TIME DEPENDENT BEHAVIOUR OF A SINGLE SERVER QUEUEING SYSTEM WITH DIFFERENTIATED WORKING VACATIONS SUBJECT TO SYSTEM DISASTER

V Karthick, V Suvitha

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Department of Mathematics,
College of Engineering and Technology,
SRM Institute of Science and Technology,
Kattankulthur – 603 203, Tamilnadu, India.

kv0994@srmist.edu.in, suvithav@srmist.edu.in

Abstract

This study investigates the time dependent behaviour of the single server queue with differentiated working vacations. The model also takes into account the possibility of a disaster happening during busy periods and working vacations, with the repair procedure starting right away. The time-dependent probabilities of system size are described in terms of modified Bessel functions in the paper using explicit equations that were generated using generating functions. Numeric instances have been added to support the theoretical findings even more.

Keywords: Transient Analysis, Differentiated Vacations, Quasi birth death process, Disaster, Repair

1. Introduction

Many queueing systems allow servers to go offline when the system is empty for any period of time. This random period of server absence, known as a server vacation, could indicate the server can take a break or perform an additional task during this period. In 1975, Levy and Yechiali first presented the vacation queuing model. Numerous researchers have worked on queues with vacationing servers during the past few decades. Doshi [5], Takagi [19], Upadhyaya [23], Tian and Zhang [20] and Ke et al. [9] have conducted comprehensive surveys on vacation queueing models, considering various contexts. The queueing model can be applied to a variety of real-world stochastic service systems since server vacations are especially advantageous for systems where the server can use idle time for other activities. Recent research on vacation queueing models has also been done by Sapkota [13] and Tian et al. [21]. The working vacation queue is the queue in which the server serves customers at a rate that is lower than the busy time service rate. This kind of technology has a wide range of practical applications, including the rate at which employees perform their official work both in the office and at home. Servi and Finn [14] introduced the M/M/1 queueing model with working vacations, where a customer isserved at a lower service rate instead of stopping the service completely. M/M/1 queueing model with working vacation and two types of server failure was discussed by Agrawal et al. [1]. Recently, Tian et al. (2021) conducted an analysis of Markovian queues with Bernoulli interruptions and single working vacations. Recently, Kumar et al. [11] presented a transient analysis of working vacation queueing system.

Differentiated vacations are one of the various types of queueing vacation types. In this case, whenever the system becomes empty, the server initiates a type I vacation, which has a random length. When a server returns from vacation and discovers there is no queue, a new type II random vacation is started. When the server returns from either type I or type II vacations and there are customers in the system, the server starts providing services to them right away and continues doing so until there are no more users in it. The queueing model with differentiated queueing vacations were initially proposed by Ibe and Isijola [6]. Since then, a number of scholars have examined differentiated vacation queueing systems, including Suranga Sampath et al. [17], Suranga Sampath and Jicheng [18], and Jain and Sigman [7]. Vijayashree and Janani [24] analyse a single server differentiated working vacation queueing model's transient behaviour. Recently, a transient analysis of a single server differentiated queuing system is given by Azhagappan and Deepa [3].

In computer systems, telecommunication networks, and other queueing systems, congestion and blocking are frequently predicted using queuing theory and network analysis. Disasters can happen as a result of unforeseen situations, and these systems are frequently prone to unreliability. All jobs sitting in the buffer, including the one being processed by the server, are lost when a calamity happens since the system is rendered inoperable. Towsley and Tripathi [22] were the first to analyse queueing systems that were subject to disasters. This phenomenon was examined by Chen et al. [4], who called it a "mass exodus". As part of their "stochastic clearing" research, Artaljo and Gomez-Coral [2] examined queueing systems with catastrophes. Single-server queueing systems with disasters have been the subject of transient analysis by Kumar et al. [10], Sudhesh and Vairthiyanathan (2019), and Jain and Singh [8]. Recently, Sudhesh et al. [15] gave the transient analysis of single server queue with disaster. We take into account an $M/M/1$ queue with differentiated working vacations subject to system disaster and server repair. In this sense, we have seen that the service rate is different, but arrival rate is same for all the states.

The proposed queueing model is motivated real-time application with power-saving features in a smart home automation system.The smart home automation system monitors various sensors and devices within a home, controlling tasks such as lighting, temperature, security, and appliances. It continuously processes sensor data and user commands to maintain an optimal and comfortable environment. When no user commands are received and there are no sensor-triggered events for a certain period, the system switches to an type I power save mode to conserve power. In this state, non-essential components are turned off or put into low-power mode.After the type I power save mode duration expires, the system periodically wakes up to check for any new user commands or sensor-triggered events. If there are pending actions or events, it resumes normal operation and executes the necessary tasks. If there are no pending commands or events, the system enters a deeper power-saving state known as type II power saving mode. In this mode, only essential components remain active to maintain basic system functionality and listen for any incoming commands or events.The smart home automation system may be susceptible to security attacks, such as unauthorized access, data breaches, or control manipulation. These attacks can compromise the integrity and privacy of the system and disrupt its normal operation. The repair process starts immediately.The power-saving features in this smart home automation system help reduce energy consumption during periods of inactivity, contributing to energy efficiency and cost savings. The security considerations highlight the importance of safeguarding the system against potential attacks to protect the privacy, safety, and functionality of the smart home environment.

