A NOVEL METHOD TO GENERATE A FAMILY OF BATHTUB-SHAPED FAILURE RATES FROM A FAMILY OF UPSIDE DOWN BATHTUB-SHAPED FAILURE RATES AND VICE-VERSA

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Abstract

It is indeed a matter of great significance for system engineers and scientists to derive new classes of lifetime distributions for providing a better statistical model which will fit a given lifetime data set. It is known that many real time data have varied characteristics and can be modeled by distributions with bathtub and upside down bathtub failure rates viz., Weibull, Modified Weibull, Inverse Weibull. This paper proposes a method which generates a family of distributions having bathtub (BT)-shaped failure rate from a distribution having upside down bathtub (UBT)-shaped failure rate and vice-versa. The proposed method is validated with the help of a few statistical distributions. The closure properties of the proposed model under various reliability operations are studied.

Keywords: Aging phenomenon, hazard rate, bathtub-shaped failure rate, upside down bathtub-shaped failure rate.

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1. Introduction

Lifetime distributions are usually categorized based on their failure pattern. Given that a device has survived till time $t > 0$, the hazard (failure) rate provides instantaneous failure rate in a very small (future) time interval. The shape of the hazard rate function can be strictly decreasing, strictly increasing, constant, BT and UBT. Increasing failure rate often occurs in the real life situations, where devices are more likely to fail with respect to age. Decreasing hazard rate appears when materials become harder with respect to time. The concept of bathtub (resp. upside bathtub) hazard rate distribution is discussed in the literature based on whether the corresponding hazard rate is decreasing (resp. increasing) in the region $(0, T_0]$, constant in $[T_0, T_1]$, and increasing (resp. decreasing) in $[T_1, \infty)$ where T_0 and T_1 are non-negative real numbers. In that case,

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the random variable X is said to be BT (resp. UBT). Here, T_0 and T_1 are considered as change points of the hazard rate function. This concept holds even if $T_0 = T_1$. BT-shaped hazard rate is a combination of three different types of shapes, which usually appears in the study of life cycle of an industrial product or in the whole life span of a biological entity. Due to design error or installation problem, there is a high chance that a device has high likelihood of failure in first few weeks of operation. After initial period, the failure rate becomes relatively low, known as normal wear period. Then, the device reaches at the end of its life and the failure probability becomes very high with respect to time due to ageing. We refer to Rajarshi and Rajarshi [6] and Lai et al.[4] for some discussions on such kind of distributions. There are some other situations, related to the study of lifetime of a patient after major surgery, where models having unimodal hazard rate or having UBT-shaped hazard rate are useful. In biological science, it is observed in the course of a disease whose mortality reaches a peak after some finite period and then declines gradually. The commonly used distributions with UBT-shaped hazard rate are inverse Gaussian distribution, log-normal distribution, etc.

There are various transformations used by researchers to convert a baseline distribution into a new statistical distribution to get better flexibility. For example, inverted family of distributions can be obtained from a baseline distribution after using an inverse transformation. It has been shown by Keller et al.[3] that for pistons, crankshaft, main bearings failure data sets, the inverse Weibull distribution provides a better fit than the exponential and Weibull distributions. Akg*ü*l et al. [1] explored that the wind speed data can be modelled by inverse Weibull distribution, which gives a better output than Weibull distribution. This paper aims to provide a new method for the generation of a family of BT-shaped failure rates from a family of UBT-shaped failure rates and vice-versa. It is well-known that a series system formed with independent components each having BT-shaped failure rate with different change points has a BT-shaped failure rate with an arbitrary change point. In this article, we propose a new transformation so as to have a common (specific) change point of the resulting BT-shaped failure rate. Some of the resulting mathematical avenues are also explored for reverse model. Let X be a non-negative absolutely continuous random variable with probability density function (PDF) $f(.)$ and cumulative distribution function (CDF) $F(.)$. Then, the hazard rate of X is denoted by $r(t) = f(t)/\bar{F}(t)$, where $\bar{F}(t) = 1 - F(t),\ t > 0$. Throughout the paper, we assume that the derivative exists whenever, it is implemented.

The rest of the paper is arranged as follows. Section 2 provides a transformation/method for the generation of the BT-shaped failure rate distribution from UBT-shaped failure rate distribution. In addition, various properties of the resulting BT-shaped failure rate distribution are explored. Section 3 discusses a method of generating UBT-shaped failure rates from BT-shaped failure rate with some notable consequences. Finally, Section 4 concludes the paper.

2. A method to generate a BT-shaped failure rate using UBT-shaped failure rate

Let U be a non-negative absolutely continuous lifetime random variable with CDF $F_U(\cdot)$ having UBT-shaped failure rate $r_U(t)$, for $t \in (l_U, u_U)$, where l_U and u_U denote respectively the lower and the upper bounds of the support of the random variable U. In this section, we introduce an interesting method to generate a distribution with corresponding lifetime random variable B with a BT-shaped failure rate $r_B(t)$, $t \in (l_B, u_B)$, where l_B and u_B denote respectively the lower and the upper bounds of the support of B using the distribution with UBT-shaped failure rate $r_U(t)$. Throughout the paper, we assume $l_U = 0$ and $u_U = +\infty$.

