Analysis of $M, MAP/PH_1, PH_2/1$ non-preemptive priority Queueing model with Delayed working vacations, immediate feedback,impatient customer, differentiate breakdown and phase type repair

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Abstract

The arrival of high priority customers is governed by the Poisson process while that of low priority customers is governed by the Markovian Arrival Process, and the service times are determined by a distinct Phase-type distribution. When the service is finished and the system is empty, the server stays idle for a random period (delay time). If a customer arrives within the delayed period, the server resumes normal service to the customer immediately. Otherwise, at the end of the delayed period, the server will take a working vacation and will instantly provide slow service to customers (high priority customers only). The Matrix analytic method is used to investigate the system. We also discussed the steady-state vector and busy period for our concept. The estimated and visually displayed performance measures of the system

Keywords: Non-preemptive Priority, Working vacation policy, Phase-type repair, Immediate feedback, Differentiate breakdown, Delay time.

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1. Introduction

For the past two decades, the priority queue hypothesis has been used in communication strategies. Because priority does not come under FCFO, it distinguishes it from a normal queue. It is a special type of queue in which each customer is dealt with priority and served according to its priority. There are two different types of priority service available in a queueing system: preemptive and non-preemptive. Priority customers that arrive early will wait until the service is finished while regular customers are serviced. This belongs under the non-preemptive priority rule. In the event of a preemptive rule, high-priority consumers would frequently interrupt low-priority service.

Ayyappan et al. [21] looked at $M/M/1$ for retrials, with negative arrival while using non-preemptive priority service. Bhagat and Jain [5] described a multi-server, non-preemptive priority service that is susceptible to failure and maintenance. According to Jeganathan et al. [9], the inventory system and non-preemptive priority service for retrials have been discussed. Additionally, discretionary priority service is utilized, taking into account both disciplines. Ayyappan and Somasundaram [3] analyzed discretionary priority service for retrials used $M^{X1}, M^{X2}/G1, G2/1$.

Krishnamoorthy and Divya [13] examined queueing models with MAP and PH distributions, as well as working vacations under N-policy.

In many real-world queueing situations, the server can be seen working during its rest period if necessary. Working vacation means that the server offers service at a lower rate throughout the vacation period rather than entirely shutting down. In the past few decades have seen, queueing systems with server working vacation, owing to similarities between telecommunication system, manufacturing system, and computer system. Yang et al. [22] applied the spectral expansion method to deal with a single server queueing model with delayed working vacations and working breakdown: The author showed the steady-state probability vector, LST of sojourn time, and expected sojourn time. After service completion, the server is idle when there are no customers in the system for a certain amount of time (changeover time) (Pikkala et al. [19], Krishna Reddy and Anitha [11]). The server begins offering service if customers access the system during changeover time; if not, the server goes on vacation at the end of the changeover time.

After obtaining service from the server, customers may be satisfied or unsatisfied. Customers who are satisfied with the system will leave, while those who are not satisfied will get feedback right away. A single server model with starting failures, standby server, single vacation, delayed repair, breakdown, immediate feedback, and impatient customers was extensively analyzed by Ayyappan and Thilagavathy [1], who found the expected results for both the system size and orbit size. In their 2008 study, Badamchi Zadeh and Shahkar [4] examined queuing systems that included two phases of heterogeneous service, optional second service, and feedback for each service. In contrast to the current study, when services are parallel, they had sequential services during their studies. Afterward, performance measures for the Poisson arrival queuing system and probability-generating functions are obtained under the assumption of exponential service times. Ayyappan and Thilagavathy [2] explored closedown, breakdown and multiple vacation used $MAP/PH/1$.

When the system is inactive or when a customer is being served, random failures can happen. The terms "hard failure" and "soft failure" refer to two different kinds of system failure. Hard failure's typically takes a long period and needs the repairman's actual presence. On the other side, soft failure's takes less time because the system may be recovered with a simple reboot. Markovian queueing models with two different forms of server breakdown have already been studied by Jain and Jain (2010) [7], Kalyanaraman (2019) [10], Krishna Kumar(2008)[12], Li (2013) [16], and many others. Using the matrix geometric technique, stability conditions for a single server infinite capacity Markovian queue were obtained. According to Janani [8], the final value theorem of the Laplace transform is used to convert the transient state probabilities of the model into steady-state probabilities.

When customers abandon the line because they have waited too long for service, they are considered impatient customers. Kumar [14] investigated a non-Markovian queue with an unreliable server that first provides an essential service and then one of the m optional services. He has described the balking techniques as well as cost analysis for the objective of model optimization. A single server queueing system with associated reneging, feedback, and balking was investigated by Rakesh Kumar and Soodan [20]. We explored the time-dependent behavior of the model using the Runge-Kutta method. Additionally, they discovered the average waiting time and system size. In modeling, the arrival using a Markovian Arrival Process, a particular type of Versatile Markovian Point Process was proposed by Neuts [18]. Lucantoni et al. [17], with considerable VMPP as BMAP notational simplifications since it started in 1990. Due to its ability to simulate a broad spectrum of real-world events, MAP is an effective point process in stochastic modeling. Chakravarthy [6], describes two parameter matrices of m dimensions, let's say D_0 and D_1. Transitions in the MAP are determined by the generator matrix $D = D_0 + D_1$.

2. Model Formulation

Within this part, our focus is on a system for queueing with a single server, utilizing non-preemptive priority. Customers categorized as high priority (HP) arrive through a Poisson process with rate denoted by λ_2, while low priority (LP) customers arrive via a Markovian Arrival Process represented by (D_0, D_1) of order m. The matrix D_0 means no arrival LP customer, while the matrix D_1 depicts LP customer arrival. HP customers have a limited capacity of K size, while LP customers have unlimited capacity. The fundamental arrival rate, denoted as λ_1, is equivalent to $\pi_1 D_1 e$, where π_1 represents the stationary probability vector. A customer categorized as HP is assumed to have a service time that follows a phase-type distribution with the notation (γ, U) of order n, while an LP customer's service time is assumed to follow a phase-type distribution with the notation (γ', U') of order n'.

Upon completion of the service, if no customer is in the system, then the server will remain inactive for a random duration. That time is referred to as the delayed period. The delayed period follows an exponential distribution with parameter ω. when a customer arrives during the delayed period, prompt resumption of regular service is initiated by the server. However, if the delayed period ends and any customer does not arrive, the server will proceed on a working vacation. The vacation period is generated by an exponentially distributed parameter η. HP customers who arrive during this period will be served at a lower service rate and it is followed by phase-type distribution with representation $(\gamma, \theta U)$, where $0 < \theta < 1$. As such, the mean service rate in normal mode is $\mu_1 = [\gamma(-U)^{-1}e]^{-1}$, and the vacation mode of service rate is $\theta\mu_1$.

After completion of service for HP customer during working vacation, if there exists no HP customer awaiting service, then the server will doemant in vacation mode, irrespective of the presence of LP customers in the system. After the expiration of the vacation clock during a WV, the server shall revert to its normal working mode. At the end of vacation period, LP customers shall be considered for service during no HP customer present in the system. The expected service rate of an LP customer is denoted by $\mu_2 = [\gamma'(-U')^{-1}e]^{-1}$.

The server is affected by soft failure(short time) during idle period and hard failure (long time) during normal busy period (both HP and LP customers). The rates of soft and hard failure are exponentially distributed with parameters ψ_1 and ψ_2. When a soft and hard failure happens, the server repair process starts immediately. The customer who is receiving service at that point must join the front of the waiting queue. If there are any customers in line when the repair is finished, the server will start servicing them. Or else, the server remains idle and repair times follows a phase-type distribution (α, T) of order l for soft failure, where $T^0 + Te = 0$ and (α', T') of order l' for hard failure, where $T'^0 + T'e = 0$. The repair rate is indicated as $\tau_1 = [\alpha(-T)^{-1}e]^{-1}$ and $\tau_2 = [\alpha'(-T')^{-1}e]^{-1}$ respectively.

