SOME PROPERTIES OF TSALLIS ENTROPY BASED ON A DOUBLY TRUNCATED (INTERVAL) RANDOM VARIABLE

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Abstract

In this paper, we first study doubly truncated (interval) Tsallis entropy and suggest doubly truncated (interval) cumulative residual Tsallis entropy ($ICRT$), which is an extension of cumulative residual Tsallis entropy (CRT) and the dynamic CRT defined by the aid of Sati and Gupta and of Kumar, respectively. We investigate some properties and characterization of this measure, such as its relation with doubly truncated Shannon entropy, mean residual (past) life, and hazard rate (or reversed hazard rate). Also, the twin measure, doubly truncated (interval) cumulative past Tsallis entropy, is determined, and some of its properties are studied. Moreover, their monotonicity and related aging classes of distributions are expressed, and the upper (lower) bound for them is acquired. In the end, we propose four nonparametric estimators and compare their performance by utilizing simulation data. Also, being based on the best-proposed estimator, a real data set is additionally examined.

Keywords: Doubly truncated (interval) Tsallis entropy, Doubly truncated (interval) cumulative residual Tsallis entropy (ICRT), Doubly truncated (interval) cumulative past Tsallis entropy (ICPT), Hazard rate, Reversed hazard rate, Mean residual life, Mean past life, Nonparametric estimators

1. Introduction

The notion of entropy, later generalized to information theory and statistical mechanics, was initially created by physicists in the area of equilibrium thermodynamics. The most famous one is due to [22], that plays an essential role in measuring the average uncertainty of a random variable. Entropy plays an important role in measuring the index of dispersion, volatility, or uncertainty related to a random variable X. Here and during this paper, X is an absolutely continuous nonnegative random variable, with probability density function (pdf) $f(x)$ and survival function $\bar{F}(x) = P(X > x)$. Then the average amount of uncertainty associated with the random variable X as given by Shannon entropy, is

$$H(X) = -\int_0^{\infty} f(x) \ln f(x) dx.$$

Although, in certain situations, the Shannon entropy is not suitable where some generalized forms are of importance. Several generalized entropy measures are accessible in literature, which have many huge properties consisting of smoothness, big dynamic range with respect to certain conditions, and many others, which lead them to greater flexibility in practice. One prevalent generalization is the Tsallis entropy, introduced by [24], determined as a generalization of the Boltzmann–Gibbs entropy. Inside the studying of statistical mechanics, Tsallis entropy gives a much broader view of how disorder emerges in macroscopic systems. For a continuous nonnegative random variable X, Tsallis entropy is determined as

$$T^{\alpha}(X) = \frac{1}{\alpha - 1}\left(1 - \int_0^{\infty} f(x)^{\alpha} dx\right), \tag{1}$$

where $0 < \alpha \neq 1$. Clearly, when $\alpha \to 1$, we have $T^{\alpha}(X) \to H(X)$. Tsallis exploited its nonextensive features, and it has more and more extensive applications in science and technology. This entropy measure is extra flexible because of the parameter α, and it increases the scope of application. Tsallis entropy preserves many significant characteristics of Shannon entropy except for the additivity property. From the years 2000 on, an increasingly wide spectrum of natural, artificial, and socially complicated systems were identified that verify the predictions and conclusions derived from this nonadditive entropy. Extensive or nonextensive statistical mechanics derive from the additivity or nonadditivity of the corresponding entropy measures. The Tsallis entropy is broadly utilized in physics to examine the distribution characterizing the movement of cold atoms in dissipative optical lattices [9] and signal processing [23]. More properties and applications of Tsallis entropy have been mentioned in [24, 25].
Considering the measures based on residual lifetime random variable, $X_t = (X - t|X \geq t)$ has an essential role in many grounds, including reliability theory, survival analysis, and information theory. So, [10, 6] defined the residual Tsallis entropy (RT) based on the random variable X_t by

$$RT(X;t) = \frac{1}{\alpha - 1}\left(1 - \int_t^{\infty} (\frac{f(x)}{\bar{F}(t)})^{\alpha} dx\right).$$

The expected uncertainty involved in the remaining lifetime of a component is measured basically by RT. It is clear that $RT(X;0) = T^{\alpha}(X)$. Lately, [10, 4] introduced an entropy-based measure of uncertainty in past lifetime distributions and denominated it past Tsallis entropy (PT). The uncertainty of the idle time of a component or system that is based on past lifetime random variable $X_t^* = (t - X|X \leq t)$ is indicated by PT, and it is given by

$$PT(X;t) = \frac{1}{\alpha - 1}\left(1 - \int_0^{t} (\frac{f(x)}{F(t)})^{\alpha} dx\right),$$

and also, $PT(X;\infty) = T^{\alpha}(X)$.

Currently, many researchers advanced new measures of uncertainty to overcome the limitations of traditional entropy measures and increase the applicability of information measures in diverse areas of science and engineering. With this motivation, [18] studied an alternative to Shannon differential entropy. The cumulative residual entropy (CRE) is obtained by replacing the pdf $f(x)$ in $H(X)$ with the survival function $\bar{F}(x) = P(X > x)$, given by $H(X) = -\int_0^{\infty} \bar{F}(x) \ln \bar{F}(x) dx$. The CRE is regarded to be greater stable due to the fact that the distribution function is greater regular than the pdf, and it owns more mathematical properties and special applications. Also, it is easily computable, always nonnegative, and its definition is valid in both the continuous and discrete cases. Additionally , the distribution exists despite the fact that the pdf does now not.

In information theory, numerous attempts have been made by researchers, and an eminent amount of work has been done from both theoretical and application points of view for studying and extending the notion of CRE. Motived by the extensive applicability of $H(X)$, a cumulative version of (1) studied by [19], is determined as the cumulative Tsallis entropy (CRT)

$$CRT(X) = \frac{1}{\alpha - 1}\left(1 - \int_0^{\infty} \bar{F}(x)^{\alpha} dx\right).$$

Although [19] denoted that $CRT(X)$ tends to $CRE(X)$ when $\alpha \to 1$, where $CRE(X) = -\int_0^\infty \bar{F}(x) \ln(\bar{F}(x)dx$, defined by [18], but [16] showed with a counter example that it is not true. The cumulative past Tsallis entropy (CPT) has also been introduced and studied by [16] as follows:

$$CPT(X) = \frac{1}{\alpha - 1}\left(1 - \int_0^\infty F(x)^\alpha dx\right).$$

[19] gave the dynamic version of cumulative residual Tsallis entropy (DCRT), which is the CRT of the residual random variable X_t and it is given by

$$DCRT(X;t) = \frac{1}{\alpha - 1}\left(1 - \int_t^\infty (\frac{\bar{F}(x)}{\bar{F}(t)})^\alpha dx\right),$$

and $DCRT(X;0) = CRT(X)$. Furthermore, [8] studied many properties of DCRT, and [16] introduced the dynamic version of cumulative past Tsallis entropy ($DCPT$) by

$$DCPT(X;t) = \frac{1}{\alpha - 1}\left(1 - \int_0^t (\frac{F(x)}{F(t)})^\alpha dx\right),$$

and $DCPT(X;\infty) = CPT(X)$. Occasionally, in many conditions, we just possess information between two points. Thus, we have to look at the statistical measures (particularly in information theory and reliability) under the case of doubly truncated random variables. For instance, in reliability, if X indicates the lifetime of a unit, then the random variable $X_{t_1,t_2} = (X - t_1 | t_1 \leq X \leq t_2)$ is known as the doubly truncated residual lifetime. Note that the well-known random variable, $X_t = (X - t | X \geq t)$, is the particular case of X_{t_1,t_2} when t_2 tends to ∞. Also, doubly truncated past lifetime is the random variable $X^*_{t_1,t_2} = (t_2 - X | t_1 \leq X \leq t_2)$, which in the specific case when $t_1 = 0$, it is the past lifetime random variable X^*_t. Another generalization of Tsallis entropy is based on a doubly truncated (interval) random variable [13], which reads as follows:

$$T^\alpha(X;t_1,t_2) = \frac{1}{\alpha - 1}\left(1 - \int_{t_1}^{t_2} (\frac{f(x)}{F(t_2) - F(t_1)})^\alpha dx\right), \tag{2}$$

where $(t_1,t_2) \in D = \{(t_1,t_2) : F(t_1) < F(t_2)\}$ and $T^\alpha(X;0,\infty)$ is the Tsallis entropy $T^\alpha(X)$, and $T^\alpha(X;t_1,\infty)$ is the residual entropy $RT(X;t_1)$ and also $T^\alpha(X;0,t_2)$ is the past entropy $PT(X;t_2)$. Also, when $\alpha \to 1$, we have $T^\alpha(X;t_1,t_2) \to H(X;t_1,t_2) = -\int_{t_1}^{t_2} \frac{f(x)}{F(t_2)-F(t_1)} \ln(\frac{f(x)}{F(t_2)-F(t_1)})$.

