WEIGHTED INTERVENED EXPONENTIAL DISTRIBUTION AS A LIFETIME DISTRIBUTION

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Abstract

This study proposes and investigates the weighted intervened exponential distribution, which is demonstrated as a generalized extension of the intervened exponential distribution. The form of the weighted intervened exponential distribution is obtained by considering a specific non-negative weighted function. The probability density function and cumulative density function of the proposed model are given, and its generalized form of reliability function and the hazard rate function is also derived. By choosing a different set of parametric values, the graphical demonstrations of the probability density function of weighted intervened exponential distribution are given where it acquires different curve shapes. The weighted intervened exponential distribution density function is then further studied in the limited form as a special case called the length-biased intervened exponential distribution. Along with the distribution of order statistics, stochastic ordering, stress-strength reliability, and entropy measure, several distributional and reliability aspects of the length-biased intervened exponential distribution are derived. For estimating the unidentified parameters of the length-biased variant, the most suggested approach known as the maximum likelihood estimation technique is implemented. To explore the behavior of the parameter estimates for various sample sizes, a sample data generation technique is required to carry out the process. Since the quantile function of the length biased intervened exponential distribution is not in closed form. So, the alternative data generation algorithm is employed which is known as the acceptance-rejection algorithm technique, and a Monte-Carlo simulation study is done. The absolute average bias and mean square error of the estimated parameters of the length-biased version model are calculated and it is noticed that both the calculated measures decrease simultaneously on increasing the sample size. In order to determine if the model is appropriate, a real-life time-to-event data set is examined as an example, and length biased distribution is juxtaposed with several other common available lifetime distributions for comparison purposes.

Keywords: Acceptance-rejection algorithm; Entropy; Weighted intervened distribution; Monte Carlo simulation

1. Introduction

The literature witnesses the dominance of exponential distribution among all existing lifetime models over data analysis in essential fields like reliability theory, survival analysis, and several other branches of statistical as well as the applied sciences. As it is well observed the exponential model has been extended in different ways with the help of new model development methods and transformation techniques, such as inverted gamma distribution by Lin *et al.*, [11], generalized exponential distribution by Gupta and Kundu [9], weighted exponential distribution by Gupta and Kundu [8], etc. are few examples. Apart from these extensions, a new development in statistics called the intervention was brought into existence by Shanmugam [21] in the form of a new discrete intervention model called the intervened Poisson distribution. Later on, in another successful attempt Shanmugam *et al.* [22] developed the continuous intervened exponential model, which was later introduced as a lifetime distribution in the field of reliability theory and survival analysis by Bhat and Pundir [2]. The cumulative and probability density functions ($c.d.f$ & $p.d.f$) of I_vED are given by

$$F_{I_vED}(x;\Theta) = \begin{cases} 1 - \frac{\rho e^{-(x-\sigma)/\rho\theta} - e^{-(x-\sigma)/\theta}}{(\rho-1)} & \rho \neq 1 \\ 1 - \left(1 + \frac{x-\sigma}{\theta}\right) e^{-(x-\sigma)/\theta} & \rho = 1 \end{cases} \tag{1}$$

and,

$$f_{I_vED}(x;\Theta) = \begin{cases} \frac{e^{-(x-\sigma)/\rho\theta} - e^{-(x-\sigma)/\theta}}{(\rho-1)\theta} & \rho \neq 1 \\ \frac{(x-\sigma)}{\theta^2} e^{-(x-\sigma)/\theta} & \rho = 1 \end{cases} \tag{2}$$

where, $\sigma < x < \infty$, and $\Theta = \{(\sigma,\theta,\rho) : \sigma > 0, \theta > 0, \rho > 0\}$ which is commonly known as the parameter space with intervention parameter ρ and rate parameter θ. The wider applicability of the model motivated us to extend the intervened exponential distribution (I_vED) further in the direction of the weighted distributions and the distribution obtained is known as weighted intervened exponential distribution (WI_vED). Thus, the newly proposed WI_vED is a type of generalization of I_vED. To trace the history of weighted distributions, Fisher [7] investigated the influence of ascertainment procedures on frequency estimates and developed the notion of weighted distributions. When diversifying Fisher's core theories, Rao ([18], [19]) realized the need for a unifying notion by attempting to identify numerous sampling conditions that might be handled according to what he named weighted distributions. To know the history and applicability of weighted distributions Patil published several articles that he wrote with co-authors refer Taillie *et al.* [23], Patil and Taillie [12], Patil and Rao [15], Denis and Patil [5], Laird *et al.* [10], Patil and Ord [14], etc. moreover, see more references Patil [16]. In a definition, Patil *et al.* [13] proposed the methodology used to determine the weighted version probability density function ($p.d.f$) is depicted below:

$$f(x;\Theta) = \frac{w(x) f_*(x;\Theta)}{E\left[w(X)\right]}$$

where, $f_*(x;\Theta)$ is the natural density function of existing distribution with parameter space Θ and $w(x)$ is the non-negative weighted function by choosing $w(X) = X^r$, then $E\left[w(X)\right] = E\left[X^r\right]$ is the r^{th} moment about origin. The r^{th} raw moment of I_vED derived by Bhat and Pundir [2] is presented in a simplified form as given by

$$\mu_r' = \frac{1}{\theta(\rho-1)} \sum_{k=0}^{r} \binom{r}{k} \sigma^k \theta^{r-k+1} \left(\rho^{r-k+1} - 1\right). \tag{3}$$

Therefore, according to the definition of weighted distributions, the $c.d.f$ and $p.d.f$ of $WI_vED(\sigma,\theta,\rho)$ derived from a $r.v$ $X \sim I_vED(\sigma,\theta,\rho)$ are given by

$$F_{WI_vED}(x;\Theta) = \begin{cases} 1 - \frac{\theta^{r+1}\left\{\rho^{r+1} e^{\sigma/\rho\theta}\Gamma(r+1,x/\rho\theta) - e^{\sigma/\theta}\Gamma(r+1,x/\theta)\right\}}{\sum_{k=0}^{r}\binom{r}{k}\sigma^k\theta^{r-k+1}(\rho^{r-k+1}-1)\Gamma(r-k+1)} & \rho \neq 1 \\ 1 - \frac{\theta^{r+1}\{\theta\Gamma(r+2,x/\theta) - \sigma\Gamma(r+1,x/\theta)\}}{\sum_{k=0}^{r}\binom{r}{k}\sigma^k\theta^{r-k+2}\Gamma(r-k+2)} e^{\sigma/\theta} & \rho = 1 \end{cases} \tag{4}$$

and,

$$f_{WI_vED}(x;\Theta) = \begin{cases} \frac{x^r e^{-(x-\sigma)/\rho\theta} - x^r e^{-(x-\sigma)/\theta}}{\sum_{k=0}^{r}\binom{r}{k}\sigma^k\theta^{r-k+1}(\rho^{r-k+1}-1)\Gamma(r-k+1)} & \rho \neq 1 \\ \frac{x^r(x-\sigma)}{\sum_{k=0}^{r}\binom{r}{k}\sigma^k\theta^{r-k+2}\Gamma(r-k+2)} e^{-(x-\sigma)/\theta} & \rho = 1 \end{cases} \tag{5}$$

where, $\sigma < x < \infty$, and Θ is parameter space. The reliability function of the $WI_vED(\sigma,\theta,\rho)$ is obtained as

