DISCRETE INVERSE GAMMA DISTRIBUTION BASED LOAD-SHARE MODEL WITH APPLICATION

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Abstract

In reliability engineering, the multi-component load-sharing models are being used to amplify system's reliability. This study consists of the k-component load-sharing parallel system model considering each component's failure time distribution as discrete inverse gamma. The classical and Bayesian analysis for this model is performed. The maximum likelihood estimates along with their standard errors for the parameters, system's reliability function, hazard rate function and reversed second rate of failure function are obtained. The asymptotic confidence intervals as well as two bootstrap intervals like bootstrap-p and bootstrap-t confidence intervals are constructed. Further, Bayes estimates along with their posterior standard errors and highest posterior density credible intervals for the parameters and system's reliability characteristics are obtained by using Markov Chain Monte Carlo techniques. A detailed simulation table is formed to demonstrate the effectiveness of the theory developed. Finally, a real data set is used to illustrate the applicability of the model.

Keywords: Load-share system model, Discrete inverse gamma distribution, Bootstrapping, Bayesian estimation, MCMC technique

1. Introduction

In today's engineering world, the products with high reliability are in demand. It can be accomplished with planned maintenance, improving components' reliabilities, reassignment of components etc. In this regard, manufacturers usually use redundancy techniques. While using redundancy techniques, it is assumed that the components within the system are performing independently. However, in practice, many multi-component systems work as load-sharing model which leads the independence assumption to be not valid any more. In load-sharing model (or dynamic system model), on a component's failure, the workload will be redistributed to the other working components, which imparts increased load on them. Originally, Daniels [1] developed the first load-sharing design to study textile strength.

In last two decades, various authors have discussed load-share modeling and estimated its parameters in different contextual aspects. Kim and Kvam [3] considered equal load-share rule for estimating the load-share parameters of multi-component system in parametric setup with failure time distribution as exponential. Singh et al. [12]

performed the classical and Bayesian inference for multi-component load-share system by assuming that the components have a combination of constant and linear failure rates. Park [5, 6] performed the classical and Bayesian inference for such models by assuming the underlying lifetime distribution as exponential or Weibull.

There are various real examples where load-share modeling can be used such as in mechanical and civil engineering for welded joints (enhances the stress on other joints), in textile engineering and materials science for crack growth induced by fatigue or material degradation (bigger cracks will grow faster than smaller cracks), in a power plant for electric generators (electrical load is shared), in a human body for two kidneys (simultaneously work together) and so forth.

There are some basic continuous lifetime distributions such as exponential, Weibull, Lindley and log-normal, which have been explored for analyzing load-share models. Some distributions are recently been used for this purpose like Chen, modified Weibull, and exponentiated Pareto distributions which are discussed by Pundir and Gupta [7], Singh and Goel [11] and Zhang et al. [16], respectively. However, there are various situations in which discrete distributions can perform well like number of shocks, number of patients in a ward, etc. Singh et al. [13] have dealt with discrete load-share modeling situation using geometric distribution.

Considering this, a load-share system model is developed and analyzed with discrete inverse gamma distribution (DIGD) in the present work. Hussain and Ahmad [2] obtained DIGD by discretizing the inverse gamma distribution (IGD). Its cumulative distribution function (CDF), reliability function, probability mass function (PMF), hazard rate function (HRF) and reversed second rate of failure (RSRF) are, respectively,

$$F(x) = \int_{1/x}^{\infty} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha-1} e^{-\frac{y}{\beta}} dy = \frac{\Gamma(\alpha, \frac{1}{\beta x})}{\Gamma(\alpha)} \;\; ; x = 0, 1, \ldots ; \alpha > 0, \beta > 0$$

where $\Gamma(\alpha, \frac{1}{\beta x}) = \int_{1/x}^{\infty} \frac{1}{\beta^{\alpha}} y^{\alpha-1} e^{-\frac{y}{\beta}} dy$ and Γ represents the upper incomplete gamma function,

$$R(x) = 1 - F(x) = \int_{0}^{1/x} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha-1} e^{-\frac{y}{\beta}} dy = \frac{\gamma(\alpha, \frac{1}{\beta x})}{\Gamma(\alpha)} \;\; ; x = 0, 1, \ldots ; \alpha > 0, \beta > 0$$

where $\gamma(\alpha, \frac{1}{\beta x}) = \int_{0}^{1/x} \frac{1}{\beta^{\alpha}} y^{\alpha-1} e^{-\frac{y}{\beta}} dy$ and γ represents the lower incomplete gamma function,

$$P[X = x] = p(x) = R(x) - R(x+1) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_{1/(x+1)}^{1/x} y^{\alpha-1} e^{-y/\beta} dy$$

$$= \frac{\gamma(\alpha, \frac{1}{\beta x}, \frac{1}{\beta(x+1)})}{\Gamma(\alpha)} \;\; ; x = 0, 1, \ldots ; \alpha > 0, \beta > 0 \tag{1}$$

where $$\gamma\left(\alpha, \frac{1}{\beta x}, \frac{1}{\beta(x+1)}\right) = \gamma\left(\alpha, \frac{1}{\beta x}\right) - \gamma\left(\alpha, \frac{1}{\beta(x+1)}\right),$$

$$h(x) = \frac{p(x)}{R(x)} = \frac{\gamma(\alpha, \frac{1}{\beta x}, \frac{1}{\beta(x+1)})}{\gamma(\alpha, \frac{1}{\beta x})} \tag{2}$$

and

$$h_2(x) = log\left[\frac{F(x)}{F(x-1)}\right] = log\left[\frac{\Gamma(\alpha, \frac{1}{\beta x})}{\Gamma(\alpha, \frac{1}{\beta(x-1)})}\right]$$

where, α and β are shape and scale parameters.

