ON STRESS STRENGTH RELIABILITY ESTIMATION OF EXPONENTIAL INTERVENED POISSON DISTRIBUTION

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Abstract

Aim. Inferences on stress strength reliability has many applications in reliability theory. In this paper, we made a comparative study of Simple random sampling, Ranked set sampling and Percentile ranked set sampling by considering the estimation of stress strength reliability when the stress and strength are independently following Exponential Intervened Poisson distribution. Methods. We used the method of Maximum likelihood estimation for finding the estimate of stress strength reliability. The efficiency of the proposed estimators of stress strength reliability using three sampling schemes are compared via a Monte Carlo simulation study. Also at the end of the study a real life data set is analyzed to understand the usefulness of the study. Results. The findings in this study are the stress strength reliability estimates under Percentile ranked set sampling performs better than the corresponding ones under Simple random sampling and Ranked set sampling. Conclusion. So we can conclude that making refinements in Ranked set sampling increases the efficiency of estimators by minimizing the chance of incorrect ranking.

Keywords: maximum likelihood estimation, percentile ranked set sampling, ranked set sampling, stress strength reliability.

1. Introduction

The estimation of stress strength reliability has applications in a variety of fields like engineering, healthcare, transportation etc. The stress strength reliability is defined as $R = P(X < Y)$, where X is the strength and Y is the applied stress against strength. Obviously the system will fail if the applied stress exceeds the strength of the component. Many researchers are interested to work in this area. A review of the works related to stress strength reliability until 2001 are given in Kotz et al. [10] . Krishnamoorthy et al. [11], Kundu and Gupta [12] and Raqab et al. [20] studied the estimation of R for the Exponential, two-parameter and three-parameter generalized Exponential distributions respectively. Al-Mutairi et al. [3], Ghitany et al. [7] and Rezaei et al. [21] considered the same problem in case of Lindley, power Lindley and generalized Lindley type 5, respectively.

McIntyre [14] introduced the concept of Ranked Set Sampling (RSS). The sampling units in RSS are more representative of population than Simple Random Sampling (SRS) with same sample size. Sengupta and Mukhuti [23] and Muttlak et al. [17] considered the estimation of R when the distribution of stress and strength are Exponential under RSS. Hassan et al. [8] considered the estimation of R under RSS in case of Burr type XII distribution. Akgul and Senoglu [1], Akgul et al. [2] and Al-Omari et al. [5] addressed the same problem in case of Weibull, Lindly and Exponentiated Pareto distribution respectively.

The main characteristic which determines the performance of RSS is the chance of committing error in ranking. The error in ranking increases due to the incorrect measurement of sampling observations. To control this trouble several modifications of RSS have been suggested. see, Samawi et al. [22] suggested Extreme Ranked Set Sampling (ERSS), Muttlak [15] developed Median Ranked Set Sampling (MRSS), Al-Saleh and Al-Kadiri [6] introduced Double Ranked Set Sampling (DRSS). Also Muttlak [16] and Al-Nasser [4] suggested Percentile Ranked Set Sampling (PRSS), L Ranked Set Sampling (LRSS) respectively. Recently Zamanzade and Al-Omari [25] suggested Neoteric Ranked Set Sampling (NRSS).

Intervened distributions has wide range of applications in many areas like life testing experiments, quality control and epidemiological studies etc. Shanmugam [24] developed Intervened Poisson distribution(IPD) to study the effect of some preventive actions or interventions in a system. Recently a family of distributions is generated using IPD, which contain Marshall and Olkin [13] extended families of distribution, families of distributions generated through truncated negative binomial studied by Nadarajah et al. [18] and families of distributions generated through truncated binomial distribution as sub families, see Jayakumar and Sankaran [9]. Also they introduced Exponential Intervened Poisson (EIP) distribution, which is obtained by taking Exponential distribution as the baseline distribution in the above family. Here we consider a comparative study of SRS, RSS and PRSS based on the stress strength reliability estimation of EIP distribution. That is the stress and strength are independently following EIP distribution.

A continuous random variable X on $(0, \infty)$ is said to have an EIP distribution with parameters λ, ρ and θ and write $X \sim$ EIP (λ, ρ, θ) if its probability density function is

$$f(x; \lambda, \rho, \theta) = \frac{\lambda \theta e^{-\theta x}}{e^{\lambda \rho}(e^{\lambda} - 1)} \left[(1+\rho) e^{\lambda(1+\rho)e^{-\theta x}} - \rho e^{\lambda \rho e^{-\theta x}} \right] \tag{1}$$

where $\lambda > 0, \rho \geq 0$ and $\theta > 0$.

The cumulative distribution function of X is

$$F(x) = \left[1 - \left(\frac{e^{\lambda(1+\rho)e^{-\theta x}} - e^{\lambda \rho e^{-\theta x}}}{e^{\lambda \rho}(e^{\lambda} - 1)} \right) \right]. \tag{2}$$

The corresponding survival (or reliability) and the hazard (or failure rate) functions, at any time $x > 0$, are respectively given by

$$\bar{F}(x) = \left(\frac{e^{\lambda(1+\rho)e^{-\theta x}} - e^{\lambda \rho e^{-\theta x}}}{e^{\lambda \rho}(e^{\lambda} - 1)} \right) \tag{3}$$

and

$$h_F(x) = \lambda \theta e^{-\theta x} \left[\frac{1}{(1 - e^{-\lambda e^{-\theta x}})} + \rho \right]$$

For a detailed view of properties of EIP distribution, we refer the interested readers to [9]. From [9], we can see that the distribution is under dispersed and leptokurtic. According to the value of the parameters, the distribution behave as positively skewed or negatively skewed.

