ON GOMPERTZ EXPONENTIATED INVERSE RAYLEIGH DISRIBUTION

Sule Omeiza Bashiru[1] and Halid Omobolaji Yusuf[2]

•

[1]Prince Abubakar Audu University, Anyigba, Kogi State, Nigeria
[2]Ekiti State University, Ado-Ekiti, Ekiti State, Nigeria
Email: [1]bash0140@gmail.com, Email: [2]omobolajihalid01@gmail.com

Abstract

In this paper, we proposed a four parameter Gompertz Exponentiated Inverse Rayleigh Distribution. The proposed distribution is an extention of the Exponentiated Inverse Rayleigh Distribution which was compounded with the Gompertz generated family of distribution. Several of its statistical and mathematical properties including quantiles, median, moments, skewness and kurtosis are derived. Also, the reliability and hazard rate functions are derived. To estimate the new model parameters, the maximum likelihood technique is used. To evaluate the effectiveness of the estimators in this model, a simulation study was carried out and the result of the simulation study indicated that the model is consistent since the value of the mean square error decrease as sample size increases. Finally, the usefulness of the proposed distribution is illustrated with two datasets and it is discovered that this model is more adaptable when compared to well-known models..

Keywords: Maximum likelihood estimation; Skewness; Kurtosis; Probability density function; Cummulative probability distribution.

1. Introduction

The classical distributions frequently do not offer an appropriate match to some real data sets in real-world circumstances. In order to create novel distributions, researchers devised numerous generators by inserting one or more parameters. The newly generated distributions are more adaptable than the classical distributions.

Gompertz[10] introduced a continuous probability distribution known as the Gompertz probability distribution (GD). The GD is employed to explore nature of human mortality by determining the value of life's unexpected events. Several branches of statistics have used the Gompertz distribution where survival time is necessary such as in demography Vaupel [18], Preston et al [16] and in actuary Willemse and Koppelaar[19); in gerontology, medicine, biology, and related sciences Economos [7], Brown and Forbes [6]. In this article, the Gompertz family of distributions is used to create a novel model. Some authors that have employed the Gompertz Family of distributions include : Halid and Sule [12], Alizadeh et al. [4], and Abdal-Hameed et al. [1] . Halid and Sule [12] defined the cummulative density function (CDF) of the Gompertz family of distribution as:

$$F_X(x) = 1 - e^{\left(\frac{\varphi}{\eta}\right)\left[1-(1-G(x))^{-\eta}\right]} \tag{1}$$

and the corresponding PDF to (1) is given by

$$f_X(x) = \varphi g(x)\left[1 - G(x)\right]^{-\eta-1} e^{\left(\frac{\varphi}{\eta}\left\{1-\left[1-G(x)\right]^{-\eta}\right\}\right)} \tag{2}$$

where φ and η are the extra shape parameters

The article is broken down into the following sections: In Section 2, the new distribution GEIR's derivation is described. In Section 3, the mathematical characteristics of the new distribution are explained and the Maximum likelihood estimation of the distribution is used to estimate the parameters. In Section 4, we presented and explored the new distribution's practical applicability. Finally, Section 5 displays the concluding remarks.

2. Derivation of Gompertz Exponentiated Inverse Rayleigh Distribution

In this section, we derived the Gompertz Exponentiated Inverse Rayleigh (GEIR) distribution. Rao and Mbwambo [17], introduced the CDF and PDF of Exponentiated Inverse Rayleigh (EIR) Distribution as

$$G_X(x) = 1 - \left(1 - e^{\frac{\xi}{x}}\right)^{\alpha}; \qquad x \geq 0, \xi > 0, \alpha > 0 \tag{3}$$

The corresponding pdf is given as:

$$g_X(x) = \frac{2\alpha\xi^2}{x^3} e^{-\left(\frac{\xi}{x}\right)^2} \left(e^{-\left(\frac{\xi}{x}\right)^2}\right)^{\alpha-1}; \qquad x \geq 0, \xi > 0, \alpha > 0 \tag{4}$$

putting equation (3) into (1) we have the CDF of Gompertz Exponentiated Inverse Rayleigh (GEIR)

$$F_X(x) = 1 - e^{\left(\frac{\varphi}{\eta}\right)\left[1-\left(1-exp(-(\xi/x)^2)\right)^{-\eta\alpha}\right]} \qquad x \geq 0, \xi > 0, \alpha > 0, \eta > 0, \varphi > 0 \tag{5}$$

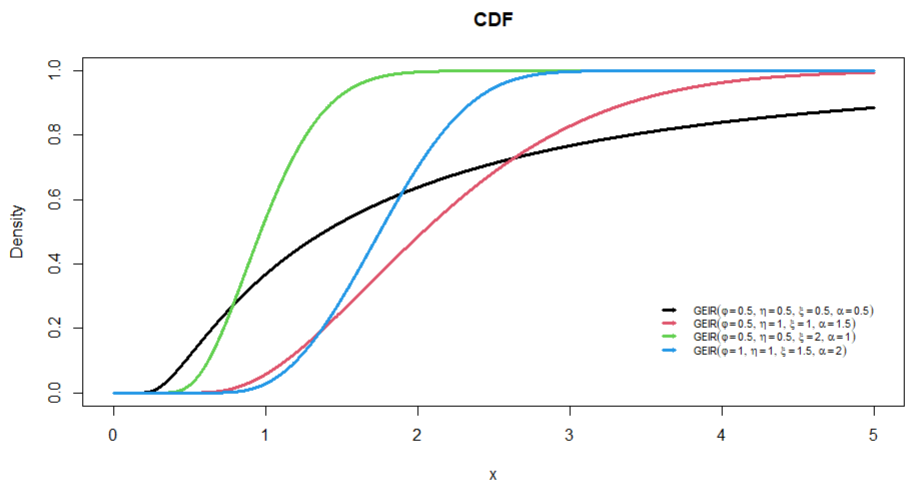

Figure 1: *CDF plot of GEIR distribution for different parameter values*

Now, putting (3)and (4) into (2), we now obtained the PDF of the proposed GEIR distribution given by

$$f_X(x) = 2\varphi\xi^2 x^{-3} e^{-\left(\frac{\xi}{x}\right)^2} \left[1 - e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\alpha\eta-1} e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\alpha\eta}\right\}} \tag{6}$$

where η and α are shape parameters, φ and ξ are scale parameters.

