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On an extension of the two-parameter Lindley distribution

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Abstract

AIM: Lindley distribution has been widely studied in statistical literature because it accommodates several interesting properties. In lifetime data analysis contexts, Lindley distribution gives a good description over exponential distribution. It has been used for analysing copious real data sets, specifically in applications of modeling stress-strength reliability. This paper proposes a new generalized two-parameter Lindley distribution and provides a comprehensive description of its statistical properties such as order statistics, limiting distributions of order statistics, Information theory measures, etc.

METHODS: We study shapes of the probability density and hazard rate functions, quantiles, moments, moment generating function, order statistic, limiting distributions of order statistics, information theory measures, and autoregressive models are among the key characteristics and properties discussed. The two-parameter Lindley distribution is then subjected to statistical analysis. The paper uses methods of maximum likelihood to estimate the parameters of the proposed distribution. The usefulness of the proposed distribution for modeling data is illustrated using a real data set by comparison with other generalizations of the exponential and Lindley distributions and is depicted graphically.

RESULTS/FINDINGS: This paper presents relevant characteristics of the proposed distribution and applications. Based on this study, we found that the proposed model can be used quite effectively to analyzing lifetime data.

CONCLUSIONS: In this article, we proffered a new customized Lindley distribution. The proposed distribution enfolds exponential and Lindley distributions as sub-models. Some properties of this distribution such as quantile function, moments, moment generating function, distributions of order statistics, limiting distributions of order statistics, entropy, and autoregressive time series models are studied. This distribution is found to be the most appropriate model to fit the carbon fibers data compared to other models. Consequently, we propose the MOTL distribution for sketching inscrutable lifetime data sets.

Keywords: Exponential distribution, Generalized family, Lindley distribution, Marshall-Olkin extended distribution, Maximum likelihood estimation

1. Introduction

Lindley distribution [16, 17] has been proposed to describe a difference between fiducial distribution and posterior distribution. The works on Lindley distribution; see, for example, [11], [14], [8], [2], [28], [33], etc. In the last decades, a lot of attempts have been made to define new probability distributions based on Lindley model, for example, three parameters-Lindley distribution [37], generalized Poisson-Lindley distribution [18], generalized Lindley distribution [23], Marshall-Olkin Lindley distribution [38], power Lindley distribution [10], two-parameter Lindley distribution [29], quasi Lindley distribution [30], transmuted Lindley distribution [20], transmuted Lindley-geometric distribution [21], beta-Lindley distribution [22], discrete Harris extended Lindley distribution [35], etc. Moreover, [36] has provided a detailed review study on

the generalizations of the Lindley distribution.
[31] introduced a new distribution, called two-parameter Lindley distribution. A random variable X is said to have the two-parameter Lindley distribution with parameters α and β if its survival function (sf) takes the form

$$\bar{F}(x,\alpha,\beta) = \frac{(\alpha+\beta+\alpha\beta x)}{\alpha+\beta}e^{-\alpha x}\,,\ x>0,\alpha>0,\beta>-\alpha \tag{1}$$

and the corresponding probability density function (pdf) can be expressed as

$$f(x,\alpha,\beta) = \frac{\alpha^2(1+\beta x)e^{-\alpha x}}{\alpha+\beta}\,,\ x>0,\alpha>0,\beta>-\alpha. \tag{2}$$

It can easily be seen that at $\beta=1$, the distribution in equation (2) reduces to the Lindley distribution and at $\beta=0$, equation (2) reduces to the exponential distribution. [12] has also studied this distribution as a new flexible form of exponential distribution is called flexible exponential distribution. Some generalizations and extensions of this flexible exponential distribution are proposed in [34] and [25].

On the other hand, there is a vast amount of statistical literature on methods of introducing new family of distributions. Notable among them are Azzalinis skewed family of distributions [4], exponentiated family of distributions [13], gamma-generated family of distributions [39, 27], Kumaraswamy family of distributions [6], Weibull generalized family of distributions [5], logistic-generated family of distributions [Torabi and Montazari(2014)], Kumaraswamy Marshal-Olkin family of distributions [1] and Marshall-Olkin Kumaraswamy family of distributions [15]. Moreover, [19] has introduced a general method for adding parameter to a baseline distribution, the resulting distribution is called Marshall-Olkin family of distributions, its sf $\bar{G}(x)$ and pdf $g(x)$ are given by the following formulae,

$$\bar{G}(x,\alpha) = \frac{\gamma\bar{F}(x)}{1-\bar{\gamma}\bar{F}(x)}\,,\ x\in R,\gamma>0 \tag{3}$$

$$g(x,\alpha) = \frac{\gamma f(x)}{(1-\bar{\gamma}\bar{F}(x))^2}\,,\ x\in R,\gamma>0 \tag{4}$$

where $\bar{F}(x)$ is sf of the random variable X to be generated, $\bar{\gamma}=1-\gamma$ and γ is a tilt parameter. If $F(x)$ has the hazard rate function (hrf) $r(x)$ then the hrf of MOE family is given by

$$h(x,\alpha) = \frac{r(x)}{1-\bar{\gamma}\bar{F}(x)}\,,\ x\in R,\gamma>0$$

The main object of this paper is to present an extension for the two-parameter Lindley distribution, that can be used as an alternative to the existing generalized exponential and Lindley distributions. The rest of this article is organized as follows: Section 2 introduces the Marshall-Olkin two-parameter Lindley distribution; its properties including quantile function, moments, moment generating function, distributions of order statistics, limiting distributions of order statistics, entropy and autoregressive time series models are presented in Section 3; Section 4 proposes parameter estimation of the proposed distribution by the method of maximum likelihood estimation; Section 5 deals with the application of the new distribution to a real data set; Section 6 presents the conclusion of the study.

2. Marshall-Olkin two-parameter Lindley distribution

If X is distributed according to equation (2), then the corresponding Marshall-Olkin (MO) generalized form of its sf and pdf using equations (3) and (4) is given by

$$\bar{G}(x,\alpha,\beta,\gamma) = \frac{\frac{\gamma(\alpha+\beta+\alpha\beta x)}{\alpha+\beta}e^{-\alpha x}}{\left[1-\bar{\gamma}\left(\frac{(\alpha+\beta+\alpha\beta x)}{\alpha+\beta}e^{-\alpha x}\right)\right]}\,,\ x>0,\alpha,\gamma>0,\beta>-\alpha \tag{5}$$

and

$$g(x,\alpha,\beta,\gamma)=\frac{\gamma\alpha^2(1+\beta x)e^{-\alpha x}}{\alpha+\beta\left[1-\bar{\gamma}\left(\frac{(\alpha+\beta+\alpha\beta x)}{\alpha+\beta}e^{-\alpha x}\right)\right]^2},\ x>0,\alpha,\gamma>0,\beta>-\alpha \tag{6}$$

respectively. The new distribution given by the pdf equation (6) is called the Marshall-Olkin two-parameter Lindley (MOTL) distribution. In addition, hrf of the MOTL distribution is given by following equation

$$h(x,\alpha,\beta,\gamma)=\frac{\alpha^2(1+\beta x)}{\left\{1-\bar{\gamma}\left[\frac{(\alpha+\beta+\alpha\beta x)}{\alpha+\beta}e^{-\alpha x}\right]\right\}(\beta+\alpha+\alpha\beta x)},\ x>0,\alpha,\gamma>0,\beta>-\alpha. \tag{7}$$

