ON THE Q-RAYLEIGH DISTRIBUTION AND ITS APPLICATIONS

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Abstract

This paper introduces the two-parameter q-Rayleigh distribution, a powerful extension of the classical Rayleigh model for analysing real-world data. Compared to the Rayleigh, the q-Rayleigh incorporates a novel pathway parameter q, offering greater flexibility in capturing diverse data patterns. We delve into the mathematical properties of the q-Rayleigh, including its hazard rate function and quantile function, and explore parameter estimation through maximum likelihood methods. We demonstrate its superior fit compared to the widely-used Rayleigh distribution for real-world data. Moreover, we explore its application in reliability analysis. This comprehensive study makes the q-Rayleigh a compelling choice for modelling data exhibiting gradual transitions and enhanced flexibility.

Keywords: q-Rayleigh distribution, Statistical properties, Parameter estimation, Modelling data

1. Introduction

The Rayleigh distribution, originally introduced by Rayleigh [13], is a notable probability distribution that serves as a specialized model and a modified variant of the Weibull distribution. Widely applicable across diverse disciplines, including medicine, engineering, finance, astronomy, and physics, the Rayleigh distribution has garnered significance due to its versatile utility in modelling various phenomena. Its pivotal role has led to extensive research, resulting in the proposal of several extensions by numerous scholars. Noteworthy examples include the truncated Rayleigh distribution, explored by Khalaf and Al-Kadim [8], and the Rayleigh Gamma-Gompertz distribution, investigated by Al-Noor and Asri [4]. Additionally, Rahman [12] introduced the Cubic Transformed Inverse Rayleigh distribution, and Adnan et al. [1] developed the Weibull Lindley Rayleigh distribution. These extensions and modifications reflect the adaptability and applicability of the Rayleigh distribution in different contexts. The probability density function (pdf) and cumulative distribution function (cdf) of the Rayleigh distribution are given respectively, by

$$f(x;\sigma) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}; \quad \sigma > 0,\ x \geqslant 0 \tag{1}$$

$$F(x;\sigma) = 1 - e^{-\frac{x^2}{2\sigma^2}}; \quad \sigma > 0,\ x \geqslant 0 \tag{2}$$

The q-distribution, a concept integral to mathematical physics and probability theory, exhibits a broader generality compared to classical distributions. Originating from the pioneering work of Tsallis [19], the landscape of probability distributions has expanded significantly through the introduction of q-type distributions. This extension involves incorporating the q Tsallis parameter, setting the stage for an extensive body of research on this topic. A notable array of q-type distributions has emerged as a result, showcasing the versatility of this concept. Notable examples include the q-exponential distributions proposed by Amari and Ohara [3], q-Gaussian distributions elucidated by Sato [16], and the q-Gamma distribution investigated by Zhang et al. [20]. Additionally, q-Weibull distributions have been introduced by researchers such as Picoli et al. [10]. These q-type distributions represent a rich and diverse set of mathematical formulations, contributing to the enhanced understanding and modelling of complex phenomena in various scientific disciplines. The cornerstone of q-type distributions is the q-exponential function:

$$\exp_q(x) = \begin{cases} [1+(1-q)x]^{\frac{1}{1-q}}, & 1+(1-q)x > 0 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

This function introduces a parameter q, that bestows a remarkable degree of adaptability in shaping the distribution, empowering it to effectively model non-trivial data patterns that often elude conventional approaches. Building upon this foundation, we introduce the q-Rayleigh distribution, a q-analogue poised to potentially expand the scope of modelling possibilities for intricate data relationships.

Recently, Gül [6] introduced the q-Rayleigh distribution for the case of $q < 1$ and discussed the estimation of unknown parameters through maximum likelihood and least squares methods. In this paper, we extend the exploration of mathematical properties to two cases: $q < 1$ and $1 < q < 2$. The analysis encompasses the survival function, hazard rate function, quantile function, limiting behaviour, and moments of the distribution. Furthermore, we delve into intriguing results concerning extreme value properties associated with the q-Rayleigh distribution. We employ the maximum likelihood estimator for parameter estimation in this new distribution. To assess its performance, we compare the q-Rayleigh distribution with the standard Rayleigh distribution using diverse real-life time data sets.

The rest of the paper is organised as follows. Section 2 introduces the novel q-Rayleigh distribution, providing a comprehensive exploration of its specific cases. Section 3 delves into the mathematical and statistical properties of this distribution, elucidating its asymptotic behaviours. Section 4, meticulously elucidates the method of maximum likelihood estimation. In Section 5, we employ the newly proposed model on two distinct datasets concerning the treatment of head and neck cancer patients with radiation plus chemotherapy, as well as COVID-19 mortality rates data from Italy. A comparative analysis with the q-Rayleigh and Rayleigh models is conducted, affirming the superior fit of the q-Rayleigh model. The conclusive Section brings together the findings, summarizing the key insights and implications derived from the exploration of the innovative q-Rayleigh distribution.

2. The q-Rayleigh distribution

2.1. Distributional characteristics

The pdf of the q-Rayleigh distribution is defined as

$$f_q(x) = (2-q)\frac{x}{\sigma^2}\exp_q\left[-\frac{x^2}{2\sigma^2}\right],\ x > 0 \quad (4)$$

where $\sigma > 0$ and $q < 2$ are shape parameters, and $\eta > 0$ is a scale parameter.

