NOVEL DISTRIBUTION FOR MODELING UNCENSORED AND CENSORED SURVIVAL TIME DATA AND REGRESSION MODEL

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Abstract

This work proposes a new one-parameter model titled the type II Topp Leone half logistic ($TIITL_{HL}$) model which is characterized by an increasing and decreasing hazard rate function quite dependent on the shape parameter. Some structural properties and basic functions used in reliability analysis are derived. Simulations are carried out for both uncensored and censored samples. The uncensored simulation results indicated that the estimators perform quite well in producing good parameter estimates at finite sample sizes. However, the Anderson Darling estimator (ADE) average estimate tend to the true parameter value faster than other methods with minimum bias. More so, simulation based on censored samples using different censoring proportions showed that the bias, MSE and MRE values decrease as the sample size increases for the different censoring proportions. Two uncensored and censored datasets from the medical and environmental sciences were analysed to show the relevance, flexibility and adaptability of the $TIITL_{HL}$ model, and the new model achieved the best performance when compared with six other competing lifetime models. In addition, the log-$TIITL_{HL}$ regression model constructed and compared with two existing models showed that this model will be a useful option in survival investigation.

Keywords: Half-logistic distribution, Classical estimation methods, Monte-Carlo simulation, type II censoring, Type-II-Topp-Leone-G class

1. Introduction

The standard half-Logistic (HL) model pioneered by [1] has gained a lot of popularity as a significant model given its extensive applicability in lifetime modeling and reliability analysis. The cumulative density function (CDF) of the standard HL model is

$$G(k) = \frac{1 - e^{-k}}{1 + e^{-k}}, k > 0, \tag{1}$$

and the corresponding probability density function (PDF) to (**??**) is

$$g(k) = \frac{2e^{-k}}{(1 + e^{-k})^2}, k > 0, \tag{2}$$

Several authors have pioneered various extensions of the HL model such as the type I HL family of distributions by [2], inverse HL model by [3], Poisson HL model by [4], type II HL family of distributions by [5], Kumaraswamy HL model by [6], extended HL model by [7], Transmuted HL model by [8], new type I HL model by [9], odd Lindley HL model by [10], weighted HL model by [11], modified HL model by [12], Poisson-logarithmic HL model by [13], extended type I HL family of distributions by [14] and Gamma power HL model by [15].

The type II Topp Leone-G (TIITL-G) class of distributions was pioneered by [16]. The TIITL-G class has just one parameter which imply that the proposed extended HL model in this study will have a single shape parameter. The CDF and PDF of the TIITL-G class are

$$F(k) = 1 - \left[1 - G^2(k)\right]^{\tau}, \tag{3}$$

and

$$f(k) = 2\tau g(k) G(k) \left[1 - G^2(k)\right]^{\tau-1}, \tag{4}$$

where $\tau > 0$ is a shape parameter, $G(k)$ and $g(k)$ are considered as the CDF and PDF of the baseline model. The novelty of this study is the creation of a new one-parameter lifetime model titled the TIITL_{HL} model, investigation of six different estimation methods for the new model with applicability to uncensored and censored survival time datasets, and introduction of a new log-TIITL_{HL} regression model for analysing censored response variable.

The remaining parts are outlined like this: Part 2 introduces the CDF and PDF of the TIITL_{HL} model. Part 3 presents reliability analysis and several important structural properties of the TIITL_{HL} model. Six classical estimation approaches are discussed in Part 4 to appreciate the parameters of the TIITL_{HL} model for uncensored sample. The maximum likelihood estimator based on the type-II right censored scheme is presented in Part 5. The finite sample performance of the TIITL_{HL} estimators is presented in Part 6 using Monte Carlo experiments. Part 7 deals with a new TIITL_{HL} regression model. The applications and empirical results are presented in Part 8. Finally, Part 9 presents the conclusion.

2. Model Genesis

This part introduces a new one-parameter model called the TIITL_{HL} model by inserting Eqs (1) and (2) into Eqs (3) and (4), then the CDF, PDF, survival function and hazard rate function (HRF) of the TIITL_{HL} model are

$$F(k,\tau) = 1 - \left[1 - \left(\frac{1-e^{-k}}{1+e^{-k}}\right)^2\right]^{\tau}, k > 0, \tau > 0, \tag{5}$$

$$f(k,\tau) = \frac{4\tau e^{-k}}{(1+e^{-k})^2}\left(\frac{1-e^{-k}}{1+e^{-k}}\right)\left[1 - \left(\frac{1-e^{-k}}{1+e^{-k}}\right)^2\right]^{\tau-1}, k > 0, \tau > 0, \tag{6}$$

Figure 1 depicts the graphical shapes of the TIITL_{HL} density function (PDF) with selected values for τ. The density function (PDF) is uni-modal, right-skewness, and heavy-tailed. The survival (Reliability) function (SF) and hazard (failure) rate (HRF) of the TIITL_{HL} model, take the forms

$$S(k,\tau) = \left[1 - \left(\frac{1-e^{-k}}{1+e^{-k}}\right)^2\right]^{\tau}, \tag{7}$$

and

$$h(k,\tau) = \frac{4\tau e^{-k}}{(1+e^{-k})^2}\left(\frac{1-e^{-k}}{1+e^{-k}}\right)\left[1 - \left(\frac{1-e^{-k}}{1+e^{-k}}\right)^2\right]^{-1}, \tag{8}$$

More so, the reversed HRF of the TIITL_{HL} model is

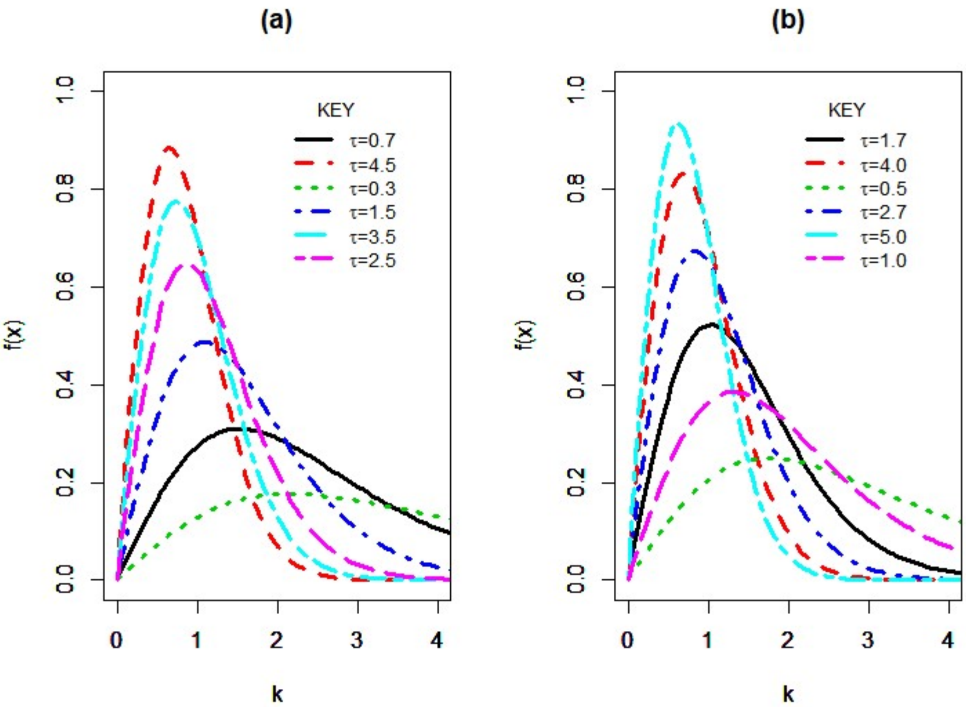

Figure 1: *The density function (PDF) plots of the $TIITL_{HL}$ model.*

$$r(k,\tau) = \frac{4\tau e^{-k}}{(1+e^{-k})^2}\left(\frac{1-e^{-k}}{1+e^{-k}}\right)\left[1-\left(\frac{1-e^{-k}}{1+e^{-k}}\right)^2\right]^{\tau-1}\left\{1-\left[1-\left(\frac{1-e^{-k}}{1+e^{-k}}\right)^2\right]^{\tau}\right\}^{-1}, \quad (9)$$

and the cumulative HRF takes the form

$$H(k,\tau) = -\tau \log\left[1-\left(\frac{1-e^{-k}}{1+e^{-k}}\right)^2\right]. \quad (10)$$

The graphical shapes of the HRF for $TIITL_{HL}$ model with various selected values of τ are depicted in Figure 2. The model is characterized by an increasing-decreasing HRF.