The arriving LP Customers may balk the system with probability b during working vacation or join the system with probability $(1 - b)$. After receiving normal service (both HP and Lp customers), the satisfied customer leave the system with probability p_1 and if the customer is not satisfied with probability q_1 then they will get feedback immediately.

3. The QBD process infinitesimal generation matrix

Notations

We will need the following notations:

- $\otimes$ -Kronecker product of two matrices of various dimensions resulting in a block matrix.
- $\oplus$ - Kronecker sum of two matrices of various dimensions resulting in a block matrix.
- I_m stand for identity matrix of m x m order.
- e - a column vector of the suitable order. Each of its entries is one.

- e_0=$e_{3m+2Knm+Kl'm+(K+1)lm}$.
- e_1=$e_{Kmn+(K+1)n'm+(K+1)l'm+(K+1)lm+Kmn+m}$.
- $N_1(t)$: the total number of LP customers in the system at epoch t.
- $N_2(t)$: the total number of HP customers in the system at epoch t.
- $J(t)$ represents the server's status at epoch t.
  As a result, the server is in one of the following states at any given time t:

$$J(t) = \begin{cases} 0, & \text{idle during normal mode,} \\ 1, & \text{if the server is offering service to HP customers during normal mode,} \\ 2, & \text{if the server is offering service to LP customers during normal mode,} \\ 3, & \text{hard failure (during normal busy mode),} \\ 4, & \text{delay time,} \\ 5, & \text{soft failure (during idle),} \\ 6, & \text{busy(HP) in working vacation mode,} \\ 7, & \text{idle in working vacation mode.} \end{cases}$$

- $R(t)$ stands for the repair process considered by phases.
- $K(t)$ stands for phases of the service.
- $A(t)$- The Markovian arrival process is considered in phases.
- Let Y=$\{Y(t) : t \geq 0\}$, where $Y(t) = \{N_1(t), N_2(t), J(t), R(t), K(t), A(t)\}$ is a $CTMC$ with state space

$$\Phi = \phi(0) \bigcup_{i=1}^{\infty} \phi(i). \tag{1}$$

where

$$\begin{aligned} \phi(0) =& \{(0,0,0,a) : \ 1 \leq a \leq m\} \cup \{(0,r,1,k_1,a) : \ 1 \leq r \leq K, \ 1 \leq k_1 \leq n, \ 1 \leq a \leq m\} \\ & \cup \{(0,r,3,k_4,a) : \ 1 \leq r \leq K, \ 1 \leq k_4 \leq l', \ 1 \leq a \leq m\} \cup \{(0,0,4,a) : \ 1 \leq a \leq m\} \\ & \cup \{(0,0,5,k_3,a) : \ 0 \leq r \leq K, \ 1 \leq k_3 \leq l, \ 1 \leq a \leq m\} \\ & \cup \{(0,1,6,k_1,a) : \ 1 \leq r \leq K, \ 1 \leq k_1 \leq n, \ 1 \leq a \leq m\} \cup \{(0,0,7,a) : \ 1 \leq a \leq m\}, \end{aligned}$$

and for $i \geq 1$,

$$\begin{aligned} \phi(i) =& \{(i,r,1,k_1,a) : \ 1 \leq r \leq K, \ 1 \leq k_1 \leq n, \ 1 \leq a \leq m\} \\ & \cup \{(i,r,2,k_2,a) : \ 0 \leq r \leq K, \ 1 \leq k_2 \leq n', \ 1 \leq a \leq m\} \\ & \cup \{(i,r,3,k_4,a) : \ 0 \leq r \leq K, \ 1 \leq k_4 \leq l', \ 1 \leq a \leq m\} \\ & \cup \{(i,r,5,k_3,a) : \ 0 \leq r \leq K, \ 1 \leq k_3 \leq l, \ 1 \leq a \leq m\} \\ & \cup \{(i,r,6,k_1,a) : \ 1 \leq r \leq K, \ 1 \leq k_1 \leq n, \ 1 \leq a \leq m\} \cup \{(i,0,7,a) : \ 1 \leq a \leq m\}. \end{aligned}$$

3.1. The Infinitesimal Generator Matrix

The quasi-birth–death process has the generator matrix Q given by

$$Q = \begin{bmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \dots \\ B_{10} & A_1 & A_0 & 0 & 0 & 0 & \dots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \dots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \dots \end{bmatrix} \tag{2}$$

$$B_{00} = \begin{bmatrix} B_{00}^{11} & B_{00}^{12} & 0 & 0 & B_{00}^{15} & 0 & 0 \\ 0 & B_{00}^{22} & B_{00}^{23} & B_{00}^{24} & 0 & 0 & 0 \\ 0 & B_{00}^{32} & B_{00}^{33} & 0 & 0 & 0 & 0 \\ 0 & B_{00}^{42} & 0 & B_{00}^{44} & 0 & 0 & B_{00}^{47} \\ B_{00}^{51} & B_{00}^{52} & 0 & 0 & B_{00}^{55} & 0 & 0 \\ 0 & B_{00}^{62} & 0 & 0 & 0 & B_{00}^{66} & B_{00}^{67} \\ B_{00}^{71} & 0 & 0 & 0 & 0 & B_{00}^{76} & B_{00}^{77} \end{bmatrix},$$

where
$B_{00}^{11} = D_0 - (\lambda_2 + \psi_1)I_m, \quad B_{00}^{12} = e_1' \otimes \alpha \otimes \lambda_2 I_m, \quad B_{00}^{15} = e_1'(K+1) \otimes \alpha \otimes \psi_1 I_m,$

$$B_{00}^{22} = \begin{bmatrix} L_1 & L_2 & 0 & \dots & 0 & 0 \\ L_3 & L_1 & L_2 & \dots & 0 & 0 \\ 0 & L_3 & L_1 & \dots & 0 & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & L_1 & L_2 \\ 0 & 0 & 0 & \dots & L_3 & L_1 + L_2 \end{bmatrix}, \ B_{00}^{23} = I_K \otimes e_n \otimes \alpha' \otimes \psi_2 I_m, \quad B_{00}^{24} = e_1 K \otimes qU^0 \otimes I_m,$$

where $L_1 = (U + pU^0\gamma) \oplus D_0 - (\lambda_2 + \psi_2)I_{nm}, \quad L_2 = \lambda_2 I_{nm}, \quad L_3 = qU^0\gamma \otimes I_m.$

$$B_{00}^{32} = I_K \otimes T'^0\gamma \otimes I_m, \ B_{00}^{33} = \begin{bmatrix} L_4 & L_5 & 0 & \dots & 0 & 0 \\ 0 & L_4 & L_5 & \dots & 0 & 0 \\ 0 & 0 & L_4 & \dots & 0 & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & L_4 & L_5 \\ 0 & 0 & 0 & \dots & 0 & L_4 + L_5 \end{bmatrix},$$

where $L_4 = T' \oplus D_0 - \lambda_2 I_{l'm}, \quad L_5 = \lambda_2 I_{l'm}.$
$B_{00}^{42} = e_1' K \otimes \alpha \otimes \lambda_2 I_m, \quad B_{00}^{44} = D_0 - (\lambda_2 + \omega)I_m, \quad B_{00}^{47} = \omega I_m, \quad B_{00}^{51} = e_1(K+1) \otimes T^0 \otimes I_m,$

$$B_{00}^{52} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ T^0\gamma \otimes I_m & 0 & 0 & \dots & 0 & 0 \\ 0 & T^0\gamma \otimes I_m & 0 & \dots & 0 & 0 \\ 0 & 0 & T^0\gamma \otimes I_m & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & T^0\gamma \otimes I_m & 0 \\ 0 & 0 & 0 & \dots & 0 & T^0\gamma \otimes I_m \end{bmatrix},$$