The distribution function estimation is not only an interesting problem by itself, but also it emerges naturally in actual problems of many scientific fields, consisting of seismology, hydrology, environmental sciences, and so on. Currently, in those disciplines, numerous methodologies have appeared for attacking statistical problems based on nonparametric ideas. With this motivation, the performance of four nonparametric estimators of $ICPT$ is compared, and also a real-life data set is illustrated based on the best-proposed estimator.

In this paper, some properties of $T^\alpha(X;t_1,t_2)$ are introduced. Additionally, we discuss the doubly truncated (interval) cumulative residual Tsallis entropy ($ICRT$) and doubly truncated (interval) cumulative past Tsallis entropy ($ICPT$), which can be general forms of the preceding findings. Some properties of $ICRT$ and $ICPT$ and their relationships with reliability measures, including hazard rate (or reversed hazard rate) and mean residual life (or mean past life), are studied. Finally, we consider four empirical and kernel-based estimators. Then, by using simulated data, we compare the behavior of the proposed estimators. In addition, a real data set from environmental monitoring is studied.

2. Doubly truncated Tsallis entropy

In this section, we express some properties and characterization results of $T^\alpha(X;t_1,t_2)$. First, for the $T^\alpha(X;t_1,t_2)$, an upper interval is acquired with respect to t_2, for any fixed t_1, in the next theorem. [13] proved a result similar to the following theorem, with respect to t_1, for any fixed

t_2. Also, it should be noted that [11] introduced the generalized failure rate (GFR) based on the doubly truncated random variables by

$$h_1(t_1,t_2)=\lim_{h\to 0^+}\left[\frac{P(t_1\le x\le t_1+h|t_1\le x\le t_2)}{h}\right]=\frac{f(t_1)}{F(t_2)-F(t_1)} \tag{3}$$

and

$$h_2(t_1,t_2)=\lim_{h\to 0^-}\left[\frac{P(t_2\le x\le t_2+h|t_1\le x\le t_2)}{h}\right]=\frac{f(t_2)}{F(t_2)-F(t_1)}, \tag{4}$$

where their relationships with $m(t_1,t_2)=E(X|t_1\le X\le t_2)=\int_{t_1}^{t_2}x\frac{f(x)}{F(t_2)-F(t_1)}dx$ for $(t_1,t_2)\in D$ are as follows:

$$h_1(t_1,t_2)=\frac{\frac{\partial m(t_1,t_2)}{\partial t_1}}{m(t_1,t_2)-t_1}, \tag{5}$$

$$h_2(t_1,t_2)=\frac{\frac{\partial m(t_1,t_2)}{\partial t_2}}{t_2-m(t_1,t_2)}. \tag{6}$$

A lower (upper) bound for the $ICRT(X;t_1,t_2)$ when increasing the ICRT property is acquired in the next theorem, for $0<\alpha<1(\alpha>1)$.

Theorem 1. The random variable X has increasing doubly truncated (interval) Tsallis entropy property if and only if the following inequalities are satisfied for all $(t_1,t_2)\in D$ and $0<\alpha<1(\alpha>1)$:

$$\frac{1}{\alpha-1}\left(1-\frac{1}{\alpha}\left(\frac{\frac{\partial m(t_1,t_2)}{\partial t_2}}{t_2-m(t_1,t_2)}\right)^{\alpha-1}\right)\le(\ge)T^\alpha(X;t_1,t_2).$$

Proof. By differentiating $T^\alpha(X;t_1,t_2)$ of the form (2) with respect to t_2, we have

$$\begin{aligned}\frac{\partial T^\alpha(X;t_1,t_2)}{\partial t_2} &= \frac{-1}{\alpha-1}\left((\frac{f(t_2)}{F(t_2)-F(t_1)})^\alpha-\alpha\frac{f(t_2)}{F(t_2)-F(t_1)}\int_{t_1}^{t_2}(\frac{f(x)}{F(t_2)-F(t_1)})^\alpha dx\right)\\ &= \frac{-1}{\alpha-1}{h_2}^\alpha(t_1,t_2)+\frac{\alpha}{\alpha-1}h_2(t_1,t_2)(1-(\alpha-1)T^\alpha(X;t_1,t_2))\\ &= h_2(t_1,t_2)\ \frac{-1}{\alpha-1}{h_2}^{\alpha-1}(t_1,t_2)+\frac{\alpha}{\alpha-1}(1-(\alpha-1)T^\alpha(X;t_1,t_2)).\end{aligned}$$

So, after suitable substitution of equation (6) and simplifying the equation we have,

$$T^\alpha(X;t_1,t_2)\le(\ge)\frac{1}{\alpha-1}(1-\frac{1}{\alpha}(h_2(t_1,t_2))^{\alpha-1}),$$

the proof is complete. ∎

We study the effect of increasing transformation on $T^\alpha(Y;t_1,t_2)$.

Lemma 1. Let X be a nonnegative continuous random variable with cumulative distribution function (cdf) F, and take $Y=\phi(X)$, where $\phi(\cdot)$ is a strictly increasing differentiable function. Then

$$T^\alpha(Y;t_1,t_2)=\frac{1}{\alpha-1}\left(1-\int_{\max\{0,\phi^{-1}(t_1)\}}^{\phi^{-1}(t_2)}\left(\frac{f(x)}{F(\phi^{-1}(t_2))-F(\phi^{-1}(t_1))}\right)^\alpha\frac{1}{(\phi'(x))^{\alpha-1}}dx\right).$$

If $Z=aX+b$, with $a>0$ and $b\ge 0$, so $F_{aX+b}(z)=F_X(\frac{z-b}{a})$, then

$$T^\alpha(Z;t_1,t_2)=\frac{a^{\alpha-1}-1}{a^{\alpha-1}(\alpha-1)}+(\frac{a^{\alpha-1}-1}{a^{\alpha-1}})T^\alpha(X;\frac{t_1-b}{a},\frac{t_2-b}{a}).$$

There are an identity and inequalities for doubly truncated (interval) Tsallis entropy based on the assumptions of the following proposition.

Proposition 1. Let X be a random variable with support in $[0,r]$ where $r>0$ and symmetric with respect to $\frac{r}{2}$; that is, $F(x)=\bar{F}(r-x)$ for $0\leq x\leq r$. Then

$$T^{\alpha}(X;t_1,t_2)=T^{\alpha}(X;r-t_1,r-t_2); \qquad 0\leq t_1,t_2\leq r.$$

Proof. We have

$$\begin{aligned}
T^{\alpha}(X;t_1,t_2) &= \frac{1}{\alpha-1}\left(1-\int_{t_1}^{t_2}(\frac{f(x)}{F(t_2)-F(t_1)})^{\alpha}dx\right)\\
&= \frac{1}{\alpha-1}\left(1-\int_{t_1}^{t_2}(\frac{f(r-x)}{\bar{F}(r-t_2)-\bar{F}(r-t_1)})^{\alpha}dx\right)\\
&= -\frac{1}{\alpha-1}\left(1-\int_{r-t_1}^{r-t_2}(\frac{f(y)}{F(r-t_1)-F(r-t_2)})^{\alpha}dy\right)\\
&= \frac{1}{\alpha-1}\left(1-\int_{r-t_2}^{r-t_1}(\frac{f(y)}{F(r-t_1)-F(r-t_2)})^{\alpha}dy\right)\\
&= T^{\alpha}(X;r-t_1,r-t_2).
\end{aligned}$$

■

Example 1. If X is uniformly distributed in $[0,r]$, then for $0\leq t_1,t_2\leq r$, we have $T^{\alpha}(X;t_1,t_2)=\frac{1}{\alpha-1}(t_2-t_1)^{1-\alpha}$, which is in agreement with Proposition 1.