$$R_{WI_vED}(x;\Theta) = \begin{cases} \frac{\theta^{r+1}\left\{\rho^{r+1} e^{\sigma/\rho\theta}\Gamma(r+1,x/\rho\theta) - e^{\sigma/\theta}\Gamma(r+1,x/\theta)\right\}}{\sum_{k=0}^{r}\binom{r}{k}\sigma^k\theta^{r-k+1}(\rho^{r-k+1}-1)\Gamma(r-k+1)} & \rho \neq 1 \\ \frac{\theta^{r+1}\{\theta\Gamma(r+2,x/\theta) - \sigma\Gamma(r+1,x/\theta)\}}{\sum_{k=0}^{r}\binom{r}{k}\sigma^k\theta^{r-k+2}\Gamma(r-k+2)} e^{\sigma/\theta} & \rho = 1 \end{cases} \tag{6}$$

and, its hazard rate function is

$$h_{WI_vED}(x;\Theta) = \begin{cases} \frac{x^r e^{-(x-\sigma)/\rho\theta} - x^r e^{-(x-\sigma)/\theta}}{\theta^{r+1}\left\{\rho^{r+1} e^{\sigma/\rho\theta}\Gamma(r+1,x/\rho\theta) - e^{\sigma/\theta}\Gamma(r+1,x/\theta)\right\}} & \rho \neq 1 \\ \frac{x^r(x-\sigma)}{\theta^{r+1}\{\theta\Gamma(r+2,x/\theta) - \sigma\Gamma(r+1,x/\theta)\}} e^{-x/\theta} & \rho = 1 \end{cases} \tag{7}$$

further, we study the special case density function of WI_vED obtained at $r = 1$ from equation (5), that is, defined as Length(size)-biased intervened exponential distribution (LBI_vED). The mean, variance several statistical and reliability properties, parameter estimation, simulation, stochastic ordering, order statistics, and real-life applicability of LBI_vED are studied in proceeding sections and subsections.

2. Distribution and its Properties

The cumulative and probability density functions ($c.d.f$ & $p.d.f$) of LBI_vED are given by

$$F_{LBI_vED}(x;\Theta) = \begin{cases} 1 - \frac{\left\{\rho(x+\rho\theta)e^{-(x-\sigma)/\rho\theta} - (x+\theta)e^{-(x-\sigma)/\theta}\right\}}{(\rho-1)[\sigma+\theta(\rho+1)]} & \rho \neq 1 \\ 1 - \frac{\theta^2 + (x+\theta)(x-\sigma+\theta)}{\theta(\sigma+2\theta)} e^{-(x-\sigma)/\theta} & \rho = 1 \end{cases} \tag{8}$$

and,

$$f_{LBI_vED}(x;\Theta) = \begin{cases} \frac{xe^{-(x-\sigma)/\rho\theta} - xe^{-(x-\sigma)/\theta}}{\theta(\rho-1)[\sigma+\theta(\rho+1)]} & \rho \neq 1 \\ \frac{x(x-\sigma)}{\theta^2(\sigma+2\theta)} e^{-(x-\sigma)/\theta} & \rho = 1 \end{cases} \tag{9}$$

where, $0 < \sigma < x < \infty$ and Θ being its parameter space. The attempt is made to derive the expressions of mean (μ_x) and its variance $\left(\sigma_x^2\right)$. Also, some of the measures of $LBI_vED(\sigma, \theta, \rho \neq 1)$ which are not in closed form are the quantile function, median and the mode. Note, for simplicity here on-wards, we take $\tau = (\rho - 1)[\sigma + \theta(\rho + 1)]$.

$$\mu_x = \frac{2\theta^2\left(\rho^3 - 1\right) + 2\sigma\theta\left(\rho^2 - 1\right) + \sigma^2(\rho - 1)}{\tau} \tag{10}$$

and

$$\sigma_x^2 = \frac{2\theta^3\left(\rho^4 - 1\right) + 2\sigma\theta^2\left(\rho^3 - 1\right) + (3\sigma + 2)\sigma\theta\left(\rho^2 - 1\right) + \sigma^3(\rho - 1)}{\tau} - (\mu_x)^2. \tag{11}$$

In Figure 1 and Figure 2, the $p.d.f$ of $LBI_vED(\sigma, \theta, \rho \neq 1)$ is plotted graphically. This is evident from the graphical representation that the distribution is positively skewed, where multiple $p.d.f$ curves are displayed for various chosen sets of parameter values.

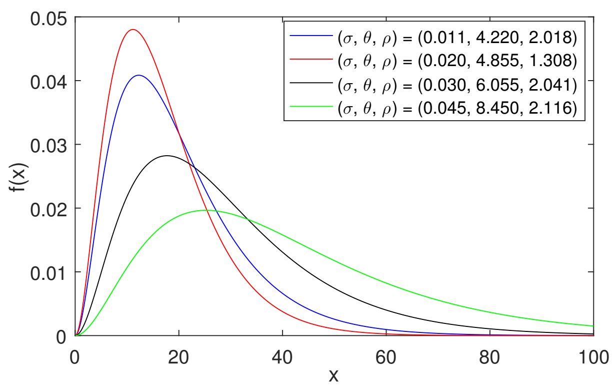

Figure 1: *P.d.f Plot*

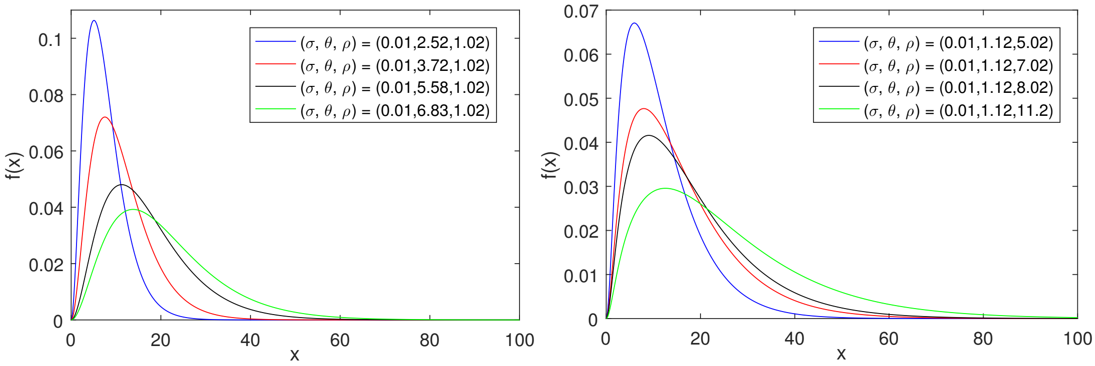

Figure 2: *P.d.f Subplots*

2.1. Moments

In this subsection, the derivation of moments particularly central and non-central are derived. So, if a random variable $X \sim LBI_vED(\sigma,\theta,\rho \neq 1)$, then central moments (μ_r) expression is obtained by

$$\mu_r = \frac{1}{\theta\tau}\int_{\sigma}^{\infty}(x-\mu_x)^r\, x\left\{e^{-(x-\sigma)/\rho\theta} - e^{-(x-\sigma)/\theta}\right\}dx$$

to solve the integral, first, we expand the term $(x-\mu_x)^r$ by using binomial expansion. After, integration and simplification the resulting equation is

$$\mu_r = \frac{1}{\tau}\sum_{k=0}^{r}\binom{r}{k}(\sigma-\mu_x)^r\,\theta^{r-k}\Gamma(r-k+1)\left\{\theta\left(\rho^{r-k+2}-1\right)(r-k+1)-\sigma\left(\rho^{r-k+1}-1\right)\right\}. \quad (12)$$