$$B_{00}^{55} = \begin{bmatrix} L_6 & L_7 & 0 & \dots & 0 & 0 \\ 0 & L_6 & L_7 & \dots & 0 & 0 \\ 0 & 0 & L_6 & \dots & 0 & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & L_6 & L_7 \\ 0 & 0 & 0 & \dots & 0 & L_6 + L_7 \end{bmatrix}, \ B_{00}^{66} = \begin{bmatrix} L_8 & L_9 & 0 & \dots & 0 & 0 \\ L_{10} & L_8 & L_9 & \dots & 0 & 0 \\ 0 & L_{10} & L_8 & \dots & 0 & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & L_8 & L_9 \\ 0 & 0 & 0 & \dots & L_{10} & L_8 + L_9 \end{bmatrix},$$

where $L_6 = T \oplus D_0 - \lambda_2 I_{lm}, \ L_7 = \lambda_2 I_{lm}, \ L_8 = \theta U \oplus (D_0 + bD_1) - (\eta + \lambda_2)I_{nm}, \ L_9 = \lambda_2 I_{nm},$ $L_{10} = \theta U' \otimes I_m.$
$B_{00}^{62} = I_K \otimes e_n \otimes \eta\gamma \otimes I_m, \ B_{00}^{67} = e_1 K \otimes \theta U^0 \otimes I_m, \quad B_{00}^{71} = \eta I_m, \quad B_{00}^{76} = e_1' K \otimes \gamma \otimes \lambda_2 I_m,$
$B_{00}^{77} = (D_0 + bD_1) - (\eta + \lambda_2)I_m.$

$$B_{01} = \begin{bmatrix} 0 & B_{01}^{12} & 0 & 0 & 0 & 0 \\ B_{01}^{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{01}^{33} & 0 & 0 & 0 \\ 0 & B_{01}^{42} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{01}^{54} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{01}^{65} & 0 \\ 0 & 0 & 0 & 0 & 00 & B_{01}^{76} \end{bmatrix},$$

where
$B_{01}^{12} = e_1'(K+1) \otimes \gamma' \otimes D_1, \quad B_{01}^{21} = I_K \otimes I_n \otimes D_1,$

$$B_{01}^{33} = \begin{bmatrix} 0 & I_{l'} \otimes D_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & I_{l'} \otimes D_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & I_{l'} \otimes D_1 & \dots & 0 & 0 \\ & & & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \dots & I_{l'} \otimes D_1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & I_{l'} \otimes D_1 \end{bmatrix}, \; B_{01}^{42} = e_1'(K+1) \otimes \gamma' \otimes D_1,$$

$B_{01}^{54} = I_{K+1} \otimes I_l \otimes D_1, \quad B_{01}^{65} = I_K \otimes I_n \otimes (1-b)D_1, \quad B_{01}^{76} = (1-b)D_1.$

$$B_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{10}^{22} & 0 & B_{10}^{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$B_{10}^{22} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ qU'^0\gamma \otimes I_m & 0 & 0 & \dots & 0 & 0 \\ 0 & qU'^0\gamma \otimes I_m & 0 & \dots & 0 & 0 \\ 0 & 0 & qU'^0\gamma \otimes I_m & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & qU'^0\gamma \otimes I_m & 0 \\ 0 & 0 & 0 & \dots & 0 & qU'^0\gamma \otimes I_m \end{bmatrix},$$

$B_{10}^{24} = e_1(K+1) \otimes qU'^0 \otimes I_m.$

$$B_{11} = \begin{bmatrix} B_{11}^{11} & B_{11}^{12} & B_{11}^{13} & 0 & 0 & 0 \\ 0 & B_{11}^{22} & B_{11}^{23} & 0 & 0 & 0 \\ B_{11}^{31} & B_{11}^{32} & B_{11}^{33} & 0 & 0 & 0 \\ B_{11}^{41} & B_{11}^{42} & 0 & B_{11}^{44} & 0 & 0 \\ B_{11}^{51} & 0 & 0 & 0 & B_{11}^{55} & B_{11}^{56} \\ 0 & B_{11}^{62} & 0 & 0 & B_{11}^{65} & B_{11}^{66} \end{bmatrix},$$

where

$$B_{11}^{11} = \begin{bmatrix} L_1 & L_2 & 0 & \dots & 0 & 0 \\ L_3 & L_1 & L_2 & \dots & 0 & 0 \\ 0 & L_3 & L_1 & \dots & 0 & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & L_1 & L_2 \\ 0 & 0 & 0 & \dots & L_3 & L_1 + L_2 \end{bmatrix}, \quad B_{11}^{12} = \begin{bmatrix} qU^0\gamma' \otimes I_m & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

$$B_{11}^{13} = \begin{bmatrix} 0 & e_n \otimes \psi_2\alpha' \otimes I_m & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & e_n \otimes \psi_2\alpha' \otimes I_m & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & e_n \otimes \psi_2\alpha' \otimes I_m & \dots & 0 & 0 \\ & & & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \dots & e_n \otimes \psi_2\alpha' \otimes I_m & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & e_n \otimes \psi_2\alpha' \otimes I_m \end{bmatrix},$$

$$B_{11}^{22} = \begin{bmatrix} L_{11} & L_{12} & 0 & \dots & 0 & 0 \\ 0 & L_{11} & L_{12} & \dots & 0 & 0 \\ 0 & 0 & L_{11} & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & L_{11} & L_{12} \\ 0 & 0 & 0 & \dots & 0 & L_{11}+L_{12} \end{bmatrix}, \quad B_{11}^{23} = I_{K+1} \otimes e_n' \otimes \psi_2\alpha' \otimes I_m,$$

where $L_{11} = (U' + pU'^0\gamma_1) \oplus D_0 - (\lambda_2 + \psi_2)I_{n'm}, \quad L_{12} = \lambda_2 I_{n'm}.$

$$B_{11}^{31} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ T'^0\gamma \otimes I_m & 0 & 0 & \dots & 0 & 0 \\ 0 & T'^0\gamma \otimes I_m & 0 & \dots & 0 & 0 \\ 0 & 0 & T'^0\gamma \otimes I_m & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & T'^0\gamma \otimes I_m & 0 \\ 0 & 0 & 0 & \dots & 0 & T'^0\gamma \otimes I_m \end{bmatrix},$$

$$B_{11}^{32} = \begin{bmatrix} T'^0\gamma' \otimes I_m & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad B_{11}^{33} = \begin{bmatrix} L_4 & L_5 & 0 & \dots & 0 & 0 \\ 0 & L_4 & L_5 & \dots & 0 & 0 \\ 0 & 0 & L_4 & \dots & 0 & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & L_4 & L_5 \\ 0 & 0 & 0 & \dots & 0 & L_4+L_5 \end{bmatrix},$$

$$B_{11}^{41} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ T^0\gamma \otimes I_m & 0 & 0 & \dots & 0 & 0 \\ 0 & T^0\gamma \otimes I_m & 0 & \dots & 0 & 0 \\ 0 & 0 & T^0\gamma \otimes I_m & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & T^0\gamma \otimes I_m & 0 \\ 0 & 0 & 0 & \dots & 0 & T^0\gamma \otimes I_m \end{bmatrix},$$

$$B_{11}^{42} = \begin{bmatrix} T^0\gamma' \otimes I_m & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, B_{11}^{44} = \begin{bmatrix} L_6 & L_7 & 0 & \dots & 0 & 0 \\ 0 & L_6 & L_7 & \dots & 0 & 0 \\ 0 & 0 & L_6 & \dots & 0 & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & L_6 & L_7 \\ 0 & 0 & 0 & \dots & 0 & L_6+L_7 \end{bmatrix},$$

$B_{11}^{51} = I_K \otimes e_n \otimes \eta\gamma \otimes I_m,$

$$B_{11}^{55} = \begin{bmatrix} L_8 & L_9 & 0 & \dots & 0 & 0 \\ L_{10} & L_8 & L_9 & \dots & 0 & 0 \\ 0 & L_{10} & L_8 & \dots & 0 & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & L_8 & L_9 \\ 0 & 0 & 0 & \dots & L_{10} & L_8 + L_9 \end{bmatrix}, \quad B_{11}^{56} = e_1 K \otimes \theta U^0 \otimes I_m,$$