Proposition 2. Let X be a nonnegative and absolutely continuous random variable. Then for $\alpha>1(0<\alpha<1)$, we have

$$1-(t_2-t_1)(\leq)T^{\alpha}(X;t_1,t_2)\leq(t_2-t_1)-1. \tag{7}$$

Proof. The upper bound and lower bound given in (7) can be obtained from the well-known inequality $\ln x\leq x-1$, where $x>0$. Let $x=\frac{f(x)}{F(t_2)-F(t_1)}$. Then $x^{\alpha-1}>0$ for $\alpha>1(0<\alpha<1)$, and by using $H(X;t_1,t_2)\leq(t_2-t_1)-1$ [15], the proof is complete. ■

Proposition 3. Let X be a nonnegative and absolutely continuous random variable with cdf $F(x)$ and pdf $f(x)$. If $f(x)$ is decreasing in x, then for $0<\alpha<1(\alpha>1)$,

$$\frac{1-h_1^{\alpha}(t_1,t_2)(t_2-t_1)}{(\alpha-1)}\geq(\leq)T^{\alpha}(X;t_1,t_2)\geq(\leq)\frac{1-h_2^{\alpha}(t_1,t_2)(t_2-t_1)}{(\alpha-1)},$$

where $h_1(t_1,t_2)$ and $h_2(t_1,t_2)$ are defined in (3) and (4).

Proof. Let $f(x)$ be decreasing in x. Then for $t_1\leq x\leq t_2$, we have

$$\frac{f(t_1)}{F(t_2)-F(t_1)}\geq\frac{f(x)}{F(t_2)-F(t_1)}\geq\frac{f(t_2)}{F(t_2)-F(t_1)}.$$

So,

$$\int_{t_1}^{t_2}(\frac{f(t_1)}{F(t_2)-F(t_1)})^{\alpha}dx\geq\int_{t_1}^{t_2}(\frac{f(x)}{F(t_2)-F(t_1)})^{\alpha}dx\geq\int_{t_1}^{t_2}(\frac{f(t_2)}{F(t_2)-F(t_1)})^{\alpha}dx.$$

Then

$$1-h_1^{\alpha}(t_1,t_2)(t_2-t_1)\leq1-\int_{t_1}^{t_2}(\frac{f(x)}{F(t_2)-F(t_1)})^{\alpha}dx\leq1-h_2^{\alpha}(t_1,t_2)(t_2-t_1).$$

Thus for $0<\alpha<1(\alpha>1)$, after some calculations, the proof is complete. ■

Example 2. Let X be a nonnegative and absolutely continuous random variable with cdf $F(x)=1-e^{-x}$ and pdf $f(x)=e^{-x}$. Then, $T^{\alpha}(X;t_1,t_2)=\frac{1}{(\alpha-1)}\left(1-\frac{\frac{1}{\alpha}(e^{-\alpha t_1}-e^{-\alpha t_2})}{(e^{t_1}-e^{t_2})^{\alpha}}\right)$, for all $\alpha>1(0<\alpha<1)$ and $t_1,t_2(t_1<t_2)$, which is in agreement with Proposition 2 and Proposition 3.

For increasing function $f(x)$, the above proposition can be similarly proved.

3. Interval cumulative residual and past Tsallis entropy

Let X be an absolutely continuous random variable and let $D = \{(x, y) : F(x) < F(y)\}$. Then we define the $ICPT$ and $ICRT$ functions, respectively, as follows:

$$ICPT(X; t_1, t_2) = \frac{1}{\alpha - 1}\left(1 - \int_{t_1}^{t_2} (\frac{F(x)}{F(t_2) - F(t_1)})^{\alpha} dx\right) \tag{8}$$

and

$$ICRT(X; t_1, t_2) = \frac{1}{\alpha - 1}\left(1 - \int_{t_1}^{t_2} (\frac{\bar{F}(x)}{\bar{F}(t_1) - \bar{F}(t_2)})^{\alpha} dx\right), \tag{9}$$

where $(t_1, t_2) \in D$. It is clear that, $ICRT(X; 0, \infty)$ is $CRT(X)$, and $ICRT(X; t_1, \infty)$ is $DCRt(X; t_1)$. Also, $ICPT(X; 0, \infty)$ is $CPT(X)$, and $ICPE(X; 0, t_2)$ is $DCPT(X; t_2)$. The applications of classes of life distributions can be demonstrated in different areas, including reliability, engineering, biological science, maintenance, and biometrics. Hence, statisticians and reliability analysts are interested in modeling survival information and classifications of life distributions based on a few aspects of aging. For instance, we refer the reader to [15, 1, 26]. So, the corresponding aging classes are defined as follows.

Definition 1. Consider the random variable X.

- X is said to have decreasing interval cumulative residual Tsallis entropy (DICRT) property if and only if for any fixed t_2, $ICRT(X; t_1, t_2)$ is decreasing with respect to t_1.
- X is said to have increasing interval cumulative past Tsallis entropy (IICPT) property if and only if for any fixed t_1, $ICPT(X; t_1, t_2)$ is increasing with respect to t_2.

An upper bound for $ICRT(X; t_1, t_2)$ with the decreasing (increasing) ICRT property is acquired in the next theorems.

Theorem 2. The random variable X has decreasing (increasing) ICRT property if and only if the following inequality is satisfied for all $(t_1, t_2) \in D$ and $0 < \alpha < 1(\alpha > 1)$:

$$ICRT(X; t_1, t_2) \leq (\geq) \frac{1}{\alpha - 1}\left(1 - \frac{1}{\alpha}(\frac{\bar{F}(t_1)}{f(t_1)})^{\alpha}\left(\frac{1 + \frac{\partial \mu(t_1, t_2)}{\partial t_1}}{\mu(t_1, t_2)}\right)^{\alpha - 1}\right).$$

Proof. By differentiating $ICRT(X; t_1, t_2)$ of the form (9) with respect to t_1, we have

$$\begin{aligned}
\frac{\partial ICRT(X; t_1, t_2)}{\partial t_1} &= \frac{1}{\alpha - 1}\left((\frac{\bar{F}(t_1)}{\bar{F}(t_1) - \bar{F}(t_2)})^{\alpha}\right.\\
&\quad \left. -\alpha \frac{f(t_1)}{\bar{F}(t_1) - \bar{F}(t_2)} \int_{t_1}^{t_2} (\frac{\bar{F}(x)}{\bar{F}(t_1) - \bar{F}(t_2)})^{\alpha} dx\right)\\
&= \frac{1}{\alpha - 1}(\frac{\bar{F}(t_1)}{f(t_1)})^{\alpha} {h_1}^{\alpha}(t_1, t_2)\\
&\quad - \frac{\alpha}{\alpha - 1} h_1(t_1, t_2)(1 - (\alpha - 1) ICRT(X; t_1, t_2)).
\end{aligned}$$

By the definition of the GFR in (3) and (4), their relationships with $\mu(t_1, t_2) = E(X - t_1 | t_1 \leq X \leq t_2)$ and $\mu^*(t_1, t_2) = E(t_2 - X | t_1 \leq X \leq t_2)$ are, respectively, as follows:

$$h_1(t_1, t_2) = \frac{1 + \frac{\partial \mu(t_1, t_2)}{\partial t_1}}{\mu(t_1, t_2)}, \tag{10}$$

$$h_2(t_1, t_2) = \frac{1 - \frac{\partial \mu^*(t_1, t_2)}{\partial t_2}}{\mu(t_1, t_2)}. \tag{11}$$

So, after suitable substitution of Eqs. (10) and (11) and simplifying the equations, we have

$$ICRT(X;t_1,t_2) \leq (\geq) \frac{1}{\alpha-1}\left(1-\frac{1}{\alpha}(\frac{\bar{F}(t_1)}{f(t_1)})^{\alpha}(h_1(t_1,t_2))^{\alpha-1}\right).$$

■

Example 3. Let X be distributed uniformly on $(0,\beta), \beta > 0$, then it can be easily verified that,

$$ICRT(X;t_1,t_2) = \frac{1}{\alpha-1}(1-\frac{(\beta-t_1)^{\alpha+1}-(\beta-t_2)^{\alpha+1}}{(t_2-t_1)^{\alpha}(1+\alpha)}),$$

$$\mu(t_1,t_2) = \frac{t_2-t_1}{2}.$$

the differentiation of $ICRT$ with respect to t_1 is negative for all $(t_1,t_2) \in D$, which shows that the uniform distribution has$DICRT$ property and theorem2 is satisfied.