The rest of this paper is organized as follows: Stress strength reliability for EIP distribution is computed in Section 2. The ML estimation of R based on SRS is considered in section 3. When RSS and PRSS are considered the ML estimation of R are considered in section 4 and section 5 respectively. An extensive Monte-Carlo simulation study is conducted in section 6. In section 7, we present a real data application. Finally conclusions are given in section 8.

2. Stress Strength Reliability

Let X and Y be the stress and strength random variables independently following $EIP(\lambda_1, \rho_1, \theta_1)$ and $EIP(\lambda_2, \rho_2, \theta_2)$, respectively. Then the system reliability is calculated as given below

$$\begin{aligned} R &= P(X < Y) \\ &= \int_0^{\infty} F_X(x) f_Y(x) dx \end{aligned}$$

$$= \int_0^\infty \left(1 - \left[\frac{e^{\lambda_1(1+\rho_1)e^{-\theta_1 x}} - e^{\lambda_1\rho_1 e^{-\theta_1 x}}}{e^{\lambda_1\rho_1(e^{\lambda_1}-1)}}\right]\right) \frac{\lambda_2\theta_2 e^{-\theta_2 x}}{e^{\lambda_2\rho_2}(e^{\lambda_2}-1)} \left[(1+\rho_2)e^{\lambda_2(1+\rho_2)e^{-\theta_2 x}} - \rho_2 e^{\lambda_2\rho_2 e^{-\theta_2 x}}\right] dx \tag{4}$$

We can not solve the above integral directly. Therefore, we use some numerical techniques to solve the equation.

3. Maximum Likelihood Estimation of R based on SRS

To obtain the Maximum likelihood estimates (MLE) of R first we need to find MLE's of the parameters. Let $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_m$ be two independent SRS samples from $EIP(\lambda_1, \rho_1, \theta_1)$ and $EIP(\lambda_2, \rho_2, \theta_2)$, respectively. Then the likelihood function based on SRS is given by,

$$\begin{aligned} L &= \prod_{i=1}^{n} f(x_i) \prod_{j=1}^{m} f(y_j) \\ &= \left(\frac{\lambda_1\theta_1}{e^{\lambda_1\rho_1}(e^{\lambda_1}-1)}\right)^n e^{-\sum_{i=1}^n \theta_1 x_i} \prod_{i=1}^{n} \left[(1+\rho_1)e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_i}} - \rho_1 e^{\lambda_1\rho_1 e^{-\theta_1 x_i}}\right] \\ & \left(\frac{\lambda_2\theta_2}{e^{\lambda_2\rho_2}(e^{\lambda_2}-1)}\right)^m e^{-\sum_{j=1}^m \theta_2 y_j} \prod_{j=1}^{m} \left[(1+\rho_2)e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_j}} - \rho_2 e^{\lambda_2\rho_2 e^{-\theta_2 y_j}}\right] \end{aligned}$$

The log likelihood function is given by,

$$\begin{aligned} &\log L \\ =& n\left[\log\lambda_1 + \log\theta_1 - \lambda_1\rho_1 - \log\left(e^{\lambda_1}-1\right)\right] - \theta_1\sum_{i=1}^{n} x_i + \\ &\sum_{i=1}^{n}\log\left[(1+\rho_1)e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_i}} - \rho_1 e^{\lambda_1\rho_1 e^{-\theta_1 x_i}}\right] + m\left[\log\lambda_2 + \log\theta_2 - \lambda_2\rho_2 - \log\left(e^{\lambda_2}-1\right)\right] - \\ &\theta_2\sum_{j=1}^{m} y_j + \sum_{j=1}^{m}\log\left[(1+\rho_2)e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_j}} - \rho_2 e^{\lambda_2\rho_2 e^{-\theta_2 y_j}}\right] \end{aligned}$$

The partial derivatives of the log likelihood function with respect to the parameters are,

$$\frac{\partial \log L}{\partial \lambda_1} = \frac{n}{\lambda_1} - n\rho_1 - \frac{ne^{\lambda_1}}{e^{\lambda_1}-1} + \sum_{i=}^{n} \frac{(1+\rho_1)^2 e^{-\theta_1 x_i} e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_i}} - \rho_1^2 e^{-\theta_1 x_i} e^{\lambda_1\rho_1 e^{-\theta_1 x_i}}}{(1+\rho_1)e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_i}} - \rho_1 e^{\lambda_1\rho_1 e^{-\theta_1 x_i}}}$$

$$\frac{\partial \log L}{\partial \rho_1} = -n\lambda_1 + \sum_{i=1}^{n} \frac{e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_i}}\left(1+\lambda_1(1+\rho_1)e^{-\theta_1 x_i}\right) - e^{\lambda_1\rho_1 e^{-\theta_1 x_i}}\left(1+\lambda_1\rho_1 e^{-\theta_1 x_i}\right)}{(1+\rho_1)e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_i}} - \rho_1 e^{\lambda_1\rho_1 e^{-\theta_1 x_i}}}$$