$B_{11}^{62} = e_1'(K+1) \otimes \eta\gamma' \otimes I_m, \quad B_{11}^{65} = e_1'(K) \otimes \alpha \otimes \lambda_2 I_m, B_{11}^{66} = (D_0 + bD_1) - (\eta + \lambda_2) I_m.$

$$B_{12} = \begin{bmatrix} B_{12}^{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{12}^{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{12}^{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{12}^{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{12}^{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{12}^{66} \end{bmatrix},$$

where
$B_{12}^{11} = I_K \otimes I_n \otimes D_1, \quad B_{12}^{22} = I_{K+1} \otimes I_{n'} \otimes D_1, \quad B_{12}^{33} = I_{K+1} \otimes I_{l'} \otimes D_1,$
$B_{12}^{44} = I_{K+1} \otimes I_l \otimes D_1, \quad B_{12}^{55} = I_K \otimes I_n \otimes (1-b)D_1, \quad B_{12}^{66} = (1-b)D_1,$

$$B_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ B_{21}^{21} & 0 & B_{21}^{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$B_{21}^{21} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ qU'^0\gamma \otimes I_m & 0 & 0 & \dots & 0 & 0 \\ 0 & qU'^0\gamma \otimes I_m & 0 & \dots & 0 & 0 \\ 0 & 0 & qU'^0\gamma \otimes I_m & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & qU'^0\gamma \otimes I_m & 0 \\ 0 & 0 & 0 & \dots & 0 & qU'^0\gamma \otimes I_m \end{bmatrix},$$

$$B_{21}^{23} = \begin{bmatrix} qU'^0\gamma' \otimes I_m & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

4. Analysis of Stability Condition

We examined our model under the assumption that the system is stable.

4.1. Condition for Stability

Let $A = A_0 + A_1 + A_2$ be the square matrix of order $Kmn + (K+1)n'm + (K+1)l'm + (K+1)lm + Kmn + m$ and it is an infinitesimal generator matrix is an irreducible. Let χ indicate the steady-state probability vector of A satisfying $\chi A = 0$ and $\chi e = 1$. The vector χ is partitioned by

$\chi = (\chi_0, \chi_1, \chi_2, \chi_3, \chi_4, \chi_5)$=$(\chi_{00}, \chi_{01}, \ldots, \chi_{0K-1}, \chi_{0K}, \chi_{11}, \chi_{12}, \ldots, \chi_{1K-1}, \chi_{1K}, \chi_{20}, \chi_{21}, \ldots, \chi_{2K-1}, \chi_{2K}, \chi_{30}, \chi_{31}, \ldots, \chi_{3K-1}, \chi_{3K}, \chi_{40}, \chi_{41}, \ldots, \chi_{4K-1}, \chi_{4K}, \chi_{50}, \chi_{51}, \ldots, \chi_{5K-1}, \chi_{5K})$, where χ_0 is of dimension Kmn, χ_1 is of dimension $(K+1)n'm$, χ_2 is of dimension $(K+1)l'm$, χ_3 is of dimension $(K+1)lm$, χ_4 is of dimension Kmn and χ_5 is of dimension m. The probability vector χ is calculated by solving the following equations:

$$\chi_{00}[(U + pU^0\gamma) \otimes I_m - (\lambda_2 + \eta)I_{nm}] + \chi_{01}(qU^0\gamma \otimes I_m) + \chi_{11}(qU'^0\gamma \otimes I_m) + \chi_{21}(T'^0\gamma \otimes I_m) + \chi_{31}(T^0\gamma \otimes I_m) + \chi_{40}(e_n \otimes \eta\gamma \otimes I_m) = 0.$$
$$\chi_{0j-1}(\lambda_2 I_{nm}) + \chi_{0j}[(U + pU^0\gamma) \otimes I_m - (\lambda_2 + \eta)I_{nm}] + \chi_{0j+1}(qU^0\gamma \otimes I_m) + \chi_{1j+1}(qU'^0\gamma \otimes I_m) + \chi_{2j+1}(T'^0\gamma \otimes I_m) + \chi_{3j+1}(T^0\gamma \otimes I_m) + \chi_{4j}(e_n \otimes \eta\gamma \otimes I_m) = 0, \quad for\ 1 \le j \le K-1.$$
$$\chi_{0K-1}(\lambda_2 I_{nm}) + \chi_{0K}[(U + pU^0\gamma) \otimes I_m - \eta I_{nm}] + \chi_{4K}(e_n \otimes \eta\gamma \otimes I_m) = 0.$$
$$\chi_{00}(qU^0\gamma' \otimes I_m) + \chi_{10}[(U' + pU'^0\gamma') \otimes I_m - (\lambda_2 + \psi_2)I_{n'm}] + \chi_{20}(T'^0\gamma' \otimes I_m) + \chi_{30}(T^0\gamma' \otimes I_m) + \chi_{50}(\eta\gamma' \otimes I_m) = 0.$$
$$\chi_{1j-1}(\lambda_2 I_{n'm}) + \chi_{1j}[(U' + pU'^0\gamma') \otimes I_m - (\lambda_2 + \psi_2)I_{n'm}] + \chi_{5j}(\eta\gamma' \otimes I_m) = 0,\ for\ 1 \le j \le K-1.$$
$$\chi_{1L-1}(\lambda_2 I_{n'm}) + \chi_{1L}[(U' + pU'^0\gamma') \otimes I_m] = 0.$$
$$\chi_{10}[((e'_n \otimes \psi_2\alpha') + qU'^0\gamma') \otimes I_m] + \chi_{20}(T' \otimes I_m - \lambda_2 I_{l'm}) = 0.$$
$$\chi_{0j-1}[e_n \otimes \psi_2\alpha' \otimes I_m] + \chi_{1j}[e'_n \otimes \psi_2\alpha' \otimes I_m] + \chi_{2j-1}(\lambda_2 I_{l'm}) + \chi_{2j}(T' \otimes I_m - \lambda_2 I_{l'm}) = 0, \quad for\ 1 \le j \le K-1.$$
$$\chi_{0K}[e_n \otimes \psi_2\alpha' \otimes I_m] + \chi_{1K}[e'_n \otimes \psi_2\alpha' \otimes I_m] + \chi_{2K-1}(\lambda_2 I_{l'm}) + \chi_{2K}(T' \otimes I_m) = 0.$$
$$\chi_{30}(T \otimes I_m - \lambda_2 I_{lm}) = 0,$$
$$\chi_{3j-1}(\lambda_2 I_{lm}) + \chi_{3j}(T \otimes I_m - \lambda_2 I_{lm}) = 0, \quad for\ 1 \le j \le K-1.$$
$$\chi_{3K-1}(\lambda_2 I_{lm}) + \chi_{3K}(T \otimes I_m) = 0.$$
$$\chi_{40}[\theta U \otimes I_m - (\eta + \lambda_2)I_{nm}] + \chi_{41}(\theta U' \otimes I_m) + \chi_{50}(\alpha \otimes \lambda_2 I_m) = 0.$$
$$\chi_{4j-1}(\lambda_2 I_{nm}) + \chi_{4j}[\theta U \otimes I_m - (\eta + \lambda_2)I_{nm}] + \chi_{4j+1}(\theta U' \otimes I_m) + \chi_{5j}(\alpha \otimes \lambda_2 I_m) = 0, \quad for\ 1 \le j \le K-1.$$
$$\chi_{4K-1}(\lambda_2 I_{nm}) + \chi_{4K}[\theta U \otimes I_m - \eta I_{nm}] + \chi_{5K}(\alpha \otimes \lambda_2 I_m) = 0.$$
$$\chi_{4K-1}(e_1 K \otimes \theta U^0 \otimes I_m) - \chi_{5K}(\eta + \lambda_2)I_m = 0.$$

Subject to normalizing condition

$$\sum_{r=1}^{K} \chi_{0r} e_{nm} + \sum_{r=0}^{K} \chi_{1r} e_{n'm} + \sum_{r=0}^{K} \chi_{2r} e_{l'm} + \sum_{r=0}^{K} \chi_{3r} e_{lm} + \sum_{r=1}^{K} \chi_{4r} e_n m + \chi_{50} e_m = 1.$$