There exist no nonnegative random variables with increasing $ICRT(IICRT)$ over the domain $[0,\infty)$, indicated in the following theorem.

Theorem 3. If X is a nonnegative nondegenerate random variable, then $ICRT(X;t_1,t_2)$ cannot be an increasing function with respect to t_1 for any real fixed t_2.

Proof. First note that, using lHopitals rule, we have

$$\begin{aligned}\lim_{t_1\to t_2} ICRT(X;t_1,t_2) &= \lim_{t_1\to t_2}\frac{1}{\alpha-1}\left(1-\int_{t_1}^{t_2}(\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)})^{\alpha}dx\right)\\ &= \frac{1}{\alpha-1}\left(1-\lim_{t_1\to t_2}\frac{\int_{t_1}^{t_2}(\bar{F}(x))^{\alpha}dx}{(\bar{F}(t_1)-\bar{F}(t_2))^{\alpha}}\right)\\ &= \frac{1}{\alpha-1}\left(1-\lim_{t_1\to t_2}\frac{(\bar{F}(t_1))^{\alpha}}{\alpha f(t_1)(F(t_2)-F(t_1))^{\alpha-1}}\right)\\ &= -\infty.\end{aligned}$$

Now, on the contrary, suppose that $ICRT(X;t_1,t_2)$ is increasing in t_1. Then for all $t_1 \leq t_2$, $ICRT(X;t_1,t_2) \leq ICRT(X;t_2,t_2) = -\infty$, which contradicts with the fact that $ICRT(X;t_1,t_2) \in \Re$ for all $(t_1,t_2) \in D$. ■

In the following proposition, we obtain a lower bound, according to $\mu(X) = \int_x^{\infty}\frac{F(x)}{F(t)}dt$, for $E(\mu(X)|t_1 \leq X \leq t_2)$.

Proposition 4. Suppose that F is an absolutely continuous distribution function with $ICRT(X;t_1,t_2) < \infty$. Then, for $0 < \alpha < 1$

$$E(\mu(X)|t_1 \leq X \leq t_2) \geq (\alpha-1)ICRT(X;t_1,t_2) - 1.$$

Proof. By using $E(\mu(X)|t_1 \leq X \leq t_2) \geq ICRE(X;t_1,t_2)$ [5], we have

$$\begin{aligned}&\int_{t_1}^{t_2}\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)}\log(\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)})dx\\ \leq\ &\int_{t_1}^{t_2}\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)}\left((\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)})-1\right)dx\\ \leq\ &\int_{t_1}^{t_2}\left((\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)})-1\right)dx\\ \leq\ &\int_{t_1}^{t_2}\left((\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)})^{\alpha}-1\right)dx\\ =\ &\int_{t_1}^{t_2}(\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)})^{\alpha}-(t_2-t_1)dx.\end{aligned}$$

Then

$$\begin{aligned}
& -\int_{t_1}^{t_2} \frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)} \log(\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)})dx \\
\geq & -\int_{t_1}^{t_2} (\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)})^{\alpha} + (t_2-t_1)dx \\
\geq & -\int_{t_1}^{t_2} (\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)})^{\alpha}dx \\
= & (\alpha-1)ICRT(X;t_1,t_2)-1.
\end{aligned}$$

■

The following theorem tries to clarify the problem, achieving when the interval entropy uniquely appoints the distribution function.

Theorem 4. Let X be a nonnegative and continuous random variable and let $ICRT(X;t_1,t_2)$ be increasing with respect to t_1 and decreasing with respect to t_2. Then $ICRT(X;t_1,t_2)$ uniquely determines $F(x)$.

Proof. By differentiating $ICRT(X;t_1,t_2)$ with respect to $t_j(j=1,2)$, we have

$$\begin{aligned}
\frac{\partial ICRT(X;t_1,t_2)}{\partial t_2} &= \frac{1}{\alpha-1}\Bigg(-(\frac{\bar{F}(t_2)}{\bar{F}(t_1)-\bar{F}(t_2)})^{\alpha} \\
&\quad +\alpha\frac{f(t_2)}{\bar{F}(t_1)-\bar{F}(t_2)}\int_{t_1}^{t_2}(\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)})^{\alpha}dx\Bigg) \\
&= \frac{-1}{\alpha-1}(\frac{\bar{F}(t_2)}{f(t_2)})^{\alpha}{h_2}^{\alpha}(t_1,t_2) \\
&\quad +\frac{\alpha}{\alpha-1}h_2(t_1,t_2)(1-(\alpha-1)ICRT(X;t_1,t_2)) \\
&= -h_2(t_1,t_2)\Bigg(\frac{1}{\alpha-1}(\frac{\bar{F}(t_2)}{f(t_2)})^{\alpha}{h_2}^{\alpha-1}(t_1,t_2) \\
&\quad -\frac{\alpha}{\alpha-1}(1-(\alpha-1)ICRT(X;t_1,t_2))\Bigg),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial ICRT(X;t_1,t_2)}{\partial t_1} &= \frac{1}{\alpha-1}\Bigg((\frac{\bar{F}(t_1)}{\bar{F}(t_1)-\bar{F}(t_2)})^{\alpha} \\
&\quad -\alpha\frac{f(t_1)}{\bar{F}(t_1)-\bar{F}(t_2)}\int_{t_1}^{t_2}(\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)})^{\alpha}dx\Bigg) \\
&= \frac{1}{\alpha-1}(\frac{\bar{F}(t_1)}{f(t_1)})^{\alpha}{h_1}^{\alpha}(t_1,t_2) \\
&\quad -\frac{\alpha}{\alpha-1}h_1(t_1,t_2)(1-(\alpha-1)ICRT(X;t_1,t_2)) \\
&= h_1(t_1,t_2)\Bigg(\frac{1}{\alpha-1}(\frac{\bar{F}(t_1)}{f(t_1)})^{\alpha}{h_1}^{\alpha-1}(t_1,t_2) \\
&\quad -\frac{\alpha}{\alpha-1}(1-(\alpha-1)ICRT(X;t_1,t_2))\Bigg).
\end{aligned}$$

Thus, for fixed t_2 and arbitrary t_1, $h_1(t_1,t_2)$ is a positive solution to the following equation:

$$\begin{aligned}
g(x_{t_2}) &= x_{t_2}\left(\frac{1}{\alpha-1}(\frac{\bar{F}(t_1)}{\bar{F}(t_1)})^{\alpha}{x_{t_2}}^{\alpha-1}-\frac{\alpha}{\alpha-1}(1-(\alpha-1)ICRT(X;t_1,t_2))\right) \\
&\quad -\frac{\partial ICRT(X;t_1,t_2)}{\partial t_1}.
\end{aligned} \tag{12}$$

Similarly, for fixed t_1 and arbitrary t_2, we have $h_2(t_1, t_2)$ as a positive solution to the following equation:

$$\gamma(y_{t_1}) = y_{t_1}\left(\frac{1}{\alpha-1}(\frac{\bar{F}(t_2)}{\bar{F}(t_2)})^{\alpha}{y_{t_1}}^{\alpha-1} - \frac{\alpha}{\alpha-1}(1-(\alpha-1)ICRT(X;t_1,t_2))\right) + \frac{\partial ICRT(X;t_1,t_2)}{\partial t_2}. \quad (13)$$

By differentiating g and γ with respect to x_{t_2} and y_{t_1}, we get

$$\frac{\partial g(x_{t_2})}{\partial x_{t_2}} = \frac{\alpha}{\alpha-1}\left((\frac{\bar{F}(t_1)}{f(t_1)})^{\alpha}{x_{t_2}}^{\alpha-1} - (1-(\alpha-1)ICRT(X;t_1,t_2))\right),$$

and

$$\frac{\partial \gamma(y_{t_1})}{\partial y_{t_1}} = \frac{\alpha}{\alpha-1}\left((\frac{\bar{F}(t_2)}{f(t_2)})^{\alpha}{y_{t_1}}^{\alpha-1} - (1-(\alpha-1)ICRT(X;t_1,t_2))\right).$$