3. Structural Properties

This part describes the statistical properties of the $TIITL_{HL}$ model..

3.1. Quantile function, Bowley's skewness and Moor's kurtosis

If the random variable (r.v) $K \sim TIITL_{HL}(\tau)$, then the quantile function by inverting Eq (5) takes the form

$$k = -\log\left\{\frac{1-[1-(1-u)^{\frac{1}{\tau}}]^{\frac{1}{2}}}{1+[1-(1-u)^{\frac{1}{\tau}}]^{\frac{1}{2}}}\right\}, \quad (11)$$

where $u \sim uniform(0,1)$. By setting $u = 0.5$ in Eq (11), the median (M) of the $TIITL_{HL}$ model takes the form

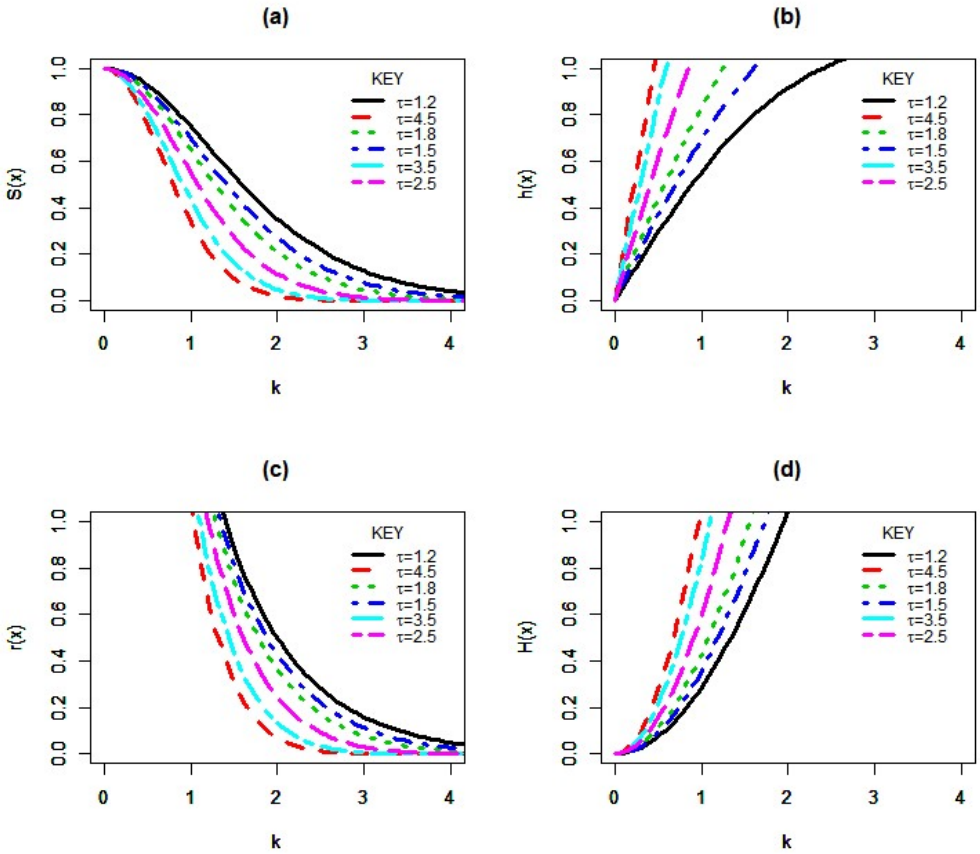

Figure 2: *Survival function plot (a), hazard rate function plot (b), reversed HRF plot (c) and cumulative HRF plot (d) of the TIITL$_{HL}$ model.*

$$k = -\log\left\{\frac{1-[1-(0.5)^{\frac{1}{\tau}}]^{\frac{1}{2}}}{1+[1-(0.5)^{\frac{1}{\tau}}]^{\frac{1}{2}}}\right\}. \tag{12}$$

The Bowley's skewness [17] and Moor's kurtosis [18] are found using the following expressions, respectively.

$$S_k = \frac{Q\left(\frac{3}{4};\tau\right) - 2Q\left(\frac{1}{2};\tau\right) + Q\left(\frac{1}{4};\tau\right)}{Q\left(\frac{3}{4};\tau\right) - Q\left(\frac{1}{4};\tau\right)}. \tag{13}$$

$$K_u = \frac{Q\left(\frac{7}{8};\tau\right) - Q\left(\frac{5}{8};\tau\right) - Q\left(\frac{3}{8};\tau\right) + Q\left(\frac{1}{8};\tau\right)}{Q\left(\frac{6}{8};\tau\right) - Q\left(\frac{2}{8};\tau\right)}. \tag{14}$$

where $Q(.)$ is the quantile function.

3.2. Dispersion index and Coefficient of variation

The dispersion index (DI) tells when a model is suitable for modeling equi-dispersed $(DI = 1)$, under-dispersed $(DI < 1)$ and over-dispersed $(DI > 1)$. The coefficient of variation (CV) is a relative measure of variability and a high CV value shows higher variability. The expressions for the DI and CV functions are

$$DI = \frac{Var(X)}{E(X)} = \frac{\frac{Q\left(\frac{3}{4};\tau\right)-Q\left(\frac{1}{4};\tau\right)}{1.35}}{\frac{Q\left(\frac{3}{4};\tau\right)+Q\left(\frac{1}{2};\tau\right)+Q\left(\frac{1}{4};\tau\right)}{3}}. \tag{15}$$

and

$$CV = \frac{(Var(X))^{\frac{1}{2}}}{E(X)} = \frac{\left(\frac{Q\left(\frac{3}{4};\tau\right)-Q\left(\frac{1}{4};\tau\right)}{1.35}\right)^{\frac{1}{2}}}{\frac{Q\left(\frac{3}{4};\tau\right)+Q\left(\frac{1}{2};\tau\right)+Q\left(\frac{1}{4};\tau\right)}{3}}. \tag{16}$$

where $Q(.)$ is the quantile function.

Table 1 reports the numerical values of the mean (ME), variance (VAR), standard deviation (STD), median (M), skewness (S_k), kurtosis (K_u), Dispersion index (DI) and Coefficient of variation (CV) for the $TIITL_{HL}$ model using selected values of τ.