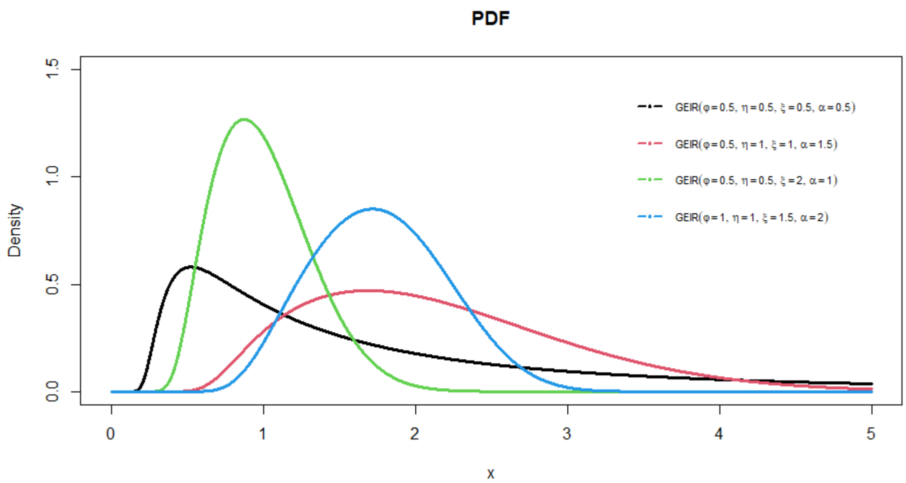

Figure 2: *PDF plot of GEIR distribution for different parameter values*

3. Properties of GEIR Distribution

3.1. Linear Mixture of GEIR

Given the CDF and PDF of GEIR distribution (5) and (6), the expressions

$$e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta\alpha}\right\}} = \sum_k^{\infty}\frac{(-1)^k}{m!}\left(\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta\alpha}\right\}\right)^k$$

$$= \sum_k^{\infty}\frac{(-1)^k}{m!}\left(\frac{\varphi}{\eta}\right)^k\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta\alpha}\right\}^k$$

$$\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta\alpha}\right\}^k = 2\varphi\alpha\xi^2x^{-3}e^{-\left(\frac{\xi}{x}\right)^2}\sum_k^{\infty}\sum_m^{\infty}\frac{(-1)^{k+m}}{m!}\binom{k}{m}\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta\alpha(m+1)-1}$$

$$f(x) = \sum_k^{\infty}\sum_m^{\infty}\frac{(-1)^{k+m}}{m!}\binom{k}{m}2\varphi\alpha\xi^2x^{-3}e^{-\left(\frac{\xi}{x}\right)^2}\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta\alpha(m+1)-1}$$

$$\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta\alpha(m+1)-1} = \sum_k^{\infty}\sum_m^{k}\sum_n^{\infty}\frac{(-1)^{k+m+n}}{m!}\binom{k}{m}\binom{-\eta\alpha(m+1)-1}{n}e^{-n\left(\frac{\xi}{x}\right)^2}$$

So therefore, the PDF of GEIR distribution can be expressed as

$$f(x) = \omega_{k,m,n}2\,(n+2)\,\varphi\alpha\xi^2x^{-3}\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)^{n+1}$$

where

$$\omega_{k,m} = \sum_k^{\infty}\sum_m^{k}\sum_n^{\infty}\frac{(-1)}{m!\,(n+2)}\binom{k}{m}\binom{-\eta\alpha(m+1)-1}{n}$$

and the CDF of GEIR distribution can be expressed as

$$F(x) = \omega_{k,m,n} \varphi \alpha \left(e^{-\left(\frac{\xi}{x}\right)^2} \right)^{n+2}$$

where $g_{n+1}(x)$ is the PDF of Gompertz Exponentiated Inverse Rayleigh distribution with shape parameter n+2

3.2. Survival Function

According to Ieren and Balogun [13], the survival function discribes the probability that a unit, or component, or individual will not fail at a given time . A survival function is generally expressed as

$$S(x) = 1 - F(x; \varphi, \alpha, \eta, \xi) \tag{7}$$

Therefore the survival function of GEIR distribution is derived by substituting (5) into (7) which resulted to

$$S_X(x) = e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\alpha\eta}\right\}} \tag{8}$$