Pundir et al. [8] obtained its reversed second rate of failure (RSRF) function and discussed its usefulness. They also discussed the classical and Bayesian inference methods for DIGD. DIGD can be applied to various applications where IGD is being used like in radar detection by Shang and Song [10], Stinco et al. [14] and in fading modeling by Yoo et al. [15], Ramirez-Epi'nosa and Lopez-Martinez [9], etc.

The current study deals with constructing a load-share parallel system model where the lifetime distribution of each component is considered as DIGD. In section 2, model description is presented. The reliability characteristics of the system are derived under section 3. In section 4, maximum likelihood (ML) estimation, bootstrapping and Bayesian estimation techniques are applied to obtain the estimates along with standard errors (SEs)/ posterior standard errors (PSEs) and confidence intervals (CIs)/ highest posterior density (HPD) credible intervals of the parameters and reliability characteristics of the load-share system model. Bayesian estimation is applied under informative as well as non-informative priors by using Markov Chain Monte Carlo (MCMC) techniques such as Gibbs sampler and Metropolis-Hastings (MH) algorithm. A simulation study is performed to compare the discussed estimation techniques and the results are given under section 5. Section 6 demonstrates the applicability of the proposed model to real data set. Finally, the article is concluded exhibiting some concluding remarks in section 7.

2. Model Description

A load-sharing system is modeled and analyzed under the following assumptions:

- A load-sharing system containing k-components is bearing a constant load which is equally shared by all its components.
- A test is prepared to record the failure times of n such i.i.d. parallel systems.
- Let T_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, k)$ be the failure time spacing between $(j-1)^{th}$ and j^{th} components in the i^{th} parallel system.
- On the successive failures of the components within the system, the load imposed on other remaining surviving components increases.
- The hazard rate of a remaining surviving component varies when the sharing load changes.
- The failure time distribution of each component in the system is independent.
- Initially, the hazard rate of each of the k-components is denoted by $h(t; \alpha, \beta_1)$. When the first component fails, the hazard rate of the remaining $(k-1)$ components changes to $h(t; \alpha, \beta_2)$ and so on. After the $(k-1)^{th}$ component failure, the hazard rate of the last surviving component changes to $h(t; \alpha, \beta_k)$.

- Each component is pertaining a failure time PMF and HRF given in equations (1) and (2), respectively.

Taking these assumptions into consideration, the hazard rate of the j^{th} component when $(j-1)$ components have already failed is

$$h(t_{ij}) = (k-j+1)h(t_{ij};\alpha,\beta_j) \ \ ; i=1,2,\dots,n; j=1,2,\dots,k$$
$$= (k-j+1)\frac{\gamma\left(\alpha,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)}\right)}{\gamma\left(\alpha,\frac{1}{\beta_j t_{ij}}\right)}$$

and the conditional failure time PMF for the j^{th} component in the i^{th} system is given by

$$p(t_{ij}) = \frac{(k-j+1)}{\Gamma(\alpha)^{(k-j+1)}}\gamma\left(\alpha,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)}\right)\left(\gamma\left(\alpha,\frac{1}{\beta_j t_{ij}}\right)\right)^{k-j} \tag{3}$$

Thus, the likelihood function for the i^{th} system is

$$L_i(t_{i1},\dots,t_{ik}|\alpha,\Lambda) = k!\prod_{j=1}^{k}\left[\frac{1}{\Gamma(\alpha)^{(k-j+1)}}\gamma\left(\alpha,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)}\right)\left(\gamma\left(\alpha,\frac{1}{\beta_j t_{ij}}\right)\right)^{k-j}\right] \tag{4}$$

where, $\Lambda = (\beta_1,\beta_2,\dots,\beta_k)$.

3. System Reliability Characteristics Measures

The reliability function of the 1-out-of-k load-share system is obtained by considering its mechanism i.e., the system works till its last component is functioning. Thus, the reliability of the system is given as

$R_s(t)$=P[at least one of k components is in operation]

=P[all k components are in operable state]
+P[any one component fails and remaining $(k-1)$ components survive]+...
+P[any $(k-1)$ components fail and last component is in operation]

$$= \left[\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{1}{\beta_1 t}\right)\right]^k + k\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{1}{\beta_1 t}\right)\right]\left[\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{1}{\beta_2 t}\right)\right]^{k-1} + \dots$$
$$+k\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{1}{\beta_1 t}\right)\right]\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{1}{\beta_2 t}\right)\right]\dots\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{1}{\beta_{k-1} t}\right)\right]\left[\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{1}{\beta_k t}\right)\right]$$

The CDF, PMF, HRF and RSRF function of the system's failure time T are respectively, given by

$$F_s(t) = 1 - R_s(t)$$
$$p_s(t) = R_s(t) - R_s(t-1)$$
$$h_s(t) = \frac{p_s(t)}{R_s(t)}$$

and

$$h_{2_s}(t) = log\frac{F_s(t)}{F_s(t-1)}.$$

4. Classical And Bayesian Inference

The likelihood function for n i.i.d. systems is obtained by using equation (4) as

$$L(T|\alpha,\Lambda) = (k!)^n \prod_{i=1}^{n}\prod_{j=1}^{k}\left[\frac{1}{\Gamma(\alpha)^{(k-j+1)}}\gamma\Big(\alpha, \frac{1}{\beta_j t_{ij}}, \frac{1}{\beta_j(t_{ij}+1)}\Big)\Big(\gamma\Big(\alpha, \frac{1}{\beta_j t_{ij}}\Big)\Big)^{k-j}\right] \quad (5)$$

and the corresponding log-likelihood function is

$$logL = nlog(k!) + \sum_{i=1}^{n}\sum_{j=1}^{k}\Bigg[-(k-j+1)\log(\Gamma(\alpha)) + \log\Big(\gamma\Big(\alpha, \frac{1}{\beta_j t_{ij}}, \frac{1}{\beta_j(t_{ij}+1)}\Big)\Big)$$
$$+(k-j)\log\Big(\gamma\Big(\alpha, \frac{1}{\beta_j t_{ij}}\Big)\Big)\Bigg] \quad (6)$$