Table 1: *The numerical values of ME, VAR, STD, M, S_k, K_u, DI and CV*

τ	ME	VAR	STD	M	S_k	K_u	DI	CV
0.2	5.282	17.356	4.166	4.836	0.238	0.629	3.286	0.789
0.5	2.784	3.531	1.879	2.634	0.177	0.484	1.268	0.675
1.0	1.832	1.293	1.137	1.763	0.135	0.370	0.706	0.621
1.5	1.459	0.769	0.877	1.412	0.117	0.321	0.527	0.601
2.0	1.247	0.543	0.737	1.212	0.108	0.294	0.436	0.591
2.5	1.107	0.419	0.647	1.078	0.102	0.277	0.378	0.584
3.0	1.006	0.341	0.584	0.980	0.097	0.266	0.339	0.581
3.5	0.928	0.287	0.536	0.905	0.095	0.257	0.310	0.578
4.0	0.865	0.248	0.498	0.845	0.092	0.251	0.287	0.576
4.5	0.814	0.218	0.467	0.795	0.091	0.246	0.268	0.574
5.0	0.771	0.194	0.441	0.753	0.089	0.242	0.252	0.572

The ME, STD, S_k and K_u values of the $TIITL_{HL}$ model decrease as the selected values of τ increase. The $TIITL_{HL}$ model is positively skewed and beneficial for over-and-under dispersed datasets. Figure 3 depict the plots of the ME, VAR, S_k and K_u of the $TIITL_{HL}$ for selected values of τ and support the conclusion reached using Table 1.

3.3. Moments and Moment generating function

The r^{th} raw moment of the $TIITL_{HL}$ model is given as

$$\mu_r' = \sum_{a=0}^{\tau-1} \sum_{b,c=0}^{\infty} \vartheta_{a,b,c} \left(b+c+1\right)^{-r-1} \Gamma\left(r+1\right). \tag{17}$$

Proof. The r^{th} raw moment of the $TIITL_{HL}$ model is found using

$$\begin{aligned} \mu_r' &= \int_0^\infty k^r f\left(k;\tau\right) dk, \\ &= 4\tau \int_0^\infty k^r \frac{e^{-k}}{(1+e^{-k})^2} \left(\frac{1-e^{-k}}{1+e^{-k}}\right) \left[1 - \left(\frac{1-e^{-k}}{1+e^{-k}}\right)^2\right]^{\tau-1} dk, \end{aligned} \tag{18}$$

By utilising Taylor series expansions in Eq (**??**), we have

$$\mu_r' = \sum_{a=0}^{\tau-1} \sum_{b,c=0}^{\infty} \vartheta_{a,b,c} \int_0^\infty k^r e^{-k(b+c+1)} dk, \tag{19}$$

where $\vartheta_{a,b,c} = 4\tau \left(-1\right)^{a+b+c} \binom{\tau-1}{a} \binom{2a+1}{b} \binom{-2\left(1+a\right)-1}{c}$.

Let
$z = k\left(b+c+1\right) \Rightarrow k = \frac{z}{(b+c+1)}$,

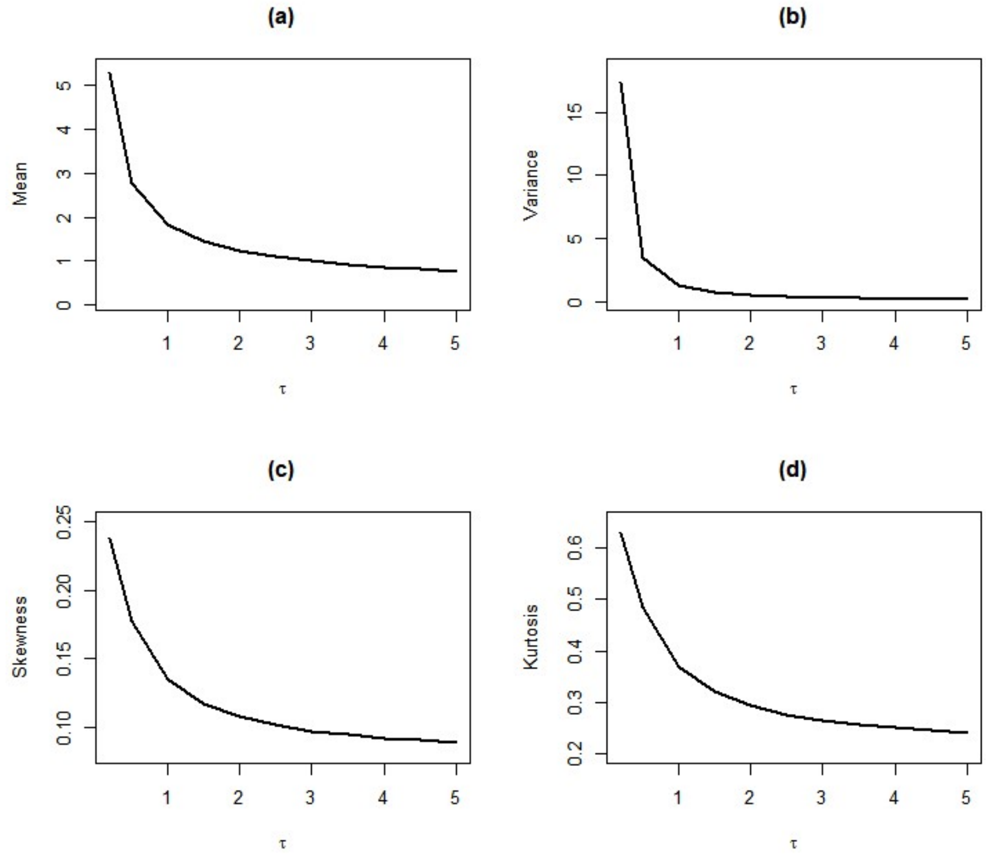

Figure 3: *The Mean, Variance, skewness and kurtosis plots of the TIITL$_{HL}$ model.*

$\frac{dk}{dz} = \frac{1}{(b+c+1)} \Rightarrow dk = \frac{dz}{(b+c+1)}$.
Hence,

$$\mu_r' = \sum_{a=0}^{\tau-1} \sum_{b,c=0}^{\infty} \vartheta_{a,b,c} \int_0^\infty \left(\frac{z}{(b+c+1)} \right)^r e^{-z} \frac{dz}{(b+c+1)}, \tag{20}$$

$$\mu_r' = \sum_{a=0}^{\tau-1} \sum_{b,c=0}^{\infty} \vartheta_{a,b,c} (b+c+1)^{-r-1} \int_0^\infty z^r e^{-z} dz, \tag{21}$$

By utilising the gamma integral function $\Gamma(\alpha+1) = \int_0^\infty z^\alpha e^{-z} dz$. The rth raw moments of the TIITL$_{HL}$ model takes the form

$$\mu_r' = \sum_{a=0}^{\tau-1} \sum_{b,c=0}^{\infty} \vartheta_{a,b,c} (b+c+1)^{-r-1} \Gamma(r+1). \tag{22}$$

The first four moments are found by inserting $r = 1, 2, 3, 4$ into Eq (22), respectively. ■
The moment generating function (MGF) of the TIITL$_{HL}$ model is given as

$$M_K(t) = \sum_{r=0}^{\infty} \sum_{a=0}^{\tau-1} \sum_{b,c=0}^{\infty} \frac{t^r \vartheta_{a,b,c}}{r!} (b+c+1)^{-r-1} \Gamma(r+1), \tag{23}$$

Proof.
The MGF of the TIITL$_{HL}$ model, say $M_K(t)$ is found using

$$M_K(t) = \mathrm{E}\left(e^{tk}\right) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} K^r f(k;\tau)\, dk = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r', \tag{24}$$

By inserting Eq (7) into Eq (24), the MGF takes the form

$$M_K(t) = \sum_{r=0}^{\infty} \sum_{a=0}^{\tau-1} \sum_{b,c=0}^{\infty} \frac{t^r \vartheta_{a,b,c}}{r!} (b+c+1)^{-r-1}\, \Gamma(r+1). \tag{25}$$