Theorem 1. Denote by $r_B(t)$ and $r_U(t)$ the BT-shaped failure rate and UBT-shaped failure rate, respectively. Then, the UBT-shaped failure rate can be obtained using an equation given by

$$r_B(t) = kM - r_U(t), \text{ for } t \geq 0, \tag{2.1}$$

where $k \geq 1$ is a real number and $M = \max_{t \geq 0} r_U(t)$.

Proof. The proof is clear from the following discussion. Note that the graphs of $r_B(t)$ and $r_U(t)$ are geometrically equivalent because one is obtained from other by reflection about horizontal axis and then by vertical translation of kM units. For a given UBT-shaped failure rate function $r_U(t)$, $-r_U(t)$ represents its vertical reflected image (or reflection about t-axis) lying in fourth quadrant, which is eventually a BT-shaped failure rate function. To shift up and to drag $-r_U(t)$, for all $t \geq 0$, back to first quadrant, we give a positive (up) shift by kM units, k being greater than or equal to one, the minimum required factor being M, where $M = \max_{t \geq 0} r_U(t)$. This completes the proof of the result. ■

Remark 1. Clearly, $\{r_U(\cdot), k\}$ completely describes the aforementioned model which satisfies the hypothesis of Theorem 1. This notation will be used throughout the article wherever required. The parameter M is derivable from $\{r_U(\cdot), k\}$.

The next theorem provides the survival function $\bar{F}_B(\cdot)$ and the density function $f_B(\cdot)$ corresponding to the newly generated distribution with BT-shaped failure rate, which is obtained from the UBT-shaped failure rate model by the method discussed in Theorem 1. The proof is omitted since it easily follows from Theorem 1 and the well-known relationship

$$\bar{F}_B(t) = e^{-\int_0^t r_B(u)du}. \tag{2.2}$$

Theorem 2. The survival and density functions of the random variable B are respectively given by

$$\bar{F}_B(t) = \frac{\exp(-kMt)}{\bar{F}_U(t)}, \quad t \geq 0 \tag{2.3}$$

and

$$f_B(t) = \frac{1}{\bar{F}_U(t)}(kM - r_U(t)) \exp(-kMt), \quad t \geq 0. \tag{2.4}$$

The method, discussed in Theorem 1 can be implemented to generate a family of BT-shaped failure rate models using a single UBT-shaped failure rate model as stated (without proof) in the next theorem.

Theorem 3. For $i = 1, \ldots, n$, the random variables B_i's have BT-shaped failure rates as given by

$$r_{B_i}(t) = k_i M - r_U(t), \quad \text{for } t \geq 0, \tag{2.5}$$

where $r_U(t)$ is the UBT-shaped failure rate, k_i's are real constants satisfying $k_i \geq 1$, and $M = \max_{t \geq 0} r_U(t)$.

Proof. The proof follows using similar arguments as in the proof of Theorem 1, and thus it is omitted. ■

Next, we consider an example to illustrate the result in Theorem 3.

Example 2.1. *Let a random variable U follow inverse Weibull distribution (see Jiang et al. [2]) with survival function* $\bar{F}_U(t) = 1 - \exp\big(-\big(\frac{\beta}{t}\big)^{\alpha}\big),\ t \geq 0,\ \alpha > 0,\ \beta > 0$. *This distribution has UBT-shaped failure rate for* $\alpha < 1$. *Taking* $\alpha = 0.5,\ \beta = 2$, *the corresponding failure rate* $r_U(t) = \frac{\alpha(\beta/t)^{\alpha}}{t\big(\exp(\beta/t)^{\alpha} - 1\big)}$ *can be shown to be UBT. Further,* $M = \max_{t \geq 0} r_U(t) = 0.35536$. *Now, the corresponding* $r_{B_i}(t)$*'s are plotted in Figure* $1(a)$, *where* $r_{B_i}(t) = k_i M - r_U(t)$, *with* $k_i = i$, *for* $i = 1, \ldots, 5$.

Below, we compare two random variables B_i and B_j having UBT-shaped hazard rates in the sense of the hazard rate order. Let X and Y be two non-negative random variables with hazard rate functions $r_X(.)$ and $r_Y(.)$, respectively. Then, X is said to be smaller than Y in the sense of the hazard rate order, denoted by $X \leq_{hr} Y$, if $r_X(x) \geq r_Y(x)$, for all $x > 0$. For various other stochastic orders, we refer to Shaked and Shanthikumar [7]. From Theorem 3, we can write $r_{B_n}(t) = k_n M - r_U(t)$, for $n = i,\ j$.