4.1. Maximum Likelihood Estimation

The ML estimates $(\hat{\alpha}, \hat{\Lambda})$ of (α, Λ) can be obtained on solving the following (k+1) normal equations:

$$\frac{\partial logL}{\partial\alpha} = -\frac{nk(k+1)}{2}\psi(\alpha) + \sum_{i=1}^{n}\sum_{j=1}^{k}\frac{1}{\gamma(\alpha, \frac{1}{\beta_j t_{ij}}, \frac{1}{\beta_j(t_{ij}+1)})}\left[\frac{\partial}{\partial\alpha}\Big(\gamma\Big(\alpha, \frac{1}{\beta_j t_{ij}}\Big)\Big) - \frac{\partial}{\partial\alpha}\Big(\gamma\Big(\alpha, \frac{1}{\beta_j(t_{ij}+1)}\Big)\Big)\right]$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{k}(k-j)\frac{1}{\gamma(\alpha, \frac{1}{\beta_j t_{ij}})}.\frac{\partial}{\partial\alpha}\Big(\gamma\Big(\alpha, \frac{1}{\beta_j t_{ij}}\Big)\Big) = 0 \quad (7)$$

$$\frac{\partial logL}{\partial\beta_j} = \frac{-n\alpha(k-j+1)}{\beta_j} + \frac{1}{\beta_j}\sum_{i=1}^{n}\frac{\gamma(\alpha+1, \frac{1}{\beta_j t_{ij}}, \frac{1}{\beta_j(t_{ij}+1)})}{\gamma(\alpha, \frac{1}{\beta_j t_{ij}}, \frac{1}{\beta_j(t_{ij}+1)})} + \frac{(k-j)}{\beta_j}\sum_{i=1}^{n}\frac{\gamma(\alpha+1, \frac{1}{\beta_j t_{ij}})}{\gamma(\alpha, \frac{1}{\beta_j t_{ij}})} = 0 \quad (8)$$

where $\psi(\alpha) = \frac{\partial}{\partial\alpha}log(\Gamma(\alpha))$ is the digamma function and $\frac{\partial}{\partial\alpha}\gamma\Big(\alpha, \frac{1}{\beta_j t_{ij}}\Big) = \int_0^{\frac{1}{\beta_j t_{ij}}} v^{(\alpha-1)}e^{-v}log(v)dv$.

Since closed form solutions cannot be obtained from the equations (7) and (8). Therefore, any numerical iterative procedure such as Newton-Raphson method can be used here. By considering the invariance property of ML estimation, one can obtain the ML estimates of the reliability characteristics $R_s(t)$, $h_s(t)$ and $h_{2s}(t)$. The asymptotic sampling distribution of $(\hat{\alpha}-\alpha, \hat{\Lambda}-\Lambda)'$ is $N_{k+1}(0, \xi^{-1})$ with ξ representing the Fisher's information matrix whose elements are given as

$$\xi_{11} = -\frac{\partial^2 logL}{\partial\alpha^2}\Big|_{\alpha=\hat{\alpha}}$$
$$\xi_{ij} = 0; i \neq j$$
$$\xi_{1j} = -\frac{\partial^2 logL}{\partial\alpha\partial\beta_j}\Big|_{\alpha=\hat{\alpha}, \beta_j=\hat{\beta}_j}$$
$$\xi_{jj} = -\frac{\partial^2 logL}{\partial\beta_j^2}\Big|_{\beta_j=\hat{\beta}_j}$$

Using equation (6), one can obtain the second-order derivatives of the log-likelihood function as

$$\frac{\partial^2 logL}{\partial\alpha^2} = \sum_{i=1}^{n}\sum_{j=1}^{k}\left[\frac{\frac{\partial^2}{\partial\alpha^2}\gamma(\alpha,\frac{1}{\beta_j t_{ij}}) - \frac{\partial^2}{\partial\alpha^2}\gamma(\alpha,\frac{1}{\beta_j(t_{ij}+1)})}{\gamma(\alpha,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)})} - \left\{\frac{\frac{\partial}{\partial\alpha}\gamma(\alpha,\frac{1}{\beta_j t_{ij}}) - \frac{\partial}{\partial\alpha}\gamma(\alpha,\frac{1}{\beta_j(t_{ij}+1)})}{\gamma(\alpha,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)})}\right\}^2\right]$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{k}(k-j)\left[\frac{\frac{\partial^2}{\partial\alpha^2}\gamma(\alpha,\frac{1}{\beta_j t_{ij}})}{\gamma(\alpha,\frac{1}{\beta_j t_{ij}})} - \left\{\frac{\frac{\partial}{\partial\alpha}\gamma(\alpha,\frac{1}{\beta_j t_{ij}})}{\gamma(\alpha,\frac{1}{\beta_j t_{ij}})}\right\}^2\right] - \frac{nk(k+1)}{2}\psi'(\alpha)$$

$$\frac{\partial^2 logL}{\partial\alpha\partial\beta_j} = \sum_{i=1}^{n}\frac{1}{\beta_j}\left[\frac{\frac{\partial}{\partial\alpha}\gamma(\alpha+1,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)})}{\gamma(\alpha,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)})} - \frac{\gamma(\alpha+1,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)})\frac{\partial}{\partial\alpha}\gamma(\alpha,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)})}{\gamma(\alpha,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)})^2}\right]$$

$$+\sum_{i=1}^{n}\frac{(k-j)}{\beta_j}\left[\frac{\frac{\partial}{\partial\alpha}\gamma(\alpha+1,\frac{1}{\beta_j t_{ij}})}{\gamma(\alpha,\frac{1}{\beta_j t_{ij}})} - \frac{\gamma(\alpha+1,\frac{1}{\beta_j t_{ij}})\frac{\partial}{\partial\alpha}\gamma(\alpha,\frac{1}{\beta_j t_{ij}})}{\gamma(\alpha,\frac{1}{\beta_j t_{ij}})^2}\right] - \frac{n(k-j+1)}{\beta_j}$$