■

3.4. Order statistics

If $k_1, k_2, \ldots, k_n$ be a random sample from the $\mathrm{TIITL_{HL}}$ model with $k_{1:n} < k_{2:n} < \ldots < k_{n:n}$ as the order statistics (O.S). The pdf of the $\mathrm{p^{th}}$ O.S of the $\mathrm{TIITL_{HL}}$ model is

$$f_{p:n}(k) = \frac{4\tau e^{-k}(1+e^{-k})^{-2}(\varphi)\, n!}{(p-1)!\,(n-p)!} \left\{1 - \left[1-(\varphi)^2\right]^{\tau}\right\}^{p-1} \left[1-(\varphi)^2\right]^{\tau[(n-p)+1]-1}. \tag{26}$$

Proof. The pdf of the $\mathrm{p^{th}}$ O.S can be found using

$$f_{p:n}(k) = \frac{n!}{(p-1)!\,(n-p)!} g(k) \left[G(k)\right]^{p-1} \left[1-G(k)\right]^{n-p}, \tag{27}$$

where B (.,.) is the beta function. By inserting Eqs (5) and (6) into Eq (27), the pdf of the $\mathrm{p^{th}}$ O.S of the $\mathrm{TIITL_{HL}}$ model after some simplification takes the form

$$f_{p:n}(k) = \frac{4\tau e^{-k}(1+e^{-k})^{-2}(\varphi)\, n!}{(p-1)!\,(n-p)!} \left\{1 - \left[1-(\varphi)^2\right]^{\tau}\right\}^{p-1} \left[1-(\varphi)^2\right]^{\tau[(n-p)+1]-1}, \tag{28}$$

where $\varphi = \left(\frac{1-e^{-k}}{1+e^{-k}}\right)$. By substituting $p = 1$ and $p = n$ into Eq (28), the lowest and highest order statistics are obtained. ■

4. Methods of estimation for uncensored sample

In this part, the parameter of the $\mathrm{TIITL_{HL}}$ model is estimated via the maximum likelihood estimation (MLE), maximum product spacing estimation (MPSE), Anderson Darling estimation (ADE), least square estimation (LSE), weighted least square estimation (WLSE), and Cramer Von Mises estimation (CVME).

4.1. The MLE

If $k_1, k_2, \ldots, k_n$ be the random observed values from $\mathrm{TIITL_{HL}}$ model. Then, the MLE function $\mathrm{L}(\tau)$ takes the form

$$L(\tau) = (4\tau)^n \prod_{i=1}^{n} \frac{e^{-k_i}}{\left(1+e^{-k_i}\right)^2} (\varphi_i) \left[1-(\varphi_i)^2\right]^{\tau-1} \tag{29}$$

where $\varphi_i = \left(\frac{1-e^{-k_i}}{1+e^{-k_i}}\right)$. The log-likelihood function of the $\mathrm{TIITL_{HL}}$ model takes the form

$$\log(L(\tau)) = n\log(4\tau) - \sum_{i=1}^{n} k_i - 2\sum_{i=1}^{n} \log\left(1+e^{-k_i}\right) + \sum_{i=1}^{n} \log(\varphi_i) + (\tau-1)\sum_{i=1}^{n} \log\left[1-(\varphi_i)^2\right], \tag{30}$$

The first derivative of Eq (30) with respect to τ is

$$\frac{\partial \log\left(L\left(\tau\right)\right)}{\partial \tau} = \frac{n}{\tau} + \sum_{i=1}^{n} \log\left[1-\left(\varphi_i\right)^2\right]. \tag{31}$$

The R (*optim function*) is employed to estimate the $\mathrm{TIITL_{HL}}$ parameter using numerical approaches.

4.2. The LSE and WLSE

Minimizing with respect to τ, the LS estimate $\hat{\tau}_{\mathrm{LS}}$ can be found using

$$LS\left(\tau\right) = \sum_{i=1}^{n}\left[1-\left[1-\left(\frac{1-e^{-k}}{1+e^{-k}}\right)^2\right]^{\tau} - \frac{i}{n+1}\right]^2. \tag{32}$$

Likewise, minimizing with respect to τ, the WLS estimates $\hat{\tau}_{\mathrm{WLS}}$ can be found using

$$WLS\left(\tau\right) = \sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{i\left(n-i+1\right)}\left[1-\left[1-\left(\frac{1-e^{-k}}{1+e^{-k}}\right)^2\right]^{\tau} - \frac{i}{n+1}\right]^2. \tag{33}$$

4.3. The MPSE

The MPS for the $\mathrm{TIITL_{HL}}$ model with ordered sample $k_{(1:n)}, k_{(2:n)}, \ldots, k_{(n:n)}$ is given as follows

$$GM\left(\tau | k_{n:n}\right) = \left[\prod_{i=1}^{n+1} D_i\left(k_i,\tau\right)\right]^{\frac{1}{n+1}}, \tag{34}$$

where $D_i\left(k_i,\tau\right) = F\left(k_{(i:n)}\left|\tau\right.\right) - F\left(k_{(i-1:n)}\left|\tau\right.\right)$; $i = 1, 2, \ldots, n+1$. and $F\left(k,\tau\right)$ is given in Eq (5).

4.4. The ADE

Minimizing with respect τ, the AD estimate $\hat{\tau}_{\mathrm{AD}}$ can be found using

$$AD\left(\tau\right) = -n - \frac{1}{n}\sum_{i=1}^{n}\left(2i-1\right)\left[\log F\left(k_{(i:n)}\left|\tau\right.\right) + \log \bar{F}\left(k_{(n+1-i:n)}\left|\tau\right.\right)\right], \tag{35}$$

where $\bar{F}\left(k,\tau\right) = 1 - F\left(k,\tau\right)$ and $F\left(k,\tau\right)$ is given in Eq (5).

4.5. The CVME

Minimizing with respect τ, the CVM estimates $\hat{\tau}_{\mathrm{CVM}}$ can be found using

$$CVM\left(\tau\right) = \frac{1}{12} + \sum_{i=1}^{n}\left[1-\left[1-\left(\frac{1-e^{-k}}{1+e^{-k}}\right)^2\right]^{\tau} - \frac{2(i-1)+1}{2n}\right]^2. \tag{36}$$

5. MLE for type II right censoring

Given that a fixed number of failed units have been observed, a life testing experiment is concluded. Then the remaining units are designated as type-II-censored. Let $k_{(1)}, k_{(2)}, \ldots, k_{(p)}, p \leq n$ denote the ordered values of a random sample (r.s) $k_1, k_2, \ldots, k_n$ (failure times) and observations cease after the $\mathrm{p^{th}}$ unsuccessful unit occurs, then the likelihood function is given

$$L\left(\tau;k\right) = \frac{n!}{(n-p)!}\left[R\left(k_p;\tau\right)\right]^{n-p}\prod_{i=1}^{p} f\left(k_i;\tau\right). \tag{37}$$

If $k_1, k_2, \ldots, k_n$ be r.s from the TIITL$_{\text{HL}}$ (τ), then the likelihood function is

$$L(\tau; k) = \frac{n!}{(n-p)!}\left[1 - \left(\frac{1-e^{-k_p}}{1+e^{-k_p}}\right)^2\right]^{\tau(n-p)} \prod_{i=1}^{p}\left\{\frac{4\tau e^{k_i}\left(\frac{1-e^{-k_i}}{1+e^{-k_i}}\right)}{(1+e^{-k_i})^2}\left[1-\left(\frac{1-e^{-k_i}}{1+e^{-k_i}}\right)^2\right]^{\tau-1}\right\}. \tag{38}$$