Notably, the classical exponential and one-parameter Lindley distributions are special cases of the MOTL distribution. Some distributions that are special cases of MOTL distribution are:

Exponential distribution : when γ=1 and β=0 in equation (6) with pdf

$$g(x,\gamma)=\alpha e^{-\alpha x}$$

One-parameter Lindley distribution : when γ=1 and β=1 in equation (6) with pdf

$$g(x,\alpha,\gamma)=\frac{\alpha^2(1+x)e^{-\alpha x}}{\alpha+1}$$

MO exponential distribution : when $\beta=0$ in equation (6) with pdf

$$g(x,\alpha,\gamma)=\frac{\gamma\alpha e^{-\alpha x}}{[1-\bar{\gamma}(e^{-\alpha x})]^2}$$

MO Lindley distribution: when $\beta=1$ in equation (6) with pdf

$$g(x,\alpha,\gamma)=\frac{\gamma\alpha^2(1+x)e^{-\alpha x}}{\alpha+1\left[1-\bar{\gamma}\frac{(\alpha+1+\alpha x)}{\alpha+1}e^{-\alpha x}\right]^2}$$

The different shapes of the pdf and hrf of the MOTL distribution are displayed in Figure 1 and Figure 2 for selected parameter values. From figures it is clear that the pdf and hrf of MOTL distribution can be increasing, decreasing, upside-down bathtub (unimodal) depending on the values of its parameters.

3. Statistical Properties

In this section, we study the statistical properties for the MOTL distribution.

3.1. Quantiles

The quantile function of the MOTL distribution is given by

$$x=G^{-1}(u)=-(\frac{\alpha+\beta}{\alpha\beta})-\frac{1}{\alpha}W_{-1}\left[-\frac{1}{\beta}\left(\frac{u-1}{1-u+u\gamma}\right)(\alpha+\beta)e^{(-\frac{\alpha+\beta}{\beta})}\right] \tag{8}$$

where $G(x)=u$ and $0\leq u\leq 1$. W_{-1} denotes the negative branch of the Lambert W function. Table 1 represents the quantiles for selected values of the parameters of the MOTL distribution using R programming language.

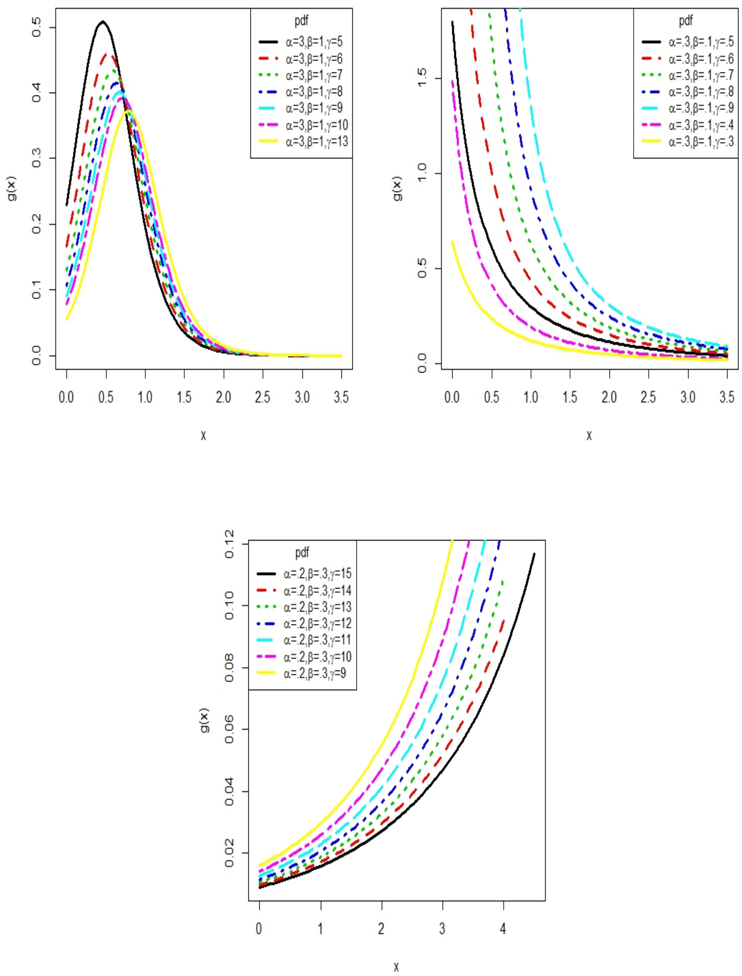

Figure 1: *Graphs of pdf of the MOTL distribution for different values of α, β and γ.*

Table 1: *The quantiles for different values of the parameters of the MOTL distribution*

u	(α,β,γ) (0.5,0.5,1.5)	(α,β,γ) (0.4,0.3,.05)	(α,β,γ) (2.1,2.2,0.5)	(α,β,γ) (3.5,4.2,2.5)	(α,β,γ) (5,6,7)
0.1	0.5784	0.2331	0.0513	0.1360	0.1515
0.2	1.1374	0.5001	0.1087	0.2503	0.2596
0.3	1.7018	0.8107	0.1741	0.3563	0.3515
0.4	2.2923	1.1796	0.2504	0.4604	0.4360
0.5	2.9324	1.6295	0.3422	0.5676	0.5202
0.6	3.6556	2.1986	0.4565	0.6838	0.6078
0.7	4.5205	2.9585	0.6074	0.8170	0.7061
0.8	5.6539	4.0696	0.8255	0.9879	0.8282
0.9	7.4520	6.0382	1.2080	1.2505	1.0135