2. Model description

In this research, a $M/M/1$ queuing model with differentiated working vacations and the possibility of disastrous breakdown and repair is taken into account. These are the main presumptions that underlie this model:

1. Customer Arrivals: Customers arrive according to a Poisson process with a rate of λ. Customers join a single waiting queue based on the sequence in which they arrive. The capacity of the system is similarly predicated on an endless number of potential clients.
2. Service Process: A single server offers the service, and service times during a normal busy period follow an exponential distribution with parameter μ.
3. Vacation Policy : The servers take a type I vacation after fully serving every customer in the system. Once the servers have attended to at least a single customer, this vacation starts. If the system is empty when the servers return after a vacation, a new random vacation of type II is started. If there are still customers in the system when the server returns from either a Type I or type II vacation, it begins serving them right away until the system is completely empty once more.
4. The servers type I and type II vacation times follow exponential distributions, and their vacation rates are indicated by the symbols θ and γ respectively.
5. Arriving customers are served at rates of $\mu_{v1}(\mu_{v2})$ during type-I (II) vacations.
6. Disastrous Breakdown and Repair: There is a chance that a disastrous breakdown will occur when the servers are either away on working vacation or busy serving customers. The breakdowns occurrence follows exponential distribution with a rate of α . When a server fails, the repair procedure begins right away at a rate of β, enabling the servers to function again as quickly as possible.

2.1. The Quasi-Birth-and-Death (QBD) process

At the time t the number of customers in the systems is consider as $H(t)$ and let $I(t)$ be the servers state, where

$$I(t) = \begin{cases} 0, & \text{the server is in busy} \\ 1, & \text{the server is in type-I vacation} \\ 2, & \text{the server is in type-II vacation} \\ 3, & \text{the server in disaster} \end{cases}$$

Then $X(t) = \{H(t), I(t)\}$, is a Continuous time Markov chain with a state space denoted by Ω as follows:
$\Omega = \{\{(i,j), i \geq 0, j = 0,1,2,3\}$.

3. Transient Analysis

Let $P_{n,j}(t)$ be the time-dependent probability for the system to be in state j with n customers at time t.

$$P'_{0,0}(t) = -(\lambda+\alpha)P_{0,0}(t) + \beta P_{0,3}(t) \tag{1}$$

$$P'_{n,0}(t) = -(\lambda+\alpha+\mu)P_{n,0}(t) + \mu P_{n+1,0}(t) + \gamma_1 P_{n,1}(t) + \gamma_2 P_{n,2}(t) + \beta P_{n,3}(t) + \lambda P_{n-1,0} \text{ for } n \geq 1 \tag{2}$$

$$P'_{0,1}(t) = -(\lambda+\alpha+\gamma_1)P_{0,1}(t) + \mu_{v1}P_{1,1}(t) + \mu P_{1,0}(t) \tag{3}$$

$$P'_{n,1}(t) = -(\lambda+\alpha+\gamma_1+\mu_{v1})P_{n+1,1}(t) + \lambda P_{n-1,1}(t) \text{ for } n \geq 1 \tag{4}$$

$$P'_{0,2}(t) = -(\lambda+\alpha)P_{0,2}(t) + \gamma_1 P_{0,1}(t) + \mu_{v2}P_{1,2}(t) \tag{5}$$

$$P'_{n,2}(t) = -(\lambda+\alpha+\mu_{v2}+\gamma_2)P_{n,2}(t) + \lambda P_{n-1,2}(t) + \mu_{v2}P_{n+1,2}(t) \text{ for } n \geq 1 \tag{6}$$

$$P'_{0,3}(t) = -(\lambda+\beta)P_{0,3}(t) + \alpha\big(1 - \sum_{n=0}^{\infty} P_{n,3}(t)\big) \tag{7}$$

$$P'_{n,3}(t) = -(\lambda+\beta)P_{n,3}(t) + \lambda P_{n-1,3}(t) \ \text{ for } n \geq 1 \tag{8}$$

We assume the initial condition as,
$P_{0,1}(0) = 1$, $P_{0,i}(0) = 0$ for $i = 0, 2, 3,$, $P_{n,i}(0) = 0$ for $n \geq 1, i = 0, 1, 2, 3,$
Taking laplace on equations (1), (3), (5), (7), (8).

$$\hat{P}_{0,0}(s) = \frac{\beta}{(s+\lambda+\alpha)}\hat{P}_{0,3}(s) \tag{9}$$

$$\hat{P}_{0,1}(s) = \frac{1}{s+\lambda+\alpha+\gamma_1} + \frac{\mu_{v1}}{s+\lambda+\alpha+\gamma_1}\hat{P}_{1,1}(s) + \frac{\mu}{(s+\lambda+\alpha+\gamma_1)}\hat{P}_{1,0}(s) \tag{10}$$

$$\hat{P}_{0,2}(s) = \frac{\gamma_1}{s+\lambda+\alpha}\hat{P}_{0,1}(s) + \frac{\mu_{v2}}{(s+\lambda+\alpha)}\hat{P}_{1,2}(s) \tag{11}$$

$$\hat{P}_{0,3}(s) = \frac{\alpha}{s(s+\lambda+\beta)} - \frac{\alpha}{(s+\lambda+\beta)}\sum_{n=0}^{\infty}\hat{P}_{n,3}(s) \tag{12}$$

$$\hat{P}_{n,3}(s) = \frac{\lambda}{(s+\lambda+\beta)}\hat{P}_{n-1,3}(s) \tag{13}$$

The above equation (13) recursively yields

$$\hat{P}_{n,3}(s) = \frac{\lambda^n}{(s+\lambda+\beta)^n}\hat{P}_{0,3}(s) \ \text{ for } n \geq 1 \tag{14}$$

Define
$Q_1(z,t) = \sum_{n=0}^{\infty} P_{n,1}(t)z^n$ then $\frac{\partial Q_1(z,t)}{\partial t} = \sum_{n=0}^{\infty} P'_{n,1}(t)z^n$
Multiplying the equations (3) and (4) by the appropriate powers of z and summing over $n \geq 1$ we obtain,

$$\frac{\partial Q_1(z,t)}{\partial t} + \left((\lambda+\alpha+\gamma_1+\mu_{v1}) - \left(\frac{\mu_{v1}}{z} + \lambda z\right)\right)Q_1(z,t) = \mu_{v1}P_{0,1}(t) - \frac{\mu_{v1}}{z}P_{0,1}(t) + \mu P_{1,0}(t)$$