The stability condition $\chi A_0 e < \chi A_2 e$ is obtained after some algebraic simplification, which turns out to be

$$\sum_{r=1}^{K} \chi_{0r}(e_n \otimes D_1 e_m) + \sum_{r=0}^{K} \chi_{1r}(e_{n'} \otimes D_1 e_m) + \sum_{r=0}^{K} \chi_{2r}(e_{l'} \otimes D_1 e_m) + \sum_{r=0}^{K} \chi_{3r}(e_l \otimes D_1 e_m) + \sum_{r=1}^{K} \chi_{4r}(e_n \otimes (1-b)D_1 e_m) + \chi_{50}(1-b)D_1 e_m < \sum_{r=0}^{K} \chi_{1r}(qU'^0 \otimes e_m).$$

4.2. The Stationary Probability Vector

Let y be the stationary probability vector of the infinitesimal generator Q of the process {Y(t): $t \ge 0$}. The subdivision of y by level as, $y = (y_0, y_1, y_2, ...)$, where y_0 is of dimension $(3m + 2Knm + Kl'm + (K+1)lm)$ for $i = 0$ and $y_1, y_2, ...$ are of dimension $Kmn + (K+1)n'm + (K+1)l'm + (K+$

$1)lm + Kmn + m$ for $i \geq 1$. As y is a stationary probability vector satisfies the relation $yQ = 0$ and $ye = 1$. Furthermore, while the stability criterion is satisfied, the equation gives the various levels.

$$y_j = y_1 R^{j-1},\ j \geq 2 \tag{3}$$

where R is the smallest non-negative solution of the quadratic equation

$$R^2 A_2 + RA_1 + A_0 = 0$$

and satisfies the relation $RA_2 e = A_0 e$ and the vector y_0, y_1 are obtained with the help of succeeding equations:

$$y_0 B_{00} + y_1 B_{10} = 0, \tag{4}$$

$$y_0 B_{01} + y_1 [A_1 + RA_2] = 0, \tag{5}$$

subject to normalizing condition

$$y_0 e_0 + y_1 [I - R]^{-1} e_1 = 1. \tag{6}$$

As a result, we can compute matrix R using Logarithmic reduction algorithm in Latouche and Ramaswami[15] and the vector y by using the special structure of something like the coefficient matrices.

5. Busy Period Analysis

• In a single-server queueing demonstration, the word busy period is characterized as the length of time between the entry of a customer into the void system and the first time from that point that the system size reaches zero. As, the first passage epoch to level zero, starting from level one. It is the first return time of level zero, taken after by a least one visit to a few other levels, which is the analog of the busy cycle.

• We have to present an outline of the fundamental period to analyze the busy period. when the QBD process is taken into thought the first passage time from level i to $i-1$, where $i \geq 2$.

• It is worth pointing out that for each level i, $i \geq 2$, there are $(3m + 3nm + lm)$ states. The state (i, j) of level i signifies the j^{th} state of level i when the states are sorted alphabetically.

• Let $\mathsf{G}_{jj'}(u, y)$ represent the conditional probability that the QBD process, starting at time $t = 0$ in the state (i, j) and keep track of the time until the first visit to the level $(i-1)$ but not later than time y. We can modify after exactly u transitions to the left and enter the state (i, j'), $t = 0$.

Let the joint transform matrix

$$\overline{\mathsf{G}}_{jj'}(z, s) = \sum_{u=1}^{\infty} z^u \int_0^{\infty} e^{-sy} d\mathsf{G}_{jj'}(u, y)\ \ ; |z| \leq 1,\ Re(s) \geq 0, \tag{7}$$

and put the matrix $\overline{\mathsf{G}}(z, s) = \overline{\mathsf{G}}_{jj'}(z, s)$. Specifically, computed the matrix $\overline{G}(z, s)$ satisfy the equation,

$$\overline{\mathsf{G}}(z, s) = z(SI - A_1)^{-1} A_2 + (SI - A_1)^{-1} A_0 \overline{\mathsf{G}}^2(z, s). \tag{8}$$

The matrix $\mathsf{G} = \mathsf{G}_{jj'} = \overline{\mathsf{G}}(1, 0)$ is concerned with negating the boundary states during the first passage times. knowing the rate matrix R allows us to use the below result to find the matrix G

$$\mathsf{G} = -(A_1 + RA_2)^{-1} A_2. \tag{9}$$

The matrix G can be found with the assistance of the Logarithmic reduction algorithm [15]. We find the matrix with the succeeding equation

$$\overline{\mathsf{G}}^{(1,0)}(z,s) = z(sI - A_1)^{-1}B_{10} + (sI - A_1)^{-1}A_0\overline{\mathsf{G}}(z,s)\overline{\mathsf{G}}^{(1,0)}(z,s) \quad (10)$$

$$\overline{\mathsf{G}}^{(0,0)}(z,s) = (sI - B_{00})^{-1}B_{01}\overline{\mathsf{G}}^{(1,0)}(z,s). \quad (11)$$

Thus, the moments that obey are calculated using the matrices G, $\overline{\mathsf{G}}^{(0,0)}(1,0)$ and $\overline{\mathsf{G}}^{(1,0)}(1,0)$ are stochastic at $z = 1$ and $s = 0$. We can find the moments as follows:

$$\vec{\mathcal{F}}_1 = -\frac{\partial}{\partial s}\overline{\mathsf{G}}(z,s)e = -[A_1 + A_0(I + \mathsf{G})]^{-1}e, \quad (12)$$

$$\vec{\mathcal{F}}_2 = \frac{\partial}{\partial z}\overline{\mathsf{G}}(z,s)e = -[A_1 + A_0(I + \mathsf{G})]^{-1}A_2e, \quad (13)$$

$$\vec{\mathcal{F}}_1^{(1,0)} = -\frac{\partial}{\partial s}\overline{\mathsf{G}}^{(1,0)}(z,s)e = -[A_1 + A_0\mathsf{G}]^{-1}(A_0\vec{\mathcal{F}}_1 + e), \quad (14)$$

$$\vec{\mathcal{F}}_2^{(1,0)} = \frac{\partial}{\partial z}\overline{\mathsf{G}}^{(1,0)}(z,s)e = -[A_1 + A_0\mathsf{G}]^{-1}(A_0\vec{\mathcal{F}}_2 + B_{10}e), \quad (15)$$

$$\vec{\mathcal{F}}_1^{(0,0)} = -\frac{\partial}{\partial s}\overline{\mathsf{G}}^{0,0}(z,s)e = -B_{00}^{-1}[B_{01}\vec{\mathcal{F}}_1^{(1,0)} + e], \quad (16)$$

$$\vec{\mathcal{F}}_2^{(0,0)} = \frac{\partial}{\partial z}\overline{\mathsf{G}}^{(0,0)}(z,s)e = -B_{00}^{-1}[B_{01}\vec{\mathcal{F}}_2^{(1,0)}]. \quad (17)$$

6. System Performance Measures

- Expected number of LP customers in the system
  $E_{LP} = \sum_{i=1}^{\infty} i y_i e.$
- Probability that the server is idle
  $P_{idle} = \sum_{a=1}^{m} y_{000a}.$
- Probability that the server busy with HP customers
  $P_{Hbusy} = \sum_{i=0}^{\infty}\sum_{r=1}^{K}\sum_{k_1=1}^{n}\sum_{a=1}^{m} y_{ir1k_1a}$
- Probability that the server is on hard failure
  $P_{HF} = \sum_{r=1}^{K}\sum_{k_4=1}^{l'}\sum_{a=1}^{m} y_{0r3k_4a} + \sum_{i=1}^{\infty}\sum_{r=0}^{K}\sum_{k_4=1}^{l'}\sum_{a=1}^{m} y_{ir3k_4a}$
- Probability that the server is Delay time to go for vacation
  $P_{DT} = \sum_{a=1}^{m} y_{004a}$
- Probability that the server is busy during working vacation
  $P_{BWV} = \sum_{i=0}^{\infty}\sum_{r=0}^{K}\sum_{k_1=1}^{n}\sum_{a=1}^{m} y_{ir6k_1a}$

7. Numerical Implementation

To compute numerical outcomes, we have employed distinct MAP representations for the arrival process in a manner that ensures their mean values are 1, as recommended by Chakravarthy [6].

