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ANALYSIS OF UNCERTAINTY WEIGHTED MEASURES FOR PARETO II DISTRIBUTION

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Abstract

Extropy is a complimentary dual of Shannon's entropy, which has many applications. The maximum likelihood and Bayesian approaches are used in this article to explore the weighted extropy and weighted residual extropy of the Pareto type II distribution. Using unified hyper-censoring data, we calculate the maximum likelihood estimation of weighted extropy and its residual measures. Based on symmetric and asymmetric loss functions, Bayesian estimators of weighted extropy and its residual measure are developed based on unified hyper censoring data. To do some complex calculations, Markov chain Monte Carlo methods are used. To test the performance of the estimators, a Monte Carlo simulation study and an illustration using real data sets were carried out. The outcomes of the study showed that as the sample size increases, maximum likelihood and Bayesian estimators of the weighted extropy and its residual measure perform well. Also, Bayesian estimators of the weighted extropy and its residual under the general entropy loss function are superior to the Bayesian estimators under the others in most cases. Theoretical and empirical findings are generally in good agreement.

Keywords: weighted extropy, weighted residual extropy, Bayesian estimate, Markov chain Monte Carlo method.

1. Introduction

The Shannon entropy, or differential entropy of a random variable Z with the probability density function (PDF) $f(z)$ is a fundamental notion in measuring discrimination and information. It is defined as

$$H(Z) = -\int_{-\infty}^{\infty} f(z) \log f(z) dz. \tag{1}$$

Shannon's entropy measure has become one of the most widely used uncertainty measurements, with several applications in various fields like reliability, survival analysis, and actuarial sciences, among others.

Entropy measurement has benefits in a variety of fields, including industrial engineering and the financial performance of the companies (see Marvizadeh [1] and Liew et al. [2]). Estimation studies for Shannon entropy with various censoring and distribution strategies can be found in Cho et al. [3], Liu and Gui [4], Hassan and Zaky [5], and Yu et al. [6]. The Bayesian estimator of dynamic residual entropy was investigated by Ahmadini et al. [7], Al-Babtain et al. [8], and Almarashi et al. [9]. Some researchers have looked at estimating entropy measures based on record data; see, for instance, Hassan and Zaki [10] and Al-Omari et al. [11]. Helmy et al. [12] studied the Shannon entropy for Lomax distribution in the context of unified hybrid censored samples. Hassan et al. [13] studied estimation of information measures for power-function distribution in the presence of outliers.

Lad et al. [14] recently demonstrated the complementary dual of entropy called extropy, a different measure of uncertainty. The positive and negative pictures of a photographic film are related to entropy and extropy metrics. Extropy is utilized in voice recognition and to score the predicting distribution (see Becerra et al. [15]). Extropy, often known as differential extropy, is defined as

$$J(Z) = \frac{-1}{2}\int_0^\infty f^2(z)dz. \tag{2}$$

Qiu [16] recently investigated the characterization results, lower bounds, monotone and symmetric features of order statistics extropy, and record values. Extropy features of ranked set sampling were investigated by Raqab and Qiu [17]. In terms of extropy, Qiu et al. [18] studied information characteristics of mixed systems. Extropy, on the other hand, is ineffective for estimating the remaining lifetime of a unit that has lived for some units of time, $Z_t = [Z - t|Z \geq t]$ is the residual life function of Z at time t. As a result, Qiu and Jia [19] suggested the residual extropy to assess the residual uncertainty of Z_t and described the characterization and monotonic features of order statistics:

$$J_t(Z) = \frac{-1}{2\overline{F}^2(t)}\int_t^\infty f^2(z)dz. \tag{3}$$

where $\overline{F}(z) = 1 - F(z)$ is the survival function of Z. The fundamental disadvantage of the preceding information measures is that they only consider the probability density of the random variable rather than the values it takes. On the right side of (2), the integrand measure is shift-independent since it is only reliant on z through $f(z)$. This shift-independent quality, on the other hand, appears to be a disadvantage in many applications, such as mathematical neurobiology and reliability. In such cases, the random variable's value, as well as the probabilities, should be taken into consideration. Analogous to the weighted entropy see (Belis and Guiasu [20]), in order to efficiently model statistical data Balakrishnan et al. [21] introduced a new measure of information named weighted extropy (WEx). For a non-negative random variable Z with PDF is $f(z)$, the WEx is defined as

$$J^w(Z) = \frac{-1}{2}\int_0^\infty z f^2(z)dz. \tag{4}$$

Now we'll look at two distributions that have the same extropy but differently weighted extropies. Let X and Y be two random variables such that $X \sim U(a,b)$, $Y \sim U(2a, a+b)$, where $a, b > 0$. We have $f_X(x) = \frac{1}{b-a}$, for $x \in (a,b)$, and $f_Y(y) = \frac{1}{b-a}$, for $y \in (2a, a+b)$, and then

$$J(X) = \frac{-1}{2}\int_a^b \frac{1}{b-a}dx = \frac{-1}{2}, J(Y) = \frac{-1}{2}\int_{2a}^{b+a} \frac{1}{b-a}dx = \frac{-1}{2}.$$

and

$$J^w(X) = \frac{-1}{2}\int_a^b x\frac{1}{b-a}dx = \frac{-1}{2(b-a)}[\frac{b^2}{2} - \frac{a^2}{2}] = \frac{-(b+a)}{4}$$
$$J^w(Y) = \frac{-1}{2}\int_{2a}^{b+a} x\frac{1}{b-a}dx = -\frac{(b+a)^2 - (2a)^2}{4(b-a)} = \frac{-(b^2 + 2ba - 3a^2)}{4(b-a)}.$$

Extropies are the same in the two cases, but weighted extropies are different, hence weighted extropies can also be used as a measure of uncertainty. The concept of residual and the past life of random variables were combined to create WEx, given the necessity of weighted measures as previously described. When an item is working at time t, it may be worthwhile to investigate its longevity beyond that time. The residual lifetime is the set of interest in such cases, thus the concept of weighted residual extropy (WREx), which is defined as follows:

$$J_t^w(Z) = \frac{-1}{2\overline{F}^2(t)}\int_t^\infty z f^2(z)dz. \tag{5}$$

In the literature, we couldn't find any research on WEx and WREx estimation problems using a unified hyper censoring scheme (UHCS). In this paper, UHCS is used for estimating weighted extropy and its residual for Pareto II distribution (P-IID). The maximum likelihood (ML) and Bayesian estimators are used to investigate WEx and WREx measures. The Bayesian estimator is provided using symmetric and asymmetric loss functions and uses the Markov chain Monte Carlo (MCMC) method to estimate the posterior distribution. Both the application to real data and simulation concerns are covered.