Corollary 1. Let B_i and B_j be two random variables with $r_{B_n}(t) = k_n M - r_U(t)$, for $n = i,\ j$ and $t \geq 0$. Then, $B_i \leq_{hr} B_j$, if and only if $k_i \geq k_j$.

Proof. The proof is straightforward, and thus it is omitted. ■

2.1. Properties of the resulting BT-shaped failure models

In this subsection, we establish an interesting property of the resulting BT-shaped failure models. The following theorem shows that a series system formed by n number of independent components each having BT-shaped failure rate obtained from a common UBT-shaped failure rate model possesses BT-shaped failure rate model. In other words, a series system is closed under the specified BT transformation as given by (2.1) and (2.5).

Theorem 4. Consider a series system formed by n components with independent lifetimes denoted by $B_i,\ i = 1, \ldots, n$. Further, let B_i have BT-shaped failure rate, say $r_{B_i}(t)$ generated from a single component with UBT-shaped failure rate $r_U(t)$ satisfying $r_{B_i}(t) = k_i M - r_U(t),\ i = 1, \ldots, n$, for all $t \geq 0$, $k_i \geq 1$, and $M = \max_{t \in (0,\infty)} r_U(t)$. Then, the system has BT-shaped failure rate, denoted by $r_{B_S}(t)$.

Proof. Note that

$$r_{B_S}(t) = \sum_{i=1}^{n} r_{B_i}(t) = \sum_{i=1}^{n} k_i M - n r_U(t) = nM\left(\frac{\sum_{i=1}^{n} k_i}{n}\right) - n r_U(t). \tag{2.6}$$

Further, $\max_{t\in(0,\infty)} nr_U(t) = nM$ and $\frac{\sum_{i=1}^{n} k_i}{n} \geq 1$. Thus, it follows from (2.1) and (2.6) that $r_{B_S}(t)$ represents a BT-shaped failure rate, implies the system has BT-shaped failure rate. This completes the proof. ■

The next remark gives an interesting fact about Theorem 4, and may be noted for independent interest.

Remark 2. For $i = 1, \ldots, n$, if $r_{B_i}(t) = k_i M - r_U(t)$, then $r_{B_i}(t)$ and $r_U(t)$ have the same change points as given by the roots of $\frac{d}{dt}(r_{B_i}(t)) = -\frac{d}{dt}(r_U(t)) = 0$, for all $t \in [0, +\infty)$. This leads to the fact that all of $r_{B_i}(t)$, for $i = 1, \ldots, n$ have the same change point as a result of which $r_{B_S}(t) = \sum_{1\leq i\leq n} r_{B_i}(t)$ has a change point equal to that of $r_{B_i}(t)$ (or $r_U(t)$) since from (2.6) we find

$$\frac{d}{dt}(r_{B_s}(t)) = -n\frac{d}{dt}(r_U(t)) = n\frac{d}{dt}(r_{B_i}(t)) = 0,$$

and hence B_S has a BT-shaped failure rate.

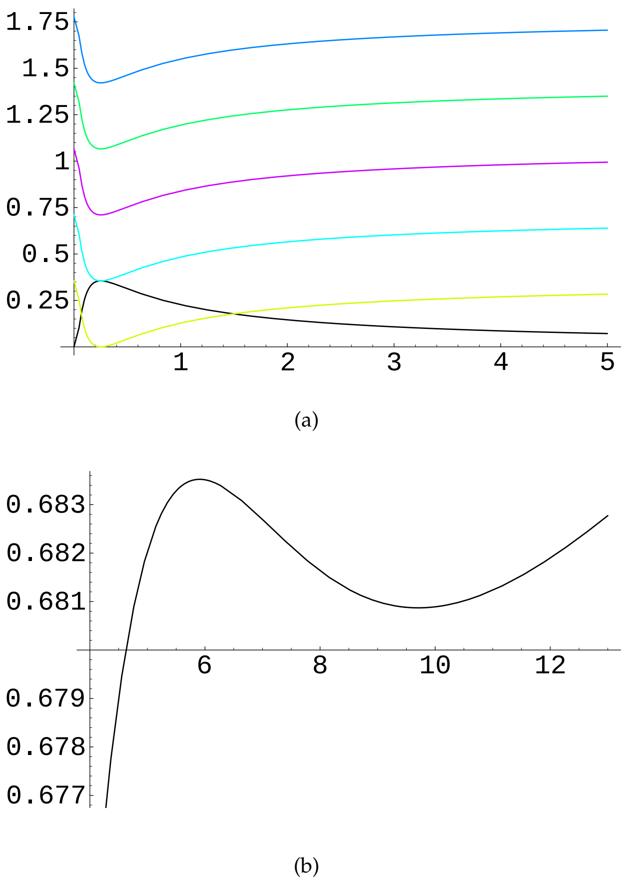


Figure 1: *(a) Plots of $r_U(t)$ (the black curve) and $r_{B_i}(t)$, for $i = 1, \ldots, 5$, respectively from bottom to top as in Example 2.1. (b) Plot of $r_X(t)$ versus t as in Counterexample 2.1*

We now state a lemma with an outline of its proof, which will be used in proving upcoming theorem.