Erlang of order 2 (ERL-A):

$$D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$$

Exponential (EXP-A):

$$D_0 = [-1], \quad D_1 = [1].$$

Hyper exponential (HYP-A):

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}.$$

MAP-Negative Correlation (MAP-NC-A):

$$D_0 = \begin{bmatrix} -1.00243 & 1.00243 & 0 \\ 0 & -1.00243 & 0 \\ 0 & 0 & -225.797 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.01002 & 0 & 0.99241 \\ 223.539 & 0 & 2.258 \end{bmatrix}.$$

MAP-Positive Correlation (MAP-PC-A):

$$D_0 = \begin{bmatrix} -1.00243 & 1.00243 & 0 \\ 0 & -1.00243 & 0 \\ 0 & 0 & -225.797 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.99241 & 0 & 0.01002 \\ 2.258 & 0 & 223.539 \end{bmatrix}.$$

Let us consider PH-distributions for the service and repair process as follows:

Erlang of order 2 (ERL-S):

$$\gamma = \gamma' = [1, 0], \quad U = U' = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}.$$

Erlang of order 2 (ERL-R):

$$\alpha = \alpha' = [1, 0], \quad T = T' = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}.$$

Exponential (EXP-S):

$$\gamma = \gamma' = [1], \quad U = U' = [-1].$$

Exponential (EXP-R):

$$\alpha = \alpha' = [1], \quad T = T' = [-1].$$

Hyper exponential (HYP-S):

$$\gamma = \gamma' = [0.8, 0.2], \quad U = U' = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}.$$

Hyper exponential (HYP-R):

$$\alpha = \alpha' = [0.8, 0.2], \quad T = T' = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}.$$

7.1. Illustration 1

We have examined the consequence of the hard failure rate ψ_2 against the Expected number of LP customers in the system(E_{LP}) in the following tables 1 - 3. Fix $\mu_1 = 20$, $\mu_2 = 15$, $K = 5$, $\lambda_1 = 1$, $\lambda_2 = 1.5$, $\eta = 8$, $\omega = 0.5$, $\psi_1 = 0.5$, $\tau_1 = 2$, $\tau_2 = 6$, $\theta = 0.6$, $b = 0.7$, $p_1 = 0.3$, $q_1 = 0.7$ such that the system is stable.

- As the hard failure rate (ψ_2) increases, the variety of arrangements of arrival and service times than the corresponding E_{LP} also increases.
- Observe the arrival times, E_{LP} increases highly in $MAP - PC - A$ and increases much slower in $ERL - A$ than all other arrival times.

7.2. Illustration 2

We investigated the impact of the vacation rate (η) against the probability of the server being idle (P_{idle}) in the following tables 4 - 6. Fix $\mu_1 = 20$, $\mu_2 = 15$, $K = 5$, $\lambda_1 = 1$, $\lambda_2 = 1.5$, $\omega = 0.5$, $\psi_1 = 0.5$, $\psi_2 = 1$, $\tau_1 = 2$, $\tau_2 = 6$, $\theta = 0.6$, $b = 0.7$, $p_1 = 0.3$, $q_1 = 0.7$ such that the system is stable.

- As the vacation rate (η) increases, the variety of arrangements of arrival and service times than the corresponding P_{idle} also increases.
- While comparing to $EXP - S$ and $HYP - S$, P_{idle} increases more rapidly for $ERL - S$. Similarly, P_{idle} increases slowly for $HYP - S$.

7.3. Illustration 3

We analyze the effect of the repair rate (τ_2) on the probability of the server being busy for HP customer (P_{Hbusy}) in the following tables 7 - 9. Fix $\mu_1 = 16$, $\mu_2 = 15$, $K = 5$, $\lambda_1 = 1$, $\lambda_2 = 1.5$, $\eta = 8$, $\omega = 0.5$, $\psi_1 = 0.5$, $\psi_2 = 1$, $\tau_1 = 2$, $\theta = 0.6$, $b = 0.7$, $p = 0.3$, $q = 0.7$ such that the system is stable.

- While maximizing the repair rate (τ_2), P_{Hbusy} minimizes for various possible arrangements of arrival and service times.
- When correlating the distinct arrival times, P_{Hbusy} decreases more quickly in the case of $MAP - PC - A$ whereas slowly in $ERL - A$. Similarly, considering the service times, P_{Hbusy} decreases gradually in $ERL - S$ and highly in $HYP - S$.

7.4. Illustration 4

To determine the existence of the service rate of HP customer (μ_2) versus the expected system size for LP customer (E_{LP}) in Figures 1 - 5. Fix $\mu_2 = 15$, $K = 5$, $\lambda_1 = 1$, $\lambda_2 = 1.5$, $\eta = 8$, $\omega = 0.5$, $\psi_1 = 0.5$, $\psi_2 = 1$, $\tau_1 = 2$, $\tau_2 = 6$, $\theta = 0.6$, $b = 0.7$, $p_1 = 0.3$, $q_1 = 0.7$ such that the system remains stable.

A quick observation from Figures 1 - 5, E_{LP} decreases while increasing the service rate of HP customers for all combinations of arrival and service time groupings. Due to the availability of the HP service rate in the system, the customers will get service successfully which leads to E_{LP} decreases. However, E_{LP} decreases slowly for $ERL - A$ with the combination of $ERL - S$ whereas slowly in $HYP - S$. Likewise, E_{LP} decreases highly for $HYP - A$ in $HYP - S$ whereas slowly in $ERL - S$.

7.5. Illustration 5

To see the features of both the HP service rate (μ_1) and repair rate of hard failure (τ_2) on the expected number of LP customers in the system (E_{LP}) in the Figures 6 - 10. Fix $\mu_2 = 15$, $K = 5$, $\lambda_1 = 1$, $\lambda_2 = 1.5$, $\eta = 8$, $\omega = 0.5$, $\psi_1 = 0.5$, $\psi_2 = 1$, $\tau_1 = 2$, $\theta = 0.6$, $b = 0.7$, $p = 0.3$, $q = 0.7$ such that stability condition is satisfied.

Observation in Figures 6 - 10, we increase the values of both the HP service rate and repair rate of hard failure, then E_{LP} decreases with various arrival groupings. Due to the HP customer increase in the service rate, E_{LP} decreases likewise increase the repair rate of hard failure decrease in the E_{LP}. Let's look at the arrival times, E_{LP} decreases slowly for $ERL - A$ and decreases fastly for $MAP - PC - A$.

Table 1: *Hard Failure rate (ψ_2) vs E_{LP} - ERL-S*

ψ_2	$ERL-A$	$EXP-A$	$HYP-A$	$NC-A$	$PC-A$
1	0.163936	0.206320	0.293281	0.330092	20.674732
1.2	0.172609	0.217533	0.312405	0.342647	21.257457
1.4	0.181624	0.229181	0.332498	0.355637	21.857954
1.6	0.190997	0.241285	0.353617	0.369083	22.477042
1.8	0.200747	0.253867	0.375822	0.383009	23.115590
2	0.210893	0.266951	0.399180	0.397438	23.774519
2.2	0.221454	0.280563	0.423761	0.412394	24.454807
2.4	0.232454	0.294730	0.449638	0.427907	25.157494
2.6	0.243915	0.309480	0.476893	0.444003	25.883685

Table 2: *Hard Failure rate (ψ_2) vs E_{LP} - EXP-S*

ψ_2	$ERL-A$	$EXP-A$	$HYP-A$	$NC-A$	$PC-A$
1	0.174277	0.217929	0.311487	0.340494	20.533617
1.2	0.182994	0.229005	0.330339	0.352724	21.059251
1.4	0.191976	0.240406	0.349917	0.365273	21.596139
1.6	0.201232	0.252143	0.370253	0.378153	22.144696
1.8	0.210771	0.264230	0.391379	0.391377	22.705353
2	0.220605	0.276679	0.413329	0.404958	23.278556
2.2	0.230744	0.289502	0.436138	0.418908	23.864775
2.4	0.241200	0.302715	0.459844	0.433243	24.464498
2.6	0.251986	0.316331	0.484485	0.447976	25.078233