Furthermore, the second-order derivatives of g and γ with respect to x_{t_2} and y_{t_1} are $\alpha(\frac{\bar{F}(t_1)}{f(t_1)})^{\alpha}{x_{t_2}}^{\alpha-2} > 0$ and $\alpha(\frac{\bar{F}(t_2)}{f(t_2)})^{\alpha}{y_{t_1}}^{\alpha-2} > 0$, respectively. Then the functions g and γ are minimized at points $x_{t_2} = \left((1-(\alpha-1)ICRT(X;t_1,t_2))(\frac{f(t_1)}{\bar{F}(t_1)})^{\alpha}\right)^{\frac{1}{\alpha-1}}$ and $y_{t_1} = \left((1-(\alpha-1)ICRT(X;t_1,t_2))(\frac{f(t_2)}{\bar{F}(t_2)})^{\alpha}\right)^{\frac{1}{\alpha-1}}$, respectively. In addition,

$$g(0) = -\frac{\partial ICRT(X;t_1,t_2)}{\partial t_1} < 0, \qquad g(\infty) = \infty,$$

and

$$\gamma(0) = -\frac{\partial ICRT(X;t_1,t_2)}{\partial t_2} < 0, \qquad \gamma(\infty) = \infty.$$

So, both functions g and γ first decrease and then increase with respect to x_{t_2} and y_{t_1}, respectively, which conclude that equations (12) and (13) have unique roots $h_1(t_1, t_2)$ and $h_2(t_1, t_2)$, respectively. Now, $ICRT(X; t_1, t_2)$ uniquely determines $GFRs$ and the distribution function, with attention to Remark 3.1 [14]. ■

Similar to Theorems 2, 3, and 4 and Proposition 4, we have the following results:

- The random variable X has decreasing (increasing) ICRT property if and only if the following inequality is satisfied for all $(t_1, t_2) \in D$ and $0 < \alpha < 1 (\alpha > 1)$:

$$ICPT(X;t_1,t_2) \leq (\geq) \frac{1}{\alpha-1}\left(1 - \frac{1}{\alpha}(\frac{\bar{F}(t_2)}{f(t_2)})^{\alpha}\left(\frac{1-\frac{\partial\mu^*(t_1,t_2)}{\partial t_2}}{\mu(t_1,t_2)}\right)^{\alpha-1}\right).$$

- If X is a nonnegative nondegenerate random variable, then $ICPT(X; t_1, t_2)$ cannot be a decreasing function with respect to t_2 for any real fixed t_1.
- Suppose that F is an absolutely continuous distribution function with $ICPT(X; t_1, t_2) < \infty$, then

$$E(\mu^*(X)|t_1 \leq X \leq t_2) \geq (\alpha-1)ICPT(X;t_1,t_2) - 1.$$

- Let X be a nonnegative and continuous random variable and let $ICPT(X; t_1, t_2)$ be increasing with respect to t_1 and decreasing with respect to t_2. Then $ICPT(X; t_1, t_2)$ uniquely determines $F(x)$.

Example 4. Let X be distributed uniformly on $(0, \beta), \beta > 0$, then it can be easily verified that,

$$ICPT(X;t_1,t_2) = \frac{1}{\alpha-1}(1 - \frac{{t_2}^{\alpha+1} - {t_1}^{\alpha+1}}{(t_2-t_1)^{\alpha}(1+\alpha)}),$$

$$\mu^*(t_1,t_2) = \frac{t_2 - t_1}{2}.$$

As the $ICPT$ is increasing with respect to t_2, X has $IICPT$ properties.

As in Lemma 1, the following theorem is proved by the same approach.

Lemma 2. Let X be a nonnegative continuous random variable with cdf F, and take $Y = \phi(X)$, where $\phi(\cdot)$ is a strictly increasing differentiable function. Then

$$ICRT(Y;t_1,t_2) = \frac{1}{\alpha-1}\left(1 - \int_{\max\{0,\phi^{-1}(t_1)\}}^{\phi^{-1}(t_2)} \left(\frac{\bar{F}(x)}{\bar{F}(\phi^{-1}(t_1)) - \bar{F}(\phi^{-1}(t_2))}\right)^{\alpha} \phi'(x)dx\right).$$

Proposition 5. If $Z = aX + b$, with $a > 0$ and $b \geq 0$, so $\bar{F}_{aX+b}(z) = \bar{F}_X(\frac{z-b}{a})$, then

$$ICRT(Z;t_1,t_2) = \frac{1-a}{\alpha-1} + aICRT(X;\frac{t_1-b}{a},\frac{t_2-b}{a}).$$

There is an identity for doubly truncated (interval) CRT in the following theorem.

Theorem 5. Let X be a random variable with support in $[0,r]$ and symmetric with respect to $\frac{r}{2}$, that is, $\bar{F}(x) = F(r-x)$ for $0 \leq x \leq r$. Then

$$ICRT(X;t_1,t_2) = ICPT(X;r-t_2,r-t_1), \qquad 0 \leq t_1,t_2 \leq r.$$

Proof. The theorem is proved by the following equation:

$$\begin{aligned} ICRT(X;t_1,t_2) &= \frac{1}{\alpha-1}\left(1 - \int_{t_1}^{t_2} (\frac{\bar{F}(x)}{\bar{F}(t_1) - \bar{F}(t_2)})^{\alpha} dx\right) \\ &= \frac{1}{\alpha-1}\left(1 - \int_{t_1}^{t_2} (\frac{F(r-x)}{F(r-t_1) - F(r-t_2)})^{\alpha} dx\right) \\ &= -\frac{1}{\alpha-1}\left(1 - \int_{r-t_1}^{r-t_2} (\frac{F(y)}{F(r-t_1) - F(r-t_2)})^{\alpha} dy\right) \\ &= \frac{1}{\alpha-1}\left(1 - \int_{r-t_2}^{r-t_1} (\frac{F(y)}{F(r-t_1) - F(r-t_2)})^{\alpha} dy\right) \\ &= ICPT(X;r-t_2,r-t_1). \end{aligned}$$

■

Example 5. If X is uniformly distributed in $[0,r]$, then for $0 \leq t_1,t_2 \leq r$, we have $ICRT(X;t_1,t_2) = ICPT(X;r-t_2,r-t_1) = \frac{1}{\alpha-1}(1 - \frac{(r-t_1)^{\alpha+1} - (r-t_2)^{\alpha+1}}{(t_2-t_1)^{\alpha}(1+\alpha)})$, which is in agreement with Theorem 5.

Similar to Lemma 2, Proposition 5, and Theorem 5, we have the following results:

- Let X be a nonnegative continuous random variable with cdf F, and take $Y = \phi(X)$, where $\phi(\cdot)$ is a strictly increasing differentiable function. Then

$$ICPT(Y;t_1,t_2) = \frac{1}{\alpha-1}\left(1 - \int_{\max\{0,\phi^{-1}(t_1)\}}^{\phi^{-1}(t_2)} \left(\frac{F(x)}{F(\phi^{-1}(t_2)) - F(\phi^{-1}(t_1))}\right)^{\alpha} \phi'(x)dx\right).$$

- If $Z = aX + b$, with $a > 0$ and $b \geq 0$, so $F_{aX+b}(z) = F_X(\frac{z-b}{a})$, then

$$ICPT(Z;t_1,t_2) = \frac{1-a}{\alpha-1} + aICPT(X;\frac{t_1-b}{a},\frac{t_2-b}{a}).$$

- Let X be a random variable with support in $[0,r]$ and symmetric with respect to $\frac{r}{2}$, that is, $F(x) = \bar{F}(r-x)$ for $0 \leq x \leq r$. Then

$$ICPT(X;t_1,t_2) = ICRT(X;r-t_2,r-t_1); \qquad 0 \leq t_1,t_2 \leq r.$$

Example 6. If X is uniformly distributed in $[0,r]$, for $0 \leq t_1, t_2 \leq r$, we have $ICPT(X;t_1,t_2) = ICRT(X;r-t_2,r-t_1) = \frac{1}{\alpha-1}(1-\frac{{t_2}^{\alpha+1}-{t_1}^{\alpha+1}}{(t_2-t_1)^{\alpha}(1+\alpha)})$, which is in agreement with Remark4 (part 1).