In a similar context, we derive the non-central moments (μ_r') expression for a random variable $X \sim LBI_vED(\sigma,\theta,\rho \neq 1)$. The procedure to obtain the final resulting expression is given by

$$\mu_r' = \frac{1}{\theta\tau}\int_{\sigma}^{\infty}x^{r+1}\left\{e^{-(x-\sigma)/\rho\theta} - e^{-(x-\sigma)/\theta}\right\}dx$$

after, simplifications the resulting equation is

$$\mu_r' = \frac{1}{\tau}\sum_{k=0}^{r+1}\binom{r+1}{k}\sigma^k\Gamma(r-k+2)\,\theta^{r-k+1}\left(\rho^{r-k+2}-1\right) \quad (13)$$

2.2. Generating Functions for Moments

In general, the moment generating function $(m.g.f)$ denoted by $M_X(t)$ of a $r.v$ X is obtained by,

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty}e^{tx}f(x)dx \quad (14)$$

so, to derive the $m.g.f$ of a $r.v$ $X \sim LBI_vED(\sigma,\theta,\rho \neq 1)$, we have

$$M_X(t) = \int_{\sigma}^{\infty}e^{tx}f_{LBI_vED}(x;\Theta)dx$$

on substituting the $p.d.f$ of LBI_vED, when $\rho \neq 1$ and proceed with the transforming technique, we get

$$= \frac{e^{tx}}{\theta\tau} \int_0^\infty (x+\sigma) \left\{ e^{-\left(\frac{1}{\theta\rho}-t\right)x} - e^{-\left(\frac{1}{\theta}-t\right)x} \right\} dx$$

on solving this further by using the gamma function, the required resulting equation of $m.g.f$ is obtained as

$$M_X(t) = \frac{\{\sigma (1-\theta t)(1-\theta\rho t) + \theta(1+\rho-2\theta\rho t)\} e^{\sigma t}}{(1-\theta t)^2 (1-\theta\rho t)^2 [\sigma + \theta(\rho+1)]}. \tag{15}$$

Next, the characteristic function $(c.f)$ of a $r.v$ $X \sim LBI_vED(\sigma, \theta, \rho \neq 1)$ is derived by

$$\phi_X(t) = E(e^{\iota t X}) = \int_\sigma^\infty e^{\iota t x} f_{LBI_vED}(x;\Theta) dx$$

$$= \frac{\{\sigma (1-\theta\iota t)(1-\theta\rho\iota t) + \theta(1+\rho-2\theta\rho\iota t)\} e^{\sigma\iota t}}{(1-\theta\iota t)^2 (1-\theta\rho\iota t)^2 [\sigma + \theta(\rho+1)]} \tag{16}$$

3. Discussion on Reliability Properties

Suppose a random variable X is such that it is non-negative, $X \sim LBI_vED(\ \sigma,\ \theta,\ \rho)$, the $p.d.f$ of X is $f_{LBI_vED}(x;\Theta)$, and its $c.d.f$ $F_{LBI_vED}(x;\Theta)$ which are given in equation (9) and (8) respectively. Then the reliability function of $LBI_vED(\sigma, \theta, \rho)$ is given by

$$R_{LBI_vED}(x;\xi) = \begin{cases} \frac{\rho(x+\rho\theta)e^{-(x-\sigma)/\rho\theta} - (x+\theta)e^{-(x-\sigma)/\theta}}{\tau} & \rho \neq 1 \\ \frac{\theta^2+(x+\theta)(x-\sigma+\theta)}{\theta(\sigma+2\theta)} e^{-(x-\sigma)/\theta} & \rho = 1 \end{cases} \tag{17}$$

the essential failure rate functions of a random variable $X \sim LBI_vED(\sigma, \theta, \rho \neq 1)$ are obtained. The hazard rate and reverse hazard rate functions are given by

$$h_{LBI_vED}(x;\xi) = \begin{cases} \frac{xe^{-(x-\sigma)/\rho\theta} - xe^{-(x-\sigma)/\theta}}{\theta\left\{\rho(x+\rho\theta)e^{-(x-\sigma)/\rho\theta} - (x+\theta)e^{-(x-\sigma)/\theta}\right\}} & \rho \neq 1 \\ \frac{x(x-\sigma)}{\theta\left\{\theta^2+(x+\theta)(x-\sigma+\theta)\right\}} & \rho = 1 \end{cases} \tag{18}$$

in Figure 3 and Figure 4, the hazard rate function of $LBI_vED(\sigma, \theta, \rho \neq 1)$ is plotted graphically. It is shown in the graphical representation the distribution is having increasing multiple hazard rate function curves that are displayed for various chosen sets of parameter values.

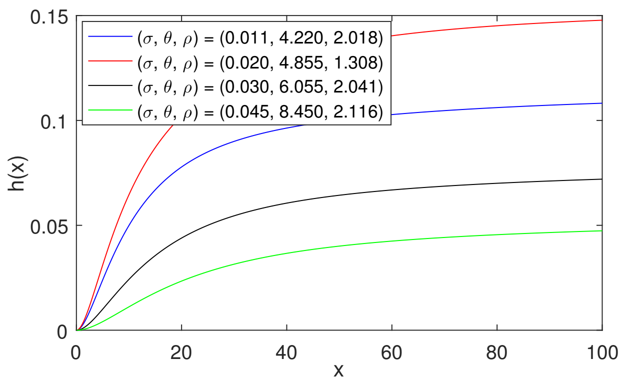

Figure 3: *Hazard Rate Function Plot*

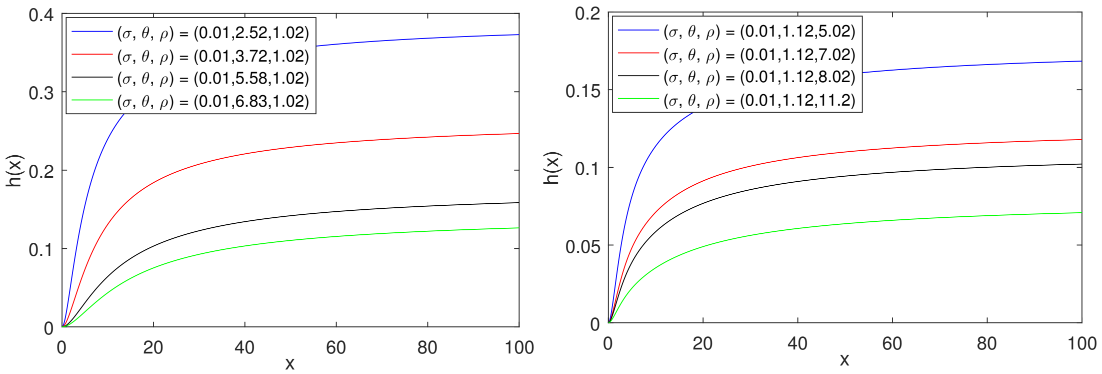

Figure 4: *Hazard Rate Function Subplots*

3.1. Mean Residual Life Function

The Mean Residual Life (MRL) function is used to investigate the aging process phenomenon of systems. To determine the MRL for a random variable X representing the component's life, is defined by

$$m_{LBI_vED}(x;\Theta) = E\left[X - x|X > x\right] = \frac{1}{R_{LBI_vED}(x;\Theta)} \int_x^\infty R_{LBI_vED}(x;\Theta)dx. \tag{19}$$