The rest of the paper is organized in the following way. The WEx and WREx expressions of the P-IID are produced in Section 2. In Section 3, using ML and Bayesian estimation methods to assess WEx and WREx for P-IID using UHCS, the study in Bayesian happened under symmetric and asymmetric loss functions using the MCMC method. Section 4 discusses the simulation problem and its application to real data. The paper comes to a close with some concluding comments in Section 5.

2. Model Description

One of the vital lifetime models is the P-IID. It was introduced by Lomax [22], which is valuable in many fields, such as actuarial science and economics. It's been beneficial in issues of reliability and life testing (see Hassan and Al-Ghamdi [23]). Harris [24], and Atkinson et al. [25] used the P-IID to analyze income and wealth data. A random variable Z has a P-IID if its PDF is given by:

$$f(z) = \frac{\theta \xi^{\theta}}{(z+\xi)^{(\theta+1)}}, \quad \theta > 0, \xi > 0, z \geq 0. \tag{6}$$

The cumulative distribution function (CDF) of Z is specified by:

$$F(z) = 1 - \frac{\xi^{\theta}}{(z+\xi)^{\theta}}. \tag{7}$$

Several authors have published studies on P-IID in the literature. Balakrishnan and Ahsanullah [26] investigated certain recurrence relations between the moments of record values from the P-IID. Singh et al. [27] used Lindley's approximation to investigate the Bayesian estimate of P-IID under T-II HCS. Estimation of reliability for P-IID using ranked set sampling was provided by Hassan et al. [28].

The WEx of the P-IID is calculated by substituting (6) in (4) as following:

$$J^{w}(z) = \frac{-1}{2} \int_0^{\infty} z \Big[\frac{\theta \xi^{\theta}}{(\xi+z)^{\theta+1}} \Big]^2 dz$$

$$= \frac{-\theta^2 \xi^{2\theta}}{2} \int_0^{\infty} z(\xi+z)^{-2\theta-2} dz.$$

Using integration by parts, then $J^{w}(z)$ is as follows

$$J^{w}(z) = \frac{-\theta^2 \xi^{2\theta}}{2(2\theta+1)} \int_0^{\infty} (\xi+z)^{-2\theta-1} dz = \frac{-\theta}{4(2\theta+1)}. \tag{8}$$

Furthermore, WREx of the P-IID is calculated by substituting (6) in (5) as below:

$$J_t^{w}(z) = \frac{-(\xi+t)^{2\theta}}{2\xi^{2\theta}} \theta^2 \xi^{2\theta} \int_t^{\infty} z(\xi+z)^{-2\theta-2} dz. \tag{9}$$

Using integration by parts, then $J_t^{w}(z)$ is as follows

$$J_t^{w}(z) = \frac{-\theta^2(\xi+t)^{2\theta}}{2} \Big[\frac{z(\xi+z)^{-2\theta-1}}{-2\theta-1} \,|_t^{\infty} - \int_t^{\infty} \frac{(\xi+z)^{-2\theta-1}}{-2\theta-1} dz \Big]$$

$$= \frac{-\theta^2(\xi+t)^{2\theta}}{2} \Big[\frac{t(\xi+t)^{-2\theta-1}}{2\theta+1} + \frac{1}{2\theta+1} \frac{(\xi+t)^{-2\theta}}{2\theta} \Big].$$

After simplification, then $J_t^{w}(z)$ takes the following form:

$$J_t^{w}(z) = \frac{-\theta(2\theta t + t + \xi)}{4(2\theta+1)(\xi+t)}. \tag{10}$$

3. Estimation Methods

In this section, we study two methods of estimations, namely the ML and Bayesian to investigate WEx and WREx of the P-IID. In the Bayesian estimation method, we obtain the WEx and WREx estimators under different types of loss functions and use the MCMC method to calculate these estimators.

3.1. Maximum Likelihood Estimator

According to many life-testing experiments, censorship is crucial to lower the expense of the experiment and to shorten the amount of time spent testing. Childs et al. [29] merged type-I (T-I) and type-II (T-II) censoring and came up with a hybrid censoring system (HCS), which includes two categories (T-I HCS & T-II HCS) and these two types have been used in many types of research. The generalized T-I HCS and generalized T-II HCS were created by Chandrasekar et al. [30] by combining these two types. The UHCS, developed by Balakrishnan et al. [31], is a combination of generalized T-I and T-II hybrid censoring schemes.
Fix $r_1, r_2 \in \{1, 2, \ldots, n\}$ and the time points $T_1, T_2 \in (0, \infty)$ in this scheme.
$T^* = min\Big(max(x_{r_2:n}, T_1), T_2\Big)$, if the r_1^{th} failure happened prior to time T_1. $T^* = min(x_{r_2:n}, T_2)$ if the r_1^{th} failure happens between T_1 and T_2, and $T^* = x_{r_1:n}$, if the r_1^{th} failure happens after T_2. Consequently, we now have six cases covered by the UHCS. We can confirm that the test will be completed in time T_2 with at least r_1 failures using this scheme technique, if not, exactly r_1 failures.
We obtain the ML estimator of the P-IID parameters under UHCS. Let $Z_{1:n}, Z_{2:n}, \ldots, Z_{n:n}$ be n identical ordered failure times have P-IID, with fixed integer $r_1, r_2 \in 1, 2, ..., n$ where $r_1 < r_2 < n$ and time points $T_1, T_2 \in (0, \infty)$ where $T_1 < T_2$. Then, the likelihood function of θ and ξ, under UHCS for 6 situations is represented by

$$L(\underline{z}|\theta, \xi) = \begin{cases} \frac{n!}{(n-D_1)!} \prod_{i=1}^{D_1} f(z_{i:n})[1 - F(T_1)]^{n-D_1}, & \text{for case 1} \\ \frac{n!}{(n-r_2)!} \prod_{i=1}^{r_2} f(z_{i:n})[1 - F(r_2)]^{n-r_2} & \text{for case 2 and 4} \\ \frac{n!}{(n-D_2)!} \prod_{i=1}^{D_2} f(z_{i:n})[1 - F(T_2)]^{n-D_2} & \text{for case 3 and 5} \\ \frac{n!}{(n-r_1)!} \prod_{i=1}^{r_1} f(z_{i:n})[1 - F(r_1)]^{n-r_1} & \text{for case 6} \end{cases}, \tag{11}$$

where D_1 and D_2 are number of failures related to T_1 and T_2, respectively and $D_1 < D_2$. The likelihood function (11) can be written as:

$$L(\underline{z}|\theta, \xi) = \frac{n!}{(n-m)!} [\prod_{i=1}^{m} f(z_{i:n})][1 - F(C)]^{n-m}, \tag{12}$$

where C denotes the experiment's end time and m denotes the number of observations made until the experiment's end time C and is given by:

$$(m, C) = \begin{cases} (D_1, T_1) & \text{at case 1} \\ (r_2, z_{r_2:n}) & \text{at case 2 and 4} \\ (D_2, T_2) & \text{at case 3 and 5} \\ (r_1, z_{r_1:n}) & \text{at case 6} \end{cases}, \tag{13}$$