The log-likelihood function without the constant term is

$$\begin{aligned} l(\tau; k) \propto\ & p\left[\log(4) + \log(\tau)\right] + \tau(n-p)\log\left[1 - \left(\tfrac{1-e^{-k_p}}{1+e^{-k_p}}\right)^2\right] - \textstyle\sum_{i=1}^{p} k_i \\ & -2\textstyle\sum_{i=1}^{p}\log(1+e^{-k_i}) + \sum_{i=1}^{p}\log\left(\tfrac{1-e^{-k_i}}{1+e^{-k_i}}\right) + (\tau-1)\sum_{i=1}^{p}\log\left[1-\left(\tfrac{1-e^{-k_i}}{1+e^{-k_i}}\right)^2\right]. \end{aligned} \tag{39}$$

Setting $\frac{\partial}{\partial\tau} l(\tau; k) = 0$. The MLE $(\hat{\tau})$ can be found as solution of

$$\frac{p}{\tau} + (n-p)\log\left[1-\left(\frac{1-e^{-k_p}}{1+e^{-k_p}}\right)^2\right] + \sum_{i=1}^{p}\log\left[1-\left(\frac{1-e^{-k_i}}{1+e^{-k_i}}\right)^2\right] \tag{40}$$

Using the R (*optim function*), the non-linear equation in Eq (40) is solved numerically to obtain the MLE $\hat{\tau}$.

6. Simulation

The Monte Carlo simulations for uncensored and censored samples are executed for the TIITL$_{\text{HL}}$ parameter (Pa.).

6.1. Simulation based on uncensored sample

The simulations using MLE, LSE, WLSE, MPSE, ADE, and CVME approaches for the TIITL$_{\text{HL}}$ parameter are presented in this subpart. The simulation is carried out as follows:

- Set the parameter value τ = 0.5, 2.5 for the Monte Carlo simulation process.
- Random samples of sizes n $= 20, 70, 150, 250, 350$ with replicates $N = 5000$ generated using Eq (11).
- The MLE, LSE, WLSE, MPSE, ADE, and CVME processes are executed to find the estimates of parameter (τ).
- Compute the average estimate (AVEs), absolute biases (ABs), mean square errors (MSEs) and mean relative error (MREs) using the information in the preceding step.

Tables 2 and 3 reports the AVEs, ABs, MSEs and MREs for the MLE, LSE, WLSE, MPSE, ADE, and CVME methods with different sample sizes. The results are graphically summarized in Figures 4 and 5. As seen from these graphs, the ABs, MSEs and MREs tend to zero as n increases for the six estimation methods. However, the ADE average estimate tend to the true parameter value faster than other estimation methods with minimum AB.

Table 2: *The six estimators AVE for $\tau = 0.5$ based on uncensored sample.*

n	Measures	MLE	MPSE	ADE	LSE	WLSE	CVME
20	AVE	0.527	0.489	0.518	0.522	0.520	0.526
70	AVE	0.507	0.492	0.504	0.505	0.505	0.506
150	AVE	0.503	0.495	0.502	0.502	0.502	0.503
250	AVE	0.502	0.496	0.501	0.501	0.501	0.501
350	AVE	0.501	0.497	0.500	0.500	0.500	0.500

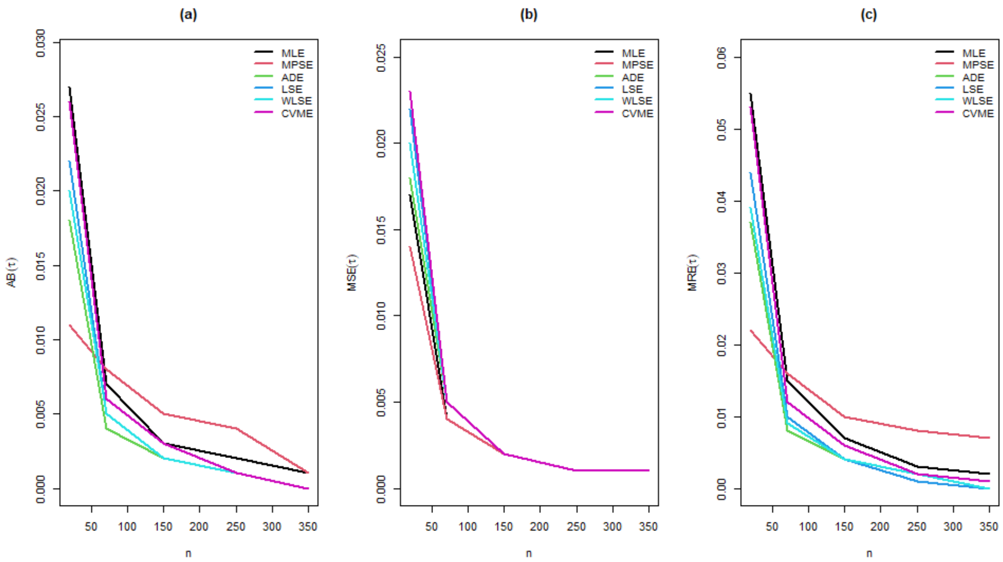

Figure 4: *AB, MSE and MRE of Estimators in Table 2.*

Table 3: *The six estimators AVE for $\tau = 2.5$ based on uncensored sample.*

n	Measures	MLE	MPSE	ADE	LSE	WLSE	CVME
20	AVE	2.637	2.444	2.591	2.610	2.598	2.632
70	AVE	2.536	2.460	2.521	2.525	2.524	2.531
150	AVE	2.517	2.475	2.509	2.511	2.511	2.514
250	AVE	2.508	2.480	2.503	2.504	2.504	2.505
350	AVE	2.504	2.483	2.500	2.501	2.501	2.502

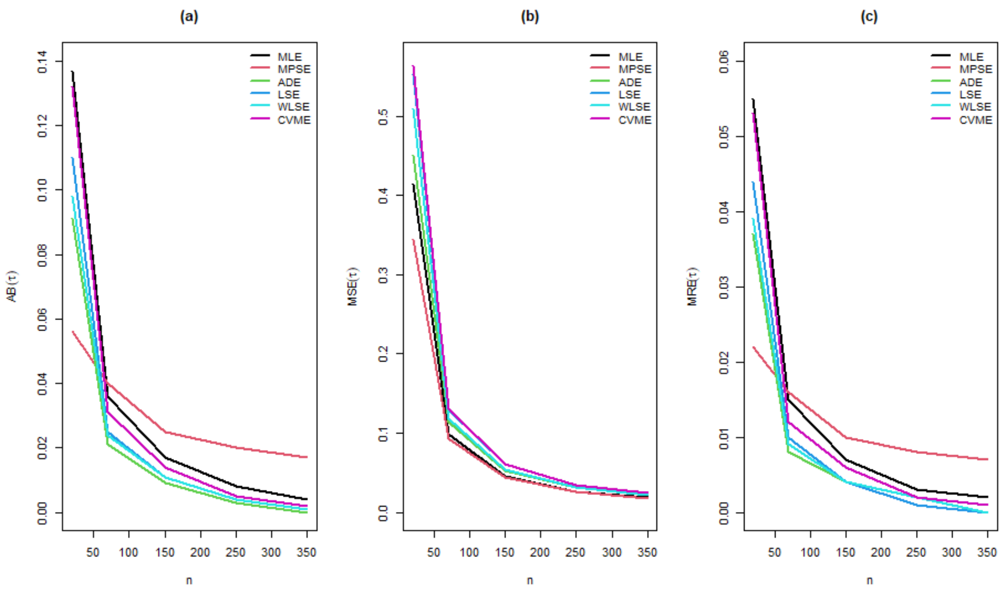

Figure 5: *AB, MSE and MRE of Estimators in Table 3.*

6.2. Simulation based on type-II-right censored sample

The simulation is executed for the MLE using different random sample sizes n = $(20, 70, 150, 250, 350)$ generated with Eq (11). The length of censored sample test is given by $p = nm$, where (m) is the censoring proportion $(0 < m < 1)$. Table 4 reports that the bias, MSE and MRE values decrease as the sample size increases for the different censoring proportions considered.