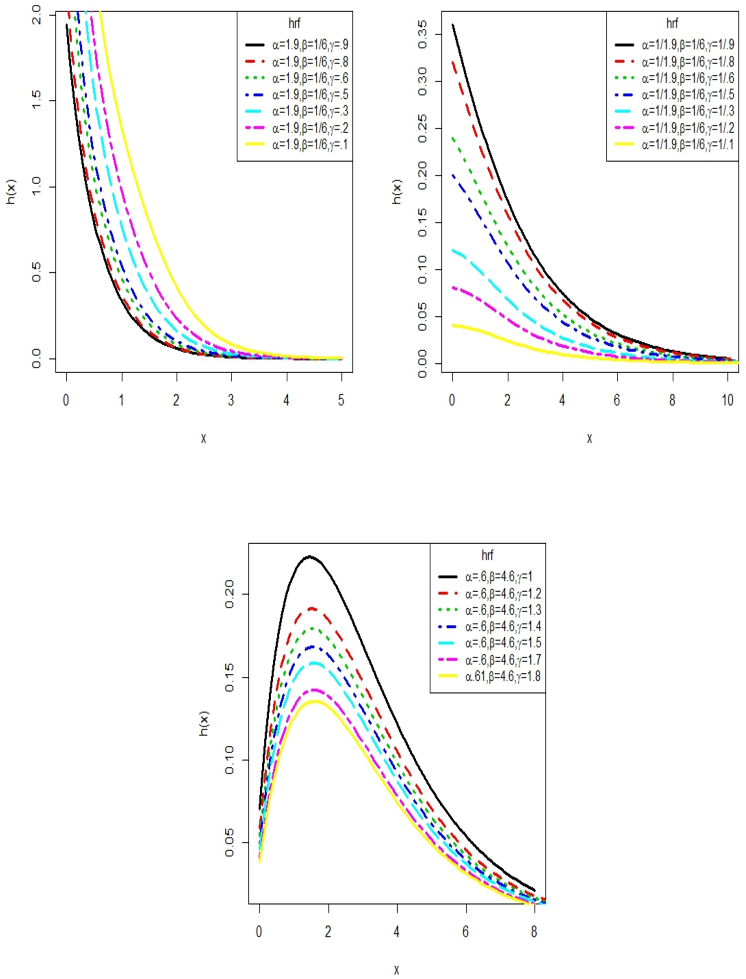

Figure 2: *Graphs of pdf of the MOTL distribution for different values of α, β and γ.*

3.2. Moments

In statistical analysis and its applications, moments have received important role. It can be used to study the most eminent features and characteristics such as tendency, dispersion, skewness and kurtosis of a distribution. We now give simple expansions for the pdf of the MOTL distribution. We have following expansion

$$(1-z)^{-r} = \sum_{i=0}^{\infty} \binom{r+i-1}{i} z^i, \; |z| < 1, r > 0$$

Put

$$S(x) \;=\; \frac{(\alpha+\beta+\alpha\beta x)}{\alpha+\beta} e^{-\alpha x}$$

When $\gamma \in (0,2)$

$$(1-(1-\gamma)(1-S(x))^{-2} \;=\; \sum_{i}^{\infty} \sum_{j=0}^{i} (i+1)(1-\gamma)^i \binom{i}{j} S(x)^j \tag{9}$$

Using the series expansion in equation (9) and the representation for the MOTL pdf in equation (6), we obtain

$$\begin{aligned} g(x) &= \gamma \sum_{i=0}^{\infty} \frac{\alpha^2}{\alpha+\beta} (i+1)(1-\gamma)^i \left\{ 1 + \frac{\alpha\beta x}{\alpha+\beta} \right\}^i (1+\beta x) e^{-(i+1)\alpha x} \\ &= \gamma \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\alpha^2}{(\alpha+\beta)} (i+1)(1-\gamma)^i \binom{i}{j} \left\{ \frac{\alpha\beta x}{\alpha+\beta} \right\}^j (1+\beta x) e^{-(i+1)\alpha x} \\ &= \gamma \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\alpha^2}{(\alpha+\beta)^{j+1}} (i+1)(1-\gamma)^i \binom{i}{j} (\alpha\beta)^j \left[x^j e^{-(i+1)\alpha x} + \beta x^{j+1} e^{-(i+1)\alpha x} \right] \end{aligned} \tag{10}$$

We have

$$E(X^r) = \int_0^{\infty} x^r g(x, \alpha, \beta, \gamma) dx. \tag{11}$$

Substituting equation (10) into the equation (11), we obtain the r^{th} moment of MOTL distribution in the form

$$\begin{aligned} E(X^r) &= \gamma \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\alpha^2}{(\alpha+\beta)^{j+1}} (i+1)(1-\gamma)^i \binom{i}{j} (\alpha\beta)^j \int_0^{\infty} x^r \left[x^j e^{-(i+1)\alpha x} + \beta x^{j+1} e^{-(i+1)\alpha x} \right] dx \\ &= \gamma \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\alpha^2}{(\alpha+\beta)^{j+1}} (i+1)(1-\gamma)^i \binom{i}{j} (\alpha\beta)^j w_{ijr} \end{aligned}$$

$$w_{ijr} = \frac{\Gamma(r+j+1)}{[(i+1)\alpha]^{r+j+1}} + \beta \frac{\Gamma(r+j+2)}{[(i+1)\alpha]^{r+j+2}}$$

Similarly, when $\gamma > 1/2$

$$g(x) \;=\; \frac{\gamma\alpha^2(1+\beta x)e^{-\alpha x}}{\alpha+\beta} \left[1 - \frac{\bar{\gamma}}{\gamma}(1-S(x)) \right]^{-2}$$

$$g(x) \;=\; \gamma \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{i} \frac{\alpha^2}{(\alpha+\beta)^{j+1}} (-1)^{i+k} \left(\frac{\bar{\gamma}}{\gamma} \right)^i \binom{i-1}{k} \binom{k}{j} (\alpha\beta)^j \left[x^j e^{-(i+1)\alpha x} + \beta x^{j+1} e^{-(i+1)\alpha x} \right]$$

The r^{th} moment of MOTL distribution is

$$E(X^r) = \gamma \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{i} \frac{\alpha^2}{(\alpha+\beta)^{j+1}} (-1)^{i+k} \left(\frac{\bar{\gamma}}{\gamma}\right)^i \binom{i-1}{k} \binom{k}{j} (\alpha\beta)^j w_{ijr}$$

Table 3.2 lists the moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of the MOTL distribution for selected values of the parameters.

Table 2: *Moments of the MOTL distribution for different values of the parameters α, β, and γ*

	(α,β,γ) (0.5,0.5,.5)	(α,β,γ) (1.5,0.5,1.5)	(α,β,γ) (2.5,0.005,3.5)	(α,β,γ) (3,5,3.5)	(α,β,γ) (5,6,7)
μ_1'	1.5273	1.0084	0.7029	0.8856	1.3452
μ_2'	3.0999	1.7819	0.7641	1.0953	2.5024
μ_3'	7.5709	4.3852	1.0910	1.6799	5.7702
μ_4'	21.4836	13.7554	1.9276	3.0565	15.7912
SD	0.8760	0.8746	0.5196	0.5577	0.8326
CV	0.5735	0.8676	0.7392	0.6297	0.6076
CS	3811.034	749.2536	211.0792	548.7451	2284.693
CK	3.8968	6.5565	5.3869	4.2850	4.3509

3.3. Moment Generating Function

Moment generating function is given by the following formula

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) \tag{12}$$

The moment generating function of MOTL distribution is obtained by using equation (12). When $\gamma \in (0,2)$, it has following form

$$M_X(t) = \gamma \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\alpha^2 t^r}{(\alpha+\beta)^{j+1} r!} (i+1)(1-\gamma)^i \binom{i}{j} (\alpha\beta)^j w_{ijr}$$