From (6) and (7) by substituting in (12) we get likelihood function of P-IID under UHCS,

$$L(\underline{z}|\theta, \xi) = \frac{n!}{(n-m)!} [\theta^m \xi^{m\theta} \prod_{i=1}^{m} \frac{1}{(\xi + z_i)^{\theta+1}}][\frac{\xi^\theta}{(C + \xi)^\theta}]^{n-m}. \tag{14}$$

Take the logarithm of both sides, denoted by l,

$$l \propto m\ln\theta + m\theta\ln\xi - (\theta+1)\sum_{i=1}^{m}\ln(\xi+z_i) + \theta(n-m)[\ln\xi - \ln(C+\xi)]. \tag{15}$$

Take derivatives of (15) with respect to θ and ξ, we can get

$$\frac{\partial l}{\partial\theta} = \frac{m}{\theta} + m\ln\xi - \sum_{i=1}^{m}\ln(\xi+z_i) + (n-m)[\ln\xi - \ln(C+\xi)], \tag{16}$$

and

$$\frac{\partial l}{\partial\xi} = \frac{m\theta}{\xi} - (\theta+1)\sum_{i=1}^{m}\frac{1}{(\xi+z_i)} + \theta(n-m)[\frac{1}{\xi} - \frac{1}{(C+\xi)}]. \tag{17}$$

Set (16) & (17) to zero and solve them to get the ML estimator of θ and ξ. Then equation (16) is written as:

$$\hat{\theta} = A(\hat{\xi}), \tag{18}$$

where

$$A(\hat{\xi}) = \frac{-m}{m\ln\hat{\xi} - \sum_{i=1}^{m}\ln(\hat{\xi}+z_i) + (n-m)[\ln\hat{\xi} - \ln(C+\hat{\xi})]}.$$

By substituting from (18) into (17) after putting them equal zero to get

$$\frac{mA(\hat{\xi})}{\hat{\xi}} - (A(\hat{\xi})+1)\sum_{i=1}^{m}\frac{1}{(\xi+z_i)} + (n-m)A(\hat{\xi})[\frac{1}{\hat{\xi}} - \frac{1}{(C+\hat{\xi})}] = 0. \tag{19}$$

We may acquire the ML estimator of θ using an iterative process by computing the ML estimator from (19) and then substituting it into (18). As a result of the invariance property of ML estimation, the estimator of $J^w(z)$ and $J_t^w(z)$, becomes

$$\hat{J}^w(z) = \frac{-\hat{\theta}}{4(2\hat{\theta}+1)}, \tag{20}$$

and

$$\hat{J}_t^w(z) = \frac{-\hat{\theta}(2\hat{\theta}t + t + \hat{\xi})}{4(2\hat{\theta}+1)(\hat{\xi}+t)}. \tag{21}$$

3.2. Bayesian Estimator

Here, we consider both θ and ξ unknown, indicating that there is no natural conjugate bivariate prior distribution. As a result, we assume that the independent priors for θ and ξ are gamma (a, b) and gamma (c, d), respectively, with $\frac{a}{b}$ and $\frac{c}{d}$ as means. The θ and ξ priors are written as follows:

$$\pi_1(\theta) \propto \left(\theta^{a-1}e^{-b\theta}\right), \qquad \theta > 0,$$

$$\pi_2(\xi) \propto \left(\xi^{c-1}e^{-\xi d}\right), \qquad \xi > 0,$$

where $a, b, c,$ and d are positive hyperparameters that carry prior knowledge. As a result, the joint prior distribution is as follows:

$$\pi(\theta, \xi) \propto \theta^{a-1}\xi^{c-1}e^{-(b\theta+d\xi)}. \tag{22}$$

The posterior distribution is given by

$$= E_1\theta^{m+a-1}\xi^{m\theta+c-1}e^{-\xi d}e^{-\theta[b+\sum_{i=1}^{m}\ln(\xi+z_i)-(n-m)\ln(\frac{\xi}{C+\xi})]}e^{-\sum_{i=1}^{m}\ln(\xi+z_i)}, \tag{23}$$

where $E_1^{-1} = \int_0^\infty \int_0^\infty L(\underline{z}|\theta,\xi)\pi(\theta,\xi)d\theta d\xi$, is the normalizing constant.

Then, the marginal posterior distributions of θ and ξ, are given below

$$\pi_1^*(\theta|\underline{z}) \propto \theta^{m+a-1}e^{-\theta b}\int_0^\infty \xi^{(m\theta+c-1)}e^{-\xi d-\theta[\sum_{i=1}^m \ln(\xi+z_i)-(n-m)\ln(\frac{\xi}{C+\xi})]}e^{-\sum_{i=1}^m \ln(\xi+z_i)}d\xi, \tag{24}$$

and

$$\pi_2^*(\xi|\underline{z}) \propto e^{-\xi d}\xi^{c-1}e^{-\sum_{i=1}^m \ln(\xi+z_i)}\int_0^\infty \theta^{m+a-1}\xi^{m\theta}e^{-\theta[b+\sum_{i=1}^m \ln(\xi+z_i)-(n-m)\ln(\frac{\xi}{C+\xi})]}d\theta. \tag{25}$$

As seen from (23) and (25) that $\pi_1^*(\theta|\xi)$ is calculated as following

$$\pi_1^*(\theta|\xi) \propto \theta^{m+a-1}e^{-\theta[b-m\ln\xi+\sum_{i=1}^m \ln(\xi+z_i)-(n-m)\ln(\frac{\xi}{C+\xi})]}. \tag{26}$$

As a result, the gamma distribution with shape parameter $(m+a-1)$ and scale parameter $[b-m\ln\xi+\sum_{i=1}^m \ln(\xi+z_i)-(n-m)\ln(\frac{\xi}{C+\xi})]$ is the posterior density function of θ given ξ. As a result, any gamma-producing technique can be used to generate θ samples with ease.
From (23) and (24) we can calculate $\pi_2^*(\xi|\theta)$, as following

$$\pi_2^*(\xi|\theta) \propto \xi^{m\theta+c-1}e^{-\xi d}e^{-\theta[\sum_{i=1}^m \ln(\xi+z_i)-(n-m)\ln(\frac{\xi}{C+\xi})]}e^{-\sum_{i=1}^m \ln(\xi+z_i)}. \tag{27}$$

Appropriate sampling techniques cannot be able to sample directly because this equation can never be solved to very well distributions. To generate an estimator for the following loss functions, use the MCMC approach.