$$\frac{\partial \log L}{\partial \theta_1} = \frac{n}{\theta_1} - \sum_{i=1}^{n} \frac{\left[\lambda_1(1+\rho_1)^2 x_i e^{\left(\lambda_1(1+\rho_1)e^{-\theta_1 x_i} - \theta_1 x_i\right)} - \lambda_1\rho_1^2 x_i e^{\left(\lambda_1\rho_1 e^{-\theta_1 x_i} - \theta_1 x_i\right)}\right]}{(1+\rho_1)e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_i}} - \rho_1 e^{\lambda_1\rho_1 e^{-\theta_1 x_i}}}$$

$$\frac{\partial \log L}{\partial \lambda_2} = \frac{m}{\lambda_2} - m\rho_2 - \frac{me^{\lambda_2}}{e^{\lambda_2}-1} + \sum_{j=}^{m} \frac{(1+\rho_2)^2 e^{-\theta_2 y_j} e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_j}} - \rho_2^2 e^{-\theta_2 y_j} e^{\lambda_2\rho_2 e^{-\theta_2 y_j}}}{(1+\rho_2)e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_j}} - \rho_2 e^{\lambda_2\rho_2 e^{-\theta_2 y_j}}}$$

$$\frac{\partial \log L}{\partial \rho_2} = -m\lambda_2 + \sum_{j=1}^{m} \frac{e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_j}}\left(1+\lambda_2(1+\rho_2)e^{-\theta_2 y_j}\right) - e^{\lambda_2\rho_2 e^{-\theta_2 y_j}}\left(1+\lambda_2\rho_2 e^{-\theta_2 y_j}\right)}{(1+\rho_2)e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_j}} - \rho_2 e^{\lambda_2\rho_2 e^{-\theta_2 y_j}}}$$

$$\frac{\partial \log L}{\partial \theta_2} = \frac{m}{\theta_2} - \sum_{j=1}^{m} \frac{\left[\lambda_2(1+\rho_2)^2 y_j e^{\left(\lambda_2(1+\rho_2)e^{-\theta_2 y_j} - \theta_2 y_j\right)} - \lambda_2\rho_2^2 y_j e^{\left(\lambda_2\rho_2 e^{-\theta_2 y_j} - \theta_2 y_j\right)}\right]}{(1+\rho_2)e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_j}} - \rho_2 e^{\lambda_2\rho_2 e^{-\theta_2 y_j}}}$$

So the ML estimates of the parameters are obtained by maximizing the log-likelihood function with respect to the parameters. Which is equivalent to the simultaneous solution of $\frac{\partial \log L}{\partial \lambda_1} = 0, \frac{\partial \log L}{\partial \rho_1} = 0, \frac{\partial \log L}{\partial \theta_1} = 0, \frac{\partial \log L}{\partial \lambda_2} = 0, \frac{\partial \log L}{\partial \rho_2} = 0$ and $\frac{\partial \log L}{\partial \theta_2} = 0$. The solutions of these equations cannot be obtained in closed form, so we used $optim()$ function in $R\ software$ to solve them numerically. Hence using the invariance property of MLE, the ML estimate of system reliability based on SRS, namely $\hat{R}_{SRS}$, is obtained by substituting the ML estimates of $(\lambda_1, \rho_1, \theta_1, \lambda_2, \rho_2, \theta_2)$ in equation 4.

4. Maximum Likelihood Estimation of R based on RSS

Let $X_{(i)ik}, (i = 1, 2, ..., m_x); (k = 1, 2, ..., r_x)$ be a ranked set sample observed from $EIP(\lambda_1, \rho_1, \theta_1)$ with sample size $n = m_x r_x$, where m_x is the set size and r_x is the number of cycles respectively. Similarly, let $Y_{(j)jl}, (j = 1, 2, ..., m_y); (l = 1, 2, ..., r_y)$ be a ranked set sample observed from $EIP(\lambda_2, \rho_2, \theta_2)$ with sample size $m = m_y r_y$, where m_y is the set size and r_y is the number of cycles respectively. Then the likelihood function based on RSS is given by,

$$
\begin{aligned}
L =& \prod_{k=1}^{r_x}\prod_{i=1}^{m_x} f(x_{ik}) \prod_{l=1}^{r_y}\prod_{j=1}^{m_y} f(y_{jl}) \\
=& C\left[\frac{\lambda_1\theta_1}{e^{\lambda_1\rho_1}(e^{\lambda_1}-1)}\right]^n \prod_{k=1}^{r_x}\prod_{i=1}^{m_x}\left[1-\frac{A_{ik}}{e^{\lambda_1\rho_1}(e^{\lambda_1}-1)}\right]^{i-1}\left[\frac{A_{ik}}{e^{\lambda_1\rho_1}(e^{\lambda_1}-1)}\right]^{m_x-i} e^{-\theta_1 x_{ik}}\left(e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_{ik}}}-\rho_1 A_{ik}\right) \\
& \left[\frac{\lambda_2\theta_2}{e^{\lambda_2\rho_2}(e^{\lambda_2}-1)}\right]^m \prod_{l=1}^{r_y}\prod_{j=1}^{m_y}\left[1-\frac{B_{jl}}{e^{\lambda_2\rho_2}(e^{\lambda_2}-1)}\right]^{j-1}\left[\frac{B_{jl}}{e^{\lambda_2\rho_2}(e^{\lambda_2}-1)}\right]^{m_y-j} e^{-\theta_2 y_{jl}}\left(e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_{jl}}}-\rho_2 B_{jl}\right)
\end{aligned}
$$

where $C = \prod_{k=1}^{r_x}\prod_{i=1}^{m_x}\frac{m_x!}{(i-1)!(m_x-i)!}\prod_{l=1}^{r_y}\prod_{j=1}^{m_y}\frac{m_y!}{(j-1)!(m_y-j)!}$, $A_{ik} = \left(e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_{ik}}} - e^{\lambda_1\rho_1 e^{-\theta_1 x_{ik}}}\right)$ and $B_{jl} = \left(e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_{jl}}} - e^{\lambda_2\rho_2 e^{-\theta_2 y_{jl}}}\right)$