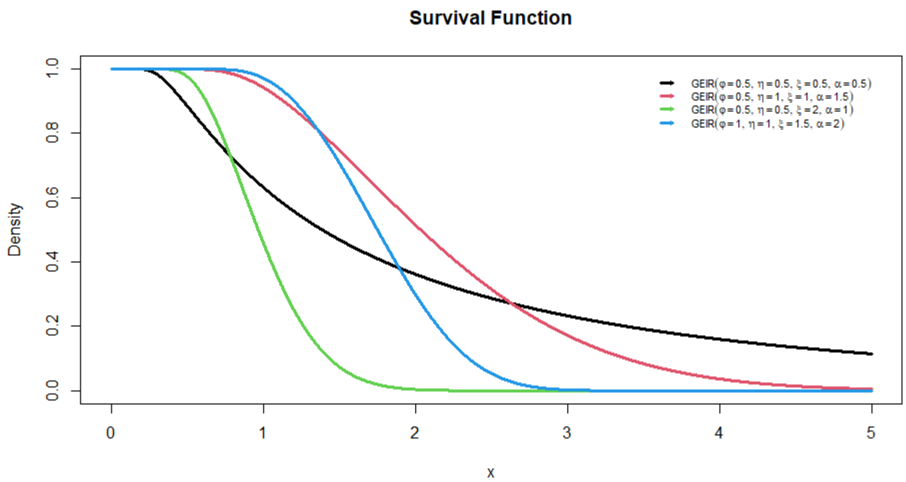

Figure 3: *GEIR distribution survival plot for various parameter values*

3.3. Hazard Function

The hazard function is given as

$$h(x) = \frac{f(x)}{1 - F(x)} \tag{9}$$

The hazard function for GEIR distribution is derived by substituting (5) and (6) into (9) and it resulted into

$$h_X(x) = \frac{2\varphi\xi^2 x^{-3} e^{-\left(\frac{\xi}{x}\right)^2} \left[1 - e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\alpha\eta-1} e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta\alpha}\right\}}}{e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta\alpha}\right\}}} \tag{10}$$

$$h_X(x) = 2\varphi\xi^2 x^{-3} e^{-\left(\frac{\xi}{x}\right)^2} \left[1 - e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\alpha\eta-1} \tag{11}$$

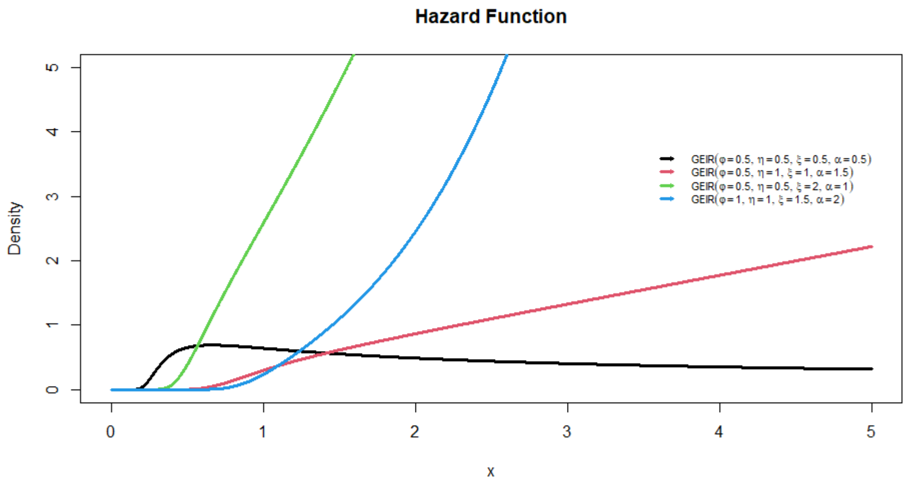

Figure 4: *Hazard plot of GEIR distribution for different parameter values*

3.4. Cumulative Hazard Function

From this definition, the cumulative hazard function, $H_X(x)$, of a continuous random variable, X, which follows the GEIR distribution is obtained.

$$H_X(x) = -log[S_X(x)] \tag{12}$$

substituting equation (8) into (12), we obtain

$$H_X(x) = -log\left[e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta\alpha}\right\}}\right] \tag{13}$$

$$H_X(x) = -\frac{\varphi}{\eta}\left\{1 - \left[1 - e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta\alpha}\right\} \tag{14}$$

3.4.1 Reversed Hazard Function

The reversed hazard function can be obtained by applying the formula below:

$$\tau(x) = \frac{f(x)}{F(x)} \tag{15}$$

Hence, we obtain the reversed hazard function by substituting (5) and (6) in (15)

$$\tau(x) = \frac{2\varphi\xi^2 x^{-3} e^{-\left(\frac{\xi}{x}\right)^2} \left[1 - e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\alpha\eta-1} e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\alpha\eta}\right\}}}{1 - e^{\left(\frac{\varphi}{\eta}\right)\left[1-(1-exp(-(\xi/x)^2))^{-\eta\alpha}\right]}} \tag{16}$$

3.5. Quantile Function, Median, Skewness and Kurtosis

The p^{th} quantile of the GEIR distribution is derived as

$$Q_X(p) = \frac{\xi}{\sqrt{-\log\left(1 - \left[1 - \frac{\eta}{\varphi}\log(1-p)\right]^{-\frac{1}{\eta\alpha}}\right)}} \tag{17}$$

we have the first three , $Q_1 1 = Q(1/4)$ and $Q_3 = Q(3/4)$, that is by substituting value of p=0.25 and p=0.75 in X_p, respectively. Also Quantile is also used in finding the skewness and kurtosis of the distribution.