By introducing $\beta = \sigma^{-2}$ and using $\exp_q(x)$ in equation (3), the pdf of the q-Rayleigh distribution, for $x > 0$ and for $q < 1$, can be rewritten as

$$f_q(x) = (2-q)\beta x\left[1-(1-q)\frac{\beta x^2}{2}\right]^{\frac{1}{1-q}},\ q < 1 \text{ and } x \in \left[0, \left(\frac{\beta}{2}(1-q)\right)^{-1/2}\right] \quad (5)$$

For $x > 0$ and $q > 1$, the pdf of the q-Rayleigh distribution is expressed as:

$$f_q(x) = (2-q)\beta x \left[1 + (q-1)\frac{\beta x^2}{2}\right]^{-\frac{1}{q-1}}, \; 1 < q < 2 \text{ and } x \in [0, +\infty) \tag{6}$$

The cumulative distribution function (cdf) of the q-Rayleigh distribution, when $q < 1$ is defined as

$$F_q(x) = 1 - \left[1 - (1-q)\frac{\beta x^2}{2}\right]^{\frac{2-q}{1-q}} \tag{7}$$

If $1 < q < 2$, the cdf function of the q-Rayleigh distribution, formulated as follows:

$$F_q(x) = 1 - \left[1 + (q-1)\frac{\beta x^2}{2}\right]^{\frac{q-2}{q-1}} \tag{8}$$

2.2. Survival function

In the context of the q-Rayleigh distribution, the survival function (sf), denoted by $S(x)$, represents the probability that an individual or entity survives beyond time t. Its mathematical expression is as follows

$$S(x) = P(X > t) = 1 - F(x)$$

$$S_q(x) = \left[1 - (1-q)\frac{\beta x^2}{2}\right]^{\frac{2-q}{1-q}}, \text{ for } q < 1,$$

$$S_q(x) = \left[1 + (q-1)\frac{\beta x^2}{2}\right]^{\frac{2-q}{1-q}}, \text{ for } 1 < q < 2$$

2.3. Hazard function

The concept of risk within the context of survival analysis is characterized by the hazard rate function (hrf), $h(x)$. This function measures the immediate risk of an event (e.g., death) for an individual who has survived until that time. Its formal representation is as follows

$$h(x) = P(X > t) = \frac{f(x)}{S(x)}$$

The hrf of q-Rayleigh distribution for $q < 1$ is defined as

$$h_q(x) = \frac{(2-q)\beta x}{1 - (1-q)\frac{\beta x^2}{2}}$$

In the case of $1 < q < 2$, the hrf of q-Rayleigh distribution is characterized by

$$h_q(x) = \frac{(2-q)\beta x}{1 + (q-1)\frac{\beta x^2}{2}}$$

2.4. Cumulative hazard function

The probability of an event occurring before a given time is quantified by the cumulative hazard function (chf), presented below

$$H(x) = -\ln(1 - F(x))$$

The chf for the q-Rayleigh distribution, with $q < 1$, is expressed as follows

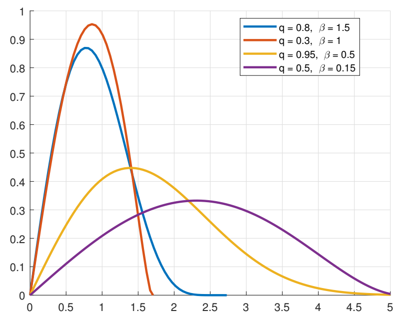
$$H_q(x) = \frac{2-q}{q-1}\ln\left[1 - (1-q)\frac{\beta x^2}{2}\right].$$

For the case where $1 < q < 2$, the chf of the q-Rayleigh distribution is given by

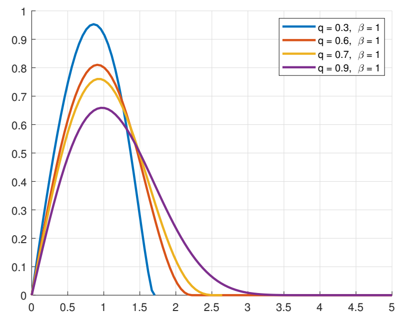
$$H_q(x) = \frac{2-q}{q-1} \ln \left[1 + (q-1)\frac{\beta x^2}{2} \right]$$

2.5. Graphical Study of q-Rayleigh distribution under various functions

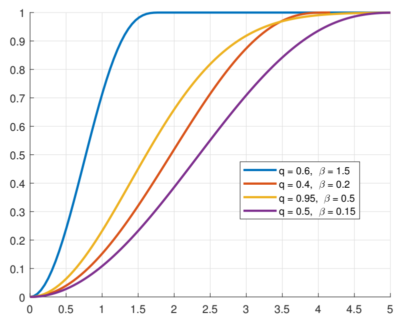
Driven by a desire to understand the nuanced behaviour of the q-Rayleigh distribution, we embark on a detailed exploration of its key functions (pdf, cdf, sf, and hrf) across a range of parameter values. By meticulously analysing the illustrative figures presented below, we uncover fascinating insights into how varying parameters sculpt the behaviour of this versatile distribution. Complementing our theoretical exploration, we presented illustrative figures to visually depict the distribution's characteristics, enhancing accessibility and understanding.



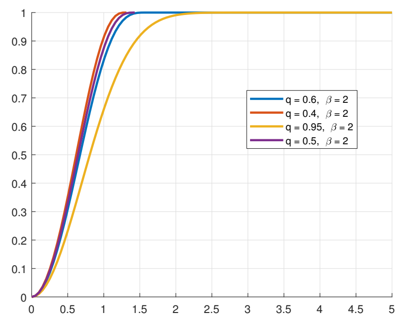
((a)) *Graph of the pdf of the q-Rayleigh distribution when all the parameters are changed*



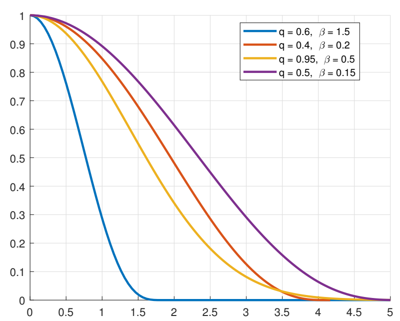
((b)) *Graph of the pdf of the q-Rayleigh distribution when changing the q values and β is fixed*