3.2.1. Loss Functions

We will look at Bayesian estimators for both symmetric and asymmetric loss functions. The squared error loss (SEL) function is one of the most extensively utilized symmetric loss functions. The following is the SEL function:

$$L_1(\phi,\delta) = (\delta-\phi)^2,$$

where δ is an estimator of ϕ. The Bayesian estimator, based on the SEL function, is calculated as follows:

$$\hat{\phi}_{SEL} = E(\phi|data). \tag{28}$$

In terms of asymmetric loss functions, we chose the linear-exponential (LINEX) and the general entropy (GE) loss functions, which are the two most often used asymmetric loss functions. The following is a definition of the LINEX loss function:

$$L_2(\phi,\delta) = e^{-h(\delta-\phi)} - h(\delta-\phi) - 1,$$

where h is the sign presenting the asymmetry (see Parsian and Kirmani, [33]). Under the LINEX loss function, the Bayesian estimator is provided by

$$\hat{\phi}_{LINEX} = \frac{-1}{h}\ln[E(e^{-h\phi}|data)]. \tag{29}$$

The following is the definition of the GE loss function:

$$L_2(\phi,\delta) = (\frac{\delta}{\phi})^h - h(\frac{\delta}{\phi}) - 1.$$

In this case, the Bayesian estimator is:

$$\hat{\phi}_{GE} = [E(\phi^{-h}|data)]^{-\frac{1}{h}}. \tag{30}$$

Under the SEL, LINEX, and GE loss functions, the Bayesian estimators of WEx and WREx are as follows:

$$\hat{g}(\theta,\xi)_{SEL} = E_1 \int_0^\infty \int_0^\infty g(\theta,\xi)\theta^{m+a-1}\xi^{m\theta+c-1}e^{-\xi d-\sum_{i=1}^{m}\ln(\xi+z_i)}e^{-\theta Y_i(\theta,\xi,z_i)}d\theta d\xi, \tag{31}$$

$$\hat{g}(\theta,\xi)_{LINEX} = \frac{-1}{h}\ln[E_1 \int_0^\infty \int_0^\infty e^{-hg(\theta,\xi)}\theta^{m+a-1}\xi^{m\theta+c-1}e^{-\xi d-\sum_{i=1}^{m}\ln(\xi+z_i)}e^{-\theta Y_i(\theta,\xi,z_i)}d\theta d\xi], \tag{32}$$

$$\hat{g}(\theta,\xi)_{GE} = \left[E_1 \int_0^\infty \int_0^\infty (g(\theta,\xi))^{-h}\theta^{m+a-1}\xi^{m\theta+c-1}e^{-\xi d-\sum_{i=1}^{m}\ln(\xi+z_i)}e^{-\theta Y_i(\theta,\xi,z_i)}d\theta d\xi\right]^{\frac{-1}{h}}, \tag{33}$$

where $Y_i(\theta,\xi,z_i) = [b + \sum_{i=1}^{m}\ln(\xi+z_i) - (n-m)\ln(\frac{\xi}{C+\xi})]$, E_1 is the normalizing constant and to calculate the WEx put $g(\theta,\xi) = J^w(z)$ and to find the WREx put $g(\theta,\xi) = J_t^w(z)$.
It should be observed that all Bayesian entropy estimators are expressed as a ratio of two integrals, which cannot be simplified or directly computed. To compute the estimators, we use the MCMC approach.

3.2.2. MCMC Method

Consider using the MCMC approach to produce samples from posterior distributions and then using the (SEL, LINEx, GE) loss functions to compute Bayesian estimates (BEs) of WEx and WREx. There are many different MCMC schemes to select from, and it can be difficult to decide which one to use. Gibbs samplers as well as Metropolis-within-Gibbs samplers are key subclasses of MCMC algorithms. The advantage of the MCMC technique over the ML method is that by creating probability intervals based on the empirical posterior distribution, we can always obtain an acceptable interval estimate of the parameters. ML estimation frequently lacks this feature.

Algorithm

1) $\xi_0 = \hat{\xi}$.
2) $\theta^{(l)}$ is obtained from Gamma $\pi_1^*(\theta|\xi)$ as shown in (26).
3) To generate $\xi^{(l)}$ from $\pi_2^*(\xi|\theta)$ using Metropolis-Hastings (M-H) algorithm, see Metropolis et al. [32].
4) Calculate $\theta^{(l)}$ and $\xi^{(l)}$.
5) Repeat steps 2-4 N times.
6) Calculate the WEx and WREx via the loss functions using the Bayesian estimators of θ and ξ.

4. Simulation Investigation and Results

In this section, we study the performance of all previously proposed estimators for WEx and WREx, so we can use a simulation study to estimate WEx and WREx and use an Illustrative example.

4.1. Simulation Study

In this subsection, we look into the efficiency of the ML estimates (MLEs) and BEs of WEx and WREx, for the P-IID in terms of mean squared errors (MSEs) under different loss functions by using the Monte Carlo simulation.
- The UHCS from the P-IID are generated for sample sizes $n = 200$ and 100, using these samples, the MSEs of MLEs and BEs are computed.
- The BEs using the suggested loss functions when $(h = -1, 1)$ are calculated.
- The simulation runs $N = 1,000$ times and for the WREx, take $t = 0.5$.
- The prior parameters used in Bayesian inference are selected $(a,b,c,d) = (3.5, 5.5, 4, 2)$.