The function $m(x;\Theta)$ is defined over any domain of a random variable, it is of special significance for a non-negative random variable explaining the lifetime of the system, and at time x for an operational system, conditional expected residual life is determined by Finkelstein [6]. Furthermore, while the shape of the hazard rate function is essential, the MRL function is considered to be much more meaningful than the hazard rate function. Since the first describes the full residual life function, whilst the other simply evaluates the possibility of immediate failure at the time x. The MRL function of the $LBI_vED(\sigma,\theta,\rho \neq 1)$ is determined by using the relation (19) and is obtained as follows:

$$m_{LBI_vED}(x;\Theta) = \frac{1}{R_{LBI_vED}(x;\Theta)} \int_x^\infty \frac{\rho\left(x+\rho\theta\right)e^{-(x-\sigma)/\rho\theta} - \left(x+\theta\right)e^{-(x-\sigma)/\theta}}{\tau}dx.$$

On solving the above integral and further simplifications gives the following required MRL function of the model.

$$m_{LBI_vED}(x;\Theta) = \frac{\theta\rho^2\left(x+2\rho\theta\right)e^{-(x-\sigma)/\rho\theta} - \theta\left(x+2\theta\right)e^{-(x-\sigma)/\theta}}{\tau R_{LBI_vED}(x;\Theta)} \tag{20}$$

3.2. Stress-Strength Reliability

It is easily noticed, that a significant amount of research work had already been carried out in the field of stress-strength modeling on the estimation of reliability $R = Pr.(X_1 > X_2)$, where the random variables X_1 and X_2 represent the strength and stress factors of the system possessing the same distributions of uni-variate family. For the vast majority of the well-known standard distributions, the algebraic representation for R has been developed. Here, $X_1 \sim LBI_vED(\sigma,\theta_2,\rho_1 \neq 1)$ and $X_2 \sim LBI_vED(\sigma,\theta_2,\rho_2 \neq 1)$ are unrelated and have the same distribution, so-called LBI_vED, we calculate the reliability R. The procedure to measure R by having the $c.d.f$ of X_2 and $p.d.f$ of X_1 is described as follows

$$F_{LBI_vED}(x;\Theta) = 1 - \frac{\rho_2\,(x+\rho_2\theta_2)\,e^{-(x-\sigma)/\rho_2\theta_2} - (x+\theta_2)\,e^{-(x-\sigma)/\theta_2}}{(\rho_2-1)\,[\sigma+\theta_2\,(\rho_2+1)]} \quad \rho \neq 1 \tag{21}$$

and,

$$f_{LBI_vED}(x;\Theta) = \frac{xe^{-(x-\sigma)/\rho_1\theta_1} - xe^{-(x-\sigma)/\theta_1}}{\theta_1(\rho_1-1)\,[\sigma+\theta_1\,(\rho_1+1)]} \qquad \rho \neq 1 \tag{22}$$

then, R is derived by

$$R = \int_\sigma^\infty \left\{ \int_\sigma^x f_{X_2}(x)dx \right\} f_{X_1}(x)dx = \int_\sigma^\infty F_{X_2}(x) f_{X_1}(x)dx \tag{23}$$

$$\begin{aligned} = \quad 1 \quad & -\frac{\theta_1\theta_2\rho_1\rho_2^2}{B\,(\theta_1\rho_1+\theta_2\rho_2)} \left\{ \sigma\,(\sigma+\theta_2\rho_2) + \frac{\theta_1\theta_2\rho_1\rho_2\,(2\sigma+\theta_2\rho_2)}{(\theta_1\rho_1+\theta_2\rho_2)} + \frac{2\,(\theta_1\theta_2\rho_1\rho_2)^2}{(\theta_1\rho_1+\theta_2\rho_2)^2} \right\} \\ + \quad & \frac{\theta_1\theta_2\rho_1}{B\,(\theta_1\rho_1+\theta_2)} \left\{ \sigma\,(\sigma+\theta_2) + \frac{\theta_1\theta_2\rho_2\,(2\sigma+\theta_2)}{(\theta_1\rho_1+\theta_2)} + \frac{2\,(\theta_1\theta_2\rho_2)^2}{(\theta_1\rho_1+\theta_2)^2} \right\} \\ + \quad & \frac{\theta_1\theta_2\rho_2^2}{B\,(\theta_1+\theta_2\rho_2)} \left\{ \sigma\,(\sigma+\theta_2\rho_2) + \frac{\theta_1\theta_2\rho_2\,(2\sigma+\theta_2\rho_2)}{(\theta_1+\theta_2\rho_2)} + \frac{2\,(\theta_1\theta_2\rho_2)^2}{(\theta_1+\theta_2\rho_2)^2} \right\} \\ - \quad & \frac{\theta_1\theta_2}{B\,(\theta_1+\theta_2)} \left\{ \sigma\,(\sigma+\theta_2) + \frac{\theta_1\theta_2\,(2\sigma+\theta_2)}{(\theta_1+\theta_2)} + \frac{2\,(\theta_1\theta_2)^2}{(\theta_1+\theta_2)^2} \right\} \end{aligned}$$

where, $B = \theta_1\,(\rho_1-1)\,(\rho_2-1)\,[\sigma+\theta_1\,(\rho_1+1)]\,[\sigma+\theta_2\,(\rho_2+1)]$.

4. Entropy Measures

In science and engineering, entropy measures have already been studied as a practical application of the system. The two uncertainty measures of variation so-called Rényi entropy by Rényi [17] and Tsallis entropy by Tsallis [24] are studied in this section, the expressions of both these measures are given in the following respective subsections:

4.1. Rényi Entropy

The Rényi entropy of order α, for a non-negative $r.v$ $X \sim LBI_vED(\,\sigma, \theta, \rho \neq 1)$ is derived by

$$\hbar_R(\alpha) = \frac{1}{1-\alpha} \log \left[\int_\sigma^\infty \{f_{LBI_vED}(x;\Theta)\}^\alpha\, dx \right]; \quad \alpha \geq 0, \alpha \neq 1$$

on substituting the $p.d.f$ of LBI_vED when $\rho \neq 1$ and making use of binomial expansion by treating α as a finite nature number. The simplified equation of Rényi entropy of order α is derived as given by

$$\hbar_R(\alpha) = \frac{1}{1-\alpha} \log \left\{ \sum_{r=0}^{\alpha} \sum_{k=0}^{\alpha} \frac{\binom{\alpha}{r}\binom{\alpha}{k}\,(-1)^r\,\sigma^k\,(\theta\rho)^{\alpha-k+1}\,\Gamma\,(\alpha-k+1)}{\{\theta\tau\}^r\,[\alpha+r\,(\rho-1)]^{\alpha-k+1}} \right\} \tag{24}$$

4.2. Tsallis Entropy

The Tsallis entropy also known as *q-entropy* of order q, for a non-negative $r.v$ $X \sim LBI_vED(\,\sigma, \theta, \rho \neq 1)$ is derived by

$$T_q(x) = \frac{1}{1-q} \left[1 - \int_\sigma^\infty \{f_{LBI_vED}(x;\Theta)\}^q\, dx \right]; \quad q \geq 0, q \neq 1$$

on substituting the $p.d.f$ of LBI_vED when $\rho \neq 1$ and making use of binomial expansion by treating α as a finite nature number. The simplified equation of Tsallis entropy of order q is derived as given by

$$T_q(x) = \frac{1}{1-q}\left[1 - \sum_{r=0}^{q}\sum_{k=0}^{q} \frac{\binom{q}{r}\binom{q}{k}(-1)^r \sigma^k (\theta\rho)^{q-k+1}\Gamma(q-k+1)}{\{\theta\tau\}^r [q + r(\rho-1)]^{q-k+1}}\right] \tag{25}$$

5. Stochastic Ordering and Order Statistics

In this section, the stochastic ordering of the model and its order statistics are discussed.