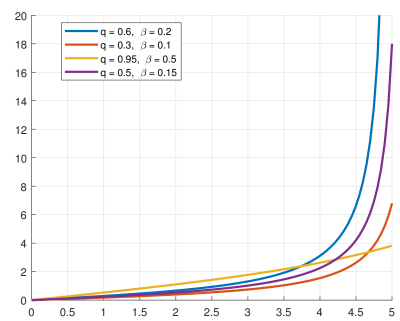
((c)) *Graph of the cdf of the q-Rayleigh distribution when all the parameters are changed*



((d)) *Graph of the cdf of the q-Rayleigh distribution when changing the q values and β is fixed*

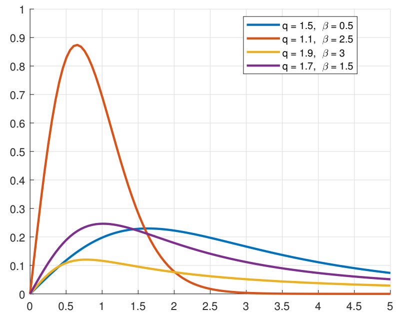


((e)) *Graph of the sf of the q-Rayleigh distribution with different parameter values*

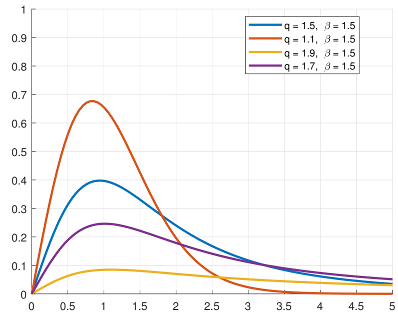


((f)) *Graph of the hrf of the q-Rayleigh distribution with different parameter values*

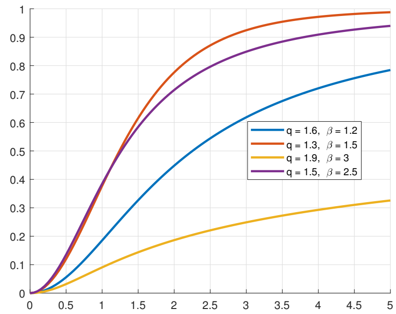
Figure 1: *Graphical representation of the key functions of the q-Rayleigh distribution: $q < 1$*



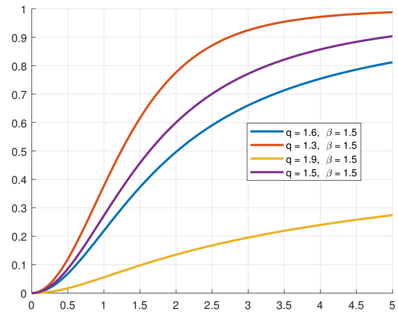
((a)) *Graph of the pdf of the q-Rayleigh distribution when all the parameters are changed*



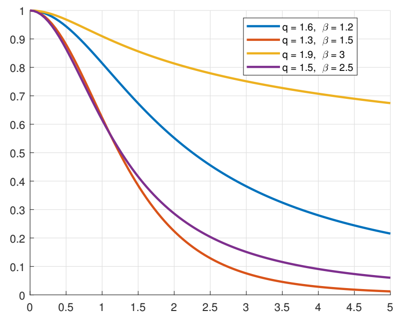
((b)) *Graph of the pdf of the q-Rayleigh distribution when changing the q values and β is fixed*



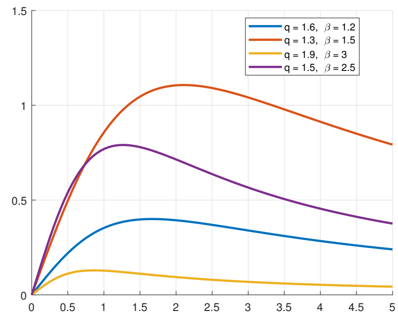
((c)) *Graph of the cdf of the q-Rayleigh distribution when all the parameters are changed*



((d)) *Graph of the cdf of the q-Rayleigh distribution when changing the q values and β is fixed*



((e)) *Graph of the sf of the q-Rayleigh distribution with different parameter values*



((f)) *Graph of the hrf of the q-Rayleigh distribution with different parameter values*

Figure 2: *Graphical representation of the key functions of the q-Rayleigh distribution:* $1 < q < 2$

Figures 1 and 2 showcase the graphical representation of the key functions of the q-Rayleigh distribution for cases where $q < 1$ and $1 < q < 2$, respectively. Examining the probability density function graphs (1(a), 1(b), 7(a),and 7(b)), it becomes evident that the distribution exhibits skewness and a high degree of adaptability to diverse parameter values.

In Figures 1(c), 1(d), 2(c) and 2(d), we observe cumulative density plots that serve to validate the distribution's suitability as a probability distribution. Additionally, Figures 1(e) and 2(e) portray the survival function, revealing distinct patterns of fast and slow decreases. The hazard rate function graphs (1(f), 2(f)) further contribute to the distribution's versatility, showcasing a range of shapes including increasing, decreasing, and constant. This variability allows for the effective fitting of datasets with diverse forms, a characteristic that the q-Rayleigh distribution adeptly demonstrates. In essence, our exploration underscores the distribution's capability to accommodate different data sets, making it a valuable tool in statistical analysis.

3. Properties

This section delves into the mathematical and statistical characteristics of the q-Rayleigh distribution.

3.1. Limiting Behaviour

Lemma 1. As the parameter q approaches 1, the pdf of the q-Rayleigh distribution, (denoted as $f_q(x)$), converges to the standard Rayleigh distribution.