Let X and Y be two random variables. Also, the distribution function and density function of X are indicated by $F(t)$ and $f(t)$ and those of Y are denoted by $G(t)$ and $g(t)$, separately . Now we compare the two random variables X and Y based on doubly truncated (interval) cumulative residual and past Tsallis entropy. So, we first need the following definitions, which can be seen in [20]

Definition 2. X is said to be less than or equal to Y in usual stochastic ordering, if $\frac{f(x)}{g(x)}$ is decreasing in $x > 0$. We write $X \overset{lr}{\leq} Y$.

Definition 3. X is said to be less than or equal to Y in likelihood ratio ordering, if $\bar{F}(x) \leq \bar{G}(x)$, for all $x > 0$. We write $X \overset{st}{\leq} Y$.

Navarro and Rubio(2011) expressed that The two random variables X and Y satisfy $X \overset{lr}{\leq} Y$ if, and only if, $[X - t_1 | t_1 \leq X \leq t_2] \leq^{st} [Y - t_1 | t_1 \leq Y \leq t_2], whenever(t_1 < t_2)$. Also, we compare two random variables X and Y based on the properties of (interval) CRT and (interval) CPT in likelihood ratio ordering.

Theorem 6. Let X and Y be two nonnegative absolutely continuous random variables with survival functions $\bar{F}(x)$ and $\bar{G}(x)$, respectively. If $X \leq (\geq)^{lr} Y$ for all $t_1, t_2 \geq 0$, then $ICRT(X;t_1,t_2) \leq (\geq) ICRT(Y;t_1,t_2)$, for $0 < \alpha < 1$; otherwise for $\alpha > 1$, $ICRT(X;t_1,t_2) \geq (\leq) ICRT(Y;t_1,t_2)$.

Proof. The assumption $X \leq (\geq)^{lr} Y$ implies that

$$\begin{aligned}
\bar{F}_{X_{t_1,t_2}} &\leq (\geq) \quad \bar{G}_{X_{t_1,t_2}}, \\
\left(\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)}\right)^{\alpha} &\leq (\geq) \quad \left(\frac{\bar{G}(x)}{\bar{G}(t_1)-\bar{G}(t_2)}\right)^{\alpha}, \\
1 - \int_{t_1}^{t_2} \left(\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)}\right)^{\alpha} dx &\geq (\leq) \quad 1 - \int_{t_1}^{t_2} \left(\frac{\bar{G}(x)}{\bar{G}(t_1)-\bar{G}(t_2)}\right)^{\alpha} dx.
\end{aligned}$$

For $\alpha > 1$, we have

$$\begin{aligned}
\frac{1}{\alpha-1}\left(1 - \int_{t_1}^{t_2} \left(\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)}\right)^{\alpha} dx\right) &\geq (\leq) \quad \frac{1}{\alpha-1}\left(1 - \int_{t_1}^{t_2} \left(\frac{\bar{G}(x)}{\bar{G}(t_1)-\bar{G}(t_2)}\right)^{\alpha} dx\right), \\
ICRT(X;t_1,t_2) &\geq (\leq) \quad ICRT(Y;t_1,t_2).
\end{aligned}$$

For $0 < \alpha < 1$, it follows that

$$\begin{aligned}
\frac{1}{\alpha-1}\left(1 - \int_{t_1}^{t_2} \left(\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)}\right)^{\alpha} dx\right) &\leq (\geq) \quad \frac{1}{\alpha-1}\left(1 - \int_{t_1}^{t_2} \left(\frac{\bar{G}(x)}{\bar{G}(t_1)-\bar{G}(t_2)}\right)^{\alpha} dx\right), \\
ICRT(X;t_1,t_2) &\leq \quad (\geq) ICRT(Y;t_1,t_2).
\end{aligned}$$

■

Theorem 7. Let X and Y be two nonnegative absolutely continuous random variables with cdfs $F(x)$ and $G(x)$, respectively. If $X \overset{st}{\leq} Y$ for all $t_1, t_2 \geq 0$, then $ICPT(X;t_1,t_2) \geq ICPT(Y;t_1,t_2)$, for $0 < \alpha < 1$; otherwise for $\alpha > 1$, $ICPT(X;t_1,t_2) \leq ICPT(Y;t_1,t_2)$.

Proof. The assumption that $X \overset{st}{\leq} Y$ implies that

$$\begin{aligned} F_{X_{t_1,t_2}} &\geq G_{X_{t_1,t_2}}, \\ (\frac{F(x)}{F(t_2)-F(t_1)})^{\alpha} &\geq (\frac{G(x)}{G(t_2)-G(t_1)})^{\alpha}, \\ 1-\int_{t_1}^{t_2}(\frac{F(x)}{F(t_2)-F(t_1)})^{\alpha}dx &\leq 1-\int_{t_1}^{t_2}(\frac{G(x)}{G(t_2)-G(t_1)})^{\alpha}dx. \end{aligned}$$

for $\alpha > 1$, we have

$$\begin{aligned} \frac{1}{\alpha-1}\left(1-\int_{t_1}^{t_2}(\frac{F(x)}{F(t_2)-F(t_1)})^{\alpha}dx\right) &\leq \frac{1}{\alpha-1}\left(1-\int_{t_1}^{t_2}(\frac{G(x)}{G(t_2)-G(t_1)})^{\alpha}dx\right), \\ ICPT(X;t_1,t_2) &\leq ICPT(Y;t_1,t_2). \end{aligned}$$

For $0 < \alpha < 1$, it follows that

$$\begin{aligned} \frac{1}{\alpha-1}\left(1-\int_{t_1}^{t_2}(\frac{F(x)}{F(t_2)-F(t_1)})^{\alpha}dx\right) &\geq \frac{1}{\alpha-1}\left(1-\int_{t_1}^{t_2}(\frac{G(x)}{G(t_2)-G(t_1)})^{\alpha}dx\right), \\ ICPT(X;t_1,t_2) &\geq ICPT(Y;t_1,t_2). \end{aligned}$$

■

Example 7. Let

$$\bar{F}(x) = \begin{cases} (\frac{x_0}{x})^{\beta_1}, & x > x_0, \\ 1, & x \leq x_0, \end{cases}$$

$$and$$

$$\bar{G}(x) = \begin{cases} (\frac{x_0}{x})^{\beta_2}, & x > x_0, \\ 1, & x \leq x_0. \end{cases}$$

That is, X and Y have Pareto distributions with parameters β_1 and β_2, respectively. If $\beta_1 \geq \beta_2$ and $0 < \beta_1, \beta_2 \leq \frac{1}{\alpha}$, hence $X \overset{lr}{\leq} Y$ for $\alpha > 1$, then $ICRT(X;t_1,t_2) \geq ICRT(Y;t_1,t_2)$. Also, the assumptions of the theorem hold, and therefore $[X - t_1|t_1 \leq X \leq t_2] \leq^{st} [Y - t_1|t_1 \leq Y \leq t_2], whenever (t_1 < t_2)$.

4. Empirical estimation of ICPT

By utilizing various empirical estimators of the cdf, we suggest four non-parametric estimators $ICPT(X;t_1,t_2)$ and also compare the implementation of the proposed estimators. For an actual-life fact set, we study the monotonicity of ICPT based totally on its kernel-smoothed estimator.

First, we introduce four nonparametric estimators, by mentioning the name $ICPT^1(X;t_1,t_2)$, $ICPT^2(X;t_1,t_2)$, $ICPT^3(X;t_1,t_2)$ and $ICPT^4(X;t_1,t_2)$, of $ICPT$ through utilizing empirical distribution function, mean empirical distribution function, median empirical distribution function, and kernel-smoothed function and their implementation by the Monte–Carlo simulation. Let $X_1, X_2, \ldots, X_n$ be an independent and identically distributed random sample drawn from a population having distribution function $F(x)$ and survival function $\bar{F}(x)$. Now, the first nonparametric estimator of $ICPT^1(X;t_1,t_2)$ may be written as

$$ICPT^1(X;t_1,t_2) = \frac{1}{\alpha-1}\left(1-\int_{t_1}^{t_2}(\frac{F_n^{(1)}(x)}{F_n^{(1)}(t_2)-F_n^{(1)}(t_1)})^{\alpha}dx\right),$$

for $0 < \alpha \neq 1$, where $F_n^{(1)}(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x), x \in R$, is the empirical distribution function and

$$I(X_i \leq x) = \begin{cases} 1 & if X \leq x, \\ 0 & otherwise, \end{cases}$$

is the indicator function of the event $X \leq x$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of random sample. Noting the sample values lying between t_1 and t_2 so that $t_1 \leq x_{(j)}, x_{(j+1)}, \ldots, x_{(k)} \leq t_2$, then

$$\begin{aligned} ICPT^1(X;t_1,t_2) &= \frac{1}{\alpha-1}\left(1-\sum_{i=j}^{k}\int_{x_{(i)}}^{x_{(i+1)}}\left(\frac{F_n^{(1)}(x)}{F_n^{(1)}(t_2)-F_n^{(1)}(t_1)}\right)^{\alpha}dx\right) \\ &= \frac{1}{\alpha-1}\left(1-\frac{1}{(F_n^{(1)}(t_2)-F_n^{(1)}(t_1))^{\alpha}}\sum_{i=j}^{k}\int_{x_{(i)}}^{x_{(i+1)}}(F_n^{(1)}(x))^{\alpha}dx\right) \\ &= \frac{1}{\alpha-1}\left(1-\frac{1}{(F_n^{(1)}(t_2)-F_n^{(1)}(t_1))^{\alpha}}\sum_{i=j}^{k}(x_{(i)}-x_{(i+1)})(F_n^{(1)}(x))^{\alpha}\right). \end{aligned} \quad (14)$$