Also $f(x_{ik})$ and $f(y_{jl})$ are defined as,

$$
\begin{aligned}
f(x_{ik}) &= \frac{m_x!}{(i-1)!(m_x-i)!}\left[F_X(x_{ik})\right]^{i-1}\left[1-F_X(x_{ik})\right]^{m_x-i} f_X(x_{ik}) \\
f(y_{jl}) &= \frac{m_y!}{(j-1)!(m_y-j)!}\left[F_Y(x_{jl})\right]^{j-1}\left[1-F_Y(x_{jl})\right]^{m_y-j} f_Y(y_{jl})
\end{aligned}
$$

The log likelihood function is given by,

$$
\begin{aligned}
\log L =& \\
& \log C + n\log\lambda_1 + n\log\theta_1 - n\lambda_1\rho_1 - n\log(e^{\lambda_1}-1) + \sum_{k=1}^{r_x}\sum_{i=1}^{m_x}(i-1)\log\left[1-\frac{A_{ik}}{e^{\lambda_1\rho_1}(e^{\lambda_1}-1)}\right] \\
& + \sum_{k=1}^{r_x}\sum_{i=1}^{m_x}(m_x-i)\log\left[\frac{A_{ik}}{e^{\lambda_1\rho_1}(e^{\lambda_1}-1)}\right] - \sum_{k=1}^{r_x}\sum_{i=1}^{m_x}\theta_1 x_{ik} + \sum_{k=1}^{r_x}\sum_{i=1}^{m_x}\log\left(e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_{ik}}}-\rho_1 A_{ik}\right) \\
& + m\log\lambda_2 + m\log\theta_2 - m\lambda_2\rho_2 - m\log(e^{\lambda_2}-1) + \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}(j-1)\log\left[1-\frac{B_{jl}}{e^{\lambda_2\rho_2}(e^{\lambda_2}-1)}\right] \\
& + \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}(m_y-j)\log\left[\frac{B_{jl}}{e^{\lambda_2\rho_2}(e^{\lambda_2}-1)}\right] - \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\theta_2 y_{jl} + \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\log\left(e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_{jl}}}-\rho_2 B_{jl}\right)
\end{aligned}
$$

Then the partial derivatives of the log likelihood function with respect to the parameters are,