Upon integrating the above linear differential equation with respect to t, we get

$$Q_1(z,t) = \int_0^t \left(\mu_{v1}P_{0,1}(t) + \frac{\mu_{v2}}{z}P_{0,1}(t) + \mu P_{0,1}(t)\right)(e^{-(\lambda+\alpha+\gamma_1+\mu_{v1})(t-y)} \times e^{((\mu_{v1}/z)+\lambda z)(t-y)})dy \tag{15}$$

If $a_i = 2\sqrt{\lambda\mu_{vi}}$ and $b_i = \sqrt{\lambda/\mu_{vi}}$ then $e^{(\mu_{vi}/z+\lambda z)t} = \sum_{-\infty}^{\infty}(b_i z)^n I_n(a_i t)$ for $i = 1, 2$ where $I_n(a_i t)$ is a bessel funtion of order n. Using that fact in equation (15) and comparing the terms coeffients of z^n for $n = 1, 2, 3, ...$

$$P_{n,1}(t) = \int_0^t \left((\mu_{v1}P_{0,1}(t) + \mu P_{1,0}(t))b_1^n I_n(.)e^{-k_1(t-y)}\right)dy + \int_0^t \left(\mu_{v1}P_{1,0}(t)b_1^{n+1}I_{n+1}(.)e^{-k_1(t-y)}\right)dy \tag{16}$$

Equating the coeffients of z^{-n} for $n = 1, 2, ...$ and applying $I_{-n}(.) = I_n(.)$ we get

$$0 = \int_0^t \left((\mu_{v1}P_{0,1}(t) + \mu P_{1,0}(t))b_1^{-n} I_n(.)e^{-k_1(t-y)}\right)dy + \int_0^t \left(\mu_{v1}P_{1,0}(t)b_1^{-n+1}I_{-n+1}(.)e^{-k_1(t-y)}\right)dy \tag{17}$$

where $k_1 = \lambda + \alpha + \gamma_1 + \mu_{v1}$ and $I_n(.) = I_n(a(t-y))$

Multiply equation (17) by b_1^{2n} and subtract from equation (16)

$$P_{n,1}(t) = \int_0^t \left(\mu_{v1} P_{0,1}(t) b_1^{n+1} \left[I_{n-1}(.) - I_{n+1}(.) \right] e^{-k_1(t-y)} \right) dy$$

Taking laplace transform on both sides

$$\hat{P}_{n,1}(s) = 2\mu_{v1} \frac{b_1^{n+1}}{a_1} \hat{\psi(s)}^n \hat{P}_{0,1}(s) \quad \text{for } n \geq 0 \tag{18}$$

In similar way using the equations (5), (6) we get

$$\hat{P}_{n,2}(s) = 2\mu_{v2} \frac{b_2^{n+1}}{a_2} \hat{\psi(s)}^n \hat{P}_{0,2}(s) \quad \text{for } n \geq 0 \tag{19}$$

Define
$Q_3(z,t) = \sum\limits_{n=0}^{\infty} P_{n,0}(t) z^n$ then $\dfrac{\partial Q_3(z,t)}{\partial t} = \sum\limits_{n=0}^{\infty} P'_{n,0}(t) z^n$
Multiplying the equations (1) and (2) by the appropriate powers of z and summing over $n \geq 0$ we obtain,

$$\begin{aligned} \frac{\partial Q_3(z,t)}{\partial t} + \left((\lambda + \alpha + \mu) - \left(\frac{\mu}{z} + \lambda z \right) \right) Q_3(z,t) &= \mu P_{0,0}(t) - \frac{\mu}{z} P_{0,0}(t) + \mu P_{1,0}(t) \\ &+ \gamma_1 \sum_{n=1}^{\infty} P_{n,1}(t) z^n + \gamma_2 \sum_{n=1}^{\infty} P_{n,2}(t) + \beta \sum_{n=0}^{\infty} P_{n,3}(t) z^n \end{aligned} \tag{20}$$

If $a_3 = 2\sqrt{\lambda\mu}$ and $b_i = \sqrt{\lambda/\mu}$ then $e^{(\mu/z+\lambda z)t} = \sum\limits_{-\infty}^{\infty} (b_3 z)^n I_n(a_3 t)$ where $I_n(a_3 t)$ is a bessel funtion of order n. Using that fact in equation (20) and comparing the terms coeffients of z^n for $n = 1, 2, 3, ...$

$$\begin{aligned} P_{n,0}(t) = &\int_0^t \left((\mu P_{0,1}(t) - \mu P_{1,0}(t)) b_3^n I_n(.) e^{-k_3(t-y)} \right) dy - \int_0^t \left(\mu P_{0,0}(t) b_3^{n+1} I_{n+1}(.) \right. \\ &\left. \times e^{-k_3(t-y)} \right) dy + \int_0^t \left(\gamma_1 \sum_{m=1}^{\infty} (\gamma_1 P_{m,1}(t) z^m + \gamma_2 P_{n,2}(t) z^m \right. \\ &\left. + \beta P_{m,3}(s)) b_3^{n-m} I_{n-m}(.) e^{-k_3(t-y)} \right) dy \end{aligned} \tag{21}$$

Equating the coeffients of z^{-n} for $n = 1, 2, ...$ and applying

$$\begin{aligned} 0 = &\int_0^t \left((\mu P_{0,1}(t) - \mu P_{1,0}(t)) b_3^{-n} I_n(.) e^{-k_3(t-y)} \right) dy - \int_0^t \left(\mu P_{0,0}(t) b_3^{-n+1} I_{-n+1}(.) \right. \\ &\left. \times e^{-k_3(t-y)} \right) dy + \int_0^t \left(\gamma_1 \sum_{m=1}^{\infty} (\gamma_1 P_{m,1}(t) z^m + \gamma_2 P_{n,2}(t) z^m \right. \\ &\left. + \beta P_{m,3}(s)) b_3^{-(n+m)} I_{n+m}(.) e^{-k_3(t-y)} \right) dy \end{aligned} \tag{22}$$