The second estimator of $ICPT^2(X;t_1,t_2)$ can be acquired by replacing mean empirical distribution function $F_n^{(2)}(x)$ in (14)as

$$ICPT^2(X;t_1,t_2) = \frac{1}{\alpha-1}\left(1-\frac{1}{(F_n^{(2)}(t_2)-F_n^{(2)}(t_1))^{\alpha}}\sum_{i=j}^{k}(x_{(i)}-x_{(i+1)})(F_n^{(2)}(x))^{\alpha}\right), \quad (15)$$

where the mean empirical distribution function is defined as

$$F_n^{(2)}(x) = \frac{1}{n+1}\sum_{i=1}^{n} I(X_i \leq x), \qquad x \in R.$$

The third nonparametric estimator of $ICPT^3(X;t_1,t_2)$ can be achieved by utilizing median empirical distribution function in (14) as follows:

$$ICPT^3(X;t_1,t_2) = \frac{1}{\alpha-1}\left(1-\frac{1}{(F_n^{(3)}(t_2)-F_n^{()}(t_1))^{\alpha}}\sum_{i=j}^{k}(x_{(i)}-x_{(i+1)})(F_n^{(3)}(x))^{\alpha}\right), \quad (16)$$

where $F_n^{(3)}(x) = \sum_{i=1}^{n} \frac{I(X_i \leq x)-0.3}{n+0.4}, x \in R$, is the median empirical distribution function.

The fourth estimator can be defined by utilizing Kernel-smoothed estimator $F_n^{(4)}(x)$ of the distribution function in (14) as follows:

$$ICPT^4(X;t_1,t_2) = \frac{1}{\alpha-1}\left(1-\frac{1}{(F_n^{(4)}(t_2)-F_n^{()}(t_1))^{\alpha}}\sum_{i=j}^{k}(x_{(i)}-x_{(i+1)})(F_n^{(4)}(x))^{\alpha}\right), \quad (17)$$

where $F_n^{(4)}(x)$, the kernel-smoothed estimator of distribution function, is defined as

$$F_n^{(4)}(x) = \frac{1}{n}\sum_{i=1}^{n} L\left(\frac{x-X_i}{h}\right),$$

where L is a distribution function of positive kernel K, that is, $L(u) = \int_u^{-\infty} K(t)dt$ and h is the bandwidth of parameter . Now, we utilize the normal kernel function $K(u) = \frac{1}{\sqrt{\pi}} \exp\left(\frac{u^2}{2}\right)$.

5. Simulation

It is widely recognized that the smoothed estimator has a better performance compared to a nonsmoothed estimator . To demonstrate the effectiveness of the empirical and kernel estimators, a Monte–Carlo simulation examination is accomplished. The estimated values are computed based on 1000 simulations from $Exp(0.5)$ (exponential distribution) each of size $n(n = 30, 35, 40, 50, 60)$ for different truncation limits and $\alpha = 0.2; 3.5$. Bias and mean square error (MSE) are also calculated. In Tables 1 and 2, we present the exact value, bias, and the

MSE of the proposed estimators of $ICPT$. The MSE of the estimators corresponding to truncation limit (0.2, 4) and for $\alpha = 0.2, 3.5$ is also displayed in Figure 1 for increasing sample size.

It is obvious that in nearly all cases $ICPT^4(X; t_1, t_2)$ (17) performs way better with less MSE than the other estimators as determined in (14) (15) and (16). Further ,for $\alpha = 0.2$, $ICPT^1(X; t_1, t_2)$ produces better result than $ICPT^3(X; t_1, t_2)$, while $ICPT^2(X; t_1, t_2)$ yields poor estimates as MSE is higher in comparison with the other estimators of ICPT. Also, for $\alpha = 3.5$, it can be seen that there is a slight difference between the first, second and third estimators and The fourth estimator is significantly better estimator . It is expected, one can depict from Tables 1 and 2 that ICPT as a measure of uncertainty declines for a shrinking interval. Generally , we can conclude that kernel smoothed estimator gives better estimates of ICPT than the other proposed estimators in terms of MSE. Also, the values of MSE of the proposed estimators are reduced by increasing sample size, which is caused by dependence of the MSE of the empirical estimators to the sample size.

It is obvious that, in nearly, all cases $ICPT^4(X; t_1, t_2)$ defined by (17) perform a way better with less MSE than the other estimators, as determined in (14), (15), and (16). Furthermore, for $\alpha = 0.2$, $ICPT^1(X; t_1, t_2)$ produces a better result than $ICPT^3(X; t_1, t_2)$, while $ICPT^2(X; t_1, t_2)$ yields poor estimates as the MSE is higher in comparison with the other estimators of ICPT. Also, for $\alpha = 3.5$, it can be seen that there is a slight difference between the first, second and third estimators and The fourth estimator is significantly better estimator . It is expected that one can depict from Tables 1 and 2 that ICPT, as a measure of uncertainty , declines for a shrinking interval. Generally , we can conclude that the kernel-smoothed estimator gives better estimates of ICPT than the other proposed estimators in terms of the MSE. Also, the values of MSE of the proposed estimators are reduced by increasing sample size, which is caused by the dependence of the MSE of the empirical estimators on the sample size.

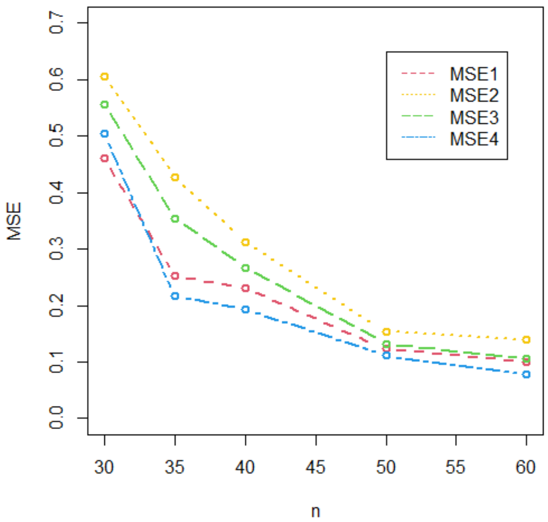
Table 1: *Bias and MSE of $ICPT^1(X; t_1, t_2)$, $ICPT^2(X; t_1, t_2)$, $ICPT^3(X; t_1, t_2)$ and $ICPT^4(X; t_1, t_2)$ for $\alpha = 3.5$ and different truncation limits ($n = 30, 35, 40, 50, 60$).*