5.1. Stochastic Ordering

Over the past several years, the usage of stochastic ordering has drastically increased across a wide range of statistical fields. These disciplines include reliability theory, queuing theory, survival analysis, and many more fields refer to Shaked and Shantikumar [20]. Let the $r.v\prime s$ X_1 and X_2, where $X_1 \sim LBI_vED(\sigma_1, \theta_1, \rho_1 \neq 1)$ and $X_2 \sim LBI_vED(\sigma_2, \theta_2, \rho_2 \neq 1)$, have $p.d.f\prime s$ denoted by $f_{X_1}(x)$, $f_{X_2}(x)$ and $c.d.f\prime s$ denoted by $F_{X_1}(x)$, $F_{X_2}(x)$ respectively. According to the model, it is believed that the random variable X_1 is smaller than X_2 in the,

(a_1) Stochastic order (mathematically $X_1 \leq_{st} X_2$), if $F_{X_1}(x;\Theta_1) \geq F_{X_2}(x;\Theta_2) \forall x$.

(a_2) Hazard rate order (mathematically $X_1 \leq_{hr} X_2$), if $h_{X_1}(x;\Theta_1) \geq h_{X_2}(x;\Theta_2) \forall x$.

(a_3) Mean residual life order (mathematically $X_1 \leq_{MRL} X_2$), if $m_{X_1}(x;\Theta_1) \geq m_{X_2}(x;\Theta_2) \forall x$.

(a_4) Likelihood ratio order (mathematically $X_1 \leq_{LR} X_2$), if $\frac{f_{X_1}(x;\Theta_1)}{f_{X_2}(x;\Theta_2)}$ decreases in x.

Since the following results show that, the four stochastic orders stated above are connected,

$$X_1 \leq_{MRL} X_2 \Leftarrow X_1 \leq_{hr} X_2 \Leftarrow X_1 \leq_{LR} X_2 \text{ and } X_1 \leq_{st} X_2 \Leftarrow X_1 \leq_{hr} X_2.$$

When the required conditions are met, the $LBI_vED(\sigma, \theta, \rho \neq 1)$ models are arranged $w.r.t$ the strongest likelihood ratio ordering, stated in the following theorem.

Theorem 1. Suppose $X_1 \sim LBI_vED(\sigma_1, \theta_1, \rho_1)$, and $X_2 \sim LBI_vED(\sigma_2, \theta_2, \rho_2)$, with the condition, that if $\sigma_1 = \sigma_2 = \sigma$, $(\rho_1 > \rho_2) > 1$, and $(\theta_1 > \theta_2)$ then $(X_1 \leq_{st} X_2)$, $(X_1 \leq_{hr} X_2)$, $(X_1 \leq_{lr} X_2)$, and $(X_1 \leq_{MRL} X_2)$.

Proof. Proof of the specified ratio is sufficient to demonstrate the outcome.

$$\frac{f_{X_1}(x;\Theta_1)}{f_{X_2}(x;\Theta_2)} = \frac{\theta_2(\rho_2-1)[\sigma_2 + \theta_2(\rho_2+1)]}{\theta_1(\rho_1-1)[\sigma_1 + \theta_1(\rho_1+1)]} \frac{\left\{e^{-(x-\sigma_1)/\rho_1\theta_1} - e^{-(x-\sigma_1)/\theta_1}\right\}}{\left\{e^{-(x-\sigma_2)/\rho_2\theta_2} - e^{-(x-\sigma_2)/\theta_2}\right\}}.$$

Now, applying log both sides and differentiate $w.r.t$ x.

$$\frac{d}{dx}\log\left\{\frac{f_{X_1}(x;\Theta_1)}{f_{X_2}(x;\Theta_2)}\right\} = \frac{\theta_1\rho_1 U(V_1 - \rho_2 V_2) - \theta_2\rho_2 V(U_1 - \rho_1 U_2)}{\theta_1\theta_2\rho_1\rho_2 UV}$$

where, $U = \left\{e^{-(x-\sigma_1)/\rho_1\theta_1} - e^{-(x-\sigma_1)/\theta_1}\right\}$, $V = \left\{e^{-(x-\sigma_2)/\rho_2\theta_2} - e^{-(x-\sigma_2)/\theta_2}\right\}$, $U_1 = e^{-(x-\sigma_1)/\rho_1\theta_1}$, $U_2 = e^{-(x-\sigma_1)/\theta_1}$, $V_1 = e^{-(x-\sigma_2)/\rho_2\theta_2}$, and $V_2 = e^{-(x-\sigma_2)/\theta_2}$. Hence, if $\sigma_1 = \sigma_2 = \sigma$, $(\rho_1 > \rho_2)$, and $(\theta_1 > \theta_2)$ then $\frac{d}{dx}\log\left\{\frac{f_{X_1}(x;\Theta_1)}{f_{X_2}(x;\Theta_2)}\right\} \leq 0$, which implies that $(X_1 \leq_{st} X_2)$, $(X_1 \leq_{hr} X_2)$, $(X_1 \leq_{lr} X_2)$, and $(X_1 \leq_{MRL} X_2)$. ■

5.2. Order Statistics

In order to discuss the order statistics of n random observations $x = \{ x_1, x_2, x_3, ..., x_n \}$ drawn from $LBI_vED(\sigma, \theta, \rho \neq 1)$. The random sample is arranged in ascending order such as $x_{1:n} \leq x_{2:n} \leq, ..., \leq x_{n:n}$. Then, we denote the $p.d.f$ of random variable which is of i^{th} order by $f_{i:n}(x_i, \Theta)$; $i = 1, 2, ..., n$, and joint $p.d.f$ of $(i, j)^{th}$ order variable pair by $f_{i:j:n}(x_i, x_j)$; $1 \leq i \leq j \leq n$, whose expressions are given by

$$f_{i:n}(x_i;\Theta) = \pi_1 [F_{LBI_vED}(x_{(i)};\Theta)]^{i-1} [1 - F_{LBI_vED}(x_{(i)};\Theta)]^{n-i} f_{LBI_vED}(x_{(i)};\Theta) \tag{26}$$