Multiply equation (22) by b_1^{2n} and subtract from equation (21)

$$\begin{aligned} P_{n,0}(t) = &\int_0^t \left(\mu P_{0,0}(t) b_3^{n+1} \left[I_{n-1}(.) - I_{n+1}(.) \right] e^{-k_3(t-y)} \right) dy + \int_0^t \left(\gamma_1 \sum_{m=1}^{\infty} (\gamma_1 P_{m,1}(t) z^m \right. \\ &\left. + \gamma_2 P_{n,2}(t) z^m + \beta P_{m,3}(s)) b_3^{n-m} \left[I_{n-m}(.) - I_{n+m}(.) \right] e^{-k_3(t-y)} \right) dy \end{aligned}$$

Taking laplace transform on both sides

$$\hat{P}_{n,0}(s) = \frac{1}{\sqrt{\omega_3^2 - a_3^2}}\left[\sum_{m=1}^{\infty}\left(\gamma_1\hat{P}_{m,1}(s) + \gamma_2\hat{P}_{m,2}(s)\right) + \sum_{m=0}^{\infty}\beta\hat{P}_{m,3}(s)\right] + 2\mu\frac{b_3^{n+1}}{a_3}\psi(\hat{s})^n\hat{P}_{0,0}(s) \text{ for } n \geq 0 \tag{23}$$

substitute $n = 1$ in (18) and (19)

$$\hat{P}_{1,1}(s) = 2\mu_{v1}b_1^2\frac{\psi(\hat{s})^s}{a1}\hat{P}_{0,1}(s) \tag{24}$$

$$\hat{P}_{1,2}(s) = 2\mu_{v2}b_2^2\frac{\psi(\hat{s})^s}{a2}\hat{P}_{0,2}(s) \tag{25}$$

Substitute (24) and (25) in $\hat{P}_{0,1}(s)$

$$\hat{P}_{0,1}(s) = \frac{1}{(s+\lambda+\alpha+\gamma_1)}\left[1 + 2\mu_{v1}^2 b_1^2\frac{\psi(\hat{s})}{a_2}\hat{P}_{0,1}(s) + \mu\hat{P}_{1,0}(s)\right]$$

$$\hat{P}_{0,1}(s) = \left[\sum_{j=0}^{\infty}\left(\frac{2\mu_{v1}^2 b_1^2\psi(\hat{s})}{(s+\lambda+\alpha+\gamma_1)a_1}\right)^j\right]\frac{1}{(s+\lambda+\alpha+\gamma_1)} + \left[\sum_{j=0}^{\infty}\left(\frac{2\mu_{v1}^2 b_1^2\psi(\hat{s})}{(s+\lambda+\alpha+\gamma_1)a_1}\right)^j\right]\frac{\mu}{(s+\lambda+\alpha+\gamma_1)}\hat{P}_{1,0}(s)$$

$$\hat{P}_{0,1}(s) = \hat{A}_1(s) + \hat{A}_1(s)\mu\hat{P}_{1,0}(s) \tag{26}$$

Substitue (25) and (26) in $\hat{P}_{0,2}(s)$

$$\hat{P}_{0,2}(s) = \frac{\gamma_1}{(s+\lambda+\alpha)}\hat{A}_1(s) + \frac{\gamma_2}{(s+\lambda+\alpha)}\hat{A}_2(s)\hat{P}_{1,0}(s) + \frac{2\mu_{v2}^2 b_2^2}{(s+\lambda+\alpha)a_2}\hat{\psi}(s)\hat{P}_{0,2}(s)$$

$$\hat{P}_{0,2}(t) = \left[\sum_{j=0}^{\infty}\frac{2\mu_{v2}^2 b_2^2}{(s+\lambda+\alpha)a_2}\hat{\psi}(s)\right]^j\frac{\gamma_1}{(s+\lambda+\alpha)}\hat{A}_1(s) + \left[\sum_{j=0}^{\infty}\frac{2\mu_{v2}^2 b_2^2}{(s+\lambda+\alpha)a_2}\hat{\psi}(s)\right]^j\frac{\gamma_2}{(s+\lambda+\alpha)}\hat{A}_2(s)\hat{P}_{1,0}(s)$$

$$\hat{P}_{0,2}(s) = \gamma_1\hat{A}_2(s) + \gamma_2\hat{A}_2(s)\hat{P}_{1,0}(s) \tag{27}$$

Substitue (26) and (27) in (18) and (19)

$$\hat{P}_{n,1}(s) = 2\mu_{v1}b_1^{n+1}\frac{\hat{\psi}(s)^n}{a_1}\left(\hat{A}_1(s) + \mu\hat{A}_1(s)\hat{P}_{1,0}(s)\right) \tag{28}$$

$$\hat{P}_{n,2}(s) = 2\mu_{v2}b_2^{n+1}\frac{\hat{\psi}(s)^n}{a_2}\left(\gamma_1\hat{A}_2(s) + \gamma_2\hat{A}_2(s)\hat{P}_{1,0}(s)\right) \tag{29}$$

Substitute (14), (28), (29) in (23)

$$\hat{P}_{n,0}(s) = \frac{1}{\sqrt{\omega_3^2 - a_3^2}}\left[\sum_{m=1}^{\infty}\left(2\gamma_1\mu_{v1}b_1^{m+1}\frac{\hat{\psi}(s)}{a_1}\left(\hat{A}_1(s) + \mu\hat{A}_1(s)\hat{P}_{1,0}(s)\right) + 2\gamma_2\mu_{v2}b_2^{m+1}\frac{\hat{\psi}(s)}{a_2}\left(\gamma_1\hat{A}_2(s) + \gamma_2\hat{A}_2(s)\hat{P}_{1,0}(s)\right)\right) + \sum_{m=0}^{\infty}\frac{\beta\lambda^m}{(s+\lambda+\beta)^m\hat{P}_{0,3}(s)}\right] \times\hat{X}_3(s) + 2\frac{\mu b_3^{n+1}}{a_3}\hat{\psi}(s)^n\frac{\beta}{(s+\lambda+\alpha)}\hat{P}_{0,3}(s) \tag{30}$$