Similarly, when $\gamma > 1/2$

$$M_X(t) = \gamma \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{i} \frac{\alpha^2 t^r}{(\alpha+\beta)^{j+1} r!} (-1)^{i+j} \left(\frac{1-\gamma}{\gamma}\right)^i \binom{i-1}{j} \binom{k}{j} (\alpha\beta)^j w_{ijr}$$

where

3.4. Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample taken from the MOTL distribution and $X_{1:n}, X_{2:n}, ..$ $., X_{n:n}$ be the corresponding order statistics. The pdf $g_{r:n}(x, \alpha, \beta, \gamma)$ of the r^{th} order statistics $X_{r:n}$ is given by

$$g_{r:n}(x,\alpha,\beta,\gamma) = \frac{n!}{(r-1)(n-r)!} g(x,\alpha,\beta,\gamma)G(x,\alpha,\beta,\gamma)^{r-1}[1-G(x,\alpha,\beta,\gamma)]^{n-r} \quad (13)$$

where g(x), G(x) are pdf and cdf of MOTL distribution by equations (5) and (6). We can use the binomial expansion of $[1-G(x)]^{n-i}$ given as follows

$$[1-G(x)]^{n-i} = \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i G(x,\alpha,\beta,\gamma)^i \quad (14)$$

substitute equation (15) into the equation (14), we get

$$g_{r:n}(x,\alpha,\beta,\gamma) = \sum_{i=0}^{n-r} \binom{n-r}{i} \frac{n!(-1)^i}{(r-1)(n-r)!} g(x,\alpha,\beta,\gamma)G(\alpha,\beta,\gamma)^{i+r-1} \quad (15)$$

We get pdf of r^{th} order statistics for MOTL distribution from equation (15), by using above equations (5) and (6). We can express the k^{th} ordinary moment of the r^{th} order statistics $X_{r:n}$ ($E(X^k_{r:n})$) as a liner combination of the k^{th} moments of the MOTL distribution with different shape parameters. The r^{th} order statistic for MOTL distribution can be expressed as

$$g_{r:n}(x,\alpha,\beta,\gamma) = \sum_{i=0}^{n-r} \binom{n-r}{i} \frac{n!(-1)^i}{(r-1)(n-r)!} \frac{\gamma \frac{\alpha^2(1+\beta x)e^{-\alpha x}}{\alpha+\beta}}{\left[1-\bar{\gamma}\frac{\alpha+\beta-(\beta+\alpha+\alpha\beta x)}{\alpha+\beta}e^{-\alpha x}\right]^2} \left\{\frac{\frac{\alpha+\beta-(\beta+\alpha+\alpha\beta x)}{\alpha+\beta}}{\left[\gamma+(1-\gamma)\frac{\alpha+\beta-(\beta+\alpha+\alpha\beta x)}{\alpha+\beta}e^{-\alpha x}\right]}\right\}^{i+r-1} \quad (16)$$

When r=1 and r=n in equation (16), we get the equations of pdf of the smallest and largest order statistics, respectively.

3.5. Limiting distributions of order statistics

Theorem: 1. *If $X_{1:n}$ be the minimum and $X_{n:n}$ be the maximum of a random sample $x = (x_1, ..., x_n)$ from MOTL distribution, then*

(a) $\lim_{n\to\infty} P\left(\frac{X_{1:n}-p_n}{q_n} \le x\right) = 1-e^{-x}$

(b) $\lim_{n\to\infty} P\left(\frac{X_{n:n}-p^*_n}{q^*_n} \le t\right) = \exp(-e^{-t})$

where $p_n = 0$, $q_n = G^{-1}(\frac{1}{n})$, $p^*_n = G^{-1}(1-\frac{1}{n})$, $q^*_n = 1$ and $G^{-1}(\cdot)$ is given in (8).

Proof

For MOTL distribution, by applying L$'$ Hospital rule, we obtain

$$\lim_{\epsilon\to 0+} \frac{G(\epsilon x)}{G(\epsilon)} = \lim_{\epsilon\to 0+} \frac{xg(\epsilon x)}{g(\epsilon)} = x$$

Hence, the minimal domain of attraction of the MOTL distribution can be the standard exponential distribution (see Theorem 8.3.6 [3]).
Subsequently, for the MOTL distribution, we can express

$$\lim_{x\to\infty+} \frac{d}{dx}\left(\frac{1}{h(x)}\right) = 0$$

Therefore, the maximal domain of attraction of MOLT distribution can be the standard Gumbel distribution (see Theorem 8.3.3 [3]).

3.6. Information Theory Measures

The concept of entropy has played a vital role in information theory. The entropy of a random variable can be defined in terms of its probability distribution and it has been used in distinct fields in science as a measure of variation of the uncertainty. Numerous measures of entropy have been mentioned and compared in the literature. The entropy and mutual information concepts have been formalized by Shannon (1948). [26] proposed a other measure of entropy namely, Rényi entropy as a generalization of Shannon entropy. The Rényi entropy is defined as $I_R(\gamma) = \frac{1}{1-\delta} \log \int_R g^{\delta}(x)dx$, $\delta > 0$ and $\delta \neq 1$. Rényi entropy of order 1 is Shannon entropy. We consider first $g^{\delta}(x)$ given by,

$$g^{\delta}(x) = \frac{\gamma^{\delta} \frac{\alpha^{2\delta}(1+\beta x)^{\delta} e^{-\alpha\delta x}}{\alpha+\beta}^{\delta}}{\left\{1 - \bar{\gamma}\frac{\alpha+\beta-(\beta+\alpha+\alpha\beta x)}{\alpha+\beta} e^{-\alpha x}\right\}^{2\delta}}$$

Suppose that $\gamma > \frac{1}{2}$. Using the series expansion

$$\begin{aligned}\left[1 - \bar{\gamma}\frac{\alpha+\beta-(\beta+\alpha+\alpha\beta x)}{\alpha+\beta} e^{-\alpha x}\right]^{-2\delta} &= \gamma^{-2\delta} \sum_{k=0}^{\infty} \frac{\Gamma(2\delta+k)}{\Gamma(2\delta)k!} \left(1 - \frac{1}{\gamma}\right)^k \\ &\quad \left[\frac{\alpha+\beta-(\beta+\alpha+\alpha\beta x)}{\alpha+\beta} e^{-\alpha x}\right]^k\end{aligned}$$

Thus we obtain that in the case $\gamma > 1/2$, the Rényi entropy is

$$\begin{aligned}I_R(\delta) &= \frac{1}{1-\delta} \log\{\frac{\gamma^{\delta}\alpha^{2\delta}}{(\alpha+\beta)^{\delta}} \sum_{i,j,k,l=0}^{\infty} \frac{\Gamma(2\delta+k)}{\Gamma(2\delta)k!(\alpha+\beta)^j} \left(1 - \frac{1}{\delta}\right)^k \\ &\quad \binom{j}{k}\binom{i+k}{i}\binom{l}{\delta}(-1)^i(\alpha\beta)^j\beta^{\delta} \int_R x^{(\delta+j)} e^{-\alpha(\delta+k)x} dx\} \\ &= \frac{1}{1-\delta} \log\{\frac{\gamma^{-\delta}\alpha^{2\delta}}{(\alpha+\beta)^{\delta}} \sum_{i,j,k,l=0}^{\infty} \frac{\Gamma(2\delta+k)}{\Gamma(2\delta)k!(\alpha+\beta)^j} \left(1 - \frac{1}{\delta}\right)^k \\ &\quad \binom{j}{k}\binom{i+k}{i}\binom{l}{\delta}(-1)^i(\alpha\beta)^j\beta^{\delta} \frac{\Gamma(\delta+j+1)}{[\alpha(\delta+k)]^{\delta+j+1}}\end{aligned}$$