and,

$$f_{i:j:n}(x_i, x_j) = \pi_2 [F_{LBI_vED}(x_{(i)};\Theta)]^{i-1} [F_{LBI_vED}(x_{(j)};\Theta) - F_{LBI_vED}(x_{(i)};\Theta)]^{j-i-1} [1 - F_{LBI_vED}(x_{(j)};\Theta)]^{n-j} f_{LBI_vED}(x_{(i)};\Theta) f_{LBI_vED}(x_{(j)};\Theta) \tag{27}$$

where $F(.)$ and $f(.)$ are $c.d.f$ and $p.d.f$. The constants π_1 and π_2 are given by

$$\pi_1 = \frac{n!}{(i-1)!(n-i)!} \text{ and } \pi_2 = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$$

on substituting $i = 1$ and $i = n$ in equation (26) the $p.d.f's$ of 1^{st} and n^{th} order statistics obtained are given as follows:

$$f_{1:n}(x_{(1)};\Theta) = n \left[1 - F_{LBI_vED}(x_{(1)};\Theta) \right]^{n-1} f_{LBI_vED}(x_{(1)};\Theta)$$

$$= \frac{n \left[\rho\phi_{(1)} e^{-(x_{(1)}-\sigma)/\rho\theta} - \psi_{(1)} e^{-(x_{(1)}-\sigma)/\theta} \right]^{n-1} \left[x_{(1)} \left\{ e^{-(x_{(1)}-\sigma)/\rho\theta} - e^{-(x_{(1)}-\sigma)/\theta} \right\} \right]}{\theta(\rho-1)^n \left[\sigma + \theta(\rho+1) \right]^n} \tag{28}$$

and,

$$f_{n:n}(x_{(n)};\Theta) = n \left[F_{LBI_vED}(x_{(n)};\Theta) \right]^{n-1} f_{LBI_vED}(x_{(n)};\Theta)$$

$$= \frac{n \left[\tau - \left\{ \rho\phi_{(n)} e^{-(x_{(n)}-\sigma)/\rho\theta} - \psi_{(n)} e^{-(x_{(n)}-\sigma)/\theta} \right\} \right]^{n-1} \left[x_{(n)} \left\{ e^{-(x_{(n)}-\sigma)/\rho\theta} - e^{-(x_{(n)}-\sigma)/\theta} \right\} \right]}{\theta(\rho-1)^n \left[\sigma + \theta(\rho+1) \right]^n}. \tag{29}$$

Similarly, the joint order statistic density function of $LBI_vED(\sigma, \theta, \rho \neq 1)$ is given as

$$\begin{aligned} f_{i:j:n}(x_{(i)}, x_{(j)}) = & \frac{\pi_2}{\theta^2(\rho-1)^n \left[\sigma + \theta(\rho+1) \right]^n} \\ & \cdot \left[(\rho-1)\left[\sigma + \theta(\rho+1) \right] - \left\{ \rho\phi_{(i)} e^{-(x_{(i)}-\sigma)/\rho\theta} - \psi_{(i)} e^{-(x_{(i)}-\sigma)/\theta} \right\} \right]^{i-1} \\ & \cdot \left[\left\{ \rho\phi_{(i)} e^{-(x_{(i)}-\sigma)/\rho\theta} - \psi_{(i)} e^{-(x_{(i)}-\sigma)/\theta} \right\} - \left\{ \rho\phi_{(j)} e^{-(x_{(j)}-\sigma)/\rho\theta} - \psi_{(j)} e^{-(x_{(j)}-\sigma)/\theta} \right\} \right]^{j-i-1} \\ & \cdot \left[\rho\phi_{(j)} e^{-(x_{(j)}-\sigma)/\rho\theta} - \psi_{(j)} e^{-(x_{(j)}-\sigma)/\theta} \right]^{n-j} \left[x_{(i)} \left\{ e^{-(x_{(i)}-\sigma)/\rho\theta} - e^{-(x_{(i)}-\sigma)/\theta} \right\} \right] \\ & \cdot \left[x_{(j)} \left\{ e^{-(x_{(j)}-\sigma)/\rho\theta} - e^{-(x_{(j)}-\sigma)/\theta} \right\} \right] \end{aligned}$$

where, $\phi_{(r)} = \left(x_{(r)} + \rho\theta \right)$ and $\psi_{(r)} = \left(x_{(r)} + \theta \right)$; $r = 1, n, i, j$

6. Parameter Estimation and Simulation

6.1. Estimation Procedure of Model Parameters

Let a random sample x_1, x_2, ..., x_n, consisting of n observations is drawn from $LBI_vED(\sigma, \theta, \rho \neq 1)$. The most frequently used technique called the method of maximum likelihood estimation approach is chosen for the estimation of the parameters. So, we write the log-likelihood function of the complete data for $LBI_vED(\sigma,\theta,\rho\neq 1)$ as

$$\log L=\sum_{i=1}^{n}\log\left\{x_ie^{-(x_i-\sigma)/\rho\theta}-x_ie^{-(x_i-\sigma)/\theta}\right\}-n\log\left\{\theta(\rho-1)\left[\sigma+\theta\left(\rho+1\right)\right]\right\}. \quad (30)$$

Setting $\phi=e^{-(x_i-\sigma)/\rho\theta}$ and $\psi=e^{-(x_i-\sigma)/\theta}$, then (30) becomes

$$\log L=\sum_{i=1}^{n}\log\left\{\phi-\psi\right\}+\sum_{i=1}^{n}\log x_i-n\log\theta-n\log(\rho-1)-n\log\left[\sigma+\theta\left(\rho+1\right)\right]. \quad (31)$$

On differentiating equation (31) with respect to $(w.r.t)$ σ, θ, and ρ, the normal equations obtained are

$$\frac{\partial\log L}{\partial\sigma}=\sum_{i=1}^{n}\frac{\phi-\rho\psi}{\rho\theta\left(\phi-\psi\right)}-\frac{n}{\left[\sigma+\theta\left(\rho+1\right)\right]} \quad (32)$$

$$\frac{\partial\log L}{\partial\theta}=\sum_{i=1}^{n}\frac{\left(x_i-\sigma\right)\left(\phi-\rho\psi\right)}{\rho\theta^2\left(\phi-\psi\right)}-\frac{n\left(\rho+1\right)}{\left[\sigma+\theta\left(\rho+1\right)\right]}-\frac{n}{\theta} \quad (33)$$

$$\frac{\partial\log L}{\partial\rho}=\sum_{i=1}^{n}\frac{\phi\left(x_i-\sigma\right)}{\rho\theta^2\left(\phi-\psi\right)}-\frac{n\theta}{\left[\sigma+\theta\left(\rho+1\right)\right]}-\frac{n}{\rho-1}. \quad (34)$$