Substitute $n = 1$ in the above equation

$$\begin{aligned}
\hat{P}_{1,0}(s) = & \frac{1}{\sqrt{\omega_3^2 - a_3^2}} \sum_{m=1}^{\infty} \left[2\gamma_1 \mu_{v1} b_1^{m+1} \frac{\hat{\psi}^m(s)}{a_1} \hat{A}_1(s) + 2\gamma_2 \mu_{v2} b_2^{m+1} \frac{\hat{\psi}^m(s)}{a_2} \gamma_1 \hat{A}_2(s) \right] \hat{X}_3(s) \\
& + \frac{1}{\sqrt{\omega_3^2 - a_3^2}} \sum_{m=1}^{\infty} \left[2\gamma_1 \mu_{v1} b_1^{m+1} \frac{\hat{\psi}^m(s)}{a_1} \hat{A}_2(s) + 2\gamma_2 \mu_{v2} b_2^{m+1} \frac{\hat{\psi}^m(s)}{a_2} \gamma_2 \hat{A}_2(s) \right] \\
& \times \hat{X}_3(s) \hat{P}_{1,0}(s) + \sum_{m=0}^{\infty} \frac{\beta \lambda^m}{(s+\lambda+\beta)^m \hat{P}_{0,3}(s)} \Bigg] \hat{X}_3(s) \\
& + 2 \frac{\mu b_3^2}{a_3} \hat{\psi}(s) \frac{\beta}{(s+\lambda+\alpha)} \hat{P}_{0,3}(s)
\end{aligned}$$

$$\begin{aligned}
\hat{P}_{1,0}(s) = & \Bigg[\sum_{j=0}^{\infty} \Bigg(\frac{1}{\sqrt{\omega_3^2 - a_3^2}} \sum_{m=1}^{\infty} \Bigg[2\gamma_1 \mu_{v1} b_1^{m+1} \frac{\hat{\psi}^m(s)}{a_1} \mu \hat{A}_1(s) + 2\gamma_2 \mu_{v2} b_2^{m+1} \frac{\hat{\psi}^m(s)}{a_2} \gamma_2 \\
& \times \hat{A}_2(s) \Bigg] \hat{X}_3(s) \Bigg)^j \Bigg] \Bigg[\frac{1}{\sqrt{\omega_3^2 - a_3^2}} \sum_{m=1}^{\infty} \Bigg[2\gamma_1 \mu_{v1} b_1^{m+1} \frac{\hat{\psi}^m(s)}{a_1} \hat{A}_1(s) + 2\gamma_2 \mu_{v2} b_2^{m+1} \\
& \times \frac{\hat{\psi}^m(s)}{a_2} \gamma_1 \hat{A}_2(s) \Bigg] \hat{X}_3(s) + \Bigg(\sum_{m=0}^{\infty} \frac{\beta \lambda^m}{(s+\lambda+\beta)^m} \hat{X}_3(s) + 2 \frac{\mu b_3^2}{a_3} \hat{\psi}(s) \\
& \times \frac{\beta}{(s+\lambda+\alpha)} \Bigg) \hat{P}_{0,3}(s) \Bigg] \\
\hat{P}_{1,0}(s) = & \hat{A}_3(s) \hat{P}_{0,3}(s) + \hat{A}_4(s)
\end{aligned} \tag{31}$$

where,

$$\begin{aligned}
A_1(s) = & \left[\sum_{j=0}^{\infty} \left(\frac{2\mu_{v1}^2 b_1^2 \psi(\hat{s})}{(s+\lambda+\alpha+\gamma_1) a_1} \right)^j \right] \frac{1}{(s+\lambda+\alpha+\gamma_1)} \\
A_2(s) = & \left[\sum_{j=0}^{\infty} \frac{2\mu_{v2}^2 b_2^2}{(s+\lambda+\alpha) a_2} \hat{\psi}(s) \right]^j \frac{1}{(s+\lambda+\alpha)} \\
A_3(s) = & \Bigg[\sum_{j=0}^{\infty} \Bigg(\frac{1}{\sqrt{\omega_3^2 - a_3^2}} \sum_{m=1}^{\infty} \Bigg[2\gamma_1 \mu_{v1} b_1^{m+1} \frac{\hat{\psi}^m(s)}{a_1} \mu \hat{A}_1(s) + 2\gamma_2 \mu_{v2} b_2^{m+1} \\
& \times \frac{\hat{\psi}^m(s)}{a_2} \gamma_2 \hat{A}_2(s) \Bigg] \hat{X}_3(s) \Bigg)^j \Bigg] \Bigg[\frac{1}{\sqrt{\omega_3^2 - a_3^2}} \sum_{m=1}^{\infty} \Bigg[2\gamma_1 \mu_{v1} b_1^{m+1} \frac{\hat{\psi}^m(s)}{a_1} \hat{A}_1(s) \\
& + 2\gamma_2 \mu_{v2} b_2^{m+1} \frac{\hat{\psi}^m(s)}{a_2} \gamma_1 \hat{A}_2(s) \Bigg] \hat{X}_3(s) \\
A_4(s) = & \Bigg[\sum_{j=0}^{\infty} \Bigg(\frac{1}{\sqrt{\omega_3^2 - a_3^2}} \sum_{m=1}^{\infty} \Bigg[2\gamma_1 \mu_{v1} b_1^{m+1} \frac{\hat{\psi}^m(s)}{a_1} \mu \hat{A}_1(s) + 2\gamma_2 \mu_{v2} b_2^{m+1} \frac{\hat{\psi}^m(s)}{a_2} \gamma_2 \\
& \times \hat{A}_2(s) \Bigg] \hat{X}_3(s) \Bigg)^j \Bigg] \Bigg(\sum_{m=0}^{\infty} \frac{\beta \lambda^m}{(s+\lambda+\beta)^m} \hat{X}_3(s) + 2 \frac{\mu b_3^2}{a_3} \hat{\psi}(s) \frac{\beta}{(s+\lambda+\alpha)} \Bigg) \\
\hat{X}_3(s) = & b_3^{n-m} \left[I_{n-m}(.) - I_{n+m}(.) \right] e^{-k_3(t-y)}
\end{aligned}$$