$\alpha = 3.5$ (t_1, t_2)	n	*Exact value*	$ICPT^1(X; t_1, t_2)$ Bias1/ MSE1	$ICPT^2(X; t_1, t_2)$ Bias2/ MSE2	$ICPT^3(X; t_1, t_2)$ Bias3/ MSE3	$ICPT^4(X; t_1, t_2)$ Bias4/ MSE4
	30		0.54832/0.325730	0.49378/0.28433	0.51636/ 0.30243	0.20438/0.17744
	35		0.549833/0.32429	0.50497/0.28408	0.52796/ 0.30195	0.23830/0.16294
(0.1,4.5)	40	-0.50468	0.55001/0.32373	0.50943/ 0.28386	0.52995/0.29999	0.28788/0.16066
	50		0.55571/0.32236	0.51445/0.28234	0.53018/0.29798	0.35550/0.17280
	60		0.55720/0.32132	0.52253/ 0.28551	0.53218/0.29726	0.38246/0.17472
	30		0.52237/0.33138	0.43733/0.27303	0.47260/0.29833	0.14750/0.25057
	35		0.53415/0.32091	0.45608/0.26707	0.48867/0.28337	0.22450/0.21577
(0.2,4)	40	-0.53930	0.53631/0.31804	0.47334/0.26538	0.49120/0.27857	0.26394/0.20544
	50		0.53835/0.31399	0.48180/0.26242	0.49168/0.27427	0.32690/0.19580
	60		0.54001/ 0.31037	0.48255/ 0.25854	0.49838/0.27144	0.36282/0.17503
	30		0.66016/0.59085	0.58444/0.46060	0.60264/0.49963	0.28883/0.38595
	35		0.67913/0.52852	0.60971/0.44713	0.61856/0.46831	0.37049/0.37528
(0.3,3.9)	40	-0.71898	0.68008/ 0.50869	0.61383/ 0.43901	0.63064/0.45793	0.42447/0.34087
	50		0.68175/0.49875	0.62145/ 0.43391	0.64163/0.45061	0.49303/0.33731
	60		0.68447/0.49486	0.62355/0.42886	0.64253/0.44791	53037/0.33580

The nonparametric estimators of the distribution function are occasionally considered as plotting positions because they supply the ordinate values in plotting the distribution function.

Table 2: *Bias and MSE of $ICPT^1(X;t_1,t_2)$, $ICPT^2(X;t_1,t_2)$, $ICPT^3(X;t_1,t_2)$ and $ICPT^4(X;t_1,t_2)$ for $\alpha = 0.2$ and different truncation limits ($n = 30, 35, 40, 50, 60$).*

$\alpha = 0.2$ (t_1,t_2)	n	*Exact value*	$ICPT^1(X;t_1,t_2)$ *Bias1/ MSE1*	$ICPT^2(X;t_1,t_2)$ *Bias2/ MSE2*	$ICPT^3(X;t_1,t_2)$ *Bias3/ MSE3*	$ICPT^4(X;t_1,t_2)$ *Bias4/ MSE4*
	30		0.01443/0.62585	0.20562/0.85397	0.15292/0.84328	0.31217/0.46912
	35		-0.08361/ 0.40997	0.10770/0.63329	0.03216/0.50496	0.19145/0.31041
(0.1,4.5)	40	3.81827	-0.13264/0.38650	0.01795/0.46097	0.00052/0.41305	0.12438/0.27354
	50		-0.2265/0.21496	-0.12186/0.27259	-0.16442/0.23459	0.00011/0.15006
	60		-0.26605/0.17343	-0.19538/0.20058	-0.20950/0.18536	-0.10140/0.11435
	30		0.0668/ 0.45989	0.21781/0.60482	0.13608/ 0.55571	0.33128/0.50470
	35		-0.02364/0.25232	0.091067/0.42669	0.07336/0.3542	0.17299/0.21602
(0.2,4)	40	3.19537	-0.04258/0.23084	0.02478/0.31252	0.00027/0.26733	0.11015/0.19296
	50		-0.14634/ 0.12285	-0.05928/0.15386	-0.07653/0.13229	0.01410/0.11127
	60		-0.15977/0.09937	-0.10696/0.13898	-0.14774/0.10505	-0.053750/0.07863
	30		0.06254/0.52412	0.15421/0.42303	0.10033/0.45151	0.21734/0.35308
	35		0.03341/0.27368	0.04638/ 0.23737	0.03507/0.26058	0.14486/0.22552
(0.3,3.9)	40	3.04031	-0.09178/0.21888	0.02148/0.15290	0.01607/0.18283	0.07261/0.20835
	50		-0.15733/0.17706	-0.10894/0.11774	-0.11561/0.13794	0.04314/0.09471
	60		-0.18802/0.10640	-0.15383/0.09217	-0.15181/0.10345	-0.10009/0.07318



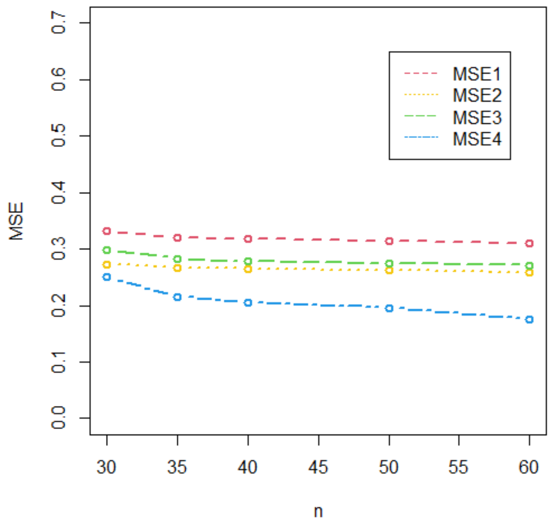


Figure 1: *Graphical showing of the MSE of four estimators. Sample size for fixed truncation limit (0.2, 4). (I) Plot of the MSE for fixed truncation limit (0.2, 4) and $\alpha = 0.2$ and (II) Plot of the MSE for fixed truncation limit (0.2, 4) and $\alpha = 3.5$.*

6. Real data

In this part, an actual life data set is examined to illustrate the applicability and usefulness of the best-proposed estimator of ICPT in actual status. For this purpose, we have taken into account the data set vinyl chloride acquired from clean upgradient groundwater monitoring wells [2]. Vinyl chloride is an organic compound that is unstable . In environmental investigations, this aspect is of extraordinary significance due to the fact that it is both anthropogenic and carcinogenic. Nonetheless, in lots of background monitoring wells, low levels of this component are determined. This compound low surface detections in clean upgradient background monitoring wells is because of cross pollution from air or gas or the analytic system itself. The data set is provided as follows. Data Set (g/l) : 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2 has been fitted with exponential distribution by [21]. They acclaimed that this data set follows $Exp(0.5320814)$ (exponential distribution). To examine the behavior of the $ICPT$, we have calculated estimated values of $ICPT^4(X;t_1,t_2)$ by means of the use of its best-proposed estimator for different truncation limits and $\alpha = 0.2, 3.5$ as shown in Table 3. It has been determined that the estimated values are decreasing in t_1 and increasing in t_2 for $\alpha = 0.2, 3.5$. So, by increasing (decreasing) the fourth estimator of ICPT(doubly truncated CPT), the amount of the dispersion of vinyl chloride obtained from clean upgradient groundwater monitoring wells increases (decreases). As expected for $0 < \alpha = 1$, $ICPT$ is an increasing function of the interval. It is worth noting that this result is according to the monotonicity of $ICPT(X; t_1, t_2)$ for $Exp(0.5320814)$ and $\alpha = 0.3, 1.5$.

Table 3: *Kernel estimates of $ICPT^4(X; t_1, t_2))$ for the Vinyl chloride data for different truncation limits (t_1, t_2) and $= 0.2; 3.5$.*

$\alpha \backslash (t_1, t_2)$	(0.4,2.9)	(0.6,2.9)	(08,2.9)	(1,2.9)	(0.2,1.8)	(0.2,2)	(0.2,2.4)	(0.2,2.8)
0.2	2.035697	1.84515	1.658554	1.47661	1.108958	1.073237	1.576643	1.94168
3.5	-0.419520	-0.508537	-0.641847	-0.919710	-1.837256	-1.312966	-0.754232	-0.437168

7. Conclusion

In information theory and also in reliability, there are several uncertainty measures that play a central role. In this paper, we first studied the notion of doubly truncated (interval) Tsallis entropy and suggested the doubly truncated (interval) cumulative residual Tsallis entropy (ICRT) and doubly truncated (interval) cumulative past Tsallis entropy (ICPT) whose some of their properties and their relations with hazard rate (reversed hazard rate) and mean residual (past) life were studied. Also, we introduced ordering classes for ICRT and ICPT and gave some characterization. In the end, we have proposed four nonparametric estimators and compared their performance by utilizing simulation data. Also, based on the best-proposed estimator, an actual data set was additionally examined.

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