The MLEs and BEs of WEx and WREx were studied under the following cases where in (Table 1) the results of WREx and in (Table 2) the results of WEx:

1- Values of n, r_2, T_2 are taken as $(n = 100, r_2 = 50, T_2 = 1.2)$ and $r_1 = (80, 70, 60)$ at different values of T_1, where $T_1 = (0.2, 0.9)$.
2- Values of n, r_2, T_2 are taken as $(n = 200, r_2 = 140, T_2 = 1.2)$ and $r_1 = (190, 170, 150)$ at different values of T_1, where $T_1 = (0.2, 0.9)$.
3- Values of n, r_2, T_1 are taken as $(n = 100, r_2 = 50, T_1 = 0.9)$ and $r_1 = (80, 70, 60)$ at different values of T_2, where $T_2 = (1.2, 3)$.
4- Values of n, r_2, T_1 are taken as $(n = 200, r_2 = 140, T_1 = 0.9)$ and $r_1 = (190, 170, 150)$ at different values of T_2, where $T_2 = (1.2, 3)$.
5- Values of n, r_1, T_1 are taken as $(n = 100, r_1 = 80, T_1 = 1.2)$ and $r_2 = (70, 50, 30)$ at $T_2 = 5$.
6- Values of n, r_1, T_1 are taken as $(n = 200, r_1 = 170, T_1 = 1.2)$ and $r_2 = (160, 150, 120)$ at $T_2 = 5$.
The MSE for N samples is calculated using

$$\text{MSE}(\hat{\phi}) = \sum_{i=1}^{N} \frac{(\hat{\phi}_i - \phi)^2}{N}, \tag{34}$$

where $\hat{\phi} = \hat{J}^w(z)$ and $\hat{J}_t^w(z)$. All results are given in Tables 1 and 2. Using tables, we may conclude the following:

1-The results in Tables 1 and 2 reveal that for all proposed estimates, the MSEs generally decrease with increasing value of n.
2- The MSE of weighted measures in ML and Bayesian estimates decreases as r_1 increases while the sample size n is fixed (see Figures 1 and 2).
3- When both the sample size n and the number of failures r_1 are fixed, the MSE of weighted measures in ML and Bayesian estimates decrease as the specified observation number r_2 increases (Figures 3 and 4).
4-The MSE of weighted measures in ML and Bayesian estimates generally decreases when the predetermined time T_1 and the extended time T_2 increase (see Figures 5 and 6).
5-The BEs of WEx and WREx viz LINEX loss function at $h = -1$ have a lot of information and the BEs using GE loss function at $h = 1$ have a lot of information since they have a small value of MSE.
6- The BEs of the $J^w(z)$ and $J_t^w(z)$ under the GE loss function are superior to the BEs under the other loss functions in most of the cases (Figures 1 and 2).
7-The BE of WEx and WREx is the best value under the GE loss function in most cases.
8-The amount of data obtained under the GE loss function is more in BEs than obtained under other loss functions.
9-By increasing the number of failures r_1 or r_2, the BEs of WEx and WREX are raised.

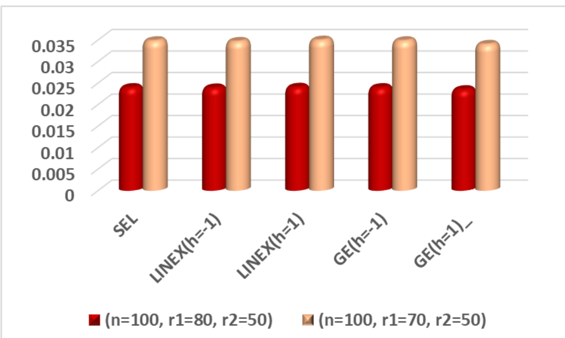

Figure 1: *The MSE of WREx estimate for various values of r_1.*

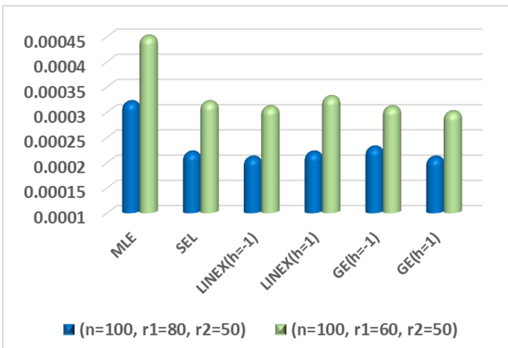

Figure 2: *MSE of WEx estimate for various values of r_1.*

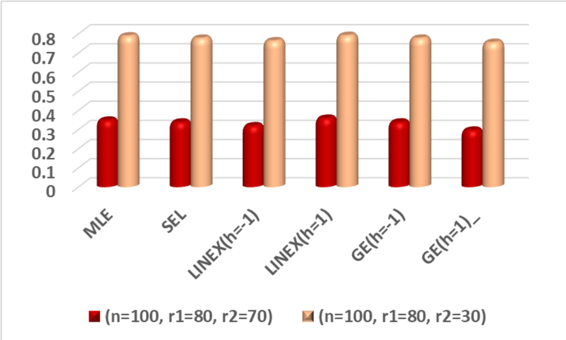

Figure 3: *The MSE of WREx estimate for different values of r_2.*

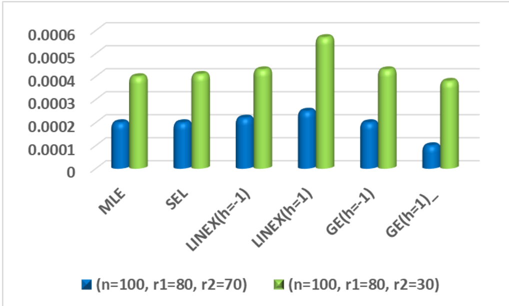

Figure 4: *The MSE of WEx estimate for various values of r_2.*

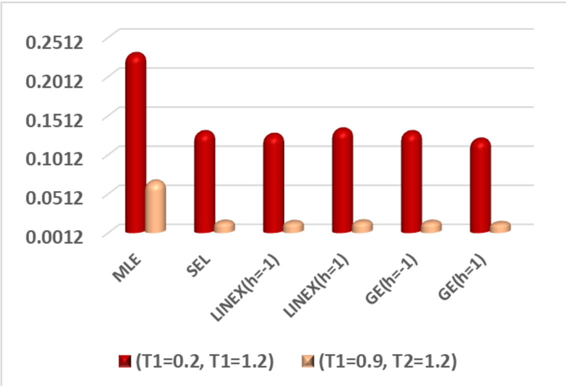

Figure 5: *The MSE of WREx estimate for various values of T_1 when $n = 200, r_1 = 190, r_2 = 140$.*