3.5.1 Median

Substitute p=0.5 in (17), we have

$$M_e = Q_X(0.5) = \frac{\xi}{\sqrt{-\log\left(1 - \left[1 - \frac{\eta}{\varphi}\log(0.5)\right]^{-\frac{1}{\eta\alpha}}\right)}} \tag{18}$$

3.5.2 Skewness and Kurtosis

According to Galton [9] and Moors[15] we can obtain the skewness (Sk) and kurtosis (Ku) measures, respectively for GEIR distribution using the following expression

$$Sk = \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2Q(\frac{1}{2})}{Q(\frac{3}{4}) - Q(\frac{1}{4})} \tag{19}$$

and

$$Ku = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})} \tag{20}$$

3.5.3 Moment

In this section, we consider the moment of the GEIR distribution. Let X = $(x_1, x_2, \ldots, x_n)$ be a sample drawn from GEIR distribution with pdf, then the r^{th} moment μ_r' can be written as

$$\mu_r' = \int_0^\infty x^r f(x)dx$$

$$\mu_r' = \int_0^\infty x^r \omega_{k,m,n} 2\,(n+2)\,\varphi\alpha\xi^2 x^{-3} \left(e^{-\left(\frac{\xi}{x}\right)^2} \right)^{n+1} dx$$

after some mathematical derivations, we obtained the r^{th} moment as

$$= \omega_{k,m,n} 2\,(n+2)\,(n+1)^{(r/2-1)} \varphi\alpha\xi^2 \Gamma(1 - \frac{r}{2}) \qquad r < 2$$

3.6. Maximum Likelihood Estimation

Due to its consistency, asymptotic efficiency, and invariance property, the Maximum Likelihood Estimation (MLE) method is frequently used to estimate unknown parameter(s). Let $x_1, x_2, \ldots, x_n$ be random sample of size n drawn from GEIR distribution, then the likelihood can be expressed as :

$$L(\varphi,\xi,\eta,\alpha) = 2^n \varphi^2 \xi^{2n} \sum_{i=1}^{n} x^{-3} e^{-\sum_{i=1}^{n}\left(\frac{\xi^2}{x^2}\right)} \prod_{i=1}^{n} \left[1-e^{-\frac{\xi^2}{x^2}}\right]^{-1-\eta\alpha} e^{-\sum_{i=1}^{n}\left(\frac{\varphi}{\eta}\left[1-\left(1-e^{-\frac{\xi^2}{x^2}}\right)^{-\alpha\eta}\right]\right)} \tag{21}$$

and the log-likelihood of expression (21) can be expressed as

$$\begin{aligned} l = \log L \quad = \quad & n\ln\left(\varphi\alpha\xi^2\right) - \eta\alpha\sum_{i=1}^{n}\ln\left(1-\mathrm{e}^{-\frac{\xi^2}{x^2}}\right) - \xi^2\sum_{i=1}^{n}\left(\frac{1}{x^2}\right) + \frac{\varphi}{\eta}\sum_{i=1}^{n}\left(1-\left(1-\mathrm{e}^{-\frac{\xi^2}{x^2}}\right)^{-\eta\alpha}\right) \\ & -3\sum_{i=1}^{n}\ln(x) - \sum_{i=1}^{n}\ln\left(1-\mathrm{e}^{-\frac{\xi^2}{x^2}}\right) \end{aligned} \tag{22}$$

Differentiating (22) with respect to φ, ξ, α and η, if equated to zero, we obtain the following estimating equations

$$\frac{\partial l}{\partial \varphi} = \frac{n}{\varphi} + \frac{1}{\eta}\sum_{i=1}^{n}\left(1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta\alpha}\right) \tag{23}$$

$$\begin{aligned} \frac{\partial l}{\partial \xi} \quad = \quad & \frac{2n}{\xi} - 2\eta\alpha\xi\sum_{i=1}^{n}\left(\frac{\mathrm{e}^{-\frac{\xi^2}{x^2}}}{x^2\left(1-\mathrm{e}^{-\frac{\xi^2}{x^2}}\right)}\right) - 2\xi\sum_{i=1}^{n}\left(\frac{2}{x^2}\right) + 2\varphi\alpha\xi\sum_{i=1}^{n}\left(\frac{\left(1-\mathrm{e}^{-\frac{\xi^2}{x^2}}\right)^{-\eta\alpha}\mathrm{e}^{-\frac{\xi^2}{x^2}}}{x^2\left(1-\mathrm{e}^{-\frac{\xi^2}{x^2}}\right)}\right) \\ & -2\xi\sum_{i=1}^{n}\left(-\frac{\mathrm{e}^{-\frac{\xi^2}{x^2}}}{x^2\left(1-\mathrm{e}^{-\frac{\xi^2}{x^2}}\right)}\right) \end{aligned} \tag{24}$$

$$\frac{\partial l}{\partial \eta} = -\alpha\sum_{i=1}^{n}\ln\left(1-\mathrm{e}^{-\frac{\xi^2}{x^2}}\right) + \frac{\alpha\varphi}{\eta}\sum_{i=1}^{n}\left[\left(1-\mathrm{e}^{-\frac{\xi^2}{x^2}}\right)^{-\eta\alpha}\ln\left(1-\mathrm{e}^{-\frac{\xi^2}{x^2}}\right)\right] - \frac{\varphi}{\eta^2}\sum_{i=1}^{n}\left(1-\left(1-\mathrm{e}^{-\frac{\xi^2}{x^2}}\right)^{-\eta\alpha}\right) \tag{25}$$

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \eta\sum_{i=1}^{n}\ln\left(1-\mathrm{e}^{-\frac{\xi^2}{x^2}}\right) + \varphi\sum_{i=1}^{n}\left[\left(1-\mathrm{e}^{-\frac{\xi^2}{x^2}}\right)^{-\eta\alpha}\ln\left(1-\mathrm{e}^{-\frac{\xi^2}{x^2}}\right)\right] \tag{26}$$