Table 3: *Hard Failure rate (ψ_2) vs E_{LP} - HYP-S*

ψ_2	$ERL-A$	$EXP-A$	$HYP-A$	$NC-A$	$PC-A$
1	0.233555	0.282009	0.402324	0.416602	19.623165
1.2	0.240760	0.290404	0.415758	0.425489	19.866887
1.4	0.247923	0.298753	0.429141	0.434323	20.109536
1.6	0.255059	0.307070	0.442498	0.443121	20.351499
1.8	0.262179	0.315369	0.455850	0.451895	20.593109
2	0.269292	0.323660	0.469217	0.460658	20.834659
2.2	0.276407	0.331955	0.482615	0.469419	21.076406
2.4	0.283532	0.340261	0.496061	0.478189	21.318580
2.6	0.290674	0.348587	0.509566	0.486975	21.561381

Table 4: *Vacation rate (η) vs P_{idle} - ERL-S*

η	$ERL-A$	$EXP-A$	$HYP-A$	$NC-A$	$PC-A$
5	0.082706	0.088338	0.098335	0.092979	0.094606
6	0.084704	0.090362	0.100285	0.094971	0.096400
7	0.086239	0.091918	0.101790	0.096500	0.097789
8	0.087467	0.093162	0.102999	0.097723	0.098910
9	0.088478	0.094188	0.103999	0.098732	0.099840
10	0.089331	0.095053	0.104845	0.099583	0.100629
11	0.090063	0.095797	0.105573	0.100315	0.101311
12	0.090700	0.096444	0.106210	0.100952	0.101907
13	0.091261	0.097015	0.106772	0.101514	0.102435

Table 5: *Vacation rate (η) vs P_{idle} - EXP-S*

η	$ERL-A$	$EXP-A$	$HYP-A$	$NC-A$	$PC-A$
5	0.084405	0.090174	0.100258	0.095251	0.096557
6	0.086462	0.092258	0.102267	0.097311	0.098407
7	0.088037	0.093853	0.103811	0.098887	0.099835
8	0.089292	0.095124	0.105045	0.100142	0.100980
9	0.090321	0.096167	0.106061	0.101172	0.101925
10	0.091185	0.097042	0.106916	0.102036	0.102722
11	0.091922	0.097789	0.107648	0.102775	0.103407
12	0.092561	0.098437	0.108283	0.103415	0.104002
13	0.093121	0.099006	0.108842	0.103977	0.104527

Table 6: *Vacation rate (η) vs P_{idle} - HYP-S*

η	$ERL-A$	$EXP-A$	$HYP-A$	$NC-A$	$PC-A$
5	0.090101	0.096426	0.106953	0.102570	0.103410
6	0.092254	0.098602	0.109047	0.104734	0.105339
7	0.093876	0.100237	0.110625	0.106360	0.106796
8	0.095147	0.101518	0.111863	0.107631	0.107942
9	0.096173	0.102552	0.112863	0.108656	0.108870
10	0.097022	0.103406	0.113691	0.109503	0.109639
11	0.097737	0.104125	0.114389	0.110217	0.110290
12	0.098349	0.104741	0.114987	0.110827	0.110848
13	0.098880	0.105275	0.115507	0.111356	0.111333

Table 7: *Repair rate (τ_2) vs P_{Hbusy} - ERL-S*

τ_2	$ERL-A$	$EXP-A$	$HYP-A$	$NC-A$	$PC-A$
5	0.105099	0.105226	0.105427	0.105573	0.105653
6	0.105061	0.105172	0.105341	0.105479	0.105548
7	0.105034	0.105135	0.105284	0.105412	0.105472
8	0.105014	0.105107	0.105243	0.105362	0.105416
9	0.104998	0.105086	0.105213	0.105323	0.105373
10	0.104986	0.105070	0.105190	0.105292	0.105339
11	0.104976	0.105056	0.105171	0.105267	0.105311
12	0.104968	0.105045	0.105156	0.105246	0.105288
13	0.104961	0.105036	0.105143	0.105228	0.105268

Table 8: *Repair rate (τ_2) vs P_{Hbusy} - EXP-S*

τ_2	$ERL-A$	$EXP-A$	$HYP-A$	$NC-A$	$PC-A$
5	0.103623	0.103745	0.103941	0.104090	0.104179
6	0.103590	0.103698	0.103864	0.104004	0.104081
7	0.103565	0.103663	0.103811	0.103941	0.104009
8	0.103545	0.103637	0.103772	0.103892	0.103954
9	0.103530	0.103616	0.103742	0.103854	0.103911
10	0.103518	0.103600	0.103719	0.103824	0.103876
11	0.103509	0.103587	0.103701	0.103798	0.103848
12	0.103500	0.103576	0.103685	0.103778	0.103824
13	0.103493	0.103567	0.103673	0.103760	0.103805

Table 9: *Repair rate (τ_2) vs P_{Hbusy} - HYP-S*

τ_2	$ERL-A$	$EXP-A$	$HYP-A$	$NC-A$	$PC-A$
5	0.095098	0.095149	0.095235	0.095317	0.095365
6	0.095158	0.095216	0.095310	0.095423	0.095481
7	0.095182	0.095241	0.095332	0.095455	0.095514
8	0.095191	0.095247	0.095335	0.095458	0.095515
9	0.095192	0.095247	0.095329	0.095449	0.095503
10	0.095190	0.095243	0.095321	0.095436	0.095487
11	0.095187	0.095238	0.095313	0.095422	0.095470
12	0.095183	0.095232	0.095304	0.095408	0.095453
13	0.095180	0.095227	0.095296	0.095394	0.095438