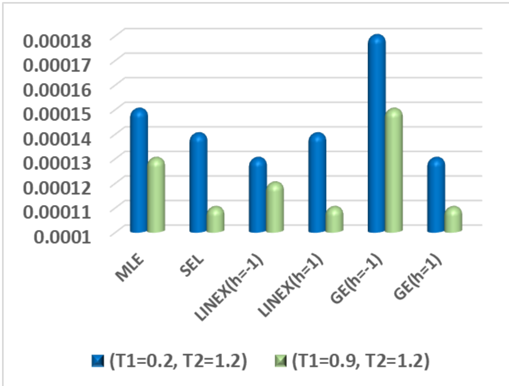

Figure 6: *The MSE of WEx estimate for various values of T_1 when $n = 200, r_1 = 190, r_2 = 140$.*

Table 1: *MLEs and BEs of WREx and associated MSE.*

n	r_1	r_2	MLE	SEL	LINEX $h=(-1)$	LINEX $h=(1)$	GE $h=(-1)$	GE $h=(1)$
					$(T_1 = 0.2$	$T_2 = 1.2)$		
100	80	50	-0.16189	-0.32887	-0.32861	-0.32914	-0.32887	-0.32731
			0.00015	0.02395	0.02387	0.02403	0.02395	0.02347
	70		-0.2097	-0.36075	-0.36034	-0.36117	-0.36075	-0.3585
			0.00127	0.03484	0.03468	0.03499	0.03484	0.034
	60		-0.16821	-0.37014	-0.36963	-0.37066	-0.37014	-0.36745
			0.00003	0.03843	0.038237	0.03863	0.03843	0.0373
200	190	140	-0.18586	-0.37053	-0.37032	-0.37073	-0.37053	-0.36944
			0.00014	0.03858	0.0385	0.03866	0.03858	0.03816
	170		-0.16035	-0.38768	-0.38744	-0.38792	-0.38768	-0.38647
			0.00019	0.04561	0.04551	0.04571	0.04561	0.0451
	150		-0.19091	-0.4324	-0.4321	-0.4327	-0.4324	-0.43103
			0.00028	0.16672	0.16656	0.16687	0.16672	0.16601
					$(T_1 = 0.9$	$T_2 = 1.2)$		
100	80	50	-0.16824	-0.30625	-0.30601	-0.30649	-0.30625	-0.30472
			0.00003	0.01746	0.0174	0.01753	0.01746	0.01706
	70		-0.17107	-0.31911	-0.3188	-0.31942	-0.31911	-0.3172
			0.00001	0.02103	0.02093	0.02112	0.02103	0.02047
	60		-0.30542	-0.35315	-0.35282	-0.3534	-0.35315	-0.351
			0.01724	0.03206	0.03194	0.03218	0.03206	0.0314
200	190	140	-0.7929	-0.73629	-0.73048	-0.74198	-0.73629	-0.71956
			0.0665	0.01465	0.01424	0.01507	0.01465	0.01294
	170		-0.64563	-0.5423	-0.5429	-0.54735	-0.54563	-0.53836
			0.0486	0.03817	0.03388	0.04233	0.03817	0.02721
	150		-0.58563	-0.48563	-0.4839	-0.48735	-0.48563	-0.47836
			0.1487	0.13817	0.13388	0.14243	0.13817	0.12601
					$(T_1 = 0.9$	$T_2 = 3)$		
100	80	50	-1.11252	-1.01252	-1.00171	-1.02322	-1.01252	-0.99011
			0.51982	0.41982	0.40594	0.4338	0.41982	0.39128
	70		-1.1324	-1.03264	-1.02677	-1.06183	-1.03264	-1.0296
			0.5188	0.4348	0.455	0.4903	0.4348	0.4112
	60		-1.1564	-1.06564	-1.05777	-1.07383	-1.06564	-1.05096
			0.5148	0.49148	0.4805	0.50303	0.49148	0.47112
200	190	140	-0.5662	-0.53222	-0.53067	-0.53375	-0.53222	-0.52614
			0.0981	0.0281	0.02759	0.02862	0.0281	0.0261
	170		-0.6997	-0.67367	-0.66779	-0.67942	-0.67367	-0.65475
			0.09774	0.09554	0.09193	0.09912	0.09554	0.0842
	150		-0.7897	-0.7767	-0.72779	-0.7752	-0.75367	-0.653
			0.11674	0.11354	0.11093	0.11812	0.11354	0.10842

Continue Table 1

n	r_1	r_2	MLE	SEL	$(T_1 = 1.2$ $T_2 = 5)$ LINEX $h = (-1)$	$h = (1)$	GE $h = (-1)$	$h = (1)$
100	80	70	-0.999	-0.94699	-0.93012	-0.9638	-0.94699	-0.90949
			0.3492	0.3392	0.31983	0.35908	0.3392	0.29693
		50	-1.12431	-1.3531	-1.34752	-1.3638	-1.35631	-1.34385
			0.5694	0.5544	0.5436	0.5774	0.5744	0.5352
		30	-1.34631	-1.24631	-1.23852	-1.25438	-1.24631	-1.23385
			0.7894	0.77744	0.76376	0.79174	0.77744	0.75562
200	170	160	-0.80514	-0.70514	-0.70141	-0.70881	-0.70514	-0.694
			0.1348	0.11598	0.11345	0.11849	0.11598	0.10852
		150	-1.6424	-1.4524	-1.4499	-1.4635	-1.4524	-1.4381
			1.222	1.1322	1.1537	1.1752	1.13332	1.1145
		120	-1.75824	-1.67524	-1.66099	-1.68935	-1.67524	-1.65781
			1.882	1.71782	1.68067	1.75502	1.71782	1.67245