The maximum likelihood estimator $\hat{\theta} = (\hat{\varphi}, \hat{\xi}, \hat{\eta}, \hat{\alpha})$ of $\theta = (\varphi, \xi, \eta, \alpha)$ is obtained by solving the nonlinear system of equations (23) - (26). In this study, we used the Newton Raphson technique, a nonlinear optimization procedure, to numerically optimize the log-likelihood function shown in (22). The asymptotic distribution of the element of the 4×4 observed information matrix of GEIR distribution can be expressed as

$$\sqrt{n}(\hat{\theta} - \theta) \sim N_4(0, \Sigma^{-1}) \tag{27}$$

where Σ is the expected information matrix. Thus, the expected information matrix is expressed as

$$\Sigma^{-1} = -E\begin{bmatrix} \frac{\partial^2 l}{\partial \varphi^2} & \frac{\partial^2 l}{\partial \varphi\partial \xi} & \frac{\partial^2 l}{\partial \varphi\partial \eta} & \frac{\partial^2 l}{\partial \varphi\partial \alpha} \\ \frac{\partial^2 l}{\partial \varphi\partial \xi} & \frac{\partial^2 l}{\partial \xi^2} & \frac{\partial^2 l}{\partial \eta\partial \xi} & \frac{\partial^2 l}{\partial \alpha\partial \xi} \\ \frac{\partial^2 l}{\partial \varphi\partial \eta} & \frac{\partial^2 l}{\partial \eta\partial \xi} & \frac{\partial^2 l}{\partial \eta^2} & \frac{\partial^2 l}{\partial \eta\partial \alpha} \\ \frac{\partial^2 l}{\partial \varphi\partial \alpha} & \frac{\partial^2 l}{\partial \alpha\partial \xi} & \frac{\partial^2 l}{\partial \eta\partial \alpha} & \frac{\partial^2 l}{\partial \alpha^2} \end{bmatrix} \tag{28}$$

The solutions to be obtained by solving (28) will yield the asymptotic variance and covariances of the parameters $\hat{\varphi}, \hat{\xi}$, $\hat{\alpha}$ and $\hat{\eta}$. Using (28), the approximate $100(1-\lambda)\%$ confidence intervals for φ, ξ, α and η can be expressed as
$\hat{\varphi} \pm Z_{\frac{\lambda}{2}}\sqrt{\hat{\Sigma}_{11}}, \hat{\xi} \pm Z_{\frac{\lambda}{2}}\sqrt{\hat{\Sigma}_{22}}, \hat{\eta} \pm Z_{\frac{\lambda}{2}}\sqrt{\hat{\Sigma}_{33}}, \hat{\alpha} \pm Z_{\frac{\lambda}{2}}\sqrt{\hat{\Sigma}_{44}}$
where $Z_{\frac{\lambda}{2}}$ is the upper λ^{th} percentile of the standard normal distribution. where

$$\frac{d^2}{d\eta^2} = \sum_{i=1}^{n}\left[\frac{2\varphi\left(1-\left(1-e^{-\frac{\xi^2}{x^2}}\right)^{-\eta\alpha}\right)}{\eta^3}\right] \tag{29}$$

$$\frac{d^2}{d\varphi^2} = -\frac{n}{\varphi^2} \tag{30}$$

$$\frac{d^2}{d\alpha^2} = -\frac{n}{\alpha^2} \tag{31}$$

$$\begin{aligned}
\frac{d^2}{d\xi^2} = & -\frac{2n}{\xi^2} - \sum_{i=1}^{n}\left[\frac{2\eta\alpha e^{-\frac{\xi^2}{x^2}}}{x^2\left(1-e^{-\frac{\xi^2}{x^2}}\right)}\right] + \sum_{i=1}^{n}\left[\frac{4\eta\alpha\xi^2 e^{-\frac{\xi^2}{x^2}}}{x^4\left(1-e^{-\frac{\xi^2}{x^2}}\right)}\right] + \sum_{i=1}^{n}\left[\frac{4\eta\alpha\xi^2\left(e^{-\frac{\xi^2}{x^2}}\right)^2}{x^4\left(1-e^{-\frac{\xi^2}{x^2}}\right)^2}\right] \\
& -\frac{2n}{x^2} - \sum_{i=1}^{n}\left[\frac{4\varphi\left(1-e^{-\frac{\xi^2}{x^2}}\right)^{-\eta\alpha}\eta\alpha^2\xi^2\left(e^{-\frac{\xi^2}{x^2}}\right)^2}{x^4\left(1-e^{-\frac{\xi^2}{x^2}}\right)^2\eta}\right] + \sum_{i=1}^{n}\left[\frac{2\varphi\left(1-e^{-\frac{\xi^2}{x^2}}\right)^{-\eta\alpha}\eta\alpha e^{-\frac{\xi^2}{x^2}}}{x^2\left(1-e^{-\frac{\xi^2}{x^2}}\right)\eta}\right] \\
& -\sum_{i=1}^{n}\left[\frac{4\varphi\left(1-e^{-\frac{\xi^2}{x^2}}\right)^{-\eta\alpha}\eta\alpha\xi^2 e^{-\frac{\xi^2}{x^2}}}{x^4\left(1-e^{-\frac{\xi^2}{x^2}}\right)\eta}\right] - \sum_{i=1}^{n}\left[\frac{4\varphi\left(1-e^{-\frac{\xi^2}{x^2}}\right)^{-\eta\alpha}\eta\alpha\xi^2\left(e^{-\frac{\xi^2}{x^2}}\right)^2}{x^4\left(1-e^{-\frac{\xi^2}{x^2}}\right)^2\eta}\right] \\
& -\sum_{i=1}^{n}\left[\frac{2e^{-\frac{\xi^2}{x^2}}}{x^2\left(1-e^{-\frac{\xi^2}{x^2}}\right)}\right] + \sum_{i=1}^{n}\left[\frac{4\xi^2 e^{-\frac{\xi^2}{x^2}}}{x^4\left(1-e^{-\frac{\xi^2}{x^2}}\right)}\right] + \sum_{i=1}^{n}\left[\frac{4n\xi^2\left(e^{-\frac{\xi^2}{x^2}}\right)^2}{x^4\left(1-e^{-\frac{\xi^2}{x^2}}\right)^2}\right]
\end{aligned} \tag{32}$$

