	0.483	96.7
		$\hat{h}(x)$	0.729	0.264	0.070	1.118	96.7
100	20	$\hat{\beta}$	0.477	0.023	0.020	0.550	95.2
		$\hat{\alpha}$	1.810	0.310	0.560	2.670	95.6
		$\hat{R}(x)$	0.909	0.237	0.056	0.184	95.0
		$\hat{h}(x)$	0.470	0.523	0.273	0.737	95.0
	50	$\hat{\beta}$	0.492	0.008	0.006	0.297	96.9
		$\hat{\alpha}$	1.626	0.126	0.159	1.484	95.5
		$\hat{R}(x)$	0.772	0.100	0.010	0.469	96.0
		$\hat{h}(x)$	0.765	0.228	0.052	1.031	96.0
	70	$\hat{\beta}$	0.495	0.005	0.004	0.237	96.5
		$\hat{\alpha}$	1.582	0.082	0.098	1.184	96.7
		$\hat{R}(x)$	0.638	0.035	0.001	0.640	97.1
		$\hat{h}(x)$	0.978	0.015	0.000	1.108	97.1
150	50	$\hat{\beta}$	0.490	0.010	0.007	0.324	96.1
		$\hat{\alpha}$	1.615	0.115	0.138	1.384	95.8
		$\hat{R}(x)$	0.835	0.162	0.026	0.306	96.0
		$\hat{h}(x)$	0.676	0.317	0.100	0.767	96.0
	70	$\hat{\beta}$	0.491	0.009	0.004	0.258	96.8
		$\hat{\alpha}$	1.599	0.099	0.103	1.198	95.7
		$\hat{R}(x)$	0.826	0.153	0.023	0.393	97.1
		$\hat{h}(x)$	0.686	0.308	0.095	0.964	97.1
	100	$\hat{\beta}$	0.495	0.005	0.003	0.202	96.7
		$\hat{\alpha}$	1.562	0.062	0.070	1.011	95.3
		$\hat{R}(x)$	0.685	0.012	0.000	0.592	97.0
		$\hat{h}(x)$	0.915	0.078	0.006	1.103	97.0
	130	$\hat{\beta}$	0.498	0.002	0.002	0.165	96.8
		$\hat{\alpha}$	1.530	0.030	0.047	0.838	96.3
		$\hat{R}(x)$	0.583	0.089	0.008	0.756	96.9
		$\hat{h}(x)$	1.046	0.053	0.003	1.173	96.9

Continued Table 1

Scheme II							
n	m	Estimate	Mean	Bias	MSE	AIL	CP
50	20	$\hat{\beta}$	0.488	0.012	0.010	0.385	96.5
		$\hat{\alpha}$	1.673	0.173	0.314	2.090	96.8
		$\hat{R}(x)$	0.533	0.139	0.019	0.906	95.0
		$\hat{h}(x)$	1.092	0.099	0.010	1.491	95.0
	30	$\hat{\beta}$	0.494	0.006	0.007	0.335	95.8
		$\hat{\alpha}$	1.652	0.152	0.263	1.921	95.2
		$\hat{R}(x)$	0.577	0.096	0.009	0.898	96.7
		$\hat{h}(x)$	1.026	0.033	0.001	1.380	96.7
100	20	$\hat{\beta}$	0.498	0.002	0.010	0.385	95.9
		$\hat{\alpha}$	1.615	0.115	0.214	1.759	96.5
		$\hat{R}(x)$	0.491	0.181	0.033	0.893	95.0
		$\hat{h}(x)$	1.133	0.140	0.020	1.393	95.0
	50	$\hat{\beta}$	0.497	0.003	0.004	0.257	96.2
		$\hat{\alpha}$	1.585	0.085	0.123	1.336	94.8
		$\hat{R}(x)$	0.512	0.160	0.026	0.967	96.0
		$\hat{h}(x)$	1.110	0.117	0.014	1.573	96.0
	70	$\hat{\beta}$	0.497	0.003	0.000	0.085	97.0
		$\hat{\alpha}$	1.559	0.059	0.004	0.084	96.8
		$\hat{R}(x)$	0.549	0.123	0.015	0.938	100.0
		$\hat{h}(x)$	1.057	0.064	0.004	1.462	100.0
150	50	$\hat{\beta}$	0.490	0.010	0.004	0.251	96.5
		$\hat{\alpha}$	1.600	0.100	0.111	1.247	96.3
		$\hat{R}(x)$	0.499	0.173	0.030	0.859	96.0
		$\hat{h}(x)$	1.169	0.176	0.031	1.156	96.0
	70	$\hat{\beta}$	0.497	0.003	0.003	0.210	96.2
		$\hat{\alpha}$	1.557	0.057	0.079	1.081	95.8
		$\hat{R}(x)$	0.569	0.104	0.011	0.913	97.1
		$\hat{h}(x)$	1.043	0.050	0.002	1.312	97.1
	100	$\hat{\beta}$	0.500	0.000	0.002	0.183	97.0
		$\hat{\alpha}$	1.538	0.038	0.059	0.939	96.5
		$\hat{R}(x)$	0.472	0.201	0.040	0.949	97.0
		$\hat{h}(x)$	1.171	0.178	0.032	1.425	97.0
	130	$\hat{\beta}$	0.498	0.002	0.002	0.160	97.1
		$\hat{\alpha}$	1.536	0.036	0.044	0.809	96.0
		$\hat{R}(x)$	0.465	0.208	0.043	0.912	96.9
		$\hat{h}(x)$	1.193	0.200	0.040	1.280	96.9