Table 2: *MLEs and BEs of WEx and associated MSE.*

n	r_1	r_2	MLE	SEL	$(T_1 = 1.2$ $T_2 = 5)$ LINEX $h = (-1)$	$h = (1)$	GE $h = (-1)$	$h = (1)$
100	80	70	-0.1182	-0.1182	-0.11823	-0.11826	-0.11823	-0.11819
			0.0002	0.0002	0.00022	0.00025	0.0002	0.0001
		50	-0.1218	-0.1215	-0.1219	-0.1221	-0.1245	-0.1215
			0.0003	0.00021	0.00023	0.00037	0.00033	0.00028
		30	-0.1248	-0.12485	-0.12495	-0.12505	-0.12495	-0.12475
			0.0004	0.00041	0.00043	0.00057	0.00043	0.00038
200	170	160	-0.1152	-0.1152	-0.11521	-0.11527	-0.11521	-0.11519
			0.00012	0.00012	0.00012	0.00019	0.00012	0.00011
		150	-0.1242	-0.12422	-0.12422	-0.12422	-0.12422	-0.12422
			0.0004	0.0008	0.0005	0.0009	0.0008	0.0004
		120	-0.1352	-0.13522	-0.13522	-0.1422	-0.1432	-0.12522
			0.0014	0.0018	0.0014	0.0016	0.0014	0.0010

Continue Table 2

n	r_1	r_2	MLE		$(T_1 = 0.2$	$T_2 = 1.2)$ LINEX	GE	
					$h = (-1)$	$h = (1)$	$h = (-1)$	$h = (1)$
100	80	50	-0.1276	-0.12482	-0.12482	-0.12492	-0.12482	-0.12470
			0.00032	0.00023	0.00023	0.00027	0.00024	0.00021
	70		-0.1015	-0.12494	-0.12494	-0.12497	-0.12494	-0.12491
			0.00006	0.00097	0.00095	0.00099	0.00091	0.0009
	60		-0.08959	-0.12476	-0.12476	-0.12478	-0.12476	-0.12465
			0.00002	0.00096	0.00096	0.00116	0.00096	0.00082
200	190	140	-0.09573	-0.1238	-0.1248	-0.1252	-0.1248	-0.1242
			0.00093	0.0009	0.00096	0.00098	0.00096	0.00091
	170		-0.09744	-0.12471	-0.12477	-0.12482	-0.12477	-0.1241
			0.00015	0.00086	0.00096	0.00106	0.00096	0.00043
	150		-0.12133	-0.12122	-0.12138	-0.12144	-0.12134	-0.12132
			0.00025	0.00025	0.00039	0.00029	0.00029	0.00027
					$(T_1 = 0.9$	$T_2 = 1.2)$		
100	80	50	-0.09412	-0.12491	-0.12493	-0.12498	-0.12493	-0.12492
			0.00057	0.00094	0.00097	0.00099	0.00097	0.00095
	70		-0.136	-0.1249	-0.12495	-0.12498	-0.12495	-0.12491
			0.00037	0.00097	0.00097	0.00099	0.00097	0.00096
	60		-0.11962	-0.12498	-0.12498	-0.12498	-0.12498	-0.1249
			0.00067	0.00098	0.00098	0.00098	0.00098	0.00098
200	190	140	-0.1172	-0.11622	-0.11623	-0.11628	-0.11623	-0.11619
			0.00013	0.00011	0.00012	0.00015	0.00015	0.00011
	170		-0.119	-0.1097	-0.10972	-0.1097	-0.10982	-0.10968
			0.00003	0.00003	0.00003	0.00004	0.00003	0.00002
	150		-0.121	-0.1117	-0.1142	-0.1137	-0.11182	-0.11168
			0.00013	0.00022	0.00019	0.00027	0.00015	0.00011
					$(T_1 = 0.9$	$T_2 = 3)$		
100	80	50	-0.13540	-0.12340	-0.12347	-0.12357	-0.12347	-0.12342
			0.00041	0.0003	0.00037	0.00047	0.00037	0.00033
	70		-0.1341	-0.12251	-0.12233	-0.12288	-0.12233	-0.12222
			0.00042	0.0003	0.00035	0.00037	0.00035	0.00032
	60		-0.1321	-0.12211	-0.12213	-0.12218	-0.12213	-0.12212
			0.00043	0.00031	0.00032	0.00035	0.00032	0.00031
200	190	140	-0.1107	-0.11079	-0.1108	-0.11087	-0.11081	-0.11078
			0.00004	0.00004	0.00004	0.00004	0.00004	0.00004
	170		-0.1142	-0.11421	-0.1145	-0.1146	-0.1146	-0.1142
			0.00008	0.00009	0.00008	0.00005	0.00009	0.00002
	150		-0.1152	-0.11521	-0.11525	-0.11526	-0.11526	-0.1152
			0.0001	0.00011	0.00012	0.00015	0.00012	0.00010

4.2. Data Analysis

Consider the data from Lee and Wang [34], which represent the remission times (in months) of a random sample of 128 bladder cancer patients. For these real data, the Kolmogorov-Smirnov (K-S) test is used, and the p-value indicates that the P-IID best matches the data where (p-value=0.336703 and K-S distance = 0.0820311). Figure 7 illustrates the estimated PDF and CDF of P-IID.