Similarly, we can show that in the case $0 < \gamma < 2$ and by using the series expansion

$$\begin{aligned}\left\{1 - \bar{\gamma}\frac{\alpha+\beta-(\beta+\alpha+\alpha\beta x)e^{-\alpha x}}{\alpha+\beta}\right\}^{-2\delta} &= \sum_{k=0}^{\infty} \frac{\Gamma(2\delta+k)}{\Gamma(2\delta)k!} (1-\gamma)^k \\ &\quad \left[\frac{(\beta+\alpha+\alpha\beta x)e^{-\alpha x}}{\alpha+\beta}\right]^k\end{aligned}$$

Corresponding Rényi entropy is

$$\begin{aligned}I_R(\delta) &= \frac{1}{1-\delta} \log\{\frac{\gamma^{-\delta}\alpha^{2\delta}}{(\alpha+\beta)^{\delta}} \sum_{i,j,k=0}^{\infty} \frac{\Gamma(2\delta+k)}{\Gamma(2\delta)k!(\alpha+\beta)^i} (1-\delta)^k \\ &\quad \binom{i}{k}\binom{j}{\delta}(\alpha\beta)^i\beta^j \int_R x^{(i+j)} e^{-\alpha(\delta+k)x} dx\} \\ &= \frac{1}{1-\delta} \log\{\frac{\gamma^{-\delta}\alpha^{2\delta}}{(\alpha+\beta)^{\delta}} \sum_{i,j,k=0}^{\infty} \frac{\Gamma(2\delta+k)}{\Gamma(2\delta)k!(\alpha+\beta)^i} (1-\delta)^k \\ &\quad \binom{i}{k}\binom{j}{\delta}(\alpha\beta)^i\beta^j \frac{\Gamma(i+j+1)}{[\alpha(\delta+k)]^{i+j+1}}\}\end{aligned}$$

3.7. Autoregressive Time Series Modeling

Autoregressive models are types of random process, has utilized to model and predict various type of natural phenomena. In other words, autoregressive models are group of linear prediction formulas which try to predict an output of a system based on the past observations. In the following Subsections we construct and explore different autoregressive models of order 1 (AR(1)), that is, MIN AR(1) model I, MIN AR(1) model II, MAX-MIN AR(1) model I and MAX-MIN AR(1) model II with MOTL as marginals.

3.7.1 MIN AR (1) Model-1 with MOTL Marginal Distribution

The first AR (1) structure is given by

$$X_n = \begin{cases} \mathfrak{e}_n, & \text{with probability } \varrho \\ min(X_{n-1}, \mathfrak{e}_n), & \text{with probability } 1-\varrho \end{cases} \tag{17}$$

where $\{\mathfrak{e}_n\}$ is a sequence of independently and identically distributed (iid) random variables independent of $\{X_n\}$ and $\varrho \in (0,1)$. Hence the process is stationary Markovian with MOTL distribution as marginal.

Theorem: 2. *$\{X_n\}$ is stationary Markovian with MOTL distribution with parameters $\varrho, \alpha, \beta \Longleftrightarrow \{\mathfrak{e}_n\}$ is distributed as two-parameter Lindley distribution, under in an AR (1) process defined in (17).*

Proof. Let $\mathfrak{e}_n$ follows two-parameter Lindley distribution with parameters α and β. Using equation (17), we can express

$$\bar{F}_{X_n}(x) = \varrho \bar{F}_{\mathfrak{e}_n}(x) + (1-\varrho)\bar{F}_{X_{n-1}}(x)\bar{F}_{\mathfrak{e}_n}(x)$$

While under stationary equilibrium,

$$\bar{F}_X(x) = \frac{\varrho \bar{F}_{\mathfrak{e}}(x)}{1-(1-\varrho)\bar{F}_{\mathfrak{e}}(x)}$$

and therefore

$$\bar{F}_{\mathfrak{e}}(x) = \frac{\bar{F}_X(x)}{\varrho + (1-\varrho)\bar{F}_X(x)}$$

When $\mathfrak{e}_n$, it follows two-parameter Lindley with parameters α and β

$$\bar{F}_{\mathfrak{e}}(x) = \frac{(\alpha+\beta+\alpha\beta x)}{\alpha+\beta} e^{-\alpha x}$$

Hence

$$\bar{F}_X(x) = \frac{\frac{\varrho(\alpha+\beta+\alpha\beta x)}{\alpha+\beta} e^{-\alpha x}}{\left[1-\bar{\varrho}\left(\frac{(\alpha+\beta+\alpha\beta x)}{\alpha+\beta} e^{-\alpha x}\right)\right]}$$

It is easy to see that this is the sf of the MOTL(α, β, ϱ).

Conversily, if

$$\bar{F}_X(x) = \frac{\frac{\varrho(\alpha+\beta+\alpha\beta x)}{\alpha+\beta} e^{-\alpha x}}{\left[1-\bar{\varrho}\left(\frac{(\alpha+\beta+\alpha\beta x)}{\alpha+\beta} e^{-\alpha x}\right)\right]}$$

then $\bar{F}_{\mathfrak{e}_n}(x)$ is distributed as two-parameter Lindley distribution with parameters α and β, and the process is stationary.

We have to prove its stationarity, so we take that $X_{n-1}\sim$ MOTL(α, β, ϱ) and $\mathfrak{e}_n$ follows two-parameter Lindley distribution with parameters α and β, then

$$\bar{F}_X(x) = \frac{\frac{\varrho(\alpha+\beta+\alpha\beta x)}{\alpha+\beta}e^{-\alpha x}}{\left[1-\bar{\varrho}\left(\frac{(\alpha+\beta+\alpha\beta x)}{\alpha+\beta}e^{-\alpha x}\right)\right]}$$

It is easy to see that X_n is distributed as MOTL(α, β, ϱ).

3.7.2 MIN AR (1) Model-II with MOTL Marginal Distribution

The second AR (1) structure is given by

$$X_n = \begin{cases} X_{n-1}, & \text{with probability } \varrho_1 \\ \mathfrak{e}_n, & \text{with probability } \varrho_2 \\ min(X_{n-1}, \mathfrak{e}_n), & \text{with probability } 1-\varrho_1-\varrho_2 \end{cases} \quad (18)$$

where ϱ_1, $\varrho_2 > 0$, $\varrho_1 + \varrho_2 < 1$ and $\{\mathfrak{e}_n\}$ is a sequence of iid random variables independent of $\{X_n\}$. This structure allows probabilistic selection of process values, innovations and combinations of both. Then the process is stationary with MOTL as marginal.