$$
\begin{aligned}
\frac{\partial \log L}{\partial \lambda_1} =& \frac{n}{\lambda_1} - n\rho_1 - \frac{ne^{\lambda_1}}{e^{\lambda_1}-1} - \\
& \sum_{k=1}^{r_x}\sum_{i=1}^{m_x}(i-1)\frac{e^{-\theta_1 x_{ik}}(e^{\lambda_1}-1)\left(e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_{ik}}}+\rho_1 A_{ik}\right) - A_{ik}\left(1+\rho_1(e^{\lambda_1}-1)\right)}{\left(e^{\lambda_1\rho_1}(e^{\lambda_1}-1)-A_{ik}\right)(e^{\lambda_1}-1)} \\
& + \sum_{k=1}^{r_x}\sum_{i=1}^{m_x}(m_x-i)\frac{e^{-\theta_1 x_{ik}}(e^{\lambda_1}-1)\left(e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_{ik}}}+\rho_1 A_{ik}\right) - A_{ik}\left(1+\rho_1(e^{\lambda_1}-1)\right)}{A_{ik}(e^{\lambda_1}-1)} \\
& + \sum_{k=1}^{r_x}\sum_{i=1}^{m_x}\frac{e^{-\theta_1 x_{ik}}\left(e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_{ik}}}-\rho_1^2 A_{ik}\right)}{e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_{ik}}}-\rho_1 A_{ik}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \log L}{\partial \rho_1} =& -n\lambda_1 + \sum_{k=1}^{r_x}\sum_{i=1}^{m_x}(i-1)\frac{\lambda_1 A_{ik}(1-e^{-\theta_1 x_{ik}})}{e^{\lambda_1\rho_1}(e^{\lambda_1}-1)-A_{ik}} + \lambda_1\sum_{k=1}^{r_x}\sum_{i=1}^{m_x}(m_x-i)(1-e^{-\theta_1 x_{ik}}) \\
& + \sum_{k=1}^{r_x}\sum_{i=1}^{m_x}\frac{\lambda_1 e^{-\theta_1 x_{ik}-\lambda_1(1+\rho_1)e^{-\theta_1 x_{ik}}} - A_{ik}(1+\lambda_1\rho_1 e^{-\theta_1 x_{ik}})}{e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_{ik}}}-\rho_1 A_{ik}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \log L}{\partial \theta_1} =& \frac{n}{\theta_1} + \lambda_1\sum_{k=1}^{r_x}\sum_{i=1}^{m_x}(i-1)\frac{x_{ik}e^{-\theta_1 x_{ik}}\left(e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_{ik}}}+\rho_1 A_{ik}\right)}{e^{\lambda_1\rho_1}(e^{\lambda_1}-1)-A_{ik}} \\
& - \lambda_1\sum_{k=1}^{r_x}\sum_{i=1}^{m_x}(m_x-i)\frac{x_{ik}e^{-\theta_1 x_{ik}}\left(e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_{ik}}}+\rho_1 A_{ik}\right)}{A_{ik}} - \sum_{k=1}^{r_x}\sum_{i=1}^{m_x}x_{ik} \\
& + \lambda_1\rho_1^2\sum_{k=1}^{r_x}\sum_{i=1}^{m_x}\frac{x_{ik}e^{-\theta_1 x_{ik}}\left(e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_{ik}}}-A_{ik}\right)}{e^{\lambda_1(1+\rho_1)e^{-\theta_1 x_{ik}}}-\rho_1 A_{ik}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \log L}{\partial \lambda_2} =& \frac{m}{\lambda_2} - m\rho_2 - \frac{me^{\lambda_2}}{e^{\lambda_2}-1} - \\
& \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}(j-1)\frac{e^{-\theta_2 y_{jl}}(e^{\lambda_2}-1)\left(e^{\lambda_2(1+\rho_2)e^{-\theta_2 x_{jl}}}+\rho_2 B_{jl}\right) - B_{jl}\left(1+\rho_2(e^{\lambda_2}-1)\right)}{\left(e^{\lambda_2\rho_2}(e^{\lambda_2}-1)-B_{jl}\right)(e^{\lambda_2}-1)} \\
& + \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}(m_y-j)\frac{e^{-\theta_2 y_{jl}}(e^{\lambda_2}-1)\left(e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_{jl}}}+\rho_2 B_{jl}\right) - B_{jl}\left(1+\rho_2(e^{\lambda_2}-1)\right)}{B_{jl}(e^{\lambda_2}-1)} \\
& + \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\frac{e^{-\theta_2 y_{jl}}\left(e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_{jl}}}-\rho_2^2 B_{jl}\right)}{e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_{jl}}}-\rho_2 B_{jl}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \log L}{\partial \rho_2} =& -m\lambda_2 + \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}(j-1)\frac{\lambda_2 B_{jl}(1-e^{-\theta_2 y_{jl}})}{e^{\lambda_2\rho_2}(e^{\lambda_2}-1)-B_{jl}} + \lambda_2\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}(m_y-j)(1-e^{-\theta_2 y_{jl}}) \\
& + \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\frac{\lambda_2 e^{-\theta_2 y_{jl}-\lambda_2(1+\rho_2)e^{-\theta_2 y_{jl}}} - B_{jl}(1+\lambda_2\rho_2 e^{-\theta_2 y_{jl}})}{e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_{jl}}}-\rho_2 B_{jl}}
\end{aligned}
$$

$$\frac{\partial \log L}{\partial \theta_2} = \frac{m}{\theta_2} + \lambda_2 \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}(j-1)\frac{y_{jl}e^{-\theta_2 y_{jl}}\left(e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_{jl}}} + \rho_2 B_{jl}\right)}{e^{\lambda_2\rho_2}(e^{\lambda_2}-1) - B_{jl}}$$
$$- \lambda_2 \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}(m_y - j)\frac{y_{jl}e^{-\theta_2 y_{jl}}\left(e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_{jl}}} + \rho_2 B_{jl}\right)}{B_{jl}} - \sum_{l=1}^{r_y}\sum_{j=1}^{m_y} y_{jl}$$
$$+ \lambda_2\rho_2^2 \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\frac{y_{jl}e^{-\theta_2 y_{jl}}\left(e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_{jl}}} - B_{jl}\right)}{e^{\lambda_2(1+\rho_2)e^{-\theta_2 y_{jl}}} - \rho_2 B_{jl}}$$

The the ML estimates of the unknown parameters under RSS are calculated by equating above equations to zero and solving simultaneously. But it is difficult so solve these equations analytically, so similar to estimation of parameters in SRS, we used $optim()$ function in $R\ software$. Then using the invariance property of MLE, the ML estimate of R based on RSS, namely $\hat{R}_{RSS}$, is obtained by substituting the ML estimates of the parameters in equation (4).

5. Maximum Likelihood Estimation of R based on PRSS

This section deals with the ML estimation of stress strength reliability measure R based on PRSS. Here we consider inference procedure for odd and even set sizes separately.

Case 1: Odd set size Let a_x, b_x, a_y and b_y are the nearest integer values of $p[m_x+1], q[m_x+1], p[m_y+1]$ and $q[m_y+1]$, where $0 < p < 1$ and $q = 1-p$. Also ϑ and ω are defined as $\frac{m_x+1}{2}$ and $\frac{m_y+1}{2}$.