Substitute (31) in (30)

$$\hat{P}_{n,0}(s) = \Big[\frac{1}{\sqrt{\omega_3^2 - a_3^2}}\Big[\sum_{m=1}^{\infty}\Big(2\gamma_1\mu_{v1}b_1^{m+1}\frac{\hat{\psi}(s)^m}{a_1}\Big)\mu\hat{A}_1(s)\hat{A}_3(s)$$
$$+\Big(2\gamma_2\mu_{v2}b_2^{m+1}\frac{\hat{\psi}(s)^m}{a_2}\Big)\gamma_2\hat{A}_2(s)\hat{A}_3(s)\Big]\hat{X}_3(s)$$
$$+\sum_{m=0}^{\infty}\frac{\beta\lambda^m}{(s+\lambda+\beta)^m} + 2\mu\frac{b_3^{n+1}}{a_3}\hat{\psi}^n(t)\frac{\beta}{(s+\lambda+\alpha)}\Big]\hat{P}_{0,3}(s)$$
$$+\Big[\frac{1}{\sqrt{\omega_3^2 - a_3^2}}\sum_{m=1}^{\infty}\Big(2\gamma_1\mu_{v1}b_1^{m+1}\frac{\hat{\psi}(s)^m}{a_1}\big(\hat{A}_1(s)+\mu\hat{A}_1(s)\hat{A}_4(s)\big)$$
$$+2\gamma_2\mu_{v2}b_2^{m+1}\frac{\hat{\psi}(s)^m}{a_2}\big(\gamma_1\hat{A}_2(s)+\gamma_2\hat{A}_2\hat{A}_4(s)\big)\Big)\Big]\hat{X}_3(s) \tag{32}$$

Substitute (31) in (28) and (29)

$$\hat{P}_{n,1}(s) = 2\mu_{v1}b_1^{n+1}\frac{\hat{\psi}(s)}{a_1}\big[\hat{A}_1(s)+\mu\hat{A}_1(s)\hat{A}_4(s)+\mu\hat{A}_1(s)\hat{A}_3(s)\hat{P}_{0,3}(s)\big] \tag{33}$$

$$\hat{P}_{n,2}(s) = 2\mu_{v2}b_2^{n+1}\frac{\hat{\psi}(s)}{a_2}\big[\gamma_1\hat{A}_2(s)+\gamma_2\hat{A}_2(s)\hat{A}_4(s)+\gamma_2\hat{A}_2(s)\hat{A}_6(s)\hat{P}_{0,3}(s)\big] \tag{34}$$

Inverting (14), (32)-(34)

$$P_{n,5}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!}e^{-(\lambda+\beta)t} * P_{0,3}(t) \text{ for } n \geq 1 \tag{35}$$

$$P_{n,0}(t) = \Big[I_0(t)\Big(\sum_{m=1}^{\infty}2\gamma_1\mu_{v1}\frac{b_1^{m+1}}{a_1}\psi(t)^m * \mu A_1(t) * A_3(t) + 2\gamma_2\mu_{v2}\frac{b_2^{m+1}}{a_2}\psi(t)^m$$
$$* \gamma_2 A_2(t) * A_3(t)\Big) * X_3(t) + \sum_{m=0}^{\infty}\frac{\lambda^m t^{m-1}}{(m-1)!}e^{-(\lambda+\beta)t} + 2\mu\frac{b_3^{n+1}}{a_3}\psi(t) * \beta e^{-(\lambda+\beta)t}\Big]$$
$$* P_{0,3}(t) + \Big[I_0(t)\Big(\sum_{m=1}^{\infty}2\gamma_1\mu_{v1}\frac{b_1^{m+1}}{a_1}\psi(t)^m * \big(\mu A_1(t)+\mu A_1(t) * A_4(t)\big)$$
$$+2\gamma_2\mu_{v2}\frac{b_2^{m+1}}{a_2}\psi(t)^m * \big(\gamma_1 A_2 + \gamma_2 A_2(t) * A_4(t)\big)\Big) * X_3(t)\Big] \text{ for } n \geq 0 \tag{36}$$

$$P_{n,1}(t) = 2\mu_{v1}\frac{b_1^{n+1}}{a_1}\psi(t)\big[A_1(t)+\mu A_1(t) * A_4(t)+\mu A_1(t) * A_3(t)P_{0,3}(t)\big] \quad \text{for } n \geq 0 \tag{37}$$

$$P_{n,2}(t) = 2\mu_{v2}\frac{b_2^{n+1}}{a_2}\psi(t)\big[\gamma_1 A_2(t)+\gamma_2 A_2(t) * A_4(t)+\gamma_2 A_2(t) * A_4(t)P_{0,3}(t)\big] \quad \text{for } n \geq 0 \tag{38}$$

Here all the probabilities are purely expressed in terms of $P_{0,3}(t)$. Using (12) we can find $P_{0,3}(t)$ in the following manner

$$\hat{P}_{0,3}(s) = \Big[\sum_{j=0}^{\infty}-\Big(\frac{\alpha}{(s+\lambda+\alpha+\beta)}\sum_{n=1}^{\infty}\frac{\lambda^n}{(s+\lambda+\beta)^n}\Big)^j\Big]\Big[\frac{\alpha}{s(s+\lambda+\alpha+\beta)}\Big] \tag{39}$$