Proof. For $q < 1$, the limiting pdf for $q = 1$ is

$$\begin{aligned}\lim_{q\to 1} f_q(x) =& \beta x \lim_{q\to 1} \left\{ \left[1-(1-q)\frac{\beta x^2}{2}\right]^{\frac{-1}{(1-q)\beta x^2/2}} \right\}^{-\beta x^2/2} \\ =& \beta x \exp\left(-\frac{\beta x^2}{2}\right)\end{aligned}$$

a Rayleigh pdf.

The established proof methodology can be directly applied to the range $1 < q < 2$, yielding an analogous conclusion. ■

3.2. Quantile Function

The quantile function of X, denoted as $Q(u)$ and defined as $Q(u) = F^{-1}(u)$, can be derived by inversely solving equations (7) and (8) as follows

$$Q_q(x) = \left[\frac{2}{\beta(1-q)}\left(1-(1-u)^{\frac{1-q}{2-q}}\right)\right]^{\frac{1}{2}}, \text{ for } q < 1,$$
$$Q_q(x) = \left[\frac{2}{\beta(q-1)}\left(-1+(u-1)^{\frac{1-q}{2-q}}\right)\right]^{\frac{1}{2}}, \text{ for } 1 < q < 2$$

3.3. Moments

This section presents the moment function for the q-Rayleigh distribution, where moments serve as quantitative indicators associated with the function's shape. The moments of the q-Rayleigh distribution can be derived as follows:

$$E(X^s) = \int_0^{+\infty} x^s f_q(x)dx$$

If $q < 1$,

$$\begin{aligned}E(X^s) &= \int_0^{\left(\frac{\beta}{2}(1-q)\right)^{-1/2}} x^s(2-q)\beta x\left[1-(1-q)\frac{\beta x^2}{2}\right]^{\frac{1}{1-q}} dx \\ &= \frac{2-q}{(1-q)^{1+s/2}}\left(\frac{2}{\beta}\right)^{s/2} B\left(\frac{1+s}{1-q}, 1+\frac{s}{2}\right)\end{aligned}$$

where,

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt = \int_0^{+\infty} \frac{t^{p-1}}{(1+t)^{p+q}}\,dt$$

denotes the beta function. It follows that the mean and variance of the q-Rayleigh random variable when $q < 1$ are

$$E(X) = \frac{2-q}{(1-q)^{1+1/2}}\left(\frac{2}{\beta}\right)^{1/2} B\left(\frac{2}{1-q},\frac{3}{2}\right)$$
$$Var(X) = \frac{2(2-q)}{(1-q)\beta}\left[B\left(\frac{3}{1-q},2\right) - \frac{2-q}{(1-q)^{3/2}}B^2\left(\frac{2}{1-q},\frac{3}{2}\right)\right]$$

If $1 < q < 2$,

$$E(X^s) = \int_0^{+\infty} x^s(2-q)\beta x\left[1+(q-1)\frac{\beta x^2}{2}\right]^{-\frac{1}{q-1}}dx$$
$$= \frac{2-q}{(q-1)^{1+s/2}}\left(\frac{2}{\beta}\right)^{s/2} B\left(\frac{1}{q-1}-\frac{s}{2}-1,\frac{s}{2}+1\right)$$

provided $\frac{1}{q-1} - \frac{s}{2} > 1$. Consequently, the mean and variance of the q-Rayleigh random variable can be expressed as follows

$$E(X) = \frac{2-q}{(q-1)^{1+1/2}}\left(\frac{2}{\beta}\right)^{1/2} B\left(\frac{1}{q-1}-\frac{3}{2},\frac{3}{2}\right)$$
$$Var(X) = \frac{2(2-q)}{(q-1)\beta}\left[B\left(\frac{1}{q-1}-2,2\right) - \frac{2-q}{(q-1)^{3/2}}B^2\left(\frac{1}{q-1}-\frac{3}{2},\frac{3}{2}\right)\right]$$

3.4. Extreme value properties

Theorem 1. Let $\{X_i, i = 1, ..., n\}$ be independent and identically distributed random variables (r.v.) following the q-Rayleigh distribution, then $U = \min_{1\leqslant i\leqslant n} X_i$ has also the same distributional form.

Proof. For $q < 1$ the survival function is $S_q(x) = \left[1-(1-q)\frac{\beta x^2}{2}\right]^{\frac{2-q}{1-q}}$. Then,

$$S_q(x) = P\left[\min_{1\leqslant i\leqslant n} X_i > x\right]$$
$$= \prod_{i=1}^{n} P\left[X_i > x\right]$$
$$= \prod_{i=1}^{n}\left[1-(1-q)\frac{\beta x^2}{2}\right]^{\frac{2-q}{1-q}}$$
$$= \left[1-(1-q)\frac{\beta x^2}{2}\right]^{n\frac{2-q}{1-q}} \longrightarrow e^{-n\frac{\beta x^2}{2}} \text{ as } q \longrightarrow 1$$

For $1 < q < 2$ the survival function is $S_q(x) = \left[1+(q-1)\frac{\beta x^2}{2}\right]^{\frac{2-q}{1-q}}$. Then,

$$S_q(x) = \left[1+(q-1)\frac{\beta x^2}{2}\right]^{-n\frac{2-q}{q-1}} \longrightarrow e^{-n\frac{\beta x^2}{2}} \text{ as } q \longrightarrow 1$$

■

Theorem 2. Let $\{X_i, i = 1, ..., n\}$ be independent and identically distributed random variables (r.v.) following the q-Rayleigh distribution, then $V = \max_{1\leqslant i\leqslant n} X_i$ has also the same distributional form.