Let $\{X_{(a_x)ik},\ i = 1,2,...,\vartheta-1;\ k = 1,2,...,r_x\} \bigcup \{X_{(\vartheta)ik},\ i = \vartheta;\ k = 1,2,...,r_x\} \bigcup \{X_{(b_x)ik},\ i = \vartheta+1,...,m_x;\ k = 1,2,...,r_x\}$ be the percentile ranked set samples selected from $EIP(\lambda_1, \rho_1, \theta_1)$ with sample size $n = m_x r_x$, where m_x and r_x be the set size and number of cycles respectively. Similarly let $\{Y_{(a_y)jl},\ j = 1,2,...,\omega-1;\ l = 1,2,...,r_y\} \bigcup \{Y_{(\omega)jl},\ j = \omega;\ l = 1,2,...,r_y\} \bigcup \{Y_{(b_y)jl},\ j = \omega+1,...,m_y;\ l = 1,2,...,r_y\}$ be the percentile ranked set samples selected from $EIP(\lambda_2, \rho_2, \theta_2)$ with sample size $m = m_y r_y$, where m_y and r_y be the set size and number of cycles respectively.

Then, the likelihood function is obtained as follows:

$$L = \prod_{k=1}^{r_x}\prod_{i=1}^{\vartheta-1} f(x_{(a_x)ik}) \prod_{k=1}^{r_x} f(x_{(\vartheta)\vartheta k}) \prod_{k=1}^{r_x}\prod_{i=\vartheta+1}^{m_x} f(x_{(b_x)ik})$$
$$\prod_{l=1}^{r_y}\prod_{j=1}^{\omega-1} f(y_{(a_y)jl}) \prod_{l=1}^{r_y} f(y_{(\omega)\omega l}) \prod_{l=1}^{r_y}\prod_{j=\omega+1}^{m_y} f(y_{(b_y)jl})$$

where

$$f(x_{(a_x)}) = \frac{m_x!}{(a_x-1)!(m_x-a_x)!}\left[F_X(x_{a_x})\right]^{a_x-1}\left[1-F_X(x_{a_x})\right]^{m_x-a_x} f_X(x_{a_x})$$
$$f(x_{(b_x)}) = \frac{m_x!}{(b_x-1)!(m_x-b_x)!}\left[F_X(x_{b_x})\right]^{b_x-1}\left[1-F_X(x_{b_x})\right]^{m_x-b_x} f_X(x_{b_x})$$
$$f(x_{(\vartheta)}) = \frac{m_x!}{(\vartheta-1)!(m_x-\vartheta)!}\left[F_X(x_{\vartheta})\right]^{\vartheta-1}\left[1-F_X(x_{\vartheta})\right]^{m_x-\vartheta} f_X(x_{\vartheta})$$

Similarly we can define $f(y_{(a_y)}), f(y_{(b_y)})$ and $f(y_{(\omega)})$.

Case 2: Even set sizes: Here, the reliability estimator is investigated when both X and Y are drawn based on PRSS from EIP with even set size.

Let $\{X_{(a_x)ik},\ i = 1,2,...,\frac{m_x}{2};\ k = 1,2,...,r_x\} \bigcup \{X_{(b_x)ik},\ i = \frac{m_x}{2}+1,...,m_x;\ k = 1,2,...,r_x\}$ and $\{Y_{(a_y)jl},\ j = 1,2,...,\frac{m_y}{2};\ l = 1,2,...,r_y\} \bigcup \{Y_{(b_y)jl},\ j = \frac{m_y}{2}+1,...,m_y;\ l = 1,2,...,r_y\}$ be percentile

ranked set samples from EIP with even set sizes.
Therefore the likelihood function is,

$$L = \prod_{k=1}^{r_x} \prod_{i=1}^{\frac{m_x}{2}} f(x_{(a_x)ik}) \prod_{k=1}^{r_x} \prod_{\frac{m_x}{2}+1}^{m_x} f(x_{(b_x)ik})$$
$$\prod_{l=1}^{r_y} \prod_{j=1}^{\frac{m_y}{2}} f(y_{(a_y)jl}) \prod_{l=1}^{r_y} \prod_{j=\frac{m_y}{2}+1}^{m_y} f(y_{(b_y)jl})$$

For finding the ML estimate of the parameters based on PRSS for both odd and even set sizes, we equate the partial derivatives of the log-likelihood equation to zero and solve them simultaneously. For this we used $optim()$ function in $R\ software$. Hence using the invariance property of MLE, the ML estimate of system reliability based on PRSS, namely $\hat{R}_{PRSS}$, is obtained by substituting the ML estimates in equation (4).

6. Simulation Study

In this section, we conducted a simulation study to assess the potentiality of system reliability estimates based on SRS, RSS and PRSS . We generate 1000 replications of the stress and strength random variables from EIP distribution with parameters $(\lambda_1, \rho_1, \theta_1, \lambda_2, \rho_2, \theta_2) = (.1, .5, 1, 1, 2, 1), (1, .5, 1, 1, 1, 1)$ and $(1, .8, 2, .5, .2, 1)$ using SRS, RSS and PRSS. Using these true values of the parameters we obtain the stress strength reliability R as $0.2634, 0.4518$ and 0.7501 respectively. For selecting samples using SRS we set the sample sizes as $(n, m) = (40, 40), (40, 60), (60, 60),$ $(60, 80)$ and $(80, 80)$. Similarly for RSS and PRSS, $(m_x, m_y) = (4, 4), (4, 6), (6, 6), (6, 8), (8, 8)$ and $r_x = r_y = 10$. Also we fix $p = .4$ for PRSS. From these generated samples we compute the estimates of stress strength reliability. Mean square error (MSE) and Relative efficiency (RE) are used to compare the estimated stress strength reliability measures. The results are reported in Table 1. In this table, RE_1, RE_2 and RE_3 is the relative efficiency of RSS over SRS, PRSS over SRS and PRSS over RSS respectively. For all sampling methods, the MSE decreases when the sample size increases, which indicates the consistency property of MLE. According to the values of relative efficiencies we can say that RSS and PRSS performs better than SRS in all cases. Moreover PRSS performs better than RSS in almost everywhere.