Lemma 1. Let S be a non-empty set and let $g_i(t)$ be defined on S such that $\max_{t\in S} g_i(t)$ exists for each $i = 1,\ldots,n$. Then, we have

$$\max_{t\in S}\sum_{i=1}^{n} g_i(t) \le \sum_{i=1}^{n}\max_{t\in S} g_i(t). \tag{2.7}$$

Proof. We know that $g_i(t) \le \max_{t\in S} g_i(t)$, for $i = 1,\ldots,n$ and $t \ge 0$. Thus,

$$\sum_{i=1}^{n} g_i(t) \le \sum_{i=1}^{n}\max_{t\in S} g_i(t).$$

Further, since $\sum_{i=1}^{n}\max_{t\in S} g_i(t)$ is an upper bound of $\sum_{i=1}^{n} g_i(t)$, it follows that

$$\max_{t\in S}\sum_{i=1}^{n} g_i(t) \le \sum_{i=1}^{n}\max_{t\in S} g_i(t).$$

Thus, the proof is completed. ■

Now, we present an example in the light of the above lemma, where strict inequality holds. It is quite easy to construct examples where equality holds.

Example 2.2. *Let $g_1,\ g_2 : \mathbb{R} \to \mathbb{R}$ be given by*

$$g_1(t) = \begin{cases} 1, & \text{for} \quad t \in \mathbb{Q} \\ 0, & \text{for} \quad t \in \mathbb{Q}^c \end{cases} \quad \text{and} \quad g_2(t) = \begin{cases} 0, & \text{for} \quad t \in \mathbb{Q} \\ 1, & \text{for} \quad t \in \mathbb{Q}^c \end{cases} \tag{2.8}$$

Clearly, $\max_{t\in[0,+\infty)} g_i(t) = 1$, for $i = 1, 2$ so that

$$\sum_{i=1}^{2}\max_{t\in[0,+\infty)} g_i(t) = 2.$$

Furthermore, since $(g_1 + g_2)(t) = 1$, for all $t \in [0, +\infty)$, we have

$$\max_{t\in[0,+\infty)}\sum_{i=1}^{2} g_i(t) = 1.$$

Thus,

$$\max_{t\in[0,+\infty)}\sum_{i=1}^{2} g_i(t) < \sum_{i=1}^{2}\max_{t\in[0,+\infty)} g_i(t)$$

is established.

In the upcoming theorem, we will observe that even though $r_{B_i}(t)$ possesses different change points yet

$$r_{B_{S^*}}(t) = \sum_{i=1}^{2} r_{B_i}(t)$$

is BT-shaped. We pause for a while and read the next remark before going to Theorem 5.

Remark 3. For $i = 1,\ldots,n$, if $r_{B_i}(t) = k_i M_i - r_{U_i}(t)$, then $r_{B_i}(t)$ possesses different change points given by the roots of $\frac{d}{dt}\big(r_{B_i}(t)\big) = -\frac{d}{dt}\big(r_{U_i}(t)\big) = 0$, for all $t \in [0, +\infty)$, provided that each $r_{U_i}(t)$ is differentiable. This leads to the fact that all of $r_{B_i}(t)$, for $i = 1,\ldots,n$ have different change points.

Theorem 5. If B_{S^*} is a random variable denoting the lifetime of a series system formed by n independent components with lifetimes B_i, for $i = 1, \ldots, n$ having BT-shaped failure rates $r_{B_i}(t)$ generated from independent components with UBT-shaped failure rates $r_{U_i}(t)$ satisfying $r_{B_i}(t) = k_i M_i - r_{U_i}(t)$, for all $t \geq 0$, $k_i \geq 1$, and $M_i = \max_{t\in(0,+\infty)} r_{U_i}(t)$, then $r_{B_{S^*}}(t)$ yields a distribution having BT-shaped failure rate.

Proof. Note that

$$r_{B_{S^*}}(t) = \sum_{i=1}^{n} r_{B_i}(t) = \sum_{i=1}^{n} (k_i M_i) - \sum_{i=1}^{n} r_{U_i}(t) = M \sum_{i=1}^{n} \left(\frac{k_i M_i}{M}\right) - \sum_{i=1}^{n} r_{U_i}(t), \tag{2.9}$$

where $M = \max_{t\in(0,\infty)} \sum_{i=1}^{n} r_{U_i}(t)$. Further, since each of $r_{U_i}(t)$, for $i \in \{1, \ldots, n\}$ is concave, $\sum_{i=1}^{n} r_{U_i}(t)$ is also concave. Moreover, the local maximizer of a concave function defined over a convex set (here $\mathbb{R}$) is the global maximizer. Thus, $\sum_{i=1}^{n} r_{U_i}(t)$ possesses a UBT-shaped failure rate with unique maximizer. So, it suffices to show that $\sum_{i=1}^{n} \frac{k_i M_i}{M} \geq 1$ to establish our claim that $r_{B_{S^*}}(t)$ represents a BT failure rate as discussed in Theorem 1. Clearly,

$$\sum_{i=1}^{n} (k_i M_i) \geq \min_{1\leq i\leq n} (k_i) \sum_{i=1}^{n} M_i \tag{2.10}$$