Proof. For $q < 1$ the cdf is $F_q(x) = 1 - \left[1 - (1-q)\frac{\beta x^2}{2}\right]^{\frac{2-q}{1-q}}$. Then,

$$\begin{aligned} F_q(x) &= P\left[\max_{1\leqslant i\leqslant n} X_i \leqslant x\right] \\ &= \prod_{i=1}^{n} P\left[X_i \leqslant x\right] \\ &= \prod_{i=1}^{n} 1 - \left[1 - (1-q)\frac{\beta x^2}{2}\right]^{\frac{2-q}{1-q}} \\ &= \left[1 - \left[1 - (1-q)\frac{\beta x^2}{2}\right]^{\frac{2-q}{1-q}}\right]^n \longrightarrow \left[1 - e^{-\frac{\beta x^2}{2}}\right]^n \text{ as } q \longrightarrow 1 \end{aligned}$$

Similarly for $1 < q < 2$ the cdf of V is

$$F_q(x) = \left[1 - \left[1 + (q-1)\frac{\beta x^2}{2}\right]^{\frac{q-2}{q-1}}\right]^n \longrightarrow \left[1 - e^{-\frac{\beta x^2}{2}}\right]^n \text{ as } q \longrightarrow 1$$

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4. Estimation of Parameters

This section explores the estimation of the unknown parameters in the q-Rayleigh distribution through the application of the maximum likelihood estimation method (MLE).

Let $x_1, x_2, \ldots, x_n$ represent a random sample obtained from the q-Rayleigh distribution. The subsequent expression outlines the logarithm of the likelihood function corresponding to the pdf represented in equation (5) for $q < 1$ is

$$\ln L = n\ln(2-q) + n\ln\beta + \sum_{i=1}^{n}\ln(x_i) + \frac{1}{1-q}\sum_{i=1}^{n}\ln\left(1 - (1-q)\frac{\beta x_i^2}{2}\right) \tag{9}$$

The maximum likelihood estimates of the parameters (q, β) are found by taking a partial derivative of $\ln L$ with respect to q and β, equating the derivatives to zero, and evaluating them at $\widehat{q}$, $\widehat{\beta}$

$$\begin{aligned} \frac{\partial \ln L}{\partial q} &= -\frac{n}{2-q} + \frac{1}{(1-q)^2}\sum_{i=1}^{n}\ln\left(1 - (1-q)\frac{\beta x_i^2}{2}\right) + \frac{1}{1-q}\sum_{i=1}^{n}\frac{\beta x_i^2}{2 - (1-q)\beta x_i^2} \\ \frac{\partial \ln L}{\partial \beta} &= \frac{n}{\beta} - \sum_{i=1}^{n}\frac{x_i^2}{2 - (1-q)\beta x_i^2} \end{aligned}$$

In the range where $1 < q < 2$, the log-likelihood corresponding to the pdf in equation (6) takes the form

$$\ln L = n\ln(2-q) + n\ln\beta + \sum_{i=1}^{n}\ln(x_i) - \frac{1}{q-1}\sum_{i=1}^{n}\ln\left(1 + (q-1)\frac{\beta x_i^2}{2}\right) \tag{10}$$

Upon differentiating the log-likelihood function in terms of the parameters q and β, one obtains the following expressions:

$$\begin{aligned} \frac{\partial \ln L}{\partial q} &= -\frac{n}{2-q} + \frac{1}{(q-1)^2}\sum_{i=1}^{n}\ln\left(1 + (q-1)\frac{\beta x_i^2}{2}\right) - \frac{1}{q-1}\sum_{i=1}^{n}\frac{\beta x_i^2}{2 + (q-1)\beta x_i^2} \\ \frac{\partial \ln L}{\partial \beta} &= \frac{n}{\beta} - \sum_{i=1}^{n}\frac{x_i^2}{2 + (q-1)\beta x_i^2} \end{aligned}$$

The partial derivatives of the log-likelihood function with respect to q and β are non-linear in both cases ($q < 1$ and $1 < q < 2$). This non-linearity poses a challenge for directly finding closed-form solutions for the MLEs of q and β. While closed-form solutions involve expressing the estimates as explicit mathematical expressions in terms of the data, numerical optimization methods often involve iterative algorithms to find approximate solutions.

$$\begin{aligned} \max \quad & \ln L \\ s.t. \quad & q < 2, \\ & \beta > 0, \end{aligned} \tag{11}$$

Despite theoretical challenges in rigorously proving the uniqueness of the solution to optimization problem (11), empirical evidence suggests a strong case for its singularity. Employing a specific optimization algorithm across a wide range of initial parameter values consistently yielded convergence to the same solution, demonstrating remarkable robustness and providing compelling support for uniqueness in practical applications. While a formal proof remains elusive, this robust empirical evidence bolsters the validity of the solution for practical applications within this domain.

5. Application to real life data

In this section, we have employed various sets of real-life failure time data to demonstrate the appropriateness of the q-Rayleigh distribution. Additionally, we have conducted a comparative analysis with the conventional Rayleigh distribution, highlighting the advantages and nuances of our proposed model. This exploration not only showcases the versatility of the q-Rayleigh distribution but also provides valuable insights into its performance in comparison to the widely accepted standard Rayleigh distribution.

To assess the flexibility of the proposed distribution, we utilized several model selection criteria, such as -log-likelihood (-LL), Kolmogorov–Smirnov (KS) statistics, and associated p-values. The analyses were carried out using Matlab software. It is important to note that a superior distribution is identified by smaller values of -LL and KS statistics. Additionally, a more favourable distribution, particularly in terms of p-values, is characterized by a significance level that aligns with the chosen threshold (<0.005), further contributing to the comprehensive evaluation of the proposed distribution's fit to the data.

Dataset 1: In medical research, the assessment and comparison of treatment regimens are commonplace. A deeper comprehension of cancer genetics has broadened the spectrum of treatment options for various cancers falling under the umbrella of head and neck cancers, including those affecting the oral cavity, throat, larynx, para-nasal sinuses, and salivary glands. The three primary types of cancer treatments encompass primary, adjuvant, and palliative approaches. Within these categories, diverse treatment regimens such as surgery, radiation, chemotherapy, hormone therapy, immune therapy, and targeted drug therapy are employed.