Table 4: *The MLE, Bias, MSE and MRE based on censored sample.*

n	m	τ	$\hat{\tau}$	Bias	MSE	MRE
20	0.3	0.5	0.881	0.381	0.145	0.762
		2.5	4.405	1.905	3.629	0.762
	0.5	0.5	0.752	0.252	0.063	0.503
		2.5	3.758	1.258	1.583	0.503
	0.7	0.5	0.781	0.281	0.079	0.562
		2.5	3.906	1.406	1.976	0.562
70	0.3	0.5	0.611	0.111	0.012	0.223
		2.5	3.057	0.557	0.310	0.223
	0.5	0.5	0.650	0.150	0.022	0.299
		2.5	3.248	0.748	0.560	0.299
	0.7	0.5	0.657	0.157	0.025	0.315
		2.5	3.287	0.787	0.620	0.315
150	0.3	0.5	0.551	0.051	0.003	0.101
		2.5	2.753	0.253	0.064	0.101
	0.5	0.5	0.554	0.054	0.003	0.108
		2.5	2.771	0.271	0.073	0.108
	0.7	0.5	0.573	0.073	0.005	0.145
		2.5	2.863	0.363	0.132	0.145
250	0.3	0.5	0.567	0.066	0.004	0.132
		2.5	2.831	0.331	0.110	0.132
	0.5	0.5	0.587	0.087	0.008	0.175
		2.5	2.936	0.436	0.190	0.175
	0.7	0.5	0.579	0.079	0.006	0.158
		2.5	2.896	0.396	0.157	0.158
350	0.3	0.5	0.485	-0.015	2E-04	0.030
		2.5	2.424	-0.076	0.006	0.030
	0.5	0.5	0.453	-0.047	0.002	0.095
		2.5	2.264	-0.236	0.056	0.095
	0.7	0.5	0.424	-0.076	0.006	0.153
		2.5	2.118	-0.382	0.146	0.153

7. The log-TIITL$_{HL}$ regression model

Let K denotes a random variable which follows the TIITL$_{HL}$ model with parameter τ. Utilizing the transformation $Y = \log(K)$ with location and scale parameters added, the density of Y is

$$f_{TIITL_{HL}}(y,\tau,\mu,\sigma) = \tfrac{4\tau}{\sigma}\exp\left[\left(\tfrac{y-\mu}{\sigma}\right) - \exp\left(\tfrac{y-\mu}{\sigma}\right)\right]\left\{1+\exp\left[-\exp\left(\tfrac{y-\mu}{\sigma}\right)\right]\right\}^{-2} \\ \times\left\{\tfrac{1-\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]}{1+\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]}\right\}\left(1-\left\{\tfrac{1-\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]}{1+\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]}\right\}^2\right)^{\tau-1}, \tag{41}$$

where $\tau, \sigma > 0$, $y, \mu \in \Re$. The random variable Y has the log-TIITL$_{HL}$ (LTIITL$_{HL}$) model with location μ and scale σ parameters, respectively. The survival function to Eq (41) is

$$S_{TIITL_{HL}}(y,\tau,\mu,\sigma) = \left(1-\left\{\frac{1-\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]}{1+\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]}\right\}^2\right)^{\tau}. \tag{42}$$

By inserting $z = (y-\mu)/\sigma$, $z \in \Re$ into Eq (41). The standardized log-TIITL$_{HL}$ density takes the form

$$f_{TIITL_{HL}}(z,\tau) = \tfrac{4\tau}{\sigma}\exp\left[(z)-\exp(z)\right]\left\{1+\exp\left[-\exp(z)\right]\right\}^{-2} \\ \times\left\{\tfrac{1-\exp[-\exp(z)]}{1+\exp[-\exp(z)]}\right\}\left(1-\left\{\tfrac{1-\exp[-\exp(z)]}{1+\exp[-\exp(z)]}\right\}^2\right)^{\tau-1}. \tag{43}$$

Let $\boldsymbol{K}_i = (k_{i1}, \ldots, k_{im})^T$ be the explanatory vector associated with the ith response variable y_i for $i = 1, \ldots, n$. A regression model based on the TIITL$_{HL}$ density function is given by

$$y_i = \boldsymbol{K}_i^T \boldsymbol{\beta} + \sigma z_i, \; i = 1, \ldots, n, \tag{44}$$

where z_i is the random error which follows the density function Eq (43), $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)^T, \sigma > 0, \; \tau > 0$ are unknown parameters and $\boldsymbol{K}_i$ is modeling $\mu_i = \boldsymbol{K}_i^T \boldsymbol{\beta}$. The density and survival functions of y_i are

$$\begin{aligned} f_{TIITL_{HL}}(y_i; \tau, \sigma, \boldsymbol{\beta}^T) = {} & \tfrac{4\tau}{\sigma} \exp\left[z_i - \exp(z_i)\right] \{1 + \exp\left[-\exp(z_i)\right]\}^{-2} \\ & \times \left\{ \tfrac{1-\exp[-\exp(z_i)]}{1+\exp[-\exp(z_i)]} \right\} \left(1 - \left\{ \tfrac{1-\exp[-\exp(z_i)]}{1+\exp[-\exp(z_i)]} \right\}^2 \right)^{\tau-1}, \end{aligned} \tag{45}$$

and

$$S_{TIITL_{HL}}(y_i; \tau, \sigma, \boldsymbol{\beta}^T) = \left(1 - \left\{ \frac{1-\exp[-\exp(z_i)]}{1+\exp[-\exp(z_i)]} \right\}^2 \right)^{\tau}, \tag{46}$$

Let F and C denote the sets of units for which y_i is the log-lifetime or log-censoring, respectively. The log-likelihood for $\boldsymbol{\vartheta} = \left(\tau, \sigma, \boldsymbol{\beta}^T\right)^T$ from Eq (44) is

$$\begin{aligned} l(\boldsymbol{\vartheta}) = {} & r \log\left(\tfrac{4\tau}{\sigma}\right) + \textstyle\sum_{i\in F} \left[z_i - \exp(z_i)\right] - 2\sum_{i\in F} \log\{1 + \exp\left[-\exp(z_i)\right]\} \\ & + \textstyle\sum_{i\in F} \log\left\{ \tfrac{1-\exp[-\exp(z_i)]}{1+\exp[-\exp(z_i)]} \right\} + (\tau - 1) \sum_{i\in F} \log\left(1 - \left\{ \tfrac{1-\exp[-\exp(z_i)]}{1+\exp[-\exp(z_i)]} \right\}^2 \right) \\ & + \tau \textstyle\sum_{i\in C} \log\left(1 - \left\{ \tfrac{1-\exp[-\exp(z_i)]}{1+\exp[-\exp(z_i)]} \right\}^2 \right). \end{aligned} \tag{47}$$

where $z_i = \left(y_i - \boldsymbol{K}_i^T \boldsymbol{\beta}\right)/\sigma$ and r is the number of uncensored observations (failures). The MLE $\hat{\vartheta}$ of ϑ can be obtained by maximizing Eq (47) using the R (*optim function*).