Again, using Lemma 1, one can show that

$$\Big(\sum_{i=1}^{n} M_i = \Big) \sum_{i=1}^{n} \max_{t\in(0,+\infty)} r_{U_i}(t) \geq \max_{t\in(0,+\infty)} \sum_{i=1}^{n} r_{U_i}(t). \tag{2.11}$$

Thus, from (2.10) and (2.11), we conclude that

$$\sum_{i=1}^{n} (k_i M_i) \geq \min_{1\leq i\leq n} (k_i) \sum_{i=1}^{n} \max_{t\in(0,\infty)} r_{U_i}(t) \geq \max_{t\in(0,\infty)} \sum_{i=1}^{n} r_{U_i}(t) = (M)$$

as $k_i \geq 1$, for all $i = 1, \ldots, n$, that is $\sum_{i=1}^{n} \frac{k_i M_i}{M} \geq 1$. This completes the proof. ■

Since this special type of construction allows the BT-shaped failure rate system to be closed under the formation of series system, a natural question that arises is whether this result can be generalized to the formation of k-out-of-n system. We recall that k-out-of-n system works if atleast k components of n number of components work. In the following counterexample, we notice that the answer of this question in negative. It shows that the BT-shaped failure rate system is not closed under the formation of parallel system.

Counterexample 2.1. *Consider a parallel system with lifetime X comprised of two components having failure rates, $r_{B_i}(t) = k_i M - r_U(t)$, $t \geq 0$ with $k_i = i + 1$, for $i = 1, 2$, and $r_U(t) = \frac{\beta(\alpha/t)^\beta}{t\left(\exp(\alpha/t)^\beta - 1\right)}$, $\alpha = 0.5$, $\beta = 2$, $M = \max_{t\geq 0} r_U(t) = 0.35536$. By Theorem 2, it follows that $\bar{F}_{B_i}(t) = \frac{\exp(-k_i M t)}{\bar{F}_U(t)}$, for $i = 1, 2$ so that $\bar{F}_X(t) = 1 - (1 - \bar{F}_{B_1}(t))(1 - \bar{F}_{B_2}(t))$, for all $t \geq 0$. The plot of $r_X(t)$ for $t \geq 0$ given in Figure 1(b) shows that it is roller coaster.*

3. A method to generate a UBT-shaped failure rate using BT-shaped failure rate

Let B^* be a continuous non-negative random variable with CDF $F_{B^*}(\cdot)$ having BT-shaped failure rate $r_{B^*}(t)$ for $t \in [0, +\infty)$. On a similar line as discussed in the earlier section, we generate a distribution with corresponding random variable U^* having UBT-shaped failure rate as given in the next theorem. The proof is omitted for the sake of conciseness.

Theorem 6. A distribution with UBT-shaped failure rate denoted by $r_{U^*}(t)$ obtained from a distribution having BT-shaped failure rate $r_{B^*}(t)$ is generated by the following equation

$$r_{U^*}(t) = \begin{cases} 0 & \text{for } \ 0 < t \leq t_1 \\ km - r_{B^*}(t) & \text{for } \ t_1 \leq t \leq t_2 \\ 0 & \text{for } \ t \geq t_2, \end{cases} \tag{3.12}$$

where t_1 and t_2 are the positive roots of $km - r_{B^*}(t) = 0$ with $t_1 \leq t_2$ and $m = \min_{t \in [0,+\infty)} r_{B^*}(t)$ and k is a real number satisfying $k \geq 2$.

The next corollary is useful to obtain the survival function $\bar{F}_{U^*}(\cdot)$, and the density function $f_{U^*}(\cdot)$ of the newly generated UBT-shaped failure rate model obtained from BT-shaped failure rate model by the approach as discussed in Theorem 6.

Corollary 2. With reference to the hypothesis as in Theorem 6, it is easy to note that

(i) the survival function of the random variable U^* is

$$\bar{F}_{U^*}(t) = \begin{cases} 1 & \text{for } \ 0 < t \leq t_1 \\ \exp(-km(t - t_1)) \frac{\bar{F}_{B^*}(t_1)}{\bar{F}_{B^*}(t)} & \text{for } \ t_1 \leq t \leq t_2 \\ \exp(-km(t_2 - t_1)) \exp(-\int_{t_2}^{t} r_{B^*}(u) du) \frac{\bar{F}_{B^*}(t_1)}{\bar{F}_{B^*}(t)} & \text{for } \ t \geq t_2; \end{cases}$$

(ii) the density function of the random variable U^* can be obtained by simply differentiating $-\bar{F}_{U^*}(t)$ with respect to t.