Theorem: 3. *$\{X_n\}$ is stationary Markovian with MOTL distribution with parameters γ, α and $\beta \Longleftrightarrow$ $\{\mathfrak{e}_n\}$ is distributed as two-parameter Lindley distribution with parameters α and β, where $\gamma = \frac{\varrho_2}{1-\varrho_1}$, under in an AR (1) process with structure defined in equation (18).*

Proof.

Let $\mathfrak{e}_n$ follows two-parameter Lindley with parameters α and β. By using equation (18),

$$\bar{F}_{X_n}(x) = \varrho_1[1-(1-\bar{F}_{X_{n-1}}(x))(1-\bar{F}_{\mathfrak{e}_n}(x))] + \varrho_2\bar{F}_{X_{n-1}}(x)\bar{F}_{\mathfrak{e}_n}(x) + (1-\varrho_1-\varrho_2)\bar{F}_{X_{n-1}}(x)$$

Under stationary equilibrium it becomes,

$$\bar{F}_X(x) = \frac{\gamma\bar{F}_{\mathfrak{e}}(x)}{1-(1-\gamma)\bar{F}_{\mathfrak{e}}(x)}$$

where $\gamma = \frac{\varrho_2}{1-\varrho_1}$, it is evident that it has Marshall-Olkin form. Next, we assume that $\{X_n\} \sim$ MOTL(α, β, γ). By using equation (18), under stationarity, we can write

$$\bar{F}_{\mathfrak{e}}(x) = \frac{(1-\varrho_1)\bar{F}_X(x)}{\varrho_2 + (1-\varrho_1-\varrho_2)\bar{F}_X(x)}$$

Next by using X_n as MOTL(α, β, γ), it can be obtained as

$$\bar{F}_{\mathfrak{e}}(x) = \frac{(\alpha+\beta+\alpha\beta x)}{\alpha+\beta}e^{-\alpha x}$$

which is the sf of two-parameter Lindley distribution with parameters α and β.

3.7.3 MAX-MIN AR(1) model-I with MOTL Marginal Distribution

Consider now the third model with AR(1) structure

$$X_n = \begin{cases} max(X_{n-1}, \mathfrak{e}_n), & \text{with probability } \varrho_1 \\ min(X_{n-1}, \mathfrak{e}_n), & \text{with probability } \varrho_2 \\ X_{n-1}, & \text{with probability } 1-\varrho_1-\varrho_2 \end{cases} \quad (19)$$

where $0 < \varrho_1, \varrho_2 < 1$, $\varrho_2 < \varrho_1$, $\varrho_1 + \varrho_2 < 1$ and $\{\mathfrak{e}_n\}$ is a sequence of iid random variables independent of $\{X_n\}$. Then the process is stationary Markovian with MOTL distribution as marginal.

Theorem: 4. *$\{X_n\}$ is stationary Markovian AR (1) max-MIN process with MOTL distribution with parameters α, β and $\gamma \iff \{\mathfrak{e}_n\}$ is distributed as two-parameter Lindley distribution with parameters α and β, where $\gamma = \frac{\varrho_1}{\varrho_2}$, under an AR (1) MAX-MIN process with structure (19).*

Proof. Let $\mathfrak{e}_n$ follows two-parameter Lindley with parameters α and β. It is obvious from equation (19),

$$\bar{F}_{X_n}(x) = \varrho_1[1-(1-\bar{F}_{X_{n-1}}(x))(1-\bar{F}_{\mathfrak{e}_n}(x))] + \varrho_2\bar{F}_{X_{n-1}}(x)\bar{F}_{\mathfrak{e}_n}(x) + (1-\varrho_1-\varrho_2)\bar{F}_{X_{n-1}}(x)$$

Under stationary equilibrium,

$$\bar{F}_{X_n}(x) = \frac{\gamma\bar{F}_{\mathfrak{e}}(x)}{1-(1-\gamma)\bar{F}_{\mathfrak{e}}(x)}$$

where $\gamma = \frac{\varrho_1}{\varrho_2}$ and $\bar{F}_{X_n}(x)$ has Marshall-Olkin form. Let $X_n \sim$ MOTL(α, β, γ).
Then by using (19), under stationarity,

$$\bar{F}_{\mathfrak{e}}(x) = \frac{\varrho_2\bar{F}_{X_n}(x)}{\varrho_1 + (\varrho_2-\varrho_1)\bar{F}_{X_n}(x)}$$

Thus, after simplification it can be written as

$$\bar{F}_{\mathfrak{e}}(x) = \frac{(\alpha+\beta+\alpha\beta x)}{\alpha+\beta}e^{-\alpha x}$$

Consequently, which is the sf of two-parameter Lindley with parameters α and β.

3.7.4 MAX-MIN AR(1) model-II with MOTL Marginal Distribution

Finally, we consider the more general max-min process that includes minimum, maximum innovations and the process. The relating model with AR(1) structure being of the form

$$X_n = \begin{cases} max(X_{n-1}, \mathfrak{e}_n), & \text{with probability } \varrho_1 \\ min(X_{n-1}, \mathfrak{e}_n), & \text{with probability } \varrho_2 \\ \mathfrak{e}_n, & \text{with probability } \varrho_3 \\ X_{n-1}, & \text{with probability } 1-\varrho_1-\varrho_2-\varrho_3 \end{cases} \quad (20)$$

where $0 < \varrho_1, \varrho_2, \varrho_3 < 1$, $\varrho_1+\varrho_2+\varrho_3 < 1$ and $\{\mathfrak{e}_n\}$ is a sequence of iid random variables independent of $\{X_n\}$. Then the process is stationary Markovian with MOTL distribution as marginal.

Theorem: 5. *AR (1) MAX-MIN process $\{X_n\}$ with structure (20) is a stationary Markovian AR (1) MAX-MIN process with MOTL distribution (α, β, γ) $\iff \{\mathfrak{e}_n\}$ is distributed as two-parameter Lindley distribution with parameters α and β where $\gamma = \frac{\varrho_1+\varrho_3}{\varrho_2+\varrho_3}$*

Proof. Let $\mathfrak{e}_n$ follows two-parameter Lindley with parameters α and β. It is clear from equation (20),

$$\begin{aligned} \bar{F}_{X_n}(x) &= \varrho_1[1-(1-\bar{F}_{X_{n-1}}(x))(1-\bar{F}_{\mathfrak{e}_n}(x))] + \varrho_2\bar{F}_{X_{n-1}}(x)\bar{F}_{\mathfrak{e}_n}(x) + \varrho_3\bar{F}_{\mathfrak{e}_n}(x) \\ &+ (1-\varrho_1-\varrho_2-\varrho_3)\bar{F}_{X_{n-1}}(x) \end{aligned}$$

Under stationary equilibrium it gives,

$$\bar{F}_{X_n}(x) = \frac{\gamma\bar{F}_{\mathfrak{e}}(x)}{1-(1-\gamma)\bar{F}_{\mathfrak{e}}(x)}$$