Table 1: *Bias, MSE and RE of $\hat{R}$ based on SRS, RSS and PRSS.*

			SRS		RSS		PRSS				
R	(m_x, m_y)	(n, m)	Bias	MSE	Bias	MSE	Bias	MSE	RE_1	RE_2	RE_3
0.2634	$(4, 4)$	$(40, 40)$	-0.0014	0.0020	-0.0083	0.0014	-0.0066	0.0012	1.47	1.69	1.15
	$(4, 6)$	$(40, 60)$	0.0035	0.0017	-0.0032	0.0011	-0.0078	0.0009	1.51	1.87	1.24
	$(6, 6)$	$(60, 60)$	0.0162	0.0011	-0.0006	0.0007	-0.0067	0.0006	1.64	1.91	1.16
	$(6, 8)$	$(60, 80)$	-0.0003	0.0010	-0.0035	0.0006	-0.0069	0.0005	1.86	1.91	1.03
	$(8, 8)$	$(80, 80)$	0.0004	0.0009	-0.0038	0.0005	-0.0053	0.0004	1.97	2.12	1.08
0.4518	$(4, 4)$	$(40, 40)$	-0.0030	0.0025	-0.0013	0.0019	-0.0023	0.0019	1.31	1.37	1.04
	$(4, 6)$	$(40, 60)$	-0.0200	0.0023	-0.0010	0.0015	-0.0017	0.0014	1.49	1.62	1.09
	$(6, 6)$	$(60, 60)$	-0.0325	0.0018	-0.0007	0.0011	0.0004	0.0010	1.69	1.84	1.09
	$(6, 8)$	$(60, 80)$	-0.0284	0.0017	-0.0004	0.0010	-0.0016	0.0009	1.74	1.95	1.12
	$(8, 8)$	$(80, 80)$	-0.0351	0.0014	0.0004	0.0007	0.0003	0.0007	1.97	2.03	1.03
0.7502	$(4, 4)$	$(40, 40)$	-0.0023	0.0018	0.0049	0.0013	0.0073	0.0013	1.37	1.44	1.05
	$(4, 6)$	$(40, 60)$	-0.0101	0.0014	0.0009	0.0010	0.0061	0.0009	1.41	1.62	1.15
	$(6, 6)$	$(60, 60)$	0.0012	0.0013	0.0037	0.0008	0.0064	0.0007	1.66	1.88	1.13
	$(6, 8)$	$(60, 80)$	-0.0227	0.0013	0.0017	0.0007	0.0049	0.0005	1.82	2.32	1.28
	$(8, 8)$	$(80, 80)$	-0.0146	0.0011	0.0015	0.0005	0.0045	0.0005	2.19	2.38	1.09

7. Data Analysis

Here we analyzed a real life data set to illustrate the use of our proposed methodology. We consider two real life data sets which contain times to breakdown of an insulating fluid between electrodes recorded at different voltages see, [19]. These are the failure times (in minutes) for an insulating fluid between two electrodes subject to a voltage of 34 kV (X) and 36 kV (Y) are given in Table 2 and Table 3.

Table 2: *Data X: (34 kV)*

0.19	0.78	0.96	1.31	2.78	3.16	4.15	4.67	4.85	6.5
7.35	8.01	8.27	12.06	31.75	32.52	33.91	36.71	72.89	

Table 3: *Data Y: (36 kV)*

0.35	0.59	0.96	0.99	1.69	1.97	2.07	2.58	2.71	2.9
3.67	3.99	5.35	13.77	25.50					

Now to identify the behaviour of the hazard rate function of the data, we examined total time on test transform plot of the data sets. For this we use $TTT()$ function in $R\ Software$. The total time on test transform plots for both data sets are given in Figure1 and Figure 2. From these figures we can say that the hazard rate function of both data sets show decreasing nature.

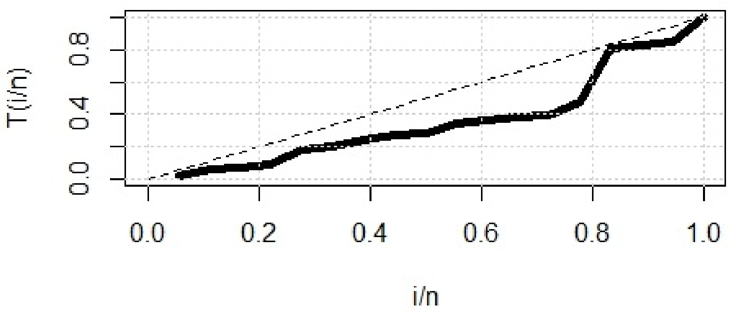

Figure 1: *The scaled TTT plot of Data X.*

Moreover the hazard rate function of EIP distribution also shows decreasing behaviour, see [9]. So we fit EIP distribution for both data sets separately. For fitting, we first find MLE's of the parameters. Also we need to check the goodness of fit of the NGP distribution for the data. For this purpose we use $-\log L$ and Kolmogorov Smirnov (KS) statistic along with p-value. The values of the estimated parameters, $-\log L$, KS, p value for both the data sets are reported in Table 4.