The following proposition, which is useful to generate a family of UBT-shaped failure rate models using a single BT-shaped model, can be easily established from Theorem 3. The proof is omitted for the sake of brevity.

Proposition 3.1. *A family of random variables U_i, for $i = 1, \ldots, n$ each with UBT-shaped failure rate, given by*

$$r_{U_i^*}(t) = \begin{cases} 0 & \textit{for } \ 0 \leq t \leq t_1^{(i)} \\ k_i m - r_{B^*}(t) & \textit{for } \ t_1^{(i)} \leq t \leq t_2^{(i)} \\ 0 & \textit{for } \ t \geq t_2^{(i)}, \end{cases} \tag{3.13}$$

is generated from a random variable B^* with BT-shaped failure rate $r_{B^*}(t)$, where $t_1^{(i)}$ and $t_2^{(i)}$ are the positive roots of $k_i m - r_{B^*}(t) = 0$, with $t_1^{(i)} \leq t_2^{(i)}$, $m = \min_{t \in [0,+\infty)} r_{B^*}(t)$ and k_i is a real number satisfying $k_i \geq 2$.

The following corollary presents condition, under which the hazard rate order between U_i^* and U_j^* exists. We omit the proof since it is a consequence of Proposition 3.1.

Corollary 3. We have $U_i^* \geq_{hr} U_j^*$, if and only if $k_i \leq k_j$.

Let us use the notation $S_i = \{t \in \mathbb{R} \mid r_{U_i^*}(t) > 0\}$. Clearly, it follows from (3.13) that $S_i = (t_1^i, t_2^i)$. Next, we state a strong result in the form of a lemma, which will be used later.

Lemma 2. If $k_i \leq k_j$, then $S_i \subseteq S_j$, for any $i, \ j \in \{1, \ldots, n\}$.

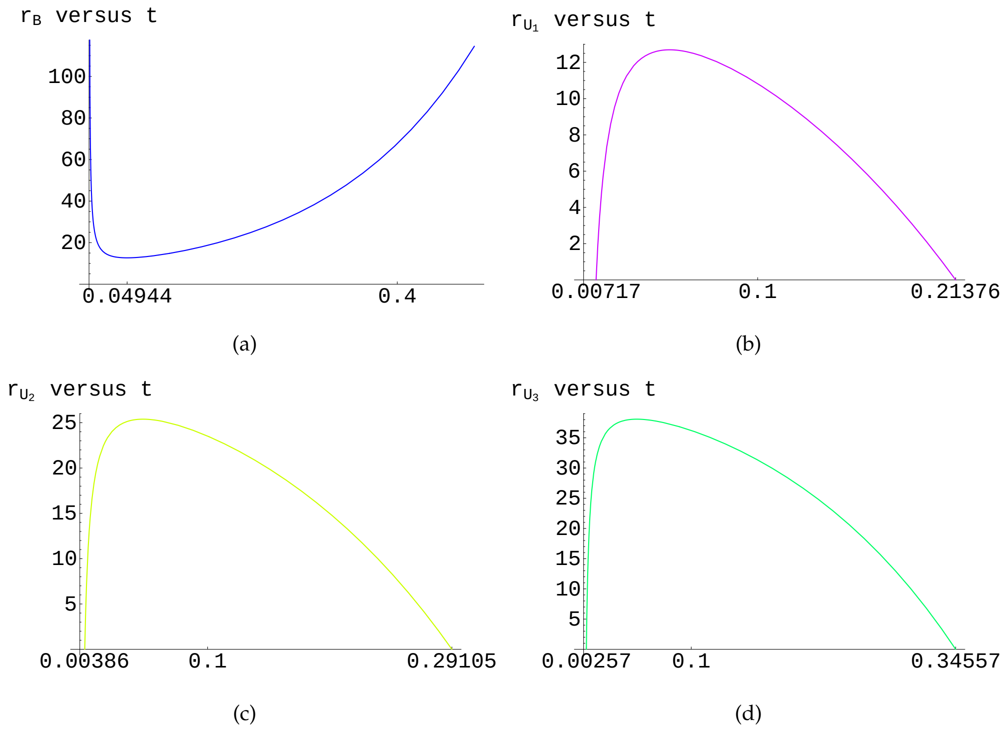


Figure 2: *Plots of* (a) $r_{B^*}(t)$, (b) $r_{U_1^*}(t)$, (c) $r_{U_2^*}(t)$, *and* (d) $r_{U_3^*}(t)$ *versus* t *as in Example 3.1.*

Proof. Since $k_i \leq k_j$, thus we have $r_{U_i^*}(t) \leq r_{U_j^*}(t)$, for $t > 0$. We claim that $S_i \subseteq S_j$ for any $i, j \in \{1, \ldots, n\}$. If $x \in S_i$ but $x \notin S_j$, then $r_{U_i^*}(x) > 0$ and $r_{U_j^*}(x) = 0$, i.e., $r_{U_i^*}(x) > r_{U_j^*}(x)$, a contradiction. Hence the result follows. ■