$$\frac{\partial^2 logL}{\partial\beta_j^2} = \frac{n(k-j+1)\alpha}{\beta_j^2} - \sum_{i=1}^{n}\frac{1}{\beta_j{}^2}\frac{\gamma(\alpha+1,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)})}{\gamma(\alpha,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)})} - \sum_{i=1}^{n}\frac{(k-j)}{\beta_j^2}\frac{\gamma(\alpha+1,\frac{1}{\beta_j t_{ij}})}{\gamma(\alpha,\frac{1}{\beta_j t_{ij}})}$$

$$+\sum_{i=1}^{n}\frac{1}{\beta_j^2}\left[\frac{\gamma(\alpha+2,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)}) - \gamma(\alpha+1,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)})}{\gamma(\alpha,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)})} - \left\{\frac{\gamma(\alpha+1,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)})}{\gamma(\alpha,\frac{1}{\beta_j t_{ij}},\frac{1}{\beta_j(t_{ij}+1)})}\right\}^2\right]$$

$$+\sum_{i=1}^{n}\frac{(k-j)}{\beta_j^2}\left[\frac{\gamma(\alpha+2,\frac{1}{\beta_j t_{ij}}) - \gamma(\alpha+1,\frac{1}{\beta_j t_{ij}})}{\gamma(\alpha,\frac{1}{\beta_j t_{ij}})} - \left\{\frac{\gamma(\alpha+1,\frac{1}{\beta_j t_{ij}})}{\gamma(\alpha,\frac{1}{\beta_j t_{ij}})}\right\}^2\right]$$

where, $\frac{\partial^2}{\partial\alpha^2}\gamma\left(\alpha,\frac{1}{\beta_j t_{ij}}\right) = \int_0^{\frac{1}{\beta_j t_{ij}}} v^{(\alpha-1)}e^{-v}(log(v))^2 dv.$

The asymptotic $100\times(1-\gamma')\%$ joint confidence ellipsoid for (α,Λ) is $(\hat{\alpha}-\alpha,\hat{\Lambda}-\Lambda)\xi^{-1}(\hat{\alpha}-\alpha,\hat{\Lambda}-\Lambda)' \le \chi^2_{(k+1),\gamma'}$, where $\chi^2_{(k+1),\gamma'}$ is the $100\times\gamma'^{th}$ percentile of χ^2-distribution with (k+1) degrees of freedom. Moreover, the asymptotic distributions of the reliability characteristics $R_s(t)$, $h_s(t)$ and $h_{2s}(t)$ are $N(0,R'\xi^{-1}R)$, $N(0,h'\xi^{-1}h)$ and $N(0,h_2'\xi^{-1}h_2)$, respectively, where,

$$R' = \left(\frac{\partial R_s(t)}{\partial\alpha},\frac{\partial R_s(t)}{\partial\Lambda}\right),\ h' = \left(\frac{\partial h_s(t)}{\partial\alpha};\frac{\partial h_s(t)}{\partial\Lambda}\right)\ and\ h_2' = \left(\frac{\partial h_{2_s}(t)}{\partial\alpha};\frac{\partial h_{2_s}(t)}{\partial\Lambda}\right).$$

4.2. Bootstrap Method

The bootstrap method is a general resampling procedure for obtaining bootstrap CIs which are an alternative to the asymptotic CIs. Two types of CIs are being used here, i.e., percentile bootstrap (boot-p) and bootstrap-t (boot-t). The procedure given in the two algorithms for boot-p and boot-t methods will be employed for obtaining the bootstrap estimates and confidence intervals for the parameters, reliability, hazard rate and RSRF functions. The algorithms for both the methods are provided in appendix A.

4.3. Bayesian Estimation Using MCMC Approach

Bayesian estimation setup usually involves generating samples from the posterior distribution and using them to summarize the knowledge about the parameters. This makes the use of prior knowledge and the available data. When the prior knowledge is not available, one can make use of the non-informative prior. Here, both the informative and non-informative priors are considered. The prior distributions regarding the load-share parameters α and β_j are taken as gamma priors as

$$g(\alpha) \propto \alpha^{a-1} e^{-\alpha/b} \ ; \alpha, a, b > 0 \tag{9}$$

and

$$h(\beta_j) \propto \beta_j^{c_j-1} e^{-\beta_j/d_j} \ ; \beta_j, c_j, d_j > 0; j = 1, 2, \ldots, k \tag{10}$$

Using the priors given in equations (9) and (10) and the likelihood function given in equation (5), the joint distribution of the parameters and the dataset is

$$K(T, \alpha, \Lambda) = L(T|\alpha, \Lambda).g(\alpha).h(\Lambda)$$

$$= (k!)^n \prod_{i=1}^{n} \prod_{j=1}^{k} \left[\frac{1}{\Gamma(\alpha)^{(k-j+1)}} \gamma\Big(\alpha, \frac{1}{\beta_j t_{ij}}, \frac{1}{\beta_j (t_{ij}+1)}\Big) \Big(\gamma\Big(\alpha, \frac{1}{\beta_j t_{ij}}\Big)\Big)^{(k-j)} \right]$$

$$\times \alpha^{a-1} e^{-\alpha/b} \times \prod_{j=1}^{k} \beta_j^{c_j-1} e^{-\beta_j/d_j}$$