To get the maximum likelihood estimates (MLE_s), we equate equations (32), (33), and (34) to zero. But the equations obtained are not in closed form. So, the recommended functions such as *nlm* or *optim* are used to maximize the log-likelihood function in the R programming language. Also, the alternative iterative technique known as the Newton-Raphson method could be employed to yield the solution for the parameters. Further to construct the Fisher information matrix there must exist the second-order partial differentials which do exists as can be proved by the continuity of first-order differentials. Also, let us suppose that the MLE_s of Θ are given by $\hat{\Theta}=\{\ (\ \sigma,\theta,\rho\ ):\ \sigma>0,\theta>0,\rho>0\ \}$ then Fisher information matrix is defined as

$$I\left(\Theta\right)=-E\begin{bmatrix}\frac{\partial^2\log L}{\partial\sigma^2} & \frac{\partial^2\log L}{\partial\sigma\partial\theta} & \frac{\partial^2\log L}{\partial\sigma\partial\rho}\\ \frac{\partial^2\log L}{\partial\theta\partial\sigma} & \frac{\partial^2\log L}{\partial\theta^2} & \frac{\partial^2\log L}{\partial\theta\partial\rho}\\ \frac{\partial^2\log L}{\partial\rho\partial\sigma} & \frac{\partial^2\log L}{\partial\rho\partial\theta} & \frac{\partial^2\log L}{\partial\rho^2}\end{bmatrix} \quad (35)$$

the partial derivatives of Fisher information matrix $I\left(\Theta\right)$ are given by

$$\frac{\partial^2\log L}{\partial\sigma^2}=\sum_{i=1}^{n}\frac{\left(\phi-\rho\psi\right)^2-\left(\phi-\psi\right)\left(\phi-\rho\psi\right)}{\rho^2\theta^2\left(\phi-\psi\right)^2}+\frac{n}{\left[\sigma+\theta\left(\rho+1\right)\right]^2} \quad (36)$$

$$\frac{\partial^2\log L}{\partial\theta^2}=\sum_{i=1}^{n}\frac{\left(x_i-\sigma\right)\left(\phi-\rho\psi\right)\left\{\left(1+3\rho\right)\psi-2\left(1+\rho\right)\phi\right\}}{\rho^2\theta^3\left(\phi-\psi\right)^2}+\frac{n\left(\rho+1\right)^2}{\left[\sigma+\theta\left(\rho+1\right)\right]^2}+\frac{n}{\theta^2} \quad (37)$$

$$\frac{\partial^2\log L}{\partial\rho^2}=\sum_{i=1}^{n}\frac{\phi\left(x_i-\sigma\right)\left\{\left(\sigma+2\rho\theta\right)\phi-2\rho\theta\psi\right\}}{\rho^4\theta^2\left(\phi-\psi\right)^2}+\frac{n\theta^2}{\left[\sigma+\theta\left(\rho+1\right)\right]^2}+\frac{n}{\left(\rho-1\right)^2} \quad (38)$$

$$\frac{\partial^2\log L}{\partial\theta\partial\sigma}=\sum_{i=1}^{n}\frac{\rho\theta\left(\phi^2+\rho\psi^2\right)-\left(\sigma+\rho\theta-2\sigma\rho+\sigma\rho^2+\rho^2\theta\right)\phi\psi}{\rho^2\theta^3\left(\phi-\psi\right)^2}+\frac{n\left(\rho+1\right)}{\left[\sigma+\theta\left(\rho+1\right)\right]^2} \quad (39)$$

$$\frac{\partial^2\log L}{\partial\rho\partial\sigma}=\sum_{i=1}^{n}\frac{\rho\theta\left(\phi^2+\rho\psi^2\right)-\left(\sigma-\sigma\rho+\rho\theta+\rho^2\theta\right)\phi\psi}{\rho^3\theta\left(\phi-\psi\right)^2}+\frac{n\theta}{\left[\sigma+\theta\left(\rho+1\right)\right]^2} \quad (40)$$

$$\frac{\partial^2\log L}{\partial\rho\partial\theta}=\sum_{i=1}^{n}\frac{\left(x_i-\sigma\right)\left\{\left(\sigma-\sigma\rho+\rho^2\theta\right)\phi\psi-\rho\theta\left(\rho+1\right)\psi^2-\rho\theta\phi^2\right\}}{\rho^3\theta^3\left(\phi-\psi\right)^2}-\frac{\sigma}{\left[\sigma+\theta\left(\rho+1\right)\right]^2}. \quad (41)$$

We drop the expectation term as it is difficult to compute for the second-order partial differential elements of the Fisher information matrix see Cohen [4]. Therefore we write $I(\Theta)$ as

$$I\left(\hat{\Theta}\right) = -\begin{bmatrix} \frac{\partial^2 \log L}{\partial \sigma^2} & \frac{\partial^2 \log L}{\partial \sigma \partial \theta} & \frac{\partial^2 \log L}{\partial \sigma \partial \rho} \\ \frac{\partial^2 \log L}{\partial \theta \partial \sigma} & \frac{\partial^2 \log L}{\partial \theta^2} & \frac{\partial^2 \log L}{\partial \theta \partial \rho} \\ \frac{\partial^2 \log L}{\partial \rho \partial \sigma} & \frac{\partial^2 \log L}{\partial \rho \partial \theta} & \frac{\partial^2 \log L}{\partial \rho^2} \end{bmatrix}_{(\sigma,\theta,\rho)=(\hat{\sigma},\hat{\theta},\hat{\rho})} \tag{42}$$

now to construct the confidence intervals $(C.I\prime s)$ for parameters of $LBI_vED(\ \sigma,\ \theta,\ \rho \neq 1\ )$, we first find inversion matrix $I(\Theta)^{-1}$ consisting of the diagonal elements as variances whereas the off-diagonal elements represent the co-variances. The $100(1-\zeta)\%$ $C.I\prime s$ for σ, θ, and ρ are $\hat{\sigma} \pm \chi_{\zeta/2}\sqrt{V(\hat{\sigma})}$, $\hat{\theta} \pm \chi_{\zeta/2}\sqrt{V(\hat{\theta})}$, and $\hat{\rho} \pm \chi_{\zeta/2}\sqrt{V(\hat{\rho})}$ respectively, where the term $\chi_{\zeta/2}$ is $(\zeta/2)th$ upper percentile of a normal variate.

6.2. Simulation Study

It is necessary for a statistical distribution, to explore the behavior of estimated parameters by performing a Monte-Carlo simulation study at various randomly selected sample sizes, as $n = \{$ 30, 90, 150, 210, 350, 500 $\}$. Since the quantile function of LBI_vED does not exist in closed form. So, to simulate the data, an alternative technique is utilized known as the acceptance-rejection algorithm. In Table 1, the average bias $(Bias = \frac{1}{n}\sum_{i=1}^{n}(\hat{\Theta}-\Theta))$ and average mean square error $(MSE = \frac{1}{n}\sum_{i=1}^{n}(\hat{\Theta}-\Theta)^2)$ of the estimated parameters are calculated. The simultaneous decrease of $Bias$ and MSE of parameters σ, θ, and ρ are reported as the sample size increases. It is clear, that the consistency property of estimated parameters holds for LBI_vED.