Example 3.1. *Let* B^* *have failure rate, given by* $r_{B^*}(t) = a(\lambda t + b)e^{\lambda t} t^{b-1}, t \geq 0$. *Taking* $a = 2$, $b = 0.2$, $\lambda = 5$, *it can be seen that* B^* *has bathtub-shaped failure rate (as* $0 < b < 1$, *(cf. Pham and Lai [5]). Here*

$$m = \min_{t \in (0, +\infty)} r_{B^*}(t) = r_{B^*}(t_m) = 12.6948,$$

where $t_m = 0.0494427$. *We construct* $r_{U_i^*}(t) = k_i m - r_{B^*}(t)$, *for* $t \in [t_i^{(1)}, t_i^{(2)}]$ *and* $= 0$ *otherwise, given that* $k_i \in \{2, 3, 4\}$, *for* $i = 1, 2$ *and* 3, *respectively. Here,* $[t_1^{(1)}, t_1^{(2)}] = [0.00717441, 0.21376]$, $[t_2^{(1)}, t_2^{(2)}] = [0.00386556, 0.291059]$, *and* $[t_3^{(1)}, t_3^{(2)}] = [0.00257858, 0.345577]$. *This example has been implemented and plotted in Figure* 2 *and Figure* 3(a).

3.1. Properties of the new UBT-shaped failure models

In this subsection, we show that the nature of the failure rate of a series system constituted by n independent components each having UBT-shaped failure rate, obtained from a single bathtub-shaped failure rate distribution can be derived using the concept in Proposition 3.1. If U_{S^*} is a random variable denoting the lifetime of a series system formed by n independent components with lifetimes U_i^* for $i = 1, \ldots, n$ with corresponding UBT-shaped failure rate functions $r_{U_i^*}(t)$, all generated from a component with BT-shaped failure rate $r_{B^*}(t)$, then $r_{U_{S^*}}(t)$ can be derived as given in the following.

Let $r_{U_i^*}(t) = k_i m - r_{B^*}(t)$, for $t \in S_i$ and $r_{U_i^*}(t) = 0$ otherwise, where $S_i = \{t \mid r_{U_i^*}(t) \neq 0\}$, $i \in A = \{1, \dots, n\}$. Let $B = \{k_1, \dots, k_n\}$. Let us define k_i^*, for all $i \in A$ as

$$k_1^* = \min_{k_i \in B} k_i, \quad k_j^* = \min_{k_i \in B - \{k_1^*, k_2^*, \dots, k_{j-1}^*\}} k_i, \quad j \in A - \{1\}.$$

Further, consider the roots $t_1^{(p)}$ and $t_2^{(p)}$ of $r_{U_p^*}(t)$, for $p \in A$ with $t_1^{(p)} < t_2^{(p)}$, where

$$r^*_{U_p^*}(t) = k_p^* m - r_{B^*}(t), \quad p \in A, \; t \in S_p,$$

with $S_p = \{t \mid r_{U_p^*}(t) \neq 0\}$. Now, one can prove that $r^*_{U_1^*}(t) < r^*_{U_2^*}(t) < \dots < r^*_{U_n^*}(t)$, for all $t \in \mathbb{R}$ as $k_j^* < k_{j+1}^*$, for all $j \in A$. Here, the two finite sets $\{r_{U_1^*}, \dots, r_{U_n^*}\}$ and $\{r^*_{U_1^*}, \dots, r^*_{U_n^*}\}$ are equal, i.e., its elements are rearrangement of each other. Each $r^*_{U_j^*}(t)$ is of UBT-shaped, for $t \in (t_1^{(j)}, t_2^{(j)})$. From lemma 2, we know that $(t_1^{(j)}, t_2^{(j)}) \subseteq (t_1^{(j+1)}, t_2^{(j+1)})$, for all $j \in A$, as shown in Figure 3(b) giving

$$r_{U_{S^*}}(t) = \begin{cases} 0 & \text{for} \quad t \in [0, t_1^{(n)}] \\ r^*_{U_n^*}(t) & \text{for} \quad t \in [t_1^{(n)}, t_1^{(n-1)}] \\ r^*_{U_n^*}(t) + r^*_{U_{n-1}^*}(t) & \text{for} \quad t \in [t_1^{(n-1)}, t_1^{(n-2)}] \\ \vdots & \\ \sum_{l=n-j}^{n} r^*_{U_l^*}(t) & \text{for} \quad t \in [t_1^{(n-j)}, t_1^{(n-j-1)}] \\ \vdots & \\ \sum_{l=1}^{n} r^*_{U_l^*}(t) & \text{for} \quad t \in [t_1^{(1)}, t_2^{(1)}] \\ \sum_{l=2}^{n} r^*_{U_l^*}(t) & \text{for} \quad t \in [t_2^{(1)}, t_2^{(2)}] \\ \vdots & \\ \sum_{l=n-j}^{n} r_{U_l^*}(t) & \text{for} \quad t \in [t_2^{(n-j-1)}, t_2^{(n-j)}] \\ \vdots & \\ r^*_{U_n}(t) & \text{for} \quad t \in [t_2^{(n-1)}, t_2^{(n)}] \\ 0 & \text{for} \quad t \in [t_2^{(n)}, +\infty), \end{cases}$$