Here, to obtain Bayes estimates of the parameters, the marginal posterior densities of the parameters are required which are difficult to obtain. Therefore, the use of MCMC techniques like MH algorithm and Gibbs sampler (given in appendix B) will be followed. For that, the full conditional densities of the parameters are obtained as:

$$\pi_1(\alpha|T, \Lambda) \propto \alpha^{a-1} e^{-\alpha/b} \prod_{i=1}^{n} \prod_{j=1}^{k} \frac{1}{\Gamma(\alpha)^{(k-j+1)}} \gamma\Big(\alpha, \frac{1}{\beta_j t_{ij}}, \frac{1}{\beta_j (t_{ij}+1)}\Big) \left[\gamma\Big(\alpha, \frac{1}{\beta_j t_{ij}}\Big)\right]^{(k-j)} \tag{11}$$

$$\pi_2(\beta_j|T, \alpha) \propto \beta_j^{c_j-1} e^{-\beta_j/d_j} \prod_{i=1}^{n} \gamma\Big(\alpha, \frac{1}{\beta_j t_{ij}}, \frac{1}{\beta_j (t_{ij}+1)}\Big) \left[\gamma\Big(\alpha, \frac{1}{\beta_j t_{ij}}\Big)\right]^{(k-j)} ; j = 1, 2, \ldots, k \tag{12}$$

Note that, sampling from equations (11) and (12) is not easy to be done directly because of its complexity. Therefore, samples are generated by using the MH algorithm.

5. Simulation Study

In this section, a simulation study is conducted for analyzing the estimates of the parameters of the proposed model by using classical as well as Bayesian approach. Sample data of size n = 30, 50, 100 and 200 are generated for $k = 3, \alpha = 10$, $\beta_1 = 0.001$, $\beta_2 = 0.005$ and $\beta_3 = 0.01$. In classical approach, the ML estimates of the parameters along with their SEs and asymptotic CIs are obtained. Two bootstrapping techniques are applied and bootstrap confidence intervals based on B = 2000 bootstrap replications are also computed. Using Bayesian approach, Bayes estimates of the parameters and reliability characteristics along with their PSEs and HPD intervals are obtained with informative as well as non-informative priors. For this, MCMC technique is employed using MH and Gibbs algorithms and 10,000 realizations are generated from each posterior density given in (11) and (12). The burn-in period of 500 realizations are removed. From the generated chains, every 5^{th} value is taken for removing the autocorrelation among the values. The trace plots for the parameters are plotted in Figs. (1) - (4) to ensure the fine mixing of the chains. For informative prior, gamma prior is considered for all the parameters α, β_1, β_2 and β_3 setting the hyperparameters as $\alpha = a \times b, \beta_1 = c_1 \times d_1, \beta_2 = c_2 \times d_2$ and $\beta_3 = c_3 \times d_3$. The values of all the hyperparameters are taken as approximately 0 under non-informative or Jeffrey's prior. All the discussed estimates along with their SEs/PSEs and CIs/HPD intervals are summarized in Table (1).

After performing the simulation study, the following results are observed:

- All the obtained estimates of the parameters and reliability characteristics become more precise (closer to true values) with an increase in the sample size.
- The SEs/PSEs magnitude of the estimates and the widths of all the intervals decreases on increasing the sample size.
- Bayes estimation with gamma priors is more precise in terms of true values and SEs than the Bayes estimation with Jeffrey's prior as well as ML estimation and bootstrapping methods for different sample sizes.
- Boot-p and boot-t confidence intervals are more precise than the asymptotic CIs as they contain the parameters in smaller width for all the sample sizes.
- Boot-p CIs are providing slightly shorter widths than boot-t CIs for all the samples sizes.

6. Real Data Application

In this section, the applicability of the model is discussed through a real dataset of plasma display devices (PDPs) which is supposed to be a load-sharing model by Kvam and Pena [4]. For PDPs, the degradation is measured in luminosity and a PDP is considered as failed when the luminosity goes below a threshold. A test is conducted with 20 items and 3 sensors spaced evenly across the test device. The dataset contains the failure times for 3 sensors on each of 20 test items.

For fitting a discrete distribution , the integer parts of the data values are taken into consideration. Now, to check whether these failure times can be modeled using load-sharing models, we setup the following hypothesis:
H_0: Load-sharing behavior exists in the dataset i.e., $\beta_1 = \beta_2 = \beta_3$
H_1: Load sharing behavior does not exist in the dataset i.e., $\beta_1 \neq \beta_2 \neq \beta_3$.
The hypothesis can be tested using the following criteria:

- Akaike information criterion (AIC): $-2logL + 2p$
- Bayesian information criterion (BIC): $-2logL + plogn$
- Deviance test statistic: $d_n = -2[logL_{H_0}(T|\hat{\alpha}, \hat{\beta}) - logL_{H_1}(T|\hat{\alpha}, \hat{\beta_1}, \hat{\beta_2}, \hat{\beta_3})]$

Table 1: *Estimates of parameters with their SEs/PSEs and CI/HPD width for values of* $\alpha = 10$, $\beta_1 = 0.001$, $\beta_2 = 0.005$, $\beta_3 = 0.01$, $R(50) = 0.9851$ *and* $h(50) = 0.0038$ *and* $RSRF(50) = 0.2381$ *for varying sample size.*