Efron [5] conducted a randomized clinical trial comparing two treatment arms for head and neck cancer patients: radiation therapy alone (Arm A) and radiation plus chemotherapy (Arm B). The study recorded survival times (in days) for 51 patients in Arm A and 44 patients in Arm B. In this investigation, we specifically focus on the data from Arm B, examining the appropriateness of fitting the data to the q-Rayleigh distribution. The results are subsequently juxtaposed with those obtained using the standard Rayleigh distribution for a comprehensive evaluation.

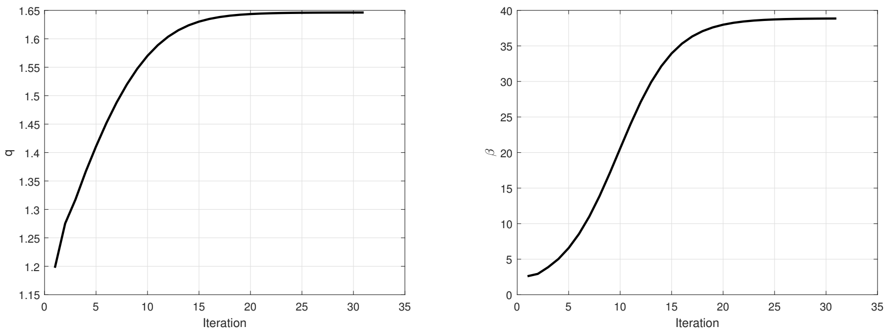
37	84	92	94	112	119	127	130	133	140	146
155	159	169	173	179	194	195	209	249	281	319
339	432	469	519	528	547	613	633	725	759	817
1092	1245	1331	1557	1771	1776	1897	2023	2146	2297	

Table 1: *Database of Arm B (Sample size 44).*

Model	Estimated Parameters		Model Selection		
	$\widehat{q}$	$\widehat{\beta}$	-LL	KS	p-value
q-Rayleigh	1.6462	38.8526	37.1631	0.10769	0.033834
Rayleigh	-	1.9505	69.9683	0.50856	1.8308×10^{-6}

Table 2: *Estimates of fitted distribution for Arm B data.*

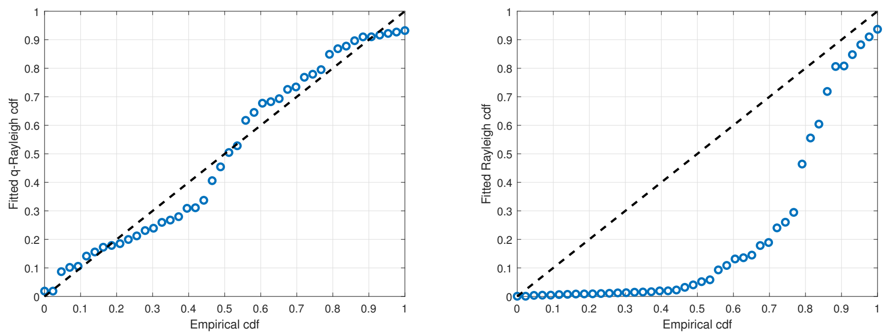
Table 2 outlines estimates for fitted distributions of Arm B data, comparing the q-Rayleigh and Rayleigh models. The negative log-likelihood values, are substantially lower for the q-Rayleigh model than for the Rayleigh model, suggesting superior fit for the former. Additionally, the KS statistic is smaller for the q-Rayleigh model compared to the Rayleigh model, reinforcing the notion that the former provides a more accurate representation of the data. The associated p-value for the KS statistic is also notably smaller for the q-Rayleigh model, underscoring its statistical significance in capturing the observed data distribution.



((a)) **((b))**

Figure 3: *Convergence of Newton-Raphson Method for Parameters Estimations q (a) and β (b) for Arm B data.*

Figure 3 illustrates the convergence of the Newton-Raphson method for parameter estimations of q and β. The convergence is achieved within 31 iterations.



((a)) **((b))**

Figure 4: *PP plot for fitted q-Rayleigh (a) and Rayleigh (b) for Arm B data.*

Figure 4 represents the Probability-Probability (PP) plot for fitted q-Rayleigh (a) and Rayleigh distribution. the PP plot for the q-Rayleigh model provides a more accurate representation of the data which implies that the former is considered better than that of the Rayleigh model. A visually superior alignment of points along the line in the PP plot for the q-Rayleigh model compared to the Rayleigh model indicates that the former better captures the distributional characteristics of the data, reinforcing the notion that the q-Rayleigh model is a more suitable fit for the observed dataset.