$$\frac{\partial^2}{\partial\xi\partial\varphi} = \sum_{i=1}^{n}\left[\frac{2n\left(1-e^{-\frac{\xi^2}{x^2}}\right)^{-\eta\alpha}\eta\alpha\xi e^{-\frac{\xi^2}{x^2}}}{x^2\left(1-e^{-\frac{\xi^2}{x^2}}\right)\eta}\right] \tag{33}$$

$$\frac{\partial^2}{\partial\alpha\partial\varphi} = 0 \tag{34}$$

$$\frac{\partial^2}{\partial\eta\partial\varphi} = -\sum_{i=1}^{n}\left[\frac{\left(1-\left(1-e^{-\frac{\xi^2}{x^2}}\right)^{-\eta\alpha}\right)}{\eta^2}\right] \tag{35}$$

$$\frac{\partial^2}{\partial\eta\partial\alpha} = 0 \tag{36}$$

$$\frac{\partial^2}{\partial\eta\partial\xi} = -\sum_{i=1}^{n}\left[\frac{2n\varphi\left(1-e^{-\frac{\xi^2}{x^2}}\right)^{-\eta\alpha}\eta\alpha\xi e^{-\frac{\xi^2}{x^2}}}{x^2\left(1-e^{-\frac{\xi^2}{x^2}}\right)\eta^2}\right] \tag{37}$$

$$\frac{\partial^2}{\partial\alpha\partial\xi} = 0 \tag{38}$$

4. Data Analysis

4.1. Simulation Studies

In this section, we simulated data set for sizes n = 30,100,200 and 500 that follows Gompertz Exponentiated Inverse Rayleigh distribution using different parameter values for the four parameters φ, η, α and ξ using the quantile function (inverse transformation method of simulation). We considered the following combinations for the parameters $(\varphi, \eta, \alpha, \xi)$ = ((0.5,1,1,0.8), (1.5,0.6,1.2,1.5), (0.5,0.5,0.5,0.5) and (1,0.5,0.5,1))at different sample sizes n = 30, 100, 200, and 500. The results presented in Table 1 displayed the true values of $(\varphi, \eta, \alpha, \xi)$ and estimated values of $(\varphi, \eta, \alpha, \xi)$ with the standard errors. The results are replicated 10,000 times and the average result were presented in the Table 1.

Table 1: *The MLE estimates and their MSE for different parameter values*

		φ	η	α	ξ	MSE_φ	MSE_η	MSE_α	MSE_ξ
30	φ = 0.5	0.4164	0.6713	1.0835	0.6710	5.9423	9.5783	15.4494	0.1593
100	η = 1	0.5453	1.4859	0.6865	0.7407	2.7936	7.6042	3.5094	0.1109
200	α = 1	0.4947	0.9795	1.0074	0.7908	2.6984	5.3368	5.4865	0.0784
500	ξ = 0.8	0.4410	1.3019	0.8647	0.7593	1.2450	3.6662	2.4340	0.0556
30	φ = 1.5	1.1616	0.7094	0.8432	1.2341	16.8549	10.3031	12.2263	0.2242
100	η = 0.6	1.0910	1.0760	0.8447	1.3581	8.7396	8.6236	6.7624	0.1484
200	α = 1.2	1.0830	0.6560	1.2001	1.4536	5.4328	3.2944	6.0152	0.1018
500	ξ = 1.5	0.8998	0.7517	1.2599	1.4447	5.4193	4.5324	7.5899	0.0701
30	φ = 0.5	1.8939	1.3197	0.1331	0.4055	14.6817	10.2453	1.0298	0.0964
100	η = 0.5	0.4297	0.5878	0.4620	0.4533	3.1636	4.3259	3.3989	0.0672
200	α = 0.5	0.4996	0.5109	0.4939	0.4895	4.1287	4.2278	4.0834	0.0474
500	ξ = 0.5	0.5095	0.7248	0.4092	0.4760	1.5357	2.1832	1.2322	0.0317
30	φ = 1	1.3023	0.5704	0.3324	0.8097	27.7198	12.1465	7.0727	0.1621
100	η = 0.5	0.9354	0.7846	0.3940	0.9009	4.7474	3.9884	1.9979	0.1091
200	α = 0.5	0.9282	0.5241	0.5117	0.9699	4.5882	2.5960	2.5287	0.0763
500	ξ = 1	0.9232	0.7251	0.4500	0.9601	5.0199	3.9469	2.4472	0.0514

4.2. Data Description

The strength data was originally reported by Badar and Priest [5] where the strength is measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows at gauge lengths of 20 mm. These data set were fitted to GEIR distribution, the Half- Logistics Inverse Rayleigh (HLIR) distribution by Almarashi et al [3] and the Type II Topp-Leone Inverse Rayleigh (T2TLIR) distribution by Mohammed and Yahia [14]. Other distributions that have been fitted to these same data are the Transmuted Inverse Rayleigh distribution (TIR) by Ahmad et al [2], the Odd Frechet Inverse Rayleigh (OFIR) distribution by Elgarhy and Alrajhi [8], one parameter Inverse Rayleigh (IR) by Trayer [20].