8. Applications

The potentiality of the introduced model is illustrated by means of five applications.

8.1. Applications to uncensored data

The first dataset consist of survival time of 72 Guinea pigs infected with virulent tubercle bacilli. The data was initially reported by [19], and analysed by [6] and [20]. The second dataset, discussed by [21] consists of 30 observations of March precipitation (in inches) in Minneapolis/St Paul.

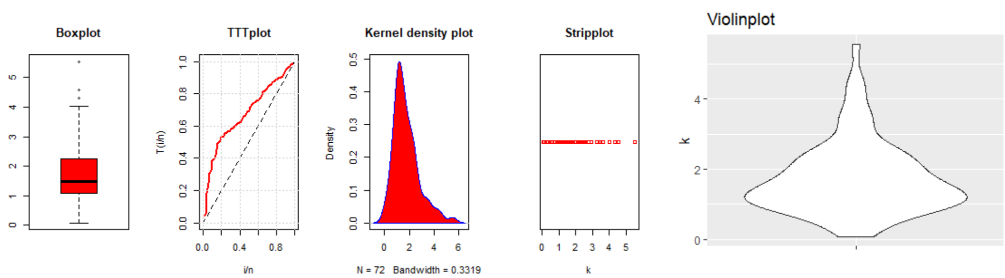

Figure 6: *Box, TTT, kernel-density, strip, and Violin plots of first uncensored data.*

Figures 6 and 7 depict the box plot, TTT plot, kernel density plot, strip plot, and Violin plot of the first and second datasets to check for outliers and symmetric nature. As depicted, both

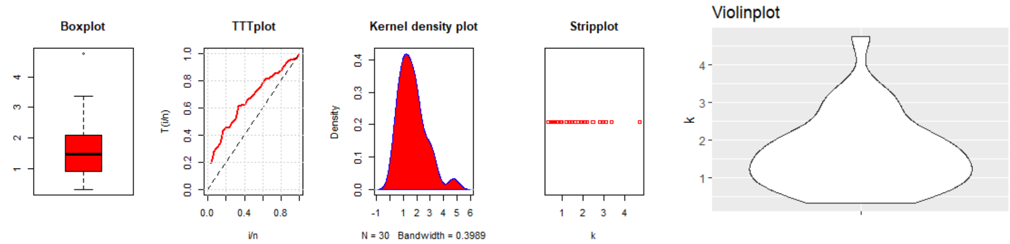

Figure 7: *Box, TTT, kernel-density, strip, and Violin plots of second uncensored data.*

datasets have certain outliers and are asymmetrical in nature. More so, the total test time (TTT) plot depicts that the utilized datasets have increasing HRF which means that the TIITL_{HL} model can be used to model these datasets.

We fit the datasets with the TIITL_{HL}, odd Lindley HL (OLIHL), Generalized HL (GHL), Inverse Lindley (ILIN), Rayleigh (R), Inverse Rayleigh (IR), and Muth (M) models. The MLEs for all the models are computed in R-software via the BFGS method (*optim* function). The W^*, A^* and KS measures (we used abbreviations) for model comparisons. The MLEs and their standard errors (SEs) in parentheses, log-likelihood (LL), and the information criteria are presented in Tables 5 and 6 for both datasets, respectively. The measures reveal that the TIITL_{HL} model provides an appropriate fit to both datasets (with lowest values of the goodness-of-fit statistics).

Table 5: *The MLEs and SEs of the fitted models with goodness-of-fit measures for first uncensored data.*

Model	MLEs (SEs)	LL	W*	A*	CAIC	AIC	BIC	HQIC	KS(P-Value)
TIITLHL	1.257(0.148)	-96.83	0.074	0.498	195.7	195.7	197.9	196.6	0.088(0.636)
OLIHL	0.289()0.024	-157.28	0.891	4.940	316.6	316.6	318.8	317.5	0.393(0.000)
GHL	0.786(0.093)	-104.94	0.080	0.531	211.9	211.9	214.2	212.8	0.218(0.002)
ILIN	1.359(0.124)	-129.20	0.929	5.634	260.5	260.4	262.7	261.3	0.232(0.001)
R	1.442(0.085)	-98.96	0.133	0.814	200.0	199.9	202.2	200.8	0.107(0.376)
IR	0.480(0.028)	-204.82	1.632	9.214	411.7	411.6	413.9	412.5	0.623(0.000)
M	0.480(0.028)	-123.46	0.118	0.735	249.0	248.9	251.2	249.8	0.434(0.000)

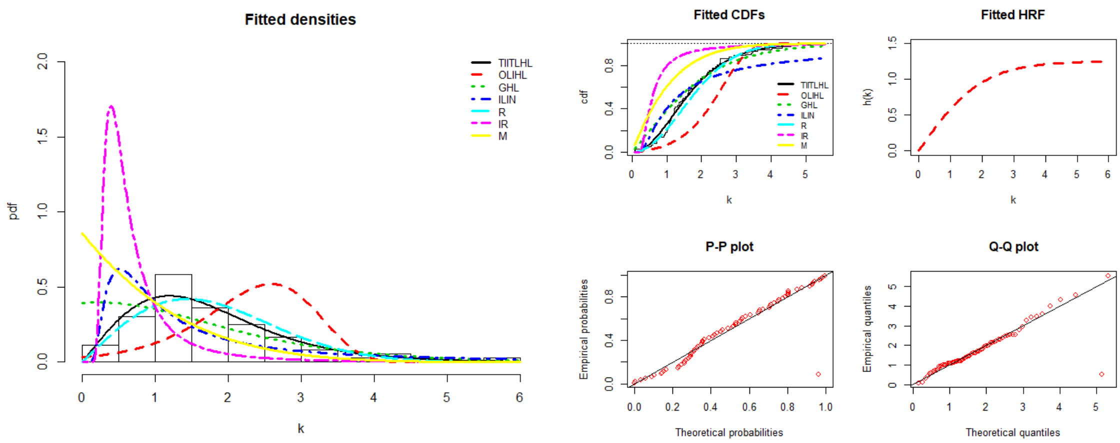

Figure 8: *The fitted pdfs (left panel) and cdfs (top-left,right panel) of competing models, fitted HRF (top-right,right panel), P-P plot (bottom-left,right panel) and Q-Q plot (bottom-right,right panel) of the TIITL_{HL} model for first uncensored data.*

Table 6: *The MLEs and SEs of the fitted models with goodness-of-fit measures for second uncensored data.*

Model	MLEs (SEs)	LL	W*	A*	CAIC	AIC	BIC	HQIC	KS(P-Value)
TIITLHL	1.351 (0.247)	-38.34	0.01483	0.1155	78.83	78.68	80.08	79.13	0.059 (0.999)
OLIHL	0.364 (0.048)	-56.22	0.25812	1.6009	114.58	114.44	115.84	114.89	0.307 (0.007)
GHL	0.828 (0.151)	-42.43	0.01530	0.1195	87.01	86.87	88.27	87.32	0.182 (0.274)
ILIN	1.583 (0.227)	-45.22	0.09750	0.6014	92.59	92.44	93.84	92.89	0.228 (0.089)
R	1.376 (0.125)	-38.92	0.02607	0.1979	79.99	79.85	81.25	80.30	0.084 (0.985)
IR	-0.927(0.085)	-44.14	0.16293	0.9881	90.42	90.27	91.67	90.72	0.240 (0.064)
M	0.185 (0.091)	-48.72	0.02471	0.1898	99.59	99.45	100.85	99.89	0.367 (0.001)

Figures 8 and 9 display the histogram and Kaplan-Meier empirical cdf in conjunction with fitted pdfs and cdfs of the TIITL$_{HL}$ and competing models for both datasets. Figures 8 and 9, also depict the fitted HRF of the TIITL$_{HL}$ model with corresponding probability-probability (P-P) and quantile-quantile (Q-Q) plots. The superiority of the TIITL$_{HL}$ model is supported by these figures.