Table 1: *Parameter Bias and MSE of* $LBI_vED(\sigma,\theta,\rho \neq 1)$

(σ,θ,ρ)	n	$Bias$			MSE		
		$\hat{\sigma}$	$\hat{\theta}$	$\hat{\rho}$	$\hat{\sigma}$	$\hat{\theta}$	$\hat{\rho}$
(0.91, 0.82, 0.78)	30	0.16580	0.37902	1.7e+01	0.027489	0.14366	3.0e+02
	90	0.06044	0.27480	2.65334	0.003654	0.07551	7.04022
	150	0.04272	0.25322	1.11791	0.001825	0.06412	1.24971
	210	0.03215	0.23397	0.97621	0.001034	0.05474	0.95299
	350	0.01838	0.20611	0.67691	0.000338	0.04248	0.45821
	500	0.01439	0.20147	0.63070	0.000207	0.04059	0.39778
(1.01, 1.87, 0.83)	30	0.39676	0.73710	1.2e+01	0.157419	0.54332	1.4e+02
	90	0.16685	0.58280	2.36629	0.027840	0.33966	5.59931
	150	0.11098	0.48364	1.03614	0.012317	0.23391	1.07359
	210	0.08229	0.43797	0.69849	0.006772	0.19182	0.48789
	350	0.05865	0.39767	0.55490	0.003440	0.15814	0.30791
	500	0.04099	0.36097	0.46926	0.001681	0.13030	0.22020
(1.02, 1.59, 1.05)	30	0.37336	0.53592	1.2e+01	0.139400	0.28721	1.6e+02
	90	0.15147	0.32432	1.97947	0.022842	0.10519	3.91829
	150	0.09869	0.25286	0.69786	0.009740	0.06394	0.48701
	210	0.07163	0.22203	0.45958	0.005131	0.04930	0.21121
	350	0.03737	0.15245	0.27079	0.001397	0.02324	0.07333
	500	0.03229	0.11726	0.19951	0.001042	0.01375	0.03980

7. Model Applicability

7.1. Real-Life Data Based Applications

In this section, an illustration of the proposed methodology is given. Real-life examples are presented to demonstrate the superior performance of the proposed LBI_vED model. The real-life data sets that are analyzed in this study are given below:

The first data set of T8 fluorescent lamps analyzed by Ahmed [1] represents the lifetime (Hours) for 50 devices given by: 0.445, 0.493, 0.285, 0.564, 0.760, 0.381, 0.690, 0.579, 0.636, 0.238, 0.149, 0.244, 0.126, 0.796, 0.405, 0.553, 0.780, 0.431, 0.184, 0.375, 0.198, 0.890, 0.192, 0.463, 0.486, 0.521, 0.366, 0.486, 0.116, 0.511, 0.612, 0.117, 0.384, 0.326, 0.057, 0.412, 0.586, 0.517, 0.570, 0.588, 0.497, 0.246, 0.234, 0.228, 0.552, 0.893, 0.403, 0.458, 0.134, 0.338.

The second data set by Bader and Priest [3] consists of 63 samples representing the strength measured in GPA for single carbon fibers with gauge lengths of 10mm given by: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

The output results of the analyzed data sets include the estimated parameters, the information measures such as Akaike information criteria $\{\ AIC = -2\log L(x;\Theta)\ \}$, Bayesian information criteria $\{\ BIC = -2\log L(x;\Theta) + k\log L(x;\Theta)\ \}$, Hannan Quen information criteria $\{\ HQIC = -2\log L(x;\Theta) + 2k\log\left[\log(n)\right]\ \}$, where constants n and k represent the sample size and the number of parameters respectively. The goodness-of-fit statistics tests are the Anderson Darling test, Cramer Von Mises test, and the Kolmogorov Smirnov test statistic with a *p-value*. The existing distributions that are compared with LBI_vED are I_vED, exponential distribution (ED), weighted exponential distribution (WED), and length-biased exponential distribution $(LBED)$. The $p.d.f/s$ are given as,

$$ED = f(x;\Theta) = \frac{1}{\sigma}e^{-x/\sigma} \tag{43}$$

$$WED = f(x;\Theta) = \frac{(\sigma+1)}{\sigma}\theta e^{-\theta x}\left(1 - e^{-\sigma\theta x}\right) \tag{44}$$

$$LBED = f(x;\Theta) = \frac{[\theta(\sigma+1)]^2}{\sigma(\sigma+2)}xe^{-\theta x}\left(1 - e^{-\sigma\theta x}\right) \tag{45}$$

Table 2: *Various measures from the first data set*

Models	*Information Measures*						
	$\hat{\sigma}$	$\hat{\theta}$	$\hat{\rho}$	$\log L$	AIC	BIC	$HQIC$
LBI_vED	0.01975	0.13979	1.00007	7.60999	-9.21998	-3.48391	-7.03565
I_vED	0.04305	0.19339	1.00033	6.10016	-6.20032	-0.46426	-4.01600
$LBED$	1.749e+4	4.65200	-	4.11157	-3.96781	-0.39908	-2.76691
WED	1.278e-04	4.65203	-	4.11157	-4.22313	-0.39908	-2.76691
ED	0.42990	-	-	-7.78987	17.5797	19.4918	18.3078
Models	*Goodness-of-fit Tests*						
	$\hat{\sigma}$	$\hat{\theta}$	$\hat{\rho}$	C_{VM}	AD	KS	*p-value*
LBI_vED	0.01975	0.13979	1.00007	0.14036	0.75075	0.12723	0.3932
I_vED	0.04305	0.19339	1.00033	0.17074	0.91949	0.15728	0.1685
$LBED$	1.749e+4	4.65200	-	0.13315	0.71331	0.16757	0.1206
WED	1.278e-04	4.65203	-	0.13315	0.71330	0.16761	0.1205
ED	0.42990	-	-	0.13395	0.71750	0.23317	0.0087

Table 3: *Various measures from the second data set*

Models	Information Measures						
	$\hat{\sigma}$	$\hat{\theta}$	$\hat{\rho}$	-log L	AIC	BIC	$HQIC$
LBI_vED	1.87153	0.50522	1.00011	59.884	125.768	132.197	128.297
I_vED	1.87408	0.59253	1.00026	60.663	127.326	133.756	129.855
$LBED$	1.58e-05	0.9806086	-	97.954	199.908	204.194	201.594
WED	1.717e-5	6.537e-01	-	110.35	224.696	228.983	226.382
ED	3.05930	-	-	133.45	268.892	271.035	269.734
Models	Goodness-of-fit Tests						
	$\hat{\sigma}$	$\hat{\theta}$	$\hat{\rho}$	C_{VM}	AD	KS	*p-value*
LBI_vED	1.87153	0.50522	1.00011	0.06424	0.42637	0.10492	0.4919
I_vED	1.87408	0.59253	1.00026	0.06944	0.46604	0.11296	0.3975
$LBED$	1.58e-05	0.9806086	-	0.05891	0.36294	0.33488	1.4e-06
WED	1.717e-5	6.537e-01	-	0.05890	0.36281	0.39019	9.3e-09
ED	3.05930	-	-	0.05885	0.36229	0.48600	2.3e-13

From Table 2 and Table 3, it is visible that all the existing distributions lack superiority against the new LBI_vED, the information measures and the goodness-of-fit test results displayed are relatively lower for LBI_vED with a higher significant *p-value*. Hence, the LBI_vED is treated as the best-fitted model.

Conclusion

In this article, the WI_vED, a generalization of the I_vED that may be used to describe time-to-event data sets as a lifetime distribution is presented and examined. The length biased variant of the I_vED is derived as a special instance of the WI_vED and its model applicability features are thoroughly examined. We find that the LBI_vED may be used as a flexible model in place of the traditional lifetime models taken in comparison that are commonly used in the literature to describe real-life time-to-event data. As evidenced by its excellent distributional, reliability, and survival qualities, we anticipate that the proposed WI_vED and the length-biased variant of I_vED will function as a competitive model for representing data from reliability analysis, survival analysis, and other domains of the statistics.

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