Inverting the above

$$P_{0,3}(t) = \left[\sum_{j=0}^{\infty} \alpha e^{-(\lambda+\alpha+\beta)t} * \sum_{n=1}^{\infty} \frac{\lambda^n t^{n-1} e^{-(\lambda+\beta)t}}{(n-1)!} \right] * \left[\delta(t) * \alpha e^{-(\lambda+\alpha+\beta)t} \right] \tag{40}$$

4. Numerical Analysis

In this section, graphs show the system's transient probabilities in various states, including busy state, type-I and type-II vacation states, as well as disaster state. Additionally, the system's mean is recorded over time. The following parameter values were used to generate the graphs: $\lambda = 2$, $\mu = 3$, $\mu_{v1} = 0.8$, $\mu_{v2} = 0.7$, $\alpha = 0.6$ and $\beta = 0.5$. Figure 2 dipicts the behaviour of the transient probability of busy period against time t for different values of n. The probability curves start at 0 and converge to a steady state over time, as shown by this graph. Figure 3 displays the behaviour of transient probabilities during the type-I vacation period, demonstrating that all probability curves begin at 0 and progressively rise to a certain extent as t rises before stabilising.

Figure 4 exhibits the graph of type-II vacation period transient probability over time t. The probability curves in that graph start at 0 and move towards a steady state over time. Furthermore, it is clear that a type-I vacation has a higher probability of having more customers than a type-II vacation does at any given time instance t. This discrepancy results from the servers in type-II vacation mode quickly switching to busy mode after the vacation is over, whereas type-I vacation takes a longer period of time. As a result, during type-II vacation, customers do not need to wait for additional processing time.

The behaviour of $P_{n,3}(t)$ appears in Figure 5, where all probability curves initially start at 0 and gradually grow to some extent as t increases, finally reaching a steady state. The mean behaviour for various disaster and repair rate values is shown in Figures 6 and 7. According to these data, an increase in repair rate results in an increase in the mean size. Similar to this, a rise in the disaster rate causes a fall in the mean size.