(a) ERL-A

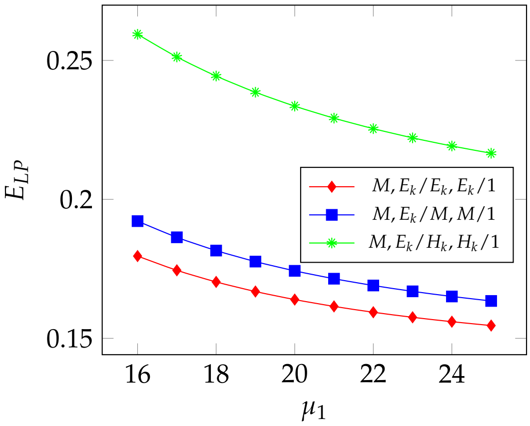

Figure 1: *High priority service rate vs.* E_{LP}

(b) EXP-A

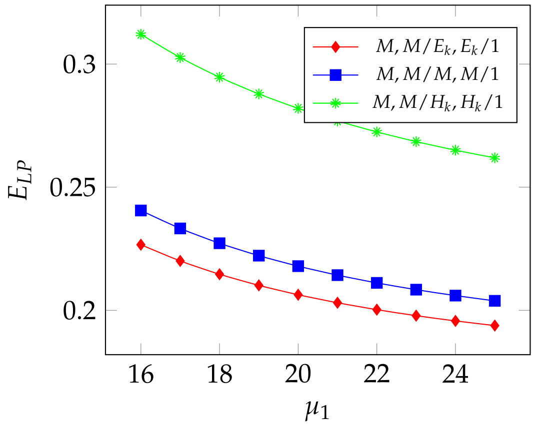

Figure 2: *High priority service rate vs.* E_{LP}

(c) HYP-A

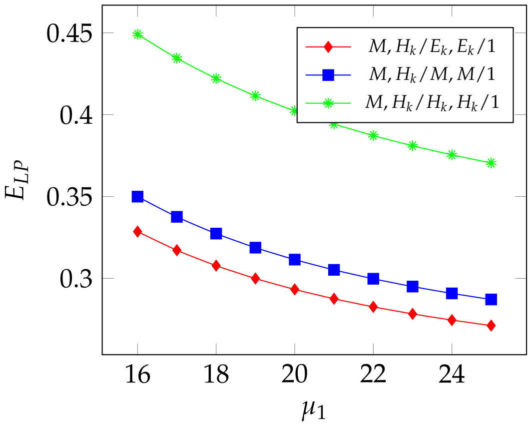

Figure 3: *High priority service rate vs.* E_{LP}

(d) MAP-NC-A

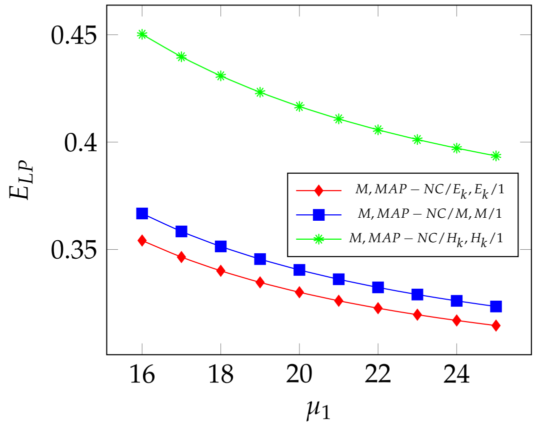

Figure 4: *High priority service rate vs.* E_{LP}

(e) MAP-PC-A

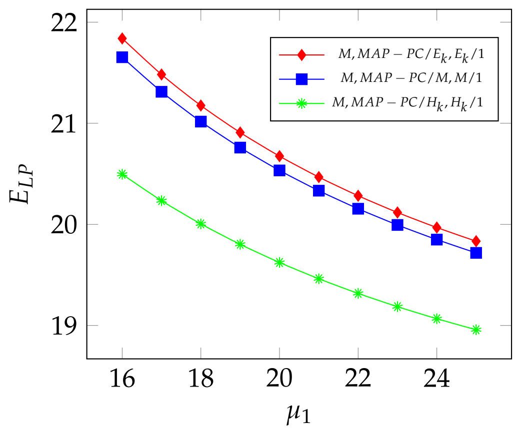

Figure 5: *High priority service rate vs.* E_{LP}

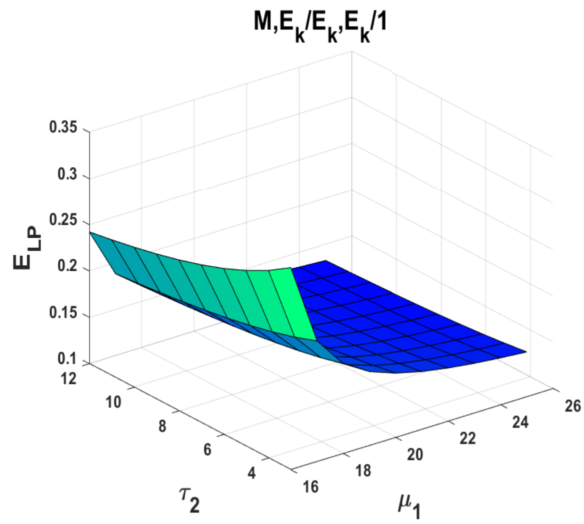

Figure 6: *HP service (μ_1) and Repair(HF) (τ_2) rates vs.* E_{LP} *- ERL-S*

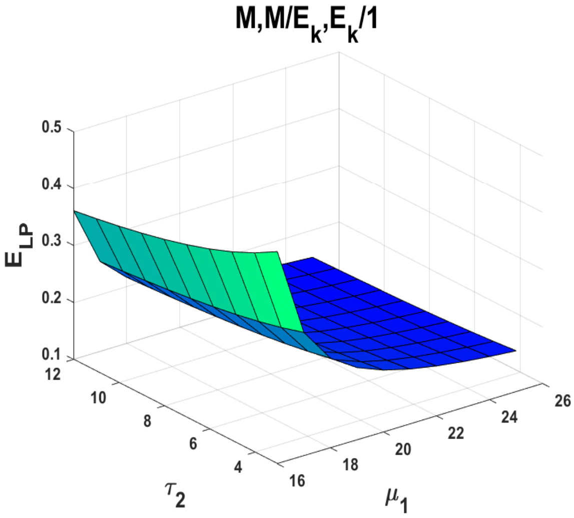

Figure 7: *HP service (μ_1) and Repair(HF) (τ_2) rates vs. E_{LP} - ERL-S*

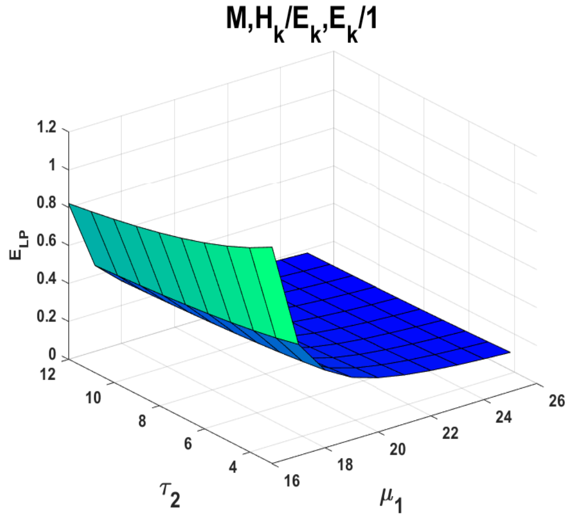

Figure 8: *HP service (μ_1) and Repair(HF) (τ_2) rates vs. E_{LP} - ERL-S*

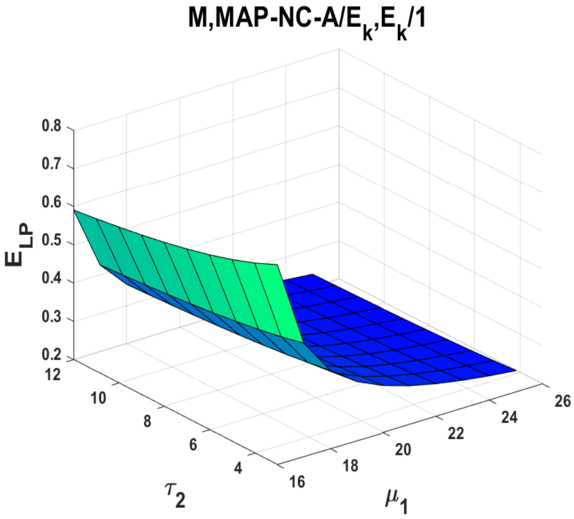

Figure 9: *HP service (μ_1) and Repair(HF) (τ_2) rates vs. E_{LP} - ERL-S*

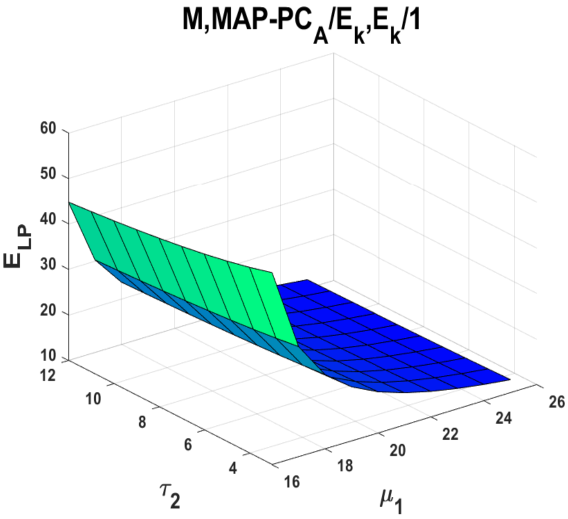

Figure 10: *HP service (μ_1) and Repair(HF) (τ_2) rates vs. E_{LP} - ERL-S*

8. Conclusion

This paper contributes by employing the Matrix analytic method to compute the stationary distribution of the number of customers in the $M, MAP/PH_1, PH_2/1$ queueing system with delayed working vacations under non-preemptive priority. We discussed some system performance measures using steady-state probabilities and also calculated busy period analysis. We used numerical examples to show how different system parameters affect performance measures.

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