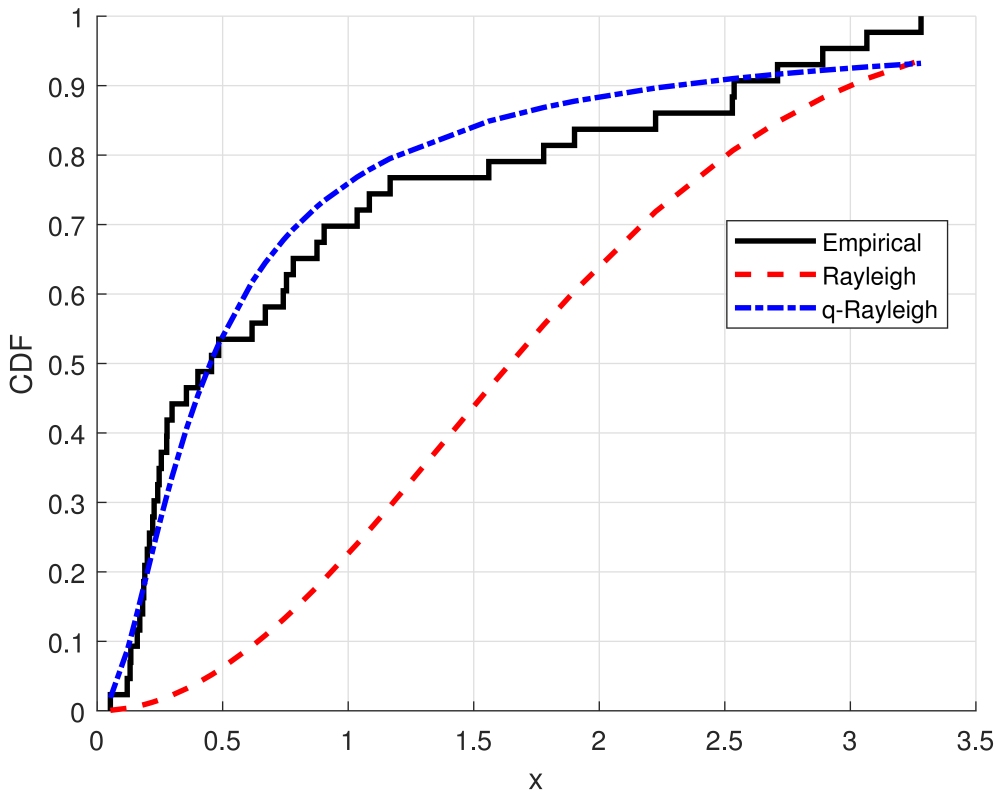


Figure 5: *Empirical, Rayleigh, and q-Rayleigh cdf's for Arm B data.*

In Figure 5, the cdf's of the Empirical, Rayleigh, and q-Rayleigh models are presented. A superior fit for the q-Rayleigh model is suggested when examining these cdf's, signifying its enhanced capability to accurately represent the observed data in comparison to the conventional Rayleigh model. This might be evidenced by a closer alignment of the q-Rayleigh cdf to the empirical cdf, suggesting that the additional parameter q improves the model's ability to capture the nuances in the data distribution.

Dataset 2: Authentic data pertaining to COVID-19 mortality rates in Italy is utilized to assess the goodness of fit of the q-Rayleigh distribution. The dataset spans a period of 59 days, commencing from February 27 to April 27, 2020, capturing the temporal evolution of mortality rates during this critical period. The detailed information, including date-specific mortality rates, is organized and presented in Table 4, forming the basis for conducting a rigorous statistical analysis to evaluate the appropriateness of the q-Rayleigh distribution in modelling the observed COVID-19 mortality trends in Italy.

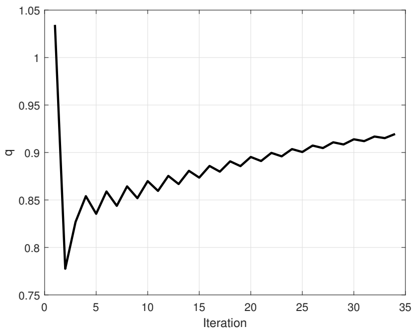
4.571	7.201	3.606	8.479	11.410	8.961	10.919	10.908	6.503	18.474	11.010	17.337
16.561	13.226	15.137	8.697	15.787	13.333	11.822	14.242	11.273	14.330	16.046	11.950
10.282	11.775	10.138	9.037	12.396	10.644	8.646	8.905	8.906	7.407	7.445	7.214
6.194	4.640	5.452	5.073	4.416	4.859	4.408	4.639	3.148	4.040	4.253	4.011
3.564	3.827	3.134	2.780	2.881	3.341	2.686	2.814	2.508	2.450	1.518	

Table 3: *COVID-19 Data in Italy from February 27 to April 27, 2020.*

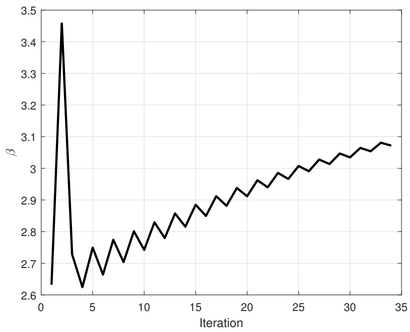
Model	Estimated Parameters		Model Selection		
	$\widehat{q}$	$\widehat{\beta}$	-LL	KS	p-value
q-Rayleigh	0.91949	3.0717	18.7225	0.14996	0.0068288
Rayleigh	-	3.6895	18.7643	0.7544	1.2615×10^{-22}

Table 4: *Estimates of fitted distribution for COVID-19 data.*

Table 4 compares two distribution models applied to COVID-19 data: the q-Rayleigh and Rayleigh distributions. The q-Rayleigh model exhibits a lower negative log-likelihood value and a smaller KS statistic compared to the Rayleigh model. Additionally, the q-Rayleigh model has a notably lower p-value, indicating a better fit to the observed COVID-19 data. These collective indicators of model performance suggest the superiority of the q-Rayleigh distribution in capturing the underlying distribution of the COVID-19 dataset during the specified period.