Data I: carbon fibers Strength (20mm) Data set

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.14, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.57, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.88, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

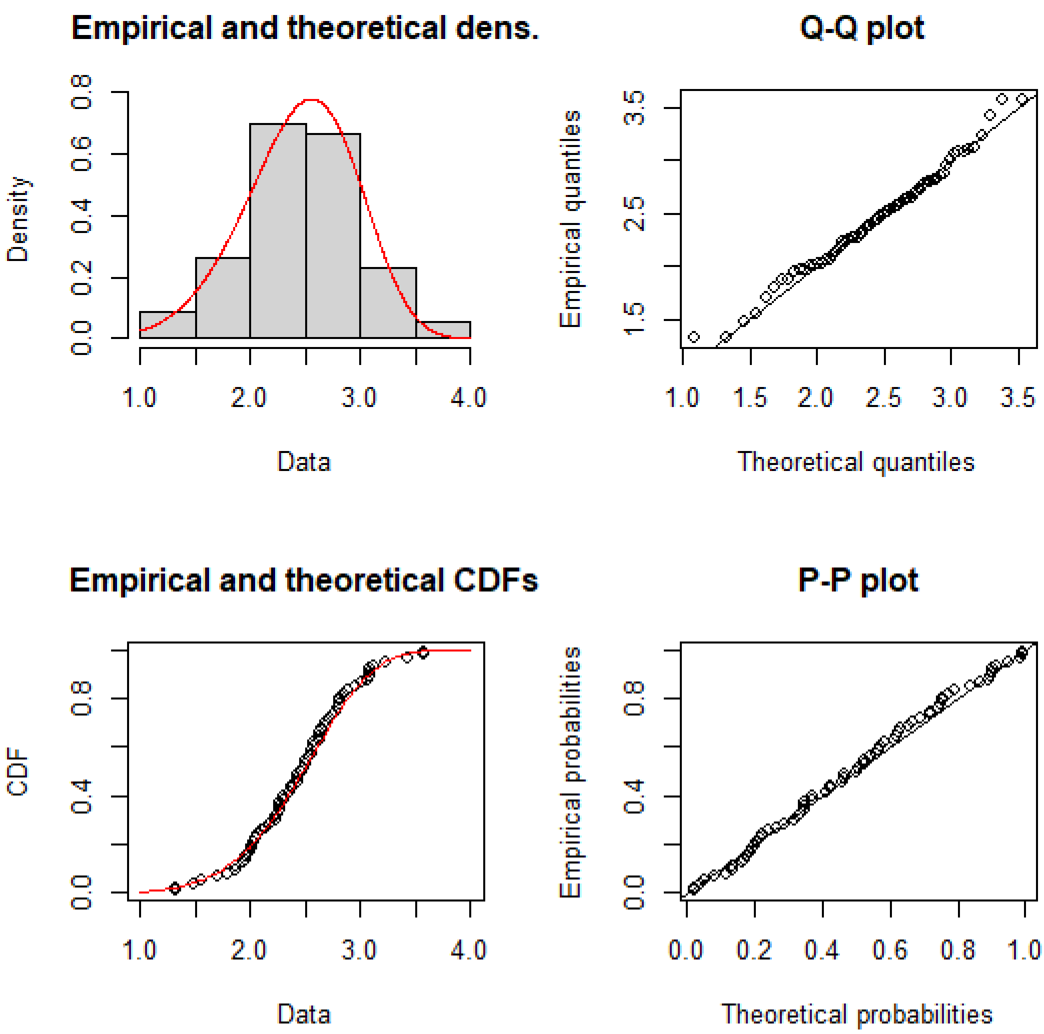

Figure 5: *Empirical and theoretical plot of Carbon*

Table 2 shows the summary statistics of the GEIR, HLIR, T2TLIR, TIR, OFIR, and IR distributions. These five distributions are fitted to data 1 using maximum likelihood estimation.

Table 2: Goodness-of-fit measures based on AIC, BIC, HQIC, K-S values for the Strength (20mm) data set (data 1)

Models	AIC	BIC	HQIC	K-S Value	P-value
GEIR(φ,η,α,ξ)	104.43	107.367	109.112	0.021	0.9321
HLIR(α,λ)	105.003	109.472	106.776	0.0596	0.9668
T2TLIR(α,θ)	108.137	112.605	109.91	0.0776	0.7993
TIR(θ,λ)	145.879	148.113	146.765	0.254	0.0002
OFIR(θ,α)	147.423	151.891	149.196	0.1801	0.0227
IR(α)	178.826	181.06	179.713	0.3549	0

Data II: Patients receiving an analgesic dataset

The data set is taken from Gross and Clark [11] which consists of 20 observations of patients receiving an analgesic 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0

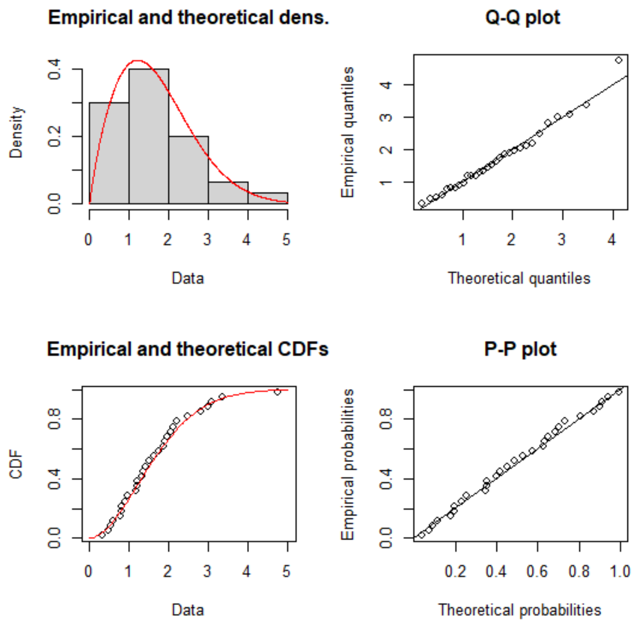

Figure 6: *Empirical and theoretical plot for Patients receiving an analgesic*

Table 3: Estimates and Goodness-of-fit measures based on AIC, BIC, HQIC, K-S values for Patients receiving an analgesic