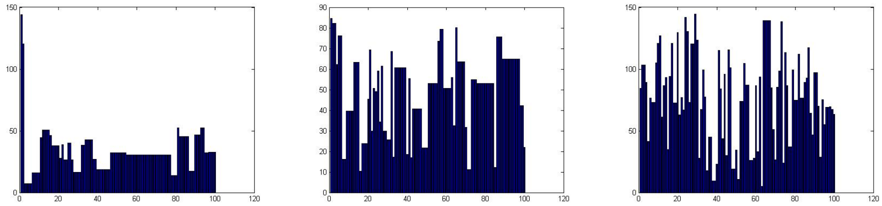

Figure 3: *Graphs of sample path of AR(1) Minification model I for different values of ϱ=0.3,0.5,0.8, α=30 and $\beta = 0.02$*

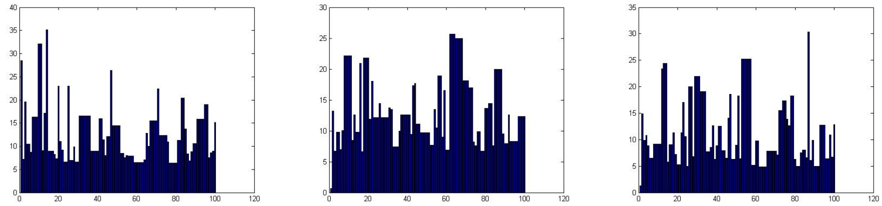

Figure 4: *Graphs of sample path of AR(1) Minification model II for different sets of (ϱ_1, ϱ_2)=(0.1,0.4), (0.4,0.1), (0.4,0.4), α=0.2 and β =0.3*

where $\gamma = \frac{\varrho_1+\varrho_3}{\varrho_2+\varrho_3}$, which has Marshall-Olkin form. Now let $X_n \sim$ MOTL(α, β, γ) and from (20), we obtain

$$\bar{F}_{\mathfrak{e}}(x) = \frac{(\varrho_2 + \varrho_3)\bar{F}_X(x)}{(\varrho_1 + \varrho_3) + (\varrho_2 - \varrho_1)\bar{F}_X(x)}$$

Thus, after simplification it reduces to

$$\bar{F}_{\mathfrak{e}}(x) = \frac{(\alpha + \beta + \alpha\beta x)}{\alpha + \beta} e^{-\alpha x}$$

which is the sf of two-parameter Lindley with parameters α and β. The sample path properties of the four AR(1) models developed in this section are displayed in Figure 3-6 and it shows how these measures vary with different values of parameters.

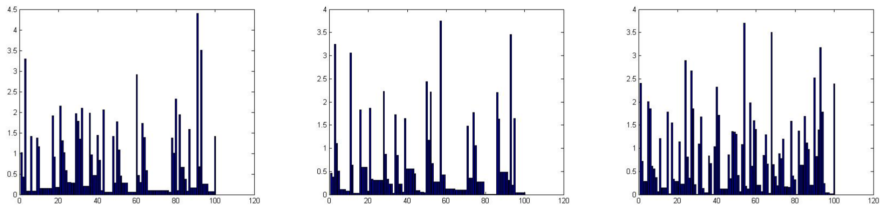

Figure 5: *Graphs of sample path of AR(1) Minification model II for different sets of (ϱ_1, ϱ_2)=(0.3,0.5), (0.5,0.3), (0.4,0.4), α=1.7 and β =1.4*

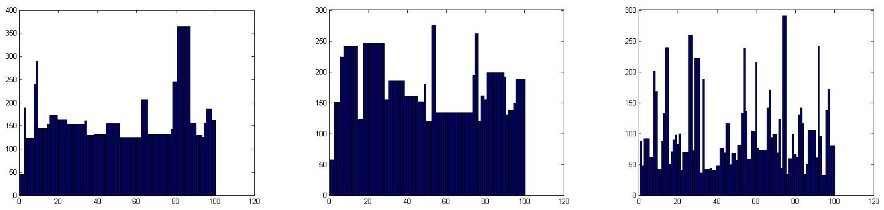

Figure 6: *Graphs of sample path of AR(1) Min-max model II for different sets of* $(\varrho_1, \varrho_2, \varrho_3)$*=(0.1,0.3,0.4), (0.2,0.2,0.4), (0.1,0.1,0.1),* α*=0.02 and* β *=0.02*

4. Estimation of Parameters

In this section, we consider maximum likelihood estimation (MLE) for a given sample of size $x_1, x_2, ..., x_n$ from MOTL(α, β, γ), then the log likelihood function is given by

$$\begin{aligned} l(\xi) &= n(\log\gamma + 2\log\alpha) + \sum_{i=1}^{n}\log(1+\beta x_i) - \alpha\sum_{i=1}^{n} x_i - n\log(\alpha+\beta) \\ &- 2\sum_{i=1}^{n}\log\left\{1-(1-\gamma)\frac{(\alpha+\beta+\alpha\beta x_i)e^{-\alpha x_i}}{\alpha+\beta}\right\} \end{aligned}$$

The partial derivative of the log likelihood functions with respect to the parameters are

$$\frac{\partial l(\xi)}{\partial\alpha} = \frac{2n}{\alpha} - \sum_{i=1}^{n} x_i - \frac{n}{\alpha+\beta} - 2\sum_{i=1}^{n}(1-\gamma)\frac{\left((\alpha^2+(\alpha+\beta)\alpha\beta x_i)x_i e^{-\alpha x_i}\right)}{\left\{1-(1-\gamma)\frac{(\alpha+\beta+\alpha\beta x_i)e^{-\alpha x_i}}{\alpha+\beta}\right\}(\alpha+\beta)^2}$$

$$\frac{\partial l(\xi)}{\partial\beta} = \sum_{i=1}^{n}\frac{x_i}{1+\beta x_i} - \frac{n}{\alpha+\beta} + 2\sum_{i=1}^{n}\frac{(1-\gamma)\alpha^2 x_i e^{-\alpha x_i}}{\left\{1-(1-\gamma)\frac{(\alpha+\beta+\alpha\beta x_i)e^{-\alpha x_i}}{\alpha+\beta}\right\}(\alpha+\beta)^2}$$

$$\frac{\partial l(\xi)}{\partial\gamma} = \frac{n}{\gamma} - 2\sum_{i=1}^{n}\frac{\frac{(\alpha+\beta+\alpha\beta x_i)e^{-\alpha x_i}}{\alpha+\beta}}{\left\{1-(1-\gamma)\frac{(\alpha+\beta+\alpha\beta x_i)e^{-\alpha x_i}}{\alpha+\beta}\right\}}$$

The MLE $\hat{\xi}=(\hat{\alpha}, \hat{\beta}, \hat{\gamma})^T$ of $\xi=(\alpha, \beta, \gamma)^T$ can be numerically obtained by solving the equations $\frac{\partial l(\xi)}{\partial\alpha} = 0$, $\frac{\partial l(\xi)}{\partial\beta} = 0$, $\frac{\partial l(\xi)}{\partial\gamma} = 0$. For this purpose, we can use functions like *nlm*, *fitdist* or *optimize* from the programming language R.