sample size	para-meter	ML	SE	CI width	boot-p	SE	width-p	width-t	Bayes (J)	PSE	HPD width	Bayes (I)	PSE	HPD width
30	α	8.0143	0.9961	3.9047	8.0238	0.9923	3.5534	3.4182	9.9002	0.0354	0.1291	9.9031	0.0353	0.1121
	β_1	0.0012	1.5×10^{-4}	5.8×10^{-4}	0.0012	1.3×10^{-4}	5.1×10^{-4}	5.7×10^{-4}	0.0009	6.2×10^{-5}	1.7×10^{-4}	0.0009	5.8×10^{-5}	1.6×10^{-4}
	β_2	0.0059	9.8×10^{-4}	0.0038	0.0057	8.6×10^{-4}	0.0031	0.0034	0.0049	3.7×10^{-4}	0.0012	0.0049	4.9×10^{-4}	0.0016
	β_3	0.0158	0.0021	0.0082	0.0154	0.0018	0.0066	0.0069	0.0091	0.0003	0.0009	0.0091	0.0003	0.0009
	$R(50)$	0.9992	0.0293	0.1148	0.9991	0.0111	0.0531	0.0542	0.9876	0.0107	0.0273	0.9875	0.0104	0.0305
	$h(50)$	0.0027	0.0005	0.0019	0.0030	0.0006	0.0022	0.0021	0.0032	0.0003	0.0009	0.0032	0.0003	0.0008
	$RSRF(50)$	0.2950	0.0911	0.3571	0.3093	0.0902	0.3260	0.3293	0.2437	0.0254	0.0961	0.2434	0.0242	0.0734
50	α	8.1604	0.7694	3.0161	8.1501	0.8469	2.8107	2.9038	9.9003	0.0261	0.0441	9.9128	0.0288	0.0586
	β_1	0.0012	1.5×10^{-4}	5.8×10^{-4}	0.0012	1.4×10^{-4}	5.6×10^{-4}	5.8×10^{-4}	0.0009	4.4×10^{-5}	1.3×10^{-4}	0.0009	4.7×10^{-5}	1.2×10^{-4}
	β_2	0.0063	8.0×10^{-4}	0.0032	0.0062	8.1×10^{-4}	0.0032	0.0031	0.0046	3.1×10^{-4}	9.1×10^{-4}	0.0049	2.5×10^{-4}	7.4×10^{-4}
	β_3	0.0145	0.0022	0.0086	0.0143	0.0022	0.0084	0.0082	0.0091	0.0004	0.0013	0.0093	0.0004	0.0014
	$R(50)$	0.9956	0.0179	0.0702	0.9951	0.0229	0.0941	0.0952	0.9871	0.0085	0.0237	0.9867	0.0075	0.0234
	$h(50)$	0.0029	0.0004	0.0015	0.0031	0.0005	0.0018	0.0016	0.0032	0.0002	0.0005	0.0033	0.0002	0.0005
	$RSRF(50)$	0.2784	0.0467	0.1831	0.2777	0.0458	0.1577	0.1558	0.2433	0.0182	0.0526	0.2429	0.0195	0.0563
100	α	8.2093	0.6708	2.6295	8.2876	0.6128	2.5204	2.6778	9.9011	0.0246	0.0527	9.9207	0.0212	0.0357
	β_1	0.0011	9.1×10^{-5}	3.5×10^{-4}	0.0011	8.1×10^{-5}	3.1×10^{-4}	3.4×10^{-4}	0.0009	1.5×10^{-5}	5.7×10^{-5}	0.0009	2.2×10^{-5}	6.4×10^{-5}
	β_2	0.0057	5.2×10^{-4}	0.0021	0.0057	5.0×10^{-4}	0.0019	0.0021	0.0048	1.6×10^{-4}	4.9×10^{-4}	0.0049	2.0×10^{-4}	5.9×10^{-4}
	β_3	0.0132	0.0013	0.0051	0.0132	0.0012	0.0043	0.0049	0.0092	0.0003	0.0012	0.0096	0.0003	0.0011
	$R(50)$	0.9985	0.0162	0.0635	0.9983	0.0081	0.0341	0.0374	0.9875	0.0035	0.0105	0.9873	0.0023	0.0086
	$h(50)$	0.0031	0.0003	0.0011	0.0032	0.0003	0.0012	0.0010	0.0034	5.3×10^{-5}	0.0002	0.0035	7.1×10^{-5}	0.0002
	$RSRF(50)$	0.2446	0.0391	0.1532	0.2481	0.0353	0.1261	0.1284	0.2422	0.0091	0.0268	0.2414	0.0064	0.0233
200	α	8.3819	0.3894	1.5264	8.6394	0.3531	1.3197	1.1781	9.9014	0.0137	0.0327	9.9374	0.0141	0.0131
	β_1	0.0011	7.2×10^{-5}	2.8×10^{-4}	0.0011	7.3×10^{-5}	2.6×10^{-4}	2.5×10^{-4}	0.0009	9.3×10^{-6}	2.5×10^{-5}	0.0009	5.6×10^{-6}	2.1×10^{-5}
	β_2	0.0055	3.9×10^{-4}	0.0015	0.0054	4.3×10^{-4}	0.0018	0.0016	0.0049	1.1×10^{-4}	2.6×10^{-4}	0.0049	1.0×10^{-4}	2.5×10^{-4}
	β_3	0.0118	0.0010	0.0039	0.0114	0.0010	0.0037	0.0037	0.0102	0.0002	0.0007	0.0101	0.0001	0.0004
	$R(50)$	0.9969	0.0092	0.0361	0.9969	0.0091	0.0347	0.0362	0.9868	0.0008	0.0029	0.9865	0.0015	0.0039
	$h(50)$	0.0035	0.0002	0.0008	0.0035	0.0002	0.0008	0.0007	0.0035	1.8×10^{-5}	0.0001	0.0036	2.3×10^{-5}	0.0001
	$RSRF(50)$	0.2426	0.0126	0.0494	0.2452	0.0135	0.0498	0.0468	0.2415	0.0039	0.0106	0.2411	0.0023	0.0082

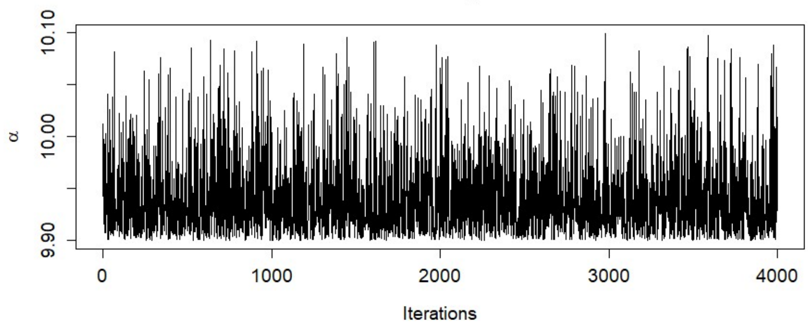

Figure 1: *Trace plot of* α.