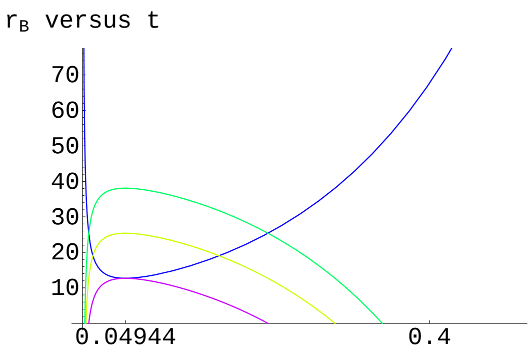
so that

$$r_{U_{S^*}}(t) \quad = \quad \begin{cases} 0 & \text{for} \quad t \in [0, t_1^{(n)}] \\ mk_n^* - r_{B^*}(t) & \text{for} \quad t \in [t_1^{(n)}, t_1^{(n-1)}] \\ m(k_n^* + k_{n-1}^*) - 2r_{B^*}(t) & \text{for} \quad t \in [t_1^{(n-1)}, t_1^{(n-2)}] \\ \vdots & \\ m\sum_{l=n-j}^{n} k_l^* - (j+1)r_{B^*}(t) & \text{for} \quad t \in [t_1^{(n-j)}, t_1^{(n-j-1)}] \\ \vdots & \\ m\sum_{l=1}^{n} k_l^* - nr_{B^*}(t) & \text{for} \quad t \in [t_1^{(1)}, t_2^{(1)}] \\ m\sum_{l=2}^{n} k_l^* - (n-1)r_{B^*}(t) & \text{for} \quad t \in [t_2^{(1)}, t_2^{2}] \\ \vdots & \\ m\sum_{l=n-j}^{n} k_l^* - (j+1)r_{B^*}(t) & \text{for} \quad t \in [t_2^{(n-j-1)}, t_2^{(n-j)}] \\ \vdots & \\ mk_n^* - r_{B^*}(t) & \text{for} \quad t \in [t_2^{(n-1)}, t_2^{(n)}] \\ 0 & \text{for} \quad t \in [t_2^{(n)}, +\infty). \end{cases}$$

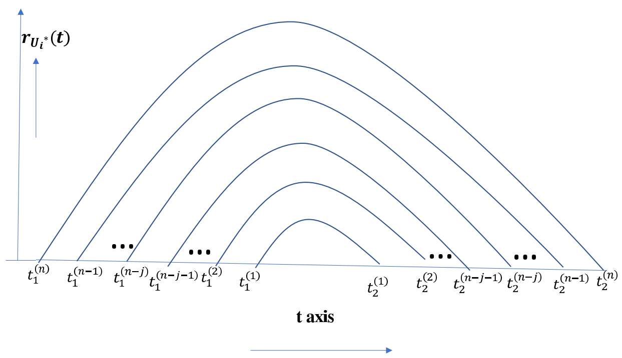
Clearly, $(j+1)r_{B^*}(t)$ represents failure rate of a bathtub distribution with

$$\min_{t\in[t_1^{(n-j)},t_1^{(n-j-1)}]} (j+1)r_{B^*}(t) = (j+1)m,$$

where $m = \min_{t\in(0,+\infty)} r_{B^*}(t)$. One can note that $r_{U_{S^*}}(t)$ represents a UBT-shaped failure model as in $r_{U_s}(t) = m\sum_{l=n-j}^{n} k_l^* - (j+1)r_{B^*}(t)$, for $t \in [t_1^{(n-j)}, t_1^{(n-j-1)}]$ and $r_{U_{S^*}}(t) = m\sum_{l=n-j}^{n} k_l^* - (j+1)r_{B^*}(t)$, for $t \in [t_2^{(n-j-1)}, t_2^{(n-j)}]$. We find that $m\sum_{l=n-j}^{n} k_l^* \geq 2(j+1)$, k_i being a real number satisfying $k_i \geq 2$ for all i.



(a)



(b)

Figure 3: *(a) Plot of $r_{B^*}(t)$ (blue color curve) and $r_{U_i^*}(t)$ versus t, for $i = 1,2,3$ (bottom to top) as in Example 3.1. (b) Plot of $r_{U_i^*}(t)$ for $i = 1,\ldots,n$ versus t.*

4. Concluding remarks

In this paper, we have proposed a novel method which yields a family of distributions with BT shaped failure rate model from a distribution having UBT-shaped failure rate and vice-versa. Few examples have been presented for the validation of the newly proposed method. In addition, the closure properties of the proposed model have been studied under various reliability operations.

Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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