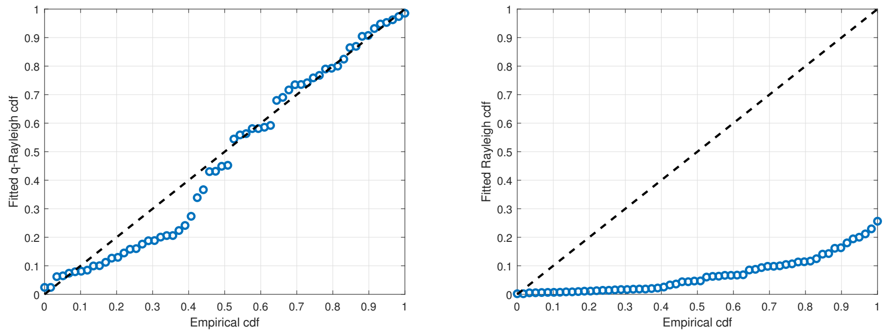
((a))



((b))

Figure 6: *Convergence of Newton-Raphson Method for Parameters Estimations q (a) and β (b) for COVID-19 data.*

Figure 6 demonstrates the convergence of the Newton-Raphson method in estimating the parameters q and β. The convergence is successfully attained after 34 iterations.



((a)) ((b))

Figure 7: *PP plot for fitted q-Rayleigh (a) and Rayleigh (b) for COVID-19 data.*

Figure 7 displays the PP plots illustrating the fitted q-Rayleigh and Rayleigh distributions concerning COVID-19 data. The PP plots provide a compelling visual diagnosis. The q-Rayleigh's points align closely with the diagonal, indicating a superior fit and capturing the nuances of the observed distribution. Conversely, the Rayleigh's deviations highlight potential inaccuracies in its representation. This comparative analysis, therefore, underscores the q-Rayleigh's superior efficacy in describing the intricacies of COVID-19 data.

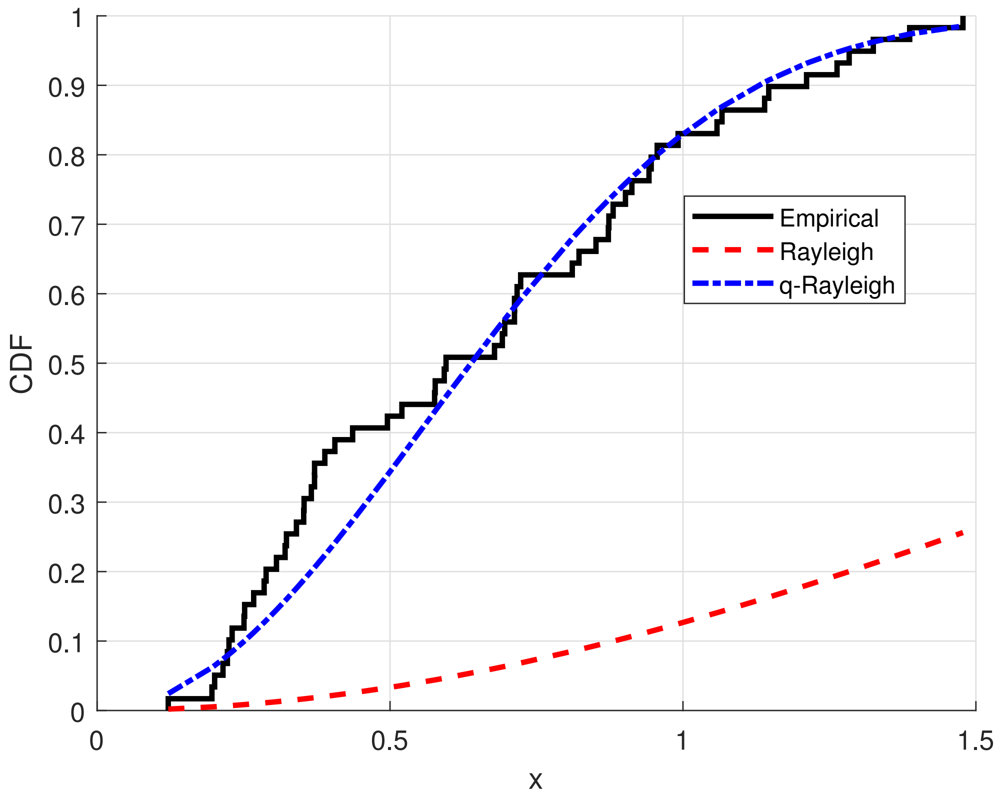


Figure 8: *Empirical, Rayleigh, and q-Rayleigh cdf's for COVID-19 data.*

Figure 8 offers a compelling insight into the process of selecting models for COVID-19 data analysis. The empirical cdf serves as the reference, with the q-Rayleigh model demonstrating a remarkable level of fidelity. Its curve closely follows the trajectory of the observed data, contrasting with the Rayleigh model's comparatively less precise fit. Consequently, the q-Rayleigh model emerges as the preferred choice, providing a more accurate and insightful representation of the pandemic's patterns.

6. Conclusion

In this research paper, we have introduced a novel category of two-parameter distributions termed as the "q-Rayleigh distribution". This distribution is formulated by utilizing the Rayleigh distribution as the foundational distribution and incorporating the q-exponential function as the generator function. To evaluate the characteristics of the model, we derived survival, hazard, and cumulative hazard functions for the q-Rayleigh distribution, analysing them graphically. Additionally, we explored extreme value properties.

The graphical examination of the q-Rayleigh distribution, employing various functions with diverse parameter values, demonstrated that the proposed distribution exhibits favourable properties in terms of its density function. We applied mathematical and statistical properties to assess the q-Rayleigh distribution, confirming its adherence to the aforementioned characteristics. The parameters of the q-Rayleigh distribution were estimated through the maximum likelihood estimation method.

To validate the goodness of fit, we employed the KS test, p-value and PP plot. Additionally, we conducted a comparison by examining the empirical cdf against those of the q-Rayleigh and Rayleigh distributions. Furthermore, we applied the q-Rayleigh distribution to cancer mortalities and COVID-19 data. The proposed distribution outperformed other distributions based on model selection criteria. In light of these findings, the q-Rayleigh distribution emerges as more adaptable and flexible in fitting real-life failure time data. We anticipate that this proposed distribution will find broader applications across diverse research domains, including reliability analysis, medical engineering, economics, and beyond.

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