5. Application

Now we use a real data set to show that the MOTL distribution can be a better model than the some other generalized exponential and Lindley distributions. The distributions are given below:

1. beta generalized exponential distribution (BGE) [9]
2. exponentiated exponential distribution (EE) [7]
3. gamma generalized exponential distribution (GGE) [39]
4. Kumaraswamy generalized exponential distribution (KGE) [6]

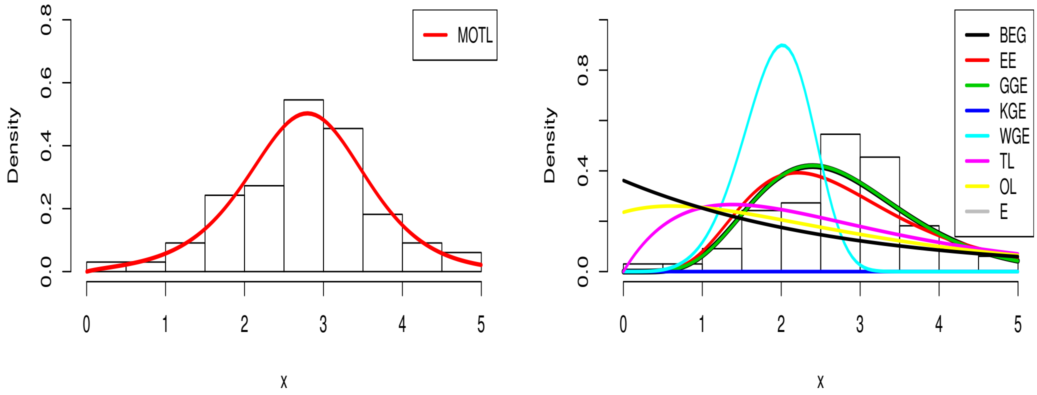

Figure 7: *pdf for fitted distributions of the breaking stress of carbon fibers data*

5. Weibull generalized exponential distribution (WGE) [1]
6. two-parameter Lindley distribution (TL) [31]
7. one-parameter Lindley distribution (OL) [16, 17]
8. classical exponential distribution (E)

The data set represents the breaking stress of carbon fibers of 50mm in length (n=66) and it has been given by [24]. The data set is given as:

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11,3.56, 4.42, 2.41, 3.19, 3.22, 1.96, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20,3.33, 2.55, 3.31, 3.31, 2.85, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88,2.82, 2.05, 3.65, 3.75, 2.43, 2.95, 2.97, 3.39,2.96, 2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53

The distribution of this data set is unimodal and slightly left skewed (skewness = 0.131 and kurtosis = 3.223). For each distribution, we estimated the unknown parameters (by the maximum likelihood method), the values of the −log-likelihood (−logL), AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), the values of the Kolmogorov-Smirnov (K-S) statistic and the corresponding p-values.

All the computations were done through the use of the R programming language. The results for these data are listed in Table 3. From the table, we can observe that MOTL distribution provides smallest −logL, AIC, BIC and K-S statistics values and highest p-value as compare to other distributions. This indicates that the MOTL distribution provides a better fit than other distributions. Plots of the histogram with fitted density functions and comparison of the cumulative distribution function for the each models with the empirical distribution function are displayed in Figure **??** and Figure **??**. From figures, one can easily identify the suitability behavior of the MOTL distribution. Therefore, the new model may be an interesting alternative to the other available generalized exponential models in the literature.

Table 3: *Estimated values, −logL, AIC, BIC, K-S statistics and p-value for data set.*

Distribution	Estimates	−logL	AIC	BIC	K-S	p-value
MOTL(α,β,γ)	$\hat{\alpha}$ = **2.2964** $\hat{\beta}$= **11012.6970** $\hat{\gamma}$= **79.8819**	**84.5821**	**175.1641**	**181.7331**	**0.0599**	**0.9717**
BEG(a,b,β)	$\hat{\beta}$= 4.323281e+06 $\hat{a}$=7.5387 $\hat{b}$=20.1066 $\hat{\beta}$=0.1176	90.9961	187.9922	194.5611	0.1309	0.2082
EE(a,β)	$\hat{a}$=9.2945 $\hat{\beta}$=1.0092	95.2187	194.4373	198.8166	0.1527	0.0921
GGE(a,β)	a=7.5710 $\hat{\beta}$=2.7395	90.9359	185.8719	190.2512	0.1303	0.2124
KGE(a,b,β)	$\hat{a}$ =4.3710 $\hat{b}$= 53.0724 $\hat{\beta}$= 0.1696	86.5579	179.1158	185.6848	0.0899	0.6603
WGE(a,b,β)	$\hat{a}$ = 3.4608 $\hat{b}$= 5.0392 $\hat{\beta}$= 1.6439 $\hat{\gamma}$=50.4204	85.7900	177.58	184.149	0.0799	0.7925
TL(α, β)	$\hat{\alpha}$= 2.2410 $\hat{\beta}$=174.011	112.0511	228.1022	232.4815	0.2510	-
OL(β)	$\hat{\beta}$=0.5895	181.7535	246.8953	249.085	0.3017	-
E(β)	$\hat{\beta}$= 0.3618	133.0921	268.1842	270.3739	0.35764	-

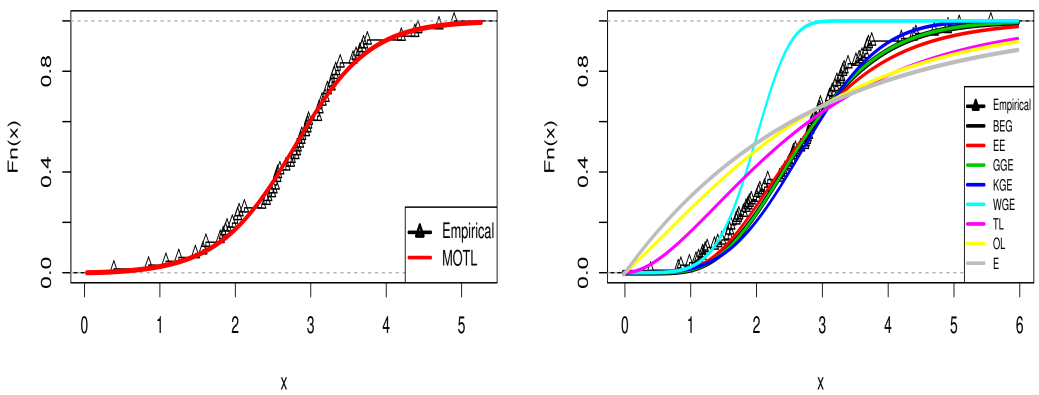

Figure 8: *Estimated cumulative distribution function for the data set*

6. Conclusions

The quality of the methods used in statistical analysis is eminently dependent on the underlying statistical distributions. In this article, we proffered a new customized Lindley distribution. The proposed distribution enfolds exponential and Lindley distributions as sub-models. Some properties of this distribution such as quantile function, moments, moment generating function, distributions of order statistics, limiting distributions of order statistics, entropy and autoregressive time series models are studied. This distribution is found to be the most appropriate model to fit the carbon fibers data compared to other models. Consequently, we propose the MOTL distribution for sketching inscrutable lifetime data sets.

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