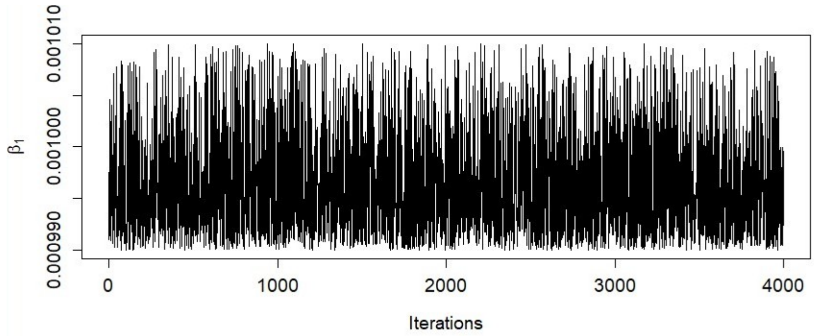

Figure 2: *Trace plot of* β_1.

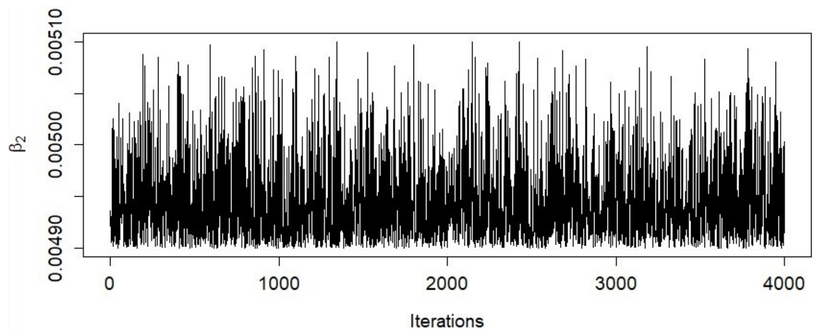

Figure 3: *Trace plot of* β_2.

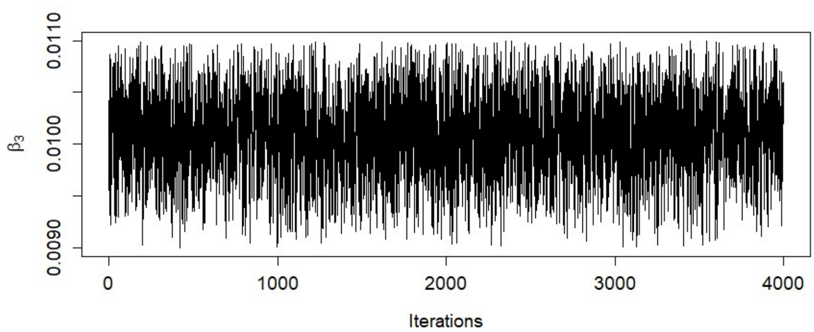

Figure 4: *Trace plot of* β_3.

The fitting summary of the dataset under both the models is provided in Table 2. The observed value of deviance test statistic d_n is 9.0385 with corresponding p-value 0.0108 (< 0.05) which suggests that H_0 cannot be accepted at 5% level of significance. The same is suggested by comparing AIC and BIC under H_0 and H_1. Hence, it is concluded that load-sharing behavior exists in the considered dataset.

Table 2: *Fitting summary of PDP dataset*

Model	-Log L	AIC	BIC	d_n
H_0	472.8236	949.6473	951.6388	9.0385
H_1	468.3044	944.6087	948.5917	

7. Concluding Remarks

In this research, a multi-component load-share parallel system is analyzed by assuming that the underlying failure time distribution of each component is DIGD. The classical as well as Bayesian estimation techniques are applied for estimating the parameters of the system. It is assumed that on a component's failure, the total workload imposed on the system will be redistributed to the other working components and this will affect their performance. Such systems exist in many engineering applications like fiber composites, power plants, manufacturing and many more. However, the study can be extended by considering non-identical components where each component is having different loads. Also, generalized IGD can be adopted instead of DIGD in future researches.

Conflict of interest: The authors declare that there is no conflict of interest.

References

[1] Daniels, H. E. (1945). The statistical theory of the strength of bundles of threads. I. *Proceedings Of The Royal Society Of London. Series A. Mathematical And Physical Sciences*, 183(995):405-435.

[2] Hussain, T. and Ahmad, M. (2012). Discrete inverse Gamma distribution. *Official Statistics And Its Impact On Good Governance And Economy Of Pakistan', 9th International Conference On Statistical Sciences, Islamic Countries Society Of Statistical Sciences, Lahore, Pakistan*, 381-394.

[3] Kim, H. and Kvam, P. H. (2004). Reliability estimation based on system data with an unknown load share rule. *Lifetime Data Analysis*, 10(1):83-94.

[4] Kvam, P. H. and Pena, E. A. (2005). Estimating load-sharing properties in a dynamic reliability system. *Journal Of The American Statistical Association*, 100(469):262-272.

[5] Park, C. (2010). Parameter estimation for the reliability of load-sharing systems. *IIE Transactions*, 42(10):753-765.

[6] Park, C. (2013). Parameter estimation from load-sharing system data using the expectation–maximization algorithm. *IIE Transactions*, 45(2):147-163.

[7] Pundir, P. S. and Gupta, P. K. (2018). Reliability estimation in load-sharing system model with application to real data. *Annals Of Data Science*, 5(1):69-91.

[8] Pundir, P. S., Srivastava, R., Agarwal, S. and Shrivastava, V. (2022). On the estimation problems for discrete inverse gamma lifetime model with real data applications. *Life Cycle Reliability And Safety Engineering*, 11(4):313-321.