Distribution	Parameters				AIC	CAIC	BIC	HQIC	Pvalue
GEIR	2.3362134	-0.3288	2.4818	2.2056	39.356	40.212	43.338	40.1332	0.4493
EIR	0.8714	3.1686			46.365	47.0709	48.3564	46.7537	0.1435
WR	11.8552	1.2364	0.0545		48.5149	50.0149	51.5021	49.098	0.4597
GR	3.2748	0.6926			40.805	41.5109	42.7965	41.1938	0.463

5. Conclusion

In this study, a proposed four parameter distributions are added to Gompterz family of distribution called Gompterz exponentiated Inverse Rayleigh (GEIR). Some structural mathematical properties; Moment, Order Statistic, Skewness and kurtosis of the derived model are obtained. A simulation study is carried out to estimate the behaviour of the shape and scale parameters, also maximum likelihood estimation method was employed to estimate the parameters of the distribution and simulation studies were performed to assess the flexibility of the proposed distribution. For the simulated dataset, the result presented in Table (1), from the result, we observed that the estimated values gotten are close to the predefined parameters and that as n increases the MSE reduces which confirms to the law of large numbers.

However, application of two real-life data set shows that the GEIR has strong and better fit than other competing models i.e., the data sets were fitted to the Half- Logistics Inverse Rayleigh (HLIR) distribution and the Type II Topp-Leone Inverse Rayleigh (T2TLIR). Other distributions that have been fitted to these same data are the Transmuted Inverse Rayleigh distribution (TIR), the Odd Frechet Inverse Rayleigh (OFIR) distribution, exponentiated inverse Rayleigh distribution (EIR), Weibul Rayleigh (WR), Gamma Rayleigh (GR) , one parameter Inverse Rayleigh (IR) distributions using goodness of fit and information criterion.

References

[1] Abdal-Hameed M.K., Khaleel2 M.A., Abdullah Z.M, Oguntade P.E., and Adejumo A.O(2018) Parameter Estimation and Reliability, Hazard Functions of Gompertz Burr Type XII Distribution. *Tikrit Journal of Administration and Ecomomics Sciences*, *14(1)*, 381 - 400

[2] Ahmad, A, Ahmad S.P.,and Ahmed A. (2014) Transmuted Inverse Rayleigh distribution: A generalization of the inverse rayleigh distribution. *Math. Theory Model*, *4(7)*, 111-131

[3] Almarashi A.M., Majdah M.B., Elgarhy M., Jamal F. and Chesneau C. (2020). Statistical Inference of the Half-Logistic Inverse Rayleigh Distribution. *Entropy*, *2020,22*, 449

[4] Alizadeh, M., Cordeiro, G. M., Pinho, L.G.B. and Ghosh, I. (2017). The Gompertz-G family of distributions. *Journal of Statistical Theory and Practice*, *11(1)*, 179-207.

[5] Bader, M. and Priest, A. (1982) Statistical Aspect of Fiber and Bundle Strength in Hybrid Composites. In: Hayashi, T., Kawata, S. and Umekawa, S., Eds., *Progress in Science and Engineering Composites, ICCM-IV, Tokyo*,, 1129-1136

[6] Brown, K. and Forbes, W.A. (1974). Mathematical model of aging processes. *Journal of Gerontology*, *29(1)*, 46-51.

[7] Economos, A. (1982). Rate of aging, rate of dying and the mechanism of mortality. *Archives of Gerontology and Geriatrics*, *1*, 46-51.

[8] Elgarhy M. and Alrajhi, S (2019) The odd Frechet inverse Rayleigh distribution: Statistical Properties and applications *J. nonlinear Sci. Appl 12*, 291-299

[9] Galton, F. (1983). Enquiries into Human Faculty and its Development. *Macmillan and Company, London.*

[10] Gompertz B. (1825). On the nature of the function expressive of the law of human mortality and on a new model of determining life contigencies. *Phil Trans. R. Soc.*, *115*, 513-585

[11] Gross, A.J. and Clark, V.A. (1975). Survival Distributions: Reliability Application in the Biometrical Sciences, *John Wiley, New York*.

[12] Halid O.Y, and Sule, O.B. (2022). A Classical and Bayesian Estimation Techniques for Gompertz Invere Rayleigh Distribution: Properties and Application *Pakistan Journal of Statistics*, *38(1)*, 49-76

[13] Ieren T.G., and Balogun O.S(2021). Exponential-Lindley Distribution: Theory and Application to Bladder Cancer Data. *Journal of Applied Probability and Statistics*, *16(2)*, 129-146.

[14] Mohammed, H.F, and Yehia N.(2019) The type II Topp-Leone generalized inverse raleigh distribution *Int. J. Contemp. Math. Sci 14(3)*, 113-122

[15] Moors, J.J. (1988). A quantile alternative for kurtosis. *The Statistician*, *37*, 25-32.

[16] Preston, S., Heuveline, P. and Guillot, M. (2001). Demograpy. Measuring and modeling population processes. *Blackwell Publisher, Oxford.*

[17] Rao G.S and Mbwambo S. (2018) Exponentiated Inverse Rayleigh Distribution and Application to Coating Weights of Iron Sheets Data. *Journal of Probability and Statistics*, *2019* 1-13

[18] Vaupel, J. (1986). How Change in Age-specific Mortality Affects Life Expentancy. *Population Studies*, *40*, 147-157

[19] Willemse, W. and Koppelaar, H. (2000). Knowledge elicitation of Gompertz law of mortality. *Scandinavian Actuarial Journal*, *2*, 168-179.

[20] Trayer, V.N.(1964) Proceedings of the Academy of Science Belarus, USSR