Continued Table 1

Scheme III							
n	m	Estimate	Mean	Bias	MSE	AIL	CP
50	20	$\hat{\beta}$	0.483	0.017	0.010	0.392	96.1
		$\hat{\alpha}$	1.693	0.193	0.319	2.081	95.9
		$\hat{R}(x)$	0.566	0.107	0.011	0.938	95.0
		$\hat{h}(x)$	1.057	0.064	0.004	1.502	95.0
	30	$\hat{\beta}$	0.495	0.005	0.007	0.332	96.3
		$\hat{\alpha}$	1.645	0.145	0.254	1.892	95.9
		$\hat{R}(x)$	0.580	0.093	0.009	0.958	96.7
		$\hat{h}(x)$	0.988	0.006	0.000	1.626	96.7
100	20	$\hat{\beta}$	0.490	0.010	0.010	0.381	96.3
		$\hat{\alpha}$	1.617	0.117	0.191	1.653	95.9
		$\hat{R}(x)$	0.603	0.069	0.005	0.952	95.0
		$\hat{h}(x)$	0.958	0.035	0.001	1.614	95.0
	50	$\hat{\beta}$	0.496	0.004	0.004	0.253	96.4
		$\hat{\alpha}$	1.588	0.088	0.123	1.333	95.4
		$\hat{R}(x)$	0.559	0.114	0.013	0.919	96.0
		$\hat{h}(x)$	1.069	0.075	0.006	1.371	96.0
	70	$\hat{\beta}$	0.494	0.006	0.003	0.209	96.8
		$\hat{\alpha}$	1.581	0.081	0.093	1.151	95.6
		$\hat{R}(x)$	0.538	0.134	0.018	0.869	97.1
		$\hat{h}(x)$	1.100	0.106	0.011	1.227	97.1
150	50	$\hat{\beta}$	0.498	0.002	0.004	0.247	96.6
		$\hat{\alpha}$	1.560	0.060	0.092	1.166	96.8
		$\hat{R}(x)$	0.462	0.210	0.044	0.932	96.0
		$\hat{h}(x)$	1.173	0.180	0.032	1.444	96.0
	70	$\hat{\beta}$	0.500	0.000	0.003	0.216	96.8
		$\hat{\alpha}$	1.546	0.046	0.071	1.030	96.3
		$\hat{R}(x)$	0.536	0.136	0.019	0.947	97.1
		$\hat{h}(x)$	1.090	0.097	0.009	1.471	97.1
	100	$\hat{\beta}$	0.496	0.004	0.002	0.185	97.2
		$\hat{\alpha}$	1.556	0.056	0.064	0.967	95.4
		$\hat{R}(x)$	0.544	0.129	0.017	0.937	97.0
		$\hat{h}(x)$	1.081	0.088	0.008	1.423	97.0
	130	$\hat{\beta}$	0.497	0.003	0.002	0.160	96.3
		$\hat{\alpha}$	1.542	0.042	0.046	0.828	95.7
		$\hat{R}(x)$	0.485	0.188	0.035	0.960	96.9
		$\hat{h}(x)$	1.153	0.160	0.025	1.560	96.9

Table 2: *Bayes estimates and associated measures for* $\alpha, \beta, R(x)$ and $h(x)$ *in case 1 using IF priors*