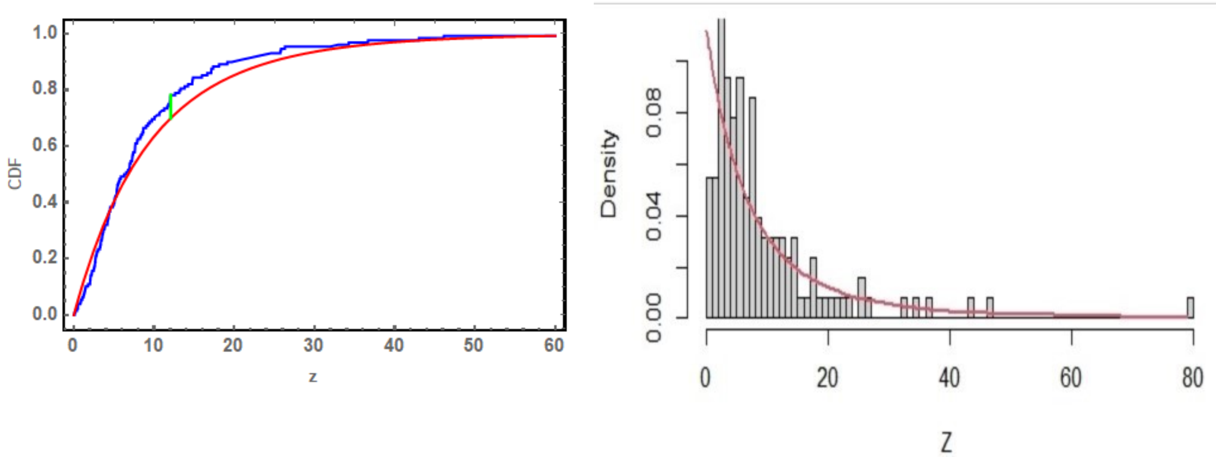

Figure 7: *Estimated PDF and CDF of P-IID*

In this application, we assume that the distribution of this data is the P-IID based on the UHCS. We take
Case 1: $(r_1 = 70, r_2 = 80, T_1 = 10, T_2 = 15)$,
Case 2: $(r_1 = 70, r_2 = 95, T_1 = 10, T_2 = 15)$,
Case 3: $(r_1 = 70, r_2 = 115, T_1 = 10, T_2 = 15)$,
Case 4: $(r_1 = 93, r_2 = 100, T_1 = 10, T_2 = 15)$,
Case 5: $(r_1 = 93, r_2 = 115, T_1 = 10, T_2 = 15)$,
Case 6: $(r_1 = 120, r_2 = 125, T_1 = 10, T_2 = 15)$.
We use a non-informative prior to calculating the BEs under (SEL, LINEX, GE) loss functions with $h = (-1, 1)$, because we don't know anything about the priors, therefore we choose $a = 0, b = 0, c = 0$, and $d = 0$. The results for the real data are listed in Tables 3 and 4.

Table 3: *Estimation of WEx when* ($T_1 = 10$, $T_2 = 15$).

Case	n	r_2	r_1	MLE	BSL	LINEX		GE	
						$h=(-1)$	$h=(1)$	$h=(-1)$	$h=(1)$
1		80	70	-0.125	-0.12183	-0.12184	-0.12183	-0.12175	-0.12183
2		95	70	-0.1252	-0.12183	-0.12184	-0.12183	-0.12175	-0.12183
3	128	115	70	-0.12	-0.12179	-0.1218	-0.12179	-0.12171	-0.12179
4		100	93	-0.125	-0.12179	-0.1218	-0.12179	-0.12171	-0.12179
5		115	93	-0.125	-0.12182	-0.12183	-0.12182	-0.12174	-0.12182
6		125	120	-0.1190	–0.11886	-0.11886	-0.11886	-0.11886	-0.11886

Table 4: *Estimation of WREx when* ($T_1 = 10$, $T_2 = 15$).

Case	n	r_2	r_1	MLE	BSL	LINEX		GE	
						$h=(-1)$	$h=(1)$	$h=(-1)$	$h=(1)$
1		80	70	-1.5305	-0.82458	-1.05544	-0.59372	-0.22025	-0.82458
2		95	70	-1.46202	-0.79034	-1.00075	-0.57993	-0.21951	-0.79034
3	128	115	70	-1.40151	-0.76005	-0.95306	-0.56705	-0.21867	-0.76005
4		100	100	-1.42467	-0.77163	-0.97124	-0.57203	-0.21895	-0.77163
5		115	115	-1.41513	-0.76689	-0.96375	-0.57003	-0.21893	-0.76689
6		125	120	-0.8634	-0.84222	-0.84318	-0.84123	-0.83977	-0.84222

The trace plot and histogram of the first 1000 MCMC results for the posterior distribution of WEx and WREx for case 1 are shown in Figures 8 and 9.

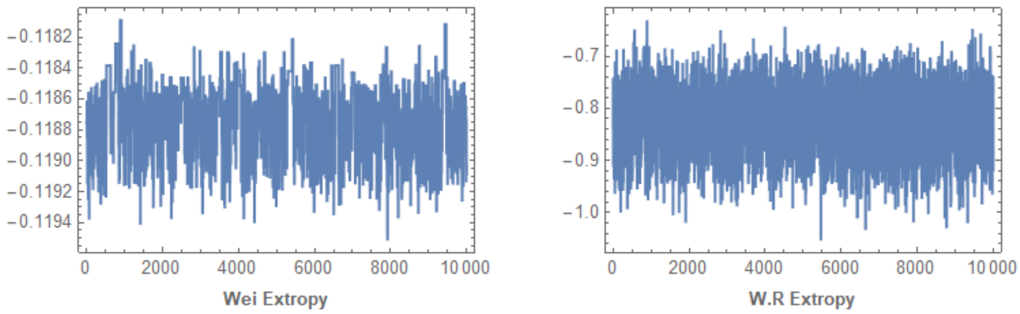

Figure 8: *The posterior sample trace plot for case* 1.

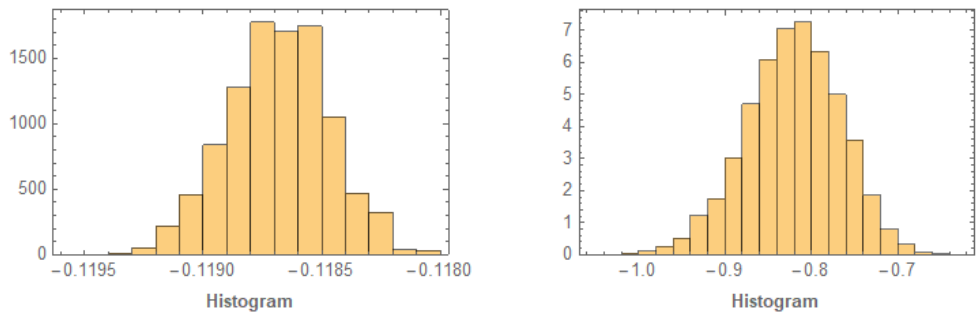

Figure 9: *The posterior sample histogram for case* 1.

We note from the study of this application that:
• The BE of WEx and WREx is less than the value of MLE in most cases.
• The BE of WEx and WREx via LINEX loss function at $h = -1$ have a small value. Furthermore, the BEs via the GE loss function at $h = -1$ have a large value. Finally, we reach the conclusion that the simulated research is supported by real data.

5. Conclusion

In this study, using UHCS, we investigated Bayesian and non-Bayesian estimators of WEx and WREx for the P-IID. For the weighted extropies measures under study, we found ML and Bayesian estimators for both symmetric and asymmetric loss functions. The MCMC techniques were used to calculate the Bayes estimates based on the M-H algorithm. In terms of accuracy measures, the performance of weighted extropy and its residual estimates for P-IID were explored. One application to real data was considered, as well as a simulation issue. In general, the MSE values of ML and Bayesian estimators of weighted measures decrease as the number of failures rises in most cases, according to the results of the study. When compared to different estimates, the Bayesian estimate of WEx and WREx under the general entropy loss function performs well in the majority of situations. By increasing the number of failures r_1 or r_2, the BEs of WEx and WREx are raised.

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Authors' contributions: All authors read and approved the final manuscript.

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