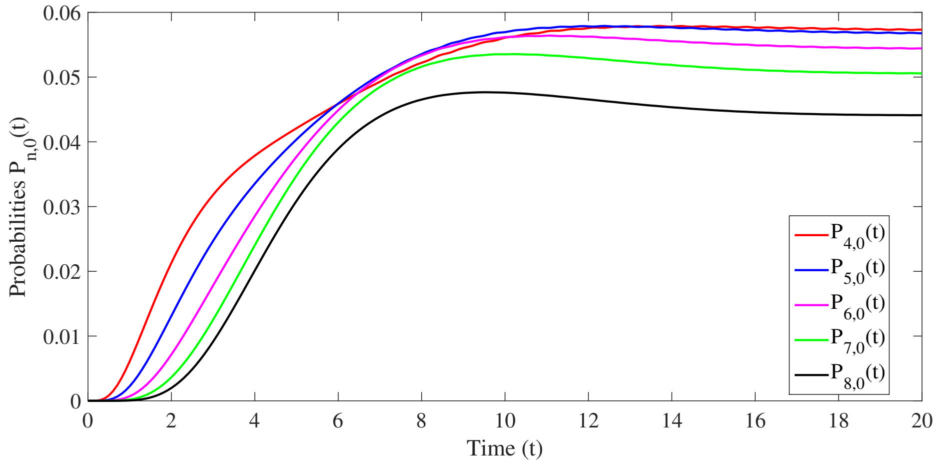

Figure 1: *Probabilities $P_{n,0}(t)$ Vs Time*

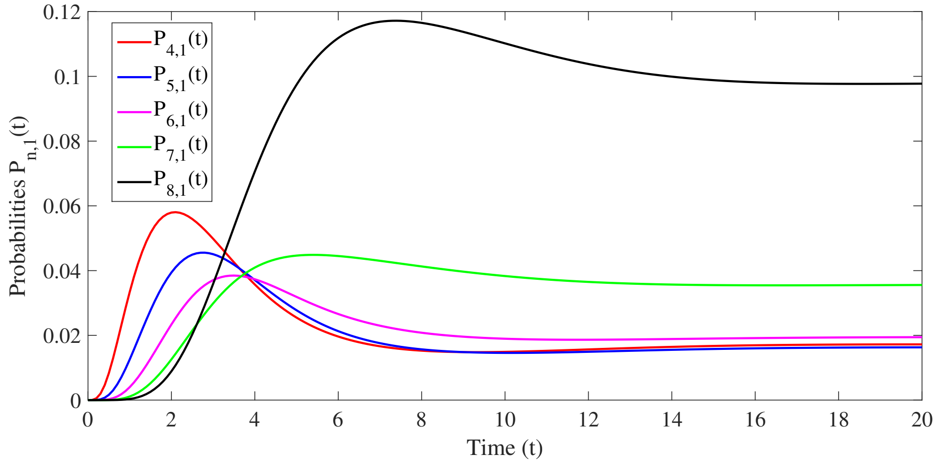

Figure 2: *Probabilities $P_{n,1}(t)$ Vs Time*

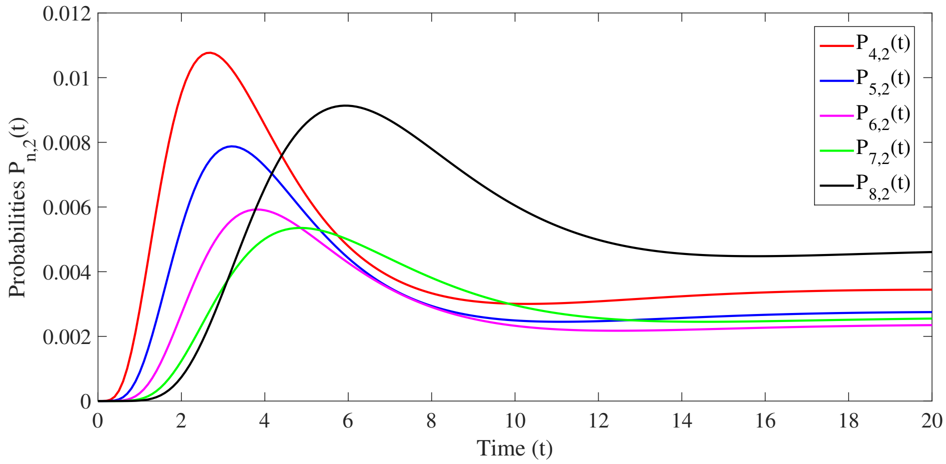

Figure 3: *Probabilities $P_{n,2}(t)$ Vs Time*

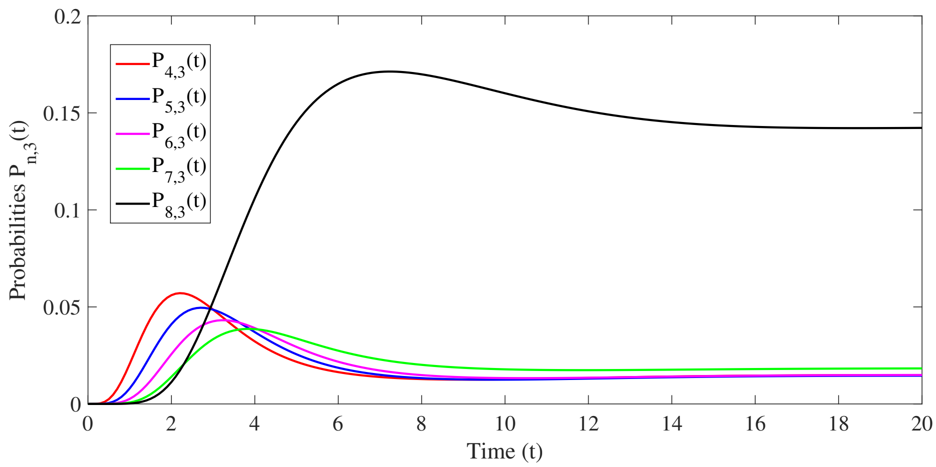

Figure 4: *Probabilities $P_{n,3}(t)$ Vs Time*

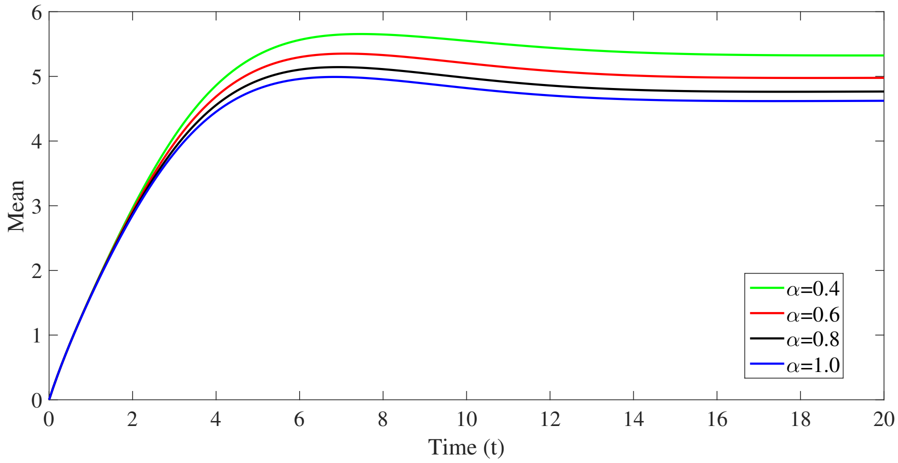

Figure 5: *Probabilities Mean Vs Time*

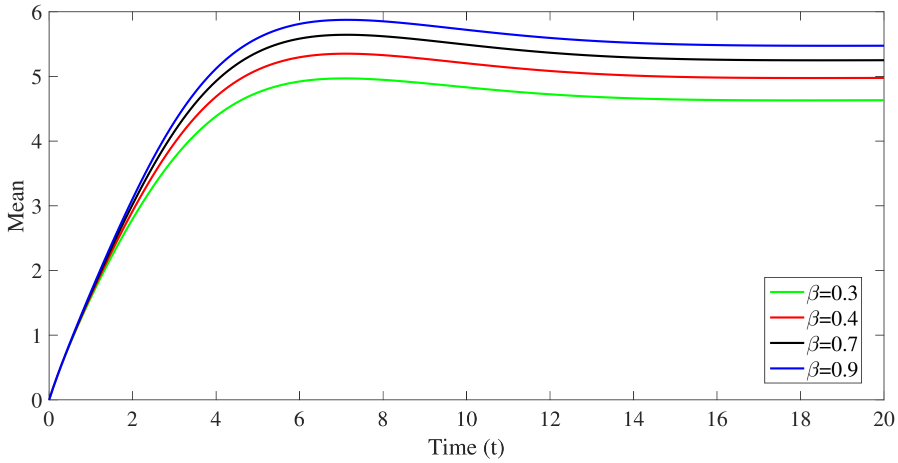

Figure 6: *Mean Vs Time*

5. Conclusion and Future work

In this study, a single server queueing system with multiple differentiated vacation, disaster, and repair periods was investigated. The modified Bessel function of the first kind was used to derive the time-dependent probability of the system size. The proposed model's numerical results indicate that the time-dependent probabilities eventually reach their respective steady-state probabilities.

By taking into account multi-server differentiated vacation queueing systems with disaster and repair, future research can build on this study. Analysing such systems would provide us with a more complete understanding of their performance and behaviour. It would also be advantageous to investigate stochastic decomposition for this model since it can provide insightful information about the dynamics of the system and aid in improving performance. These avenues of inquiry will advance queueing theory as a whole and improve our comprehension of intricate queueing systems in real-world situations.

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