Scheme I							
N	m	Estimate	Mean	Bias	MSE	CIL	CP
50	20	$\tilde{\beta}$	0.489	0.011	0.001	0.082	97.4
		$\tilde{\alpha}$	1.779	0.279	0.078	0.082	97.0
		$\tilde{R}(x)$	0.841	0.169	0.028	0.308	100.0
		$\tilde{h}(x)$	0.629	0.364	0.132	0.787	100.0
	30	$\tilde{\beta}$	0.484	0.016	0.001	0.080	97.0
		$\tilde{\alpha}$	1.734	0.234	0.055	0.082	98.5
		$\tilde{R}(x)$	0.789	0.116	0.014	0.483	96.7
		$\tilde{h}(x)$	0.732	0.261	0.068	1.100	100.0
100	20	$\tilde{\beta}$	0.477	0.023	0.001	0.084	96.7
		$\tilde{\alpha}$	1.809	0.309	0.096	0.080	96.7
		$\tilde{R}(x)$	0.909	0.237	0.056	0.186	100.0
		$\tilde{h}(x)$	0.471	0.522	0.273	0.745	100.0
	50	$\tilde{\beta}$	0.492	0.008	0.001	0.080	97.2
		$\tilde{\alpha}$	1.624	0.124	0.016	0.077	98.1
		$\tilde{R}(x)$	0.714	0.041	0.002	0.452	96.0
		$\tilde{h}(x)$	0.895	0.098	0.010	0.922	100.0
	70	$\tilde{\beta}$	0.495	0.005	0.000	0.077	97.6
		$\tilde{\alpha}$	1.580	0.080	0.007	0.084	97.4
		$\tilde{R}(x)$	0.638	0.035	0.001	0.640	97.1
		$\tilde{h}(x)$	0.977	0.016	0.000	1.099	98.6
150	50	$\tilde{\beta}$	0.491	0.009	0.001	0.081	98.9
		$\tilde{\alpha}$	1.619	0.119	0.015	0.082	96.4
		$\tilde{R}(x)$	0.793	0.120	0.014	0.281	96.0
		$\tilde{h}(x)$	0.779	0.214	0.046	0.598	100.0
	70	$\tilde{\beta}$	0.490	0.010	0.001	0.081	98.0
		$\tilde{\alpha}$	1.599	0.099	0.010	0.085	98.2
		$\tilde{R}(x)$	0.825	0.153	0.023	0.374	100.0
		$\tilde{h}(x)$	0.688	0.305	0.093	0.884	100.0
	100	$\tilde{\beta}$	0.496	0.004	0.000	0.078	96.9
		$\tilde{\alpha}$	1.549	0.049	0.003	0.084	96.8
		$\tilde{R}(x)$	0.697	0.024	0.001	0.571	99.0
		$\tilde{h}(x)$	0.899	0.094	0.009	1.114	100.0
	130	$\tilde{\beta}$	0.496	0.004	0.000	0.076	97.5
		$\tilde{\alpha}$	1.547	0.047	0.003	0.084	97.3
		$\tilde{R}(x)$	0.567	0.105	0.011	0.799	96.2
		$\tilde{h}(x)$	1.067	0.074	0.006	1.196	99.2

Continued Table 2

Scheme II							
n	m	Estimate	Mean	Bias	MSE	CIL	CP
50	20	$\tilde{\beta}$	0.486	0.014	0.001	0.080	97.4
		$\tilde{\alpha}$	1.673	0.173	0.030	0.085	97.5
		$\tilde{R}(x)$	0.532	0.140	0.020	0.965	100.0
		$\tilde{h}(x)$	1.097	0.104	0.011	1.595	100.0
	30	$\tilde{\beta}$	0.494	0.006	0.000	0.080	96.5
		$\tilde{\alpha}$	1.651	0.151	0.023	0.090	97.5
		$\tilde{R}(x)$	0.577	0.095	0.009	0.891	96.7
		$\tilde{h}(x)$	1.024	0.031	0.001	1.308	100.0
100	20	$\tilde{\beta}$	0.496	0.004	0.000	0.082	97.1
		$\tilde{\alpha}$	1.615	0.115	0.014	0.083	97.1
		$\tilde{R}(x)$	0.490	0.182	0.033	0.991	100.0
		$\tilde{h}(x)$	1.137	0.144	0.021	1.704	100.0
	50	$\tilde{\beta}$	0.493	0.007	0.000	0.083	98.8
		$\tilde{\alpha}$	1.590	0.090	0.009	0.080	97.8
		$\tilde{R}(x)$	0.522	0.151	0.023	0.922	96.0
		$\tilde{h}(x)$	1.118	0.125	0.016	1.333	100.0
	70	$\tilde{\beta}$	0.497	0.003	0.000	0.078	97.2
		$\tilde{\alpha}$	1.557	0.057	0.004	0.083	98.1
		$\tilde{R}(x)$	0.549	0.123	0.015	0.938	100.0
		$\tilde{h}(x)$	1.057	0.064	0.004	1.461	100.0
150	50	$\tilde{\beta}$	0.495	0.005	0.000	0.079	96.8
		$\tilde{\alpha}$	1.574	0.074	0.006	0.082	96.6
		$\tilde{R}(x)$	0.505	0.167	0.028	0.958	100.0
		$\tilde{h}(x)$	1.117	0.124	0.015	1.555	98.0
	70	$\tilde{\beta}$	0.496	0.004	0.000	0.082	97.7
		$\tilde{\alpha}$	1.555	0.055	0.004	0.085	96.9
		$\tilde{R}(x)$	0.567	0.105	0.011	0.911	98.6
		$\tilde{h}(x)$	1.046	0.053	0.003	1.319	97.1
	100	$\tilde{\beta}$	0.498	0.002	0.000	0.077	98.2
		$\tilde{\alpha}$	1.547	0.047	0.003	0.085	97.2
		$\tilde{R}(x)$	0.511	0.162	0.026	0.940	96.0
		$\tilde{h}(x)$	1.125	0.132	0.018	1.370	100.0
	130	$\tilde{\beta}$	0.497	0.003	0.000	0.075	97.7
		$\tilde{\alpha}$	1.540	0.040	0.002	0.082	97.2
		$\tilde{R}(x)$	0.486	0.186	0.035	0.925	98.5
		$\tilde{h}(x)$	1.164	0.170	0.029	1.330	96.9