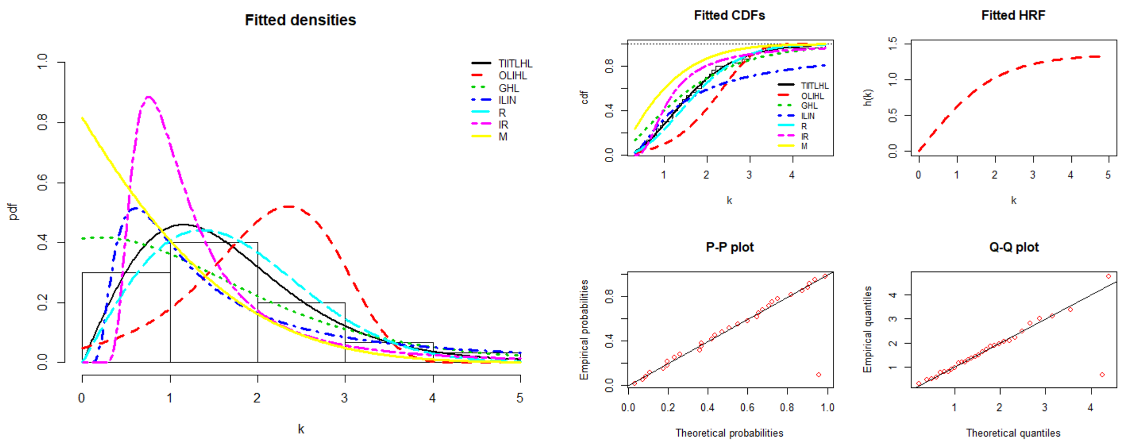

Figure 9: *The fitted pdfs (left panel) and cdfs (top-left,right panel) of competing models, fitted HRF (top-right,right panel), P-P plot (bottom-left,right panel) and Q-Q plot (bottom-right,right panel) of the TIITL$_{HL}$ model for second uncensored data.*

Tables 7 and 8 report the estimation of the TIITL$_{HL}$ parameter based on six estimation methods for the both uncensored datasets. From the results in Tables 7, the MPSE and WLSE are considered as the appropriate methods with smaller KS and larger P-values than the other methods. Likewise, the results in Table 8 attest that the MLE, ADE and WLSE are the appropriate methods with smaller KS and larger P-values than the other methods.

Table 7: *The results of the six estimation methods for first uncensored data.*

Estimate↓Method→	MLE	MPSE	CVME	ADE	LSE	WLSE
τ	1.257	1.215	1.279	1.262	1.266	1.229
KS	0.088	0.081	0.091	0.089	0.089	0.083
P-Value	0.600	0.700	0.600	0.600	0.610	0.700

Table 8: *The results of the six estimation methods for second uncensored data.*

Estimate↓Method→	MLE	MPSE	CVME	ADE	LSE	WLSE
τ	1.351	1.266	1.354	1.347	1.011	1.351
KS	0.059	0.073	0.060	0.059	0.155	0.059
P-Value	0.999	0.999	0.999	0.999	0.470	0.999

8.2. Applications to censored data

The first dataset represents the relief times (in minutes) of twenty patients receiving an analgesic. The data was initially reported by [22] and the complete sample analysed by [23] and [24]. The censored sample (number of failures) p, is chosen as 50% (censoring scheme). The MLE, KS and P-Value for the TIITL_{HL} model are reported in Table 9.

Table 9: *The MLE and performance measure for the first censored data.*

Models	MLE	KS	P-Value
TIITL_{HL}	$\tau = 0.877$	0.230	0.240

The second dataset represents the survival time of 72 Guinea pigs infected with virulent tubercle bacilli. The complete sample was analysed by [6] and [20]. The censored sample (number of failures) p, is chosen as 70% (censoring scheme). The MLE, KS and P-Value for the TIITL_{HL} model are reported in Table 10. It is evident that the TIITL_{HL} appropriately fits the two survival time censored datasets.

Table 10: *The MLE and performance measure for the second censored data.*

Models	MLE	KS	P-Value
TIITL_{HL}	$\tau = 1.174$	0.077	0.790

It is evident that the TIITL_{HL} appropriately fits the two censored samples.

8.3. Regression model application

The usefulness of log-TIITL_{HL} regression model is demonstrated by means of a real data analysis. The log-TIITL_{HL} regression model is compared with log-exponential (LE) and log-Burr-Hatke-exponential (LBHE) regression models [25]. The utilized dataset contains 100 individuals having HIV+ obtained from the **Bolstad2** package in R-software. The observed survival times (y_i), in months, with censoring indicator (0 = alive and 1 = death) is analysed with two explanatory variables: k_{i1}, (0 = no and 1 = yes) represent the history of drug usage and k_{i2} represent the ages of patients . The proposed regression model is

$$y_i = \beta_0 + \beta_1 k_{i1} + \beta_2 k_{i2} + \sigma z_i \tag{48}$$

where z_i has density Eq (43). The MLE method is utilized in estimating the unknown parameters of log-TIITL_{HL}, LE and LBHE regression models. Table 11 reports the regression models estimated parameters, -LL and performance measures (AIC, BIC, AICc and HIQC values). The results provided in Table 11 indicates that the $\text{LTIITL}_{\text{HL}}$ regression model has the lowest value of -LL and performance measures values, respectively. Hence, it is concluded that log-TIITL_{HL} regression model provides appropriate fit than LE and LBHE regression models. More so, the estimated regression parameters β_0, β_1 and β_2 are statistically significant at 5% level of significance.

Table 11: *The regression models estimated parameters and performance measures.*

	LBHE			L-E			Log-TIITL$_{HL}$		
Parameters	Estimates	SE	P-Value	Estimates	SE	P-Value	Estimates	SE	P-Value
τ	1.508	13.659	-	1.599	13.783	-	26.968	48.251	-
σ	0.778	0.067	-	0.839	0.072	-	1.684	0.143	-
β_0	6.883	7.064	0.330	6.542	7.256	0.367	2.303	0.130	<0.001
β_1	-0.091	0.014	<0.001	-0.091	0.014	<0.001	-0.023	0.009	0.020
β_2	-1.021	0.193	<0.001	-1.049	0.189	<0.001	-0.261	0.109	0.017
$-LL$		128.059			128.502			128.051	
AIC		266.12			267.00			266.10	
BIC		279.14			280.03			279.13	
AICc		266.76			267.64			266.74	
HQIC		271.39			272.28			271.37	

9. Conclusion

This work introduced a new one-parameter model titled the TIITL$_{HL}$ model and provided some of its properties. The consistency of the maximum likelihood estimator and five other estimators are proven by uncensored and censored simulation studies. Applications to real medical and environmental sciences datasets revealed its flexibility and adaptability. The log-TIITL$_{HL}$ regression model constructed and fitted to HIV+ data, and compared with other existing models showed that the model will be a useful choice in survival investigation for practitioners. Overall, the five applications showed the usefulness of the new model for asymmetric, uncensored and censored data. In future works, the Bayesian analysis of TIITL$_{HL}$ accelerated failure time model, the TIITL$_{HL}$–G family of distributions and the discrete case of the TIITL$_{HL}$ model will be addressed.

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