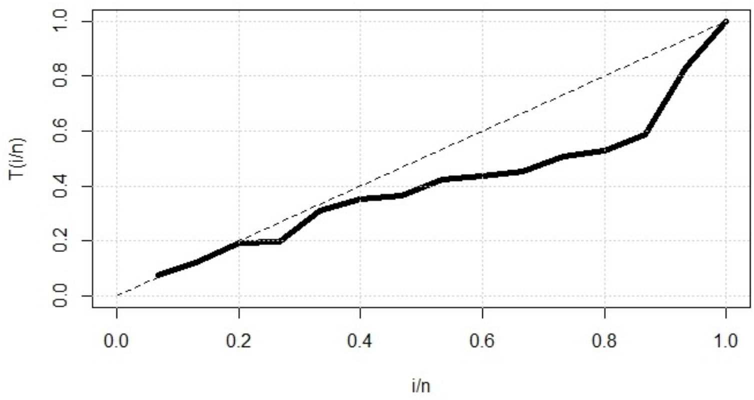

Figure 2: *The scaled TTT plot of Data Y.*

Table 4: *Estimates of the parameters, -log L, KS and P values for data sets.*

Data Set	Sample Size	λ	ρ	θ	$-\log L$	K-S	p value
X	19	0.96842829	0.54290810	0.04633776	68.54817	0.16834	0.5963
Y	15	0.9729146	0.9806698	0.1147315	36.97626	0.16411	0.7559

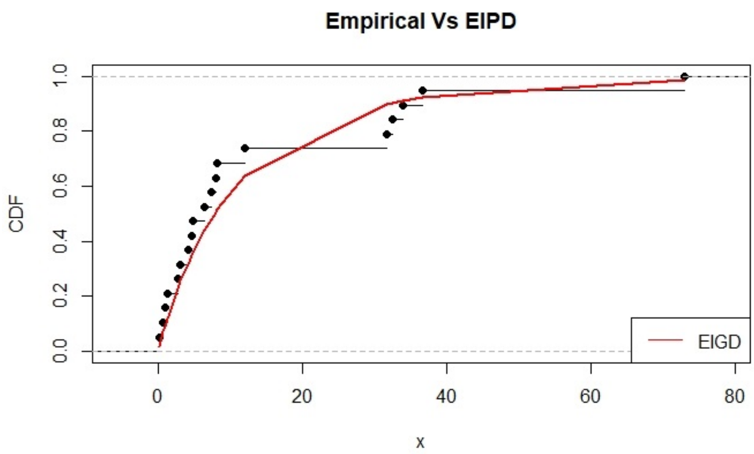

Figure 3: *The empirical distribution function and fitted distribution functions for Data X.*

From Table 4 and Figures 2 and 3, we can say that EIP distribution fits well for both data sets. So we are choosing these data sets to select samples from EIP distribution based on SRS, RSS and PRSS. For selecting the samples via SRS we take the sample sizes for X and Y as $n = 12, m = 8$. In case of RSS and PRSS, we take $m_x = 4$ and $r_x = 3$ for data X and $m_y = 2$ and $r_y = 4$ for data Y. Also R based on $n = 19$ and $m = 15$ observations is calculated as 0.27257. The mean, bias and MSEs of the estimates of R based on 10,000 replications of each sampling method is given in Table 5.

From Table 5 we can say that the estimated values of R based on $n = 12$ and $m = 8$ sampling

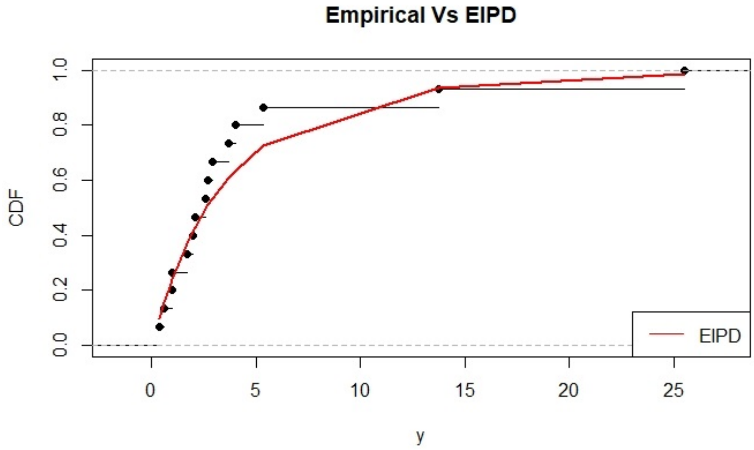

Figure 4: *The empirical distribution function and fitted distribution functions for Data Y.*

units using SRS, RSS and PRSS are close to the estimated value of R calculated from the entire data set. However, in view of MSEs, we can seen that $\hat{R}_{PRSS}$ and $\hat{R}_{RSS}$ perform better than $\hat{R}_{SRS}$.

8. Conclusions

In this paper, the ML estimates of the stress strength reliability R based on SRS, RSS and PRSS are obtained, when the stress and strength are independently following EIP distribution. The performance of the proposed estimators are compared using a Monte Carlo simulation study. From the simulation study it is clear that PRSS performs better than RSS and SRS. Also we can see that, the efficiency of all estimates increases as the set size increases. The results from the simulation study is supported by a real life data set. So if our aim is to choose a sampling procedure which minimizes the error in ranking, then we can consider PRSS than RSS and SRS.

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