Continued Table 2

Scheme III							
n	m	Estimate	Mean	Bias	MSE	CIL	CP
50	20	$\tilde{\beta}$	0.483	0.017	0.001	0.081	97.1
		$\tilde{\alpha}$	1.693	0.193	0.038	0.088	98.2
		$\tilde{R}(x)$	0.565	0.107	0.012	0.968	100.0
		$\tilde{h}(x)$	1.059	0.066	0.004	1.590	100.0
	30	$\tilde{\beta}$	0.494	0.006	0.000	0.083	98.9
		$\tilde{\alpha}$	1.644	0.144	0.021	0.080	97.4
		$\tilde{R}(x)$	0.579	0.094	0.009	0.959	100.0
		$\tilde{h}(x)$	0.991	0.003	0.000	1.626	100.0
100	20	$\tilde{\beta}$	0.488	0.012	0.001	0.079	96.4
		$\tilde{\alpha}$	1.617	0.117	0.014	0.084	97.1
		$\tilde{R}(x)$	0.602	0.070	0.005	0.973	100.0
		$\tilde{h}(x)$	0.962	0.031	0.001	1.672	100.0
	50	$\tilde{\beta}$	0.498	0.002	0.000	0.079	96.8
		$\tilde{\alpha}$	1.562	0.062	0.004	0.084	96.9
		$\tilde{R}(x)$	0.457	0.215	0.046	0.927	98.0
		$\tilde{h}(x)$	1.189	0.196	0.038	1.427	98.0
	70	$\tilde{\beta}$	0.494	0.006	0.000	0.077	97.1
		$\tilde{\alpha}$	1.580	0.080	0.007	0.089	97.3
		$\tilde{R}(x)$	0.538	0.135	0.018	0.865	100.0
		$\tilde{h}(x)$	1.102	0.109	0.012	1.254	97.1
150	50	$\tilde{\beta}$	0.497	0.003	0.000	0.079	96.7
		$\tilde{\alpha}$	1.559	0.059	0.004	0.082	97.3
		$\tilde{R}(x)$	0.472	0.200	0.040	0.912	96.0
		$\tilde{h}(x)$	1.172	0.179	0.032	1.271	100.0
	70	$\tilde{\beta}$	0.499	0.001	0.000	0.079	98.0
		$\tilde{\alpha}$	1.547	0.047	0.003	0.084	96.8
		$\tilde{R}(x)$	0.535	0.137	0.019	0.928	100.0
		$\tilde{h}(x)$	1.094	0.101	0.010	1.487	97.1
	100	$\tilde{\beta}$	0.495	0.005	0.000	0.076	97.6
		$\tilde{\alpha}$	1.552	0.052	0.003	0.082	98.0
		$\tilde{R}(x)$	0.515	0.158	0.025	0.887	99.0
		$\tilde{h}(x)$	1.127	0.134	0.018	1.416	97.0
	130	$\tilde{\beta}$	0.497	0.003	0.000	0.076	98.3
		$\tilde{\alpha}$	1.534	0.034	0.002	0.085	97.1
		$\tilde{R}(x)$	0.489	0.183	0.034	0.950	100.0
		$\tilde{h}(x)$	1.148	0.154	0.024	1.438	100.0

5. Discussion and Summary

This study uses maximum likelihood and Bayesian techniques to analyse parameter estimators, reliability function estimator, and hazard rate function estimator for EME distributions under PT2Cschemes. Gamma and uniform priors are taken into account under the squared error loss function to construct the Bayesian estimators. On the basis of IF and NIF priors, it is possible to derive approximate confidence intervals as well as Bayesian credible intervals. A simulation study is conducted to compare the effectiveness of every estimate. The Bayesian estimates using the gamma prior are, roughly speaking, generally more accurate than the MLEs, according to a numerical illustration. When compared to other schemes, Scheme I's MSEs have the highest value. Additionally, the MSEs for each estimate use the value for Scheme III that is the lowest. Comparatively speaking, the Bayesian estimates using gamma priors have the highest coverage probability.

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