[9] Ramírez-Espinosa, P. and Lopez-Martinez, F. J. (2021). Composite fading models based on inverse gamma shadowing: Theory and validation. *IEEE Transactions on Wireless Communications*, 20(8):5034-5045.

[10] Shang, X. and Song, H. (2011). Radar detection based on compound-Gaussian model with inverse gamma texture. *IET Radar, Sonar & Navigation*, 5(3):315-321.

[11] Singh, B. and Goel, R. (2019). MCMC estimation of multi-component load-sharing system model and its application. *Iranian Journal Of Science And Technology, Transactions A: Science*, 43(2):567-577.

[12] Singh, B., Sharma, K. and Kumar, A. (2008). A classical and Bayesian estimation of a k-components load-sharing parallel system. *Computational Statistics & Data Analysis*, 52(12):5175-5185.

[13] Singh, B., Sharma, K. and Kumar, A. (2009). Analyzing the dynamic system model with discrete failure time distribution. *Statistical Methods And Applications*, 18(4):521-542.

[14] Stinco, P., Greco, M. and Gini, F. (2011). Adaptive detection in compound-Gaussian clutter with inverse-gamma texture. *Proceedings Of 2011 IEEE CIE International Conference On Radar*, 1:434-437.

[15] Yoo, S. K., Bhargav, N., Cotton, S. L., Sofotasios, P. C., Matthaiou, M., Valkama, M. and Karagiannidis, G. K. (2017). The $\kappa - \mu$/inverse gamma and $\eta - \mu$/inverse gamma composite fading models: Fundamental statistics and empirical validation. *IEEE Transactions On Communications*, 69(8):5514-5530.

[16] Zhang, Z., Yang, Y. and Li, D. (2022). Estimation of parameters for load-sharing parallel systems under exponential Pareto distribution. *Proceedings Of The Institution Of Mechanical Engineers, Part O: Journal Of Risk And Reliability*, 236(2):248-255.

Appendix A

Boot-p Method

1. Generate a sample $T_{ij}(i = 1, 2, \ldots, n; j = 1, 2, \ldots, k)$ of size n by using equation (3).

2. Now, regenerate B bootstrap samples $T^*_{ij}(i = 1, 2, \ldots, n; j = 1, 2, \ldots, k)$ of size n from the original sample T_{ij} to compute B bootstrap estimates $(\hat{\alpha}^*, \hat{\Lambda}^*) \equiv \hat{\Theta}^*$ of $(\alpha, \Lambda) \equiv \Theta$.

3. Let $\hat{\Theta}^*_{(1)}, \ldots, \hat{\Theta}^*_{(B)}$ be the ordered statistics of the estimates $\hat{\Theta}^*_1, \ldots, \hat{\Theta}^*_B$. Then, $100 \times (1-\gamma)\%$ boot-p CI is: $(\hat{\Theta}^*[\gamma B/2], \hat{\Theta}^*[1-\gamma B/2])$

4. Finally, the bootstrap estimates and their corresponding variances are obtained.

Boot-t Method

1. On the basis of the genereted sample T^*_{ij}, compute the following pivots as:

$$\kappa_1^* = \frac{\hat{\alpha}^* - \hat{\alpha}}{\sqrt{\hat{V}(\hat{\alpha}^*)}} \qquad and \qquad \kappa_2^* = \frac{\hat{\Lambda}^* - \hat{\Lambda}}{\sqrt{\hat{V}(\hat{\Lambda}^*)}}$$

2. Now, repeat the step 1, B-times.

3. Consider $S_1(x) = P(\kappa_1^* \leq x)$ and $S_2(x) = P(\kappa_2^* \leq x)$ as the CDFs of κ_1^* and κ_2^*, respectively. Let for a given value x,
$\hat{\alpha}_{boot-t}(x) = \hat{\alpha}^* + \sqrt{\hat{V}(\hat{\alpha}^*)}S_1^{-1}(x)$ and $\hat{\Lambda}_{boot-t}(x) = \hat{\Lambda} + \sqrt{\hat{V}(\hat{\Lambda})}S_2^{-1}(x)$.

4. The $100 \times (1-\gamma)\%$ boot-t CIs for α and Λ are
$\hat{\alpha}_{boot-t}(\gamma/2), \hat{\alpha}_{boot-t}(1-\gamma/2)$ and $\hat{\Lambda}_{boot-t}(\gamma/2), \hat{\Lambda}_{boot-t}(1-\gamma/2)$.

Appendix B

Gibbs Algorithm

1. Generate α and β_j $(j = 1, 2, \ldots, k)$ from $\pi_1(\alpha|T, \Lambda)$ and $\pi_2(\beta_j|T, \alpha)$ as given in equations (11) and (12), respectively.

2. Repeat the above step, M times. To remove the effect of starting values, record the generated sequence of the parameters after some $N(< M)$ burn-in draws.

3. Bayes estimates and their corresponding posterior variances of the parameters as well as system reliability function, hazard rate function and RSRF function are computed by considering the means and variances of the generated values of the parameters and these three reliability characteristics.

4. Now, considering the ordered sequence of the parameters and reliability characteristics, the $100 \times (1-\gamma)\%$ HPD intervals are constructed.

Metropolis-Hastings Algorithm

1. Start with an initial value x_0 from the support of the prior distribution and consider $i = 1$.

2. Now, generate a proposal x_{prop} by using the proposal density q($x_i|x_{i-1}$).

3. Calculate the acceptance probability as

$$P_\alpha(x_{prop}|x_{i-1}) = min\left[1, \frac{q(x_{i-1}|x_{prop})f(x_{prop})}{q(x_{prop}|x_{i-1})f(x_{i-1})}\right]$$

4. Generate a random variable U from uniform distribution on $(0, 1)$.

5. The proposal point will be accepted if $u < P_\alpha$ by considering $x_i = x_{prop}$, otherwise, reject it and set $x_i = x_{i-1}$.

6. Set $i = i